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THE INDECOMPOSABLE PREPROJECTIVE AND PREINJECTIVE REPRESENTATIONS OF THE QUIVER $\widetilde{\mathbb{D}}_n$

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Abstract. Consider the quiver $\widetilde{\mathbb{D}}_n$ and its finite dimensional representations over the field k. We know due to Ringel in [7] that indecomposable representations without self extensions (called exceptional representations) can be exhibited using matrices involving as coefficients only 0 and 1, such that the number of nonzero coefficients is precisely $d-1$, where d is the global dimension of the representation. This means that the corresponding "coefficient quiver" is a tree, so we will call such a presentation a "tree presentation". In this paper we describe explicit tree presentations for the indecomposable preprojective and preinjective representations of the quiver $\widetilde{\mathbb{D}}_n$. In this way we generalize results obtained by Mróz in [5] for the quiver $\widetilde{\mathbb{D}}_4$ and by Lőrinczi and Szántó in [8] for the quiver $\widetilde{\mathbb{D}}_5$.

1. PRELIMINARIES

Let $Q = (Q_0, Q_1)$ be a tame quiver without oriented cycles (i.e. of type $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$). Suppose that the vertex set Q_0 has n elements and for an arrow $\alpha \in Q_1$ we denote by $t(\alpha), h(\alpha) \in Q_0$ the tail and head of α. The Euler form of Q is a bilinear form on $\mathbb{Z}Q_0 \cong \mathbb{Z}^n$ given by $\langle x, y\rangle = \sum_{i\in Q_0} x_i y_i - \sum_{\alpha\in Q_1} x_{t(\alpha)} y_{h(\alpha)}$. Its quadratic form q_Q (called Tits form) is independent from the orientation of Q and in the tame case it is positive semidefinite with radical $\mathbb{Z}\delta$, where δ is a minimal positive imaginary root of the corresponding Kac-Moody root system. The defect of $x \in \mathbb{Z}Q_0$ is then $\partial x = \langle \delta, x\rangle$.

Let k be a field and consider the path algebra kQ. The category mod-kQ of finite dimensional right modules over kQ will be identified with the category rep-kQ of the finite dimensional k-representations of the quiver Q. Recall that a k-representation of Q is defined as a set of finite dimensional k-spaces $\{M_i | i \in Q_0\}$ corresponding to the vertices together with k-linear maps $M_\alpha : M_{t(\alpha)} \to M_{h(\alpha)}$ corresponding to the arrows. Given two representations $M = (M_i, M_\alpha)$ and $N = (N_i, N_\alpha)$ of the quiver Q a morphism $f : M \to N$ between them consists of a family of k-linear maps (corresponding to the vertices) $f_i : M_i \to N_i$, such that $N_\alpha f_{t(\alpha)} = f_{h(\alpha)} M_\alpha$ for all $\alpha \in Q_1$.

The dimension vector of a representation $M = (M_i, M_\alpha)$ is

$$\underline{\dim} M = (d_i)_{i\in Q_0} \in \mathbb{Z}^n, \text{ where } d_i = \dim_k M_i.$$

The global dimension of M is $d = \sum_{i\in Q_0} d_i$. We will denote by $\partial M = \partial(\underline{\dim} M)$ the defect of M.

Following Ringel in [7] a basis $B = (B_i)$ of a representation $M = (M_i, M_\alpha)$ consists of a fixed basis B_i for each space M_i, where $i \in Q_0$. Let us assume that such a basis B of M is given. For any arrow α, we may replace the linear application M_α by the corresponding $d_{h(\alpha)} \times d_{t(\alpha)}$ matrix $M_{\alpha,B}$. Given $b \in B_{t(\alpha)}$ and $b' \in B_{h(\alpha)}$ we denote by $M_{\alpha,B}(b, b')$ the corresponding matrix coefficient, so $M_\alpha(b) = \sum_{b' \in B_{h(\alpha)}} \in M_{\alpha,B}(b, b')b'$. By definition, the coefficient quiver of M with respect to B has the set of vertices the disjoint union of all the bases B_i, and there is an arrow (α, b, b') if $M_{\alpha,B}(b, b') \neq 0$. We will call an indecomposable representation M of Q a tree module provided there exists a basis B of M such that the corresponding coefficient quiver is a tree. Note that for a tree module M of global dimension d, there is a basis B of M such that precisely $d-1$ matrix coefficients are non-zero, and one may assume that all these coefficients are equal to 1 (see [7] for details). Thus, any tree module can be exhibited by 0-1-matrices such that the number of 1-s is precisely $d-1$. Such a presentation is called a tree presentation.

An indecomposable module M is called exceptional if it has no self extensions (i.e. $\mathrm{Ext}^1(M, M) = 0$).

It is well known that in the tame cases the indecomposable modules in mod-kQ are of three types: preprojectives (having negative defect), preinjectives (having positive defect) and regulars (having zero defect). For all the details we refer to [2],[3],[1],[4]. What is important to notice, that indecomposable preprojectives and preinjectives are exceptional.

Having in mind all the notions above we are now able to formulate the main problem on which this article focuses.

In [7] Ringel proves that any exceptional representation of Q over a field k is a tree module, so it has a tree presentation. However in many cases these presentations are not known explicitly.

The aim of this article is to describe explicitly these existing tree presentations in case of preprojective and preinjective indecomposable representations over tame quivers of type $\widetilde{\mathbb{D}}_n$. In this way we generalize results obtained by Mróz in [5] for the quiver $\widetilde{\mathbb{D}}_4$ and by Lőrinczi and Szántó in [8] for the quiver $\widetilde{\mathbb{D}}_5$.

One can see that the general case $\widetilde{\mathbb{D}}_n$ can be traced back to the cases $\widetilde{\mathbb{D}}_4, \widetilde{\mathbb{D}}_5, \widetilde{\mathbb{D}}_6$.

2. REPRESENTATIONS OF THE QUIVER $\widetilde{\mathbb{D}}_n$ CONSTRUCTED FROM $\widetilde{\mathbb{D}}_6$ REPRESENTATIONS

In this section we will show how to get all the indecomposable preprojective and preinjective $\widetilde{\mathbb{D}}_n$ representations from $\widetilde{\mathbb{D}}_6$.

We may consider special orientations for the quiver $\widetilde{\mathbb{D}}_n$, having a unique sink on the central axis. The representations for the other orientations can be derived from this one, using reflection functors (see [3] pages 15 – 16). So we will look at the following oriented quiver of $\widetilde{\mathbb{D}}_n$ type:

$$\begin{array}{ccccccccccc}
1 & & & & & & & & n & & \\
 & \searrow & & & & & & \swarrow & & & \\
 & & 3 & \longleftarrow & 4 & \longleftarrow \cdots \longleftarrow & n-2 & \longleftarrow & n-1 & & n \geq 6 \\
 & \nearrow & & & & & & \nwarrow & & & \\
2 & & & & & & & & n+1 & &
\end{array}$$

Note that if the dimensions of the vector spaces corresponding to the vertices $3, 4, 5, \ldots, (n-1)$ are identical, then this case is equivalent to the $\widetilde{\mathbb{D}}_4$ case because we can take the identity matrix as morphisms between them. The dimensions corresponding to the other vertices are exactly the same as in the $\widetilde{\mathbb{D}}_4$ case.

Furthermore, the $P(2)$ and $P(n+1)$ representations can be easily obtained from the $P(1)$ and $P(n)$ representations just by permuting two morphisms. This also applies to the preinjective case.

The path algebra of the quiver $\widetilde{\mathbb{D}}_n$ is the following:

$$A = KQ \cong \begin{pmatrix} K & 0 & K & 0 & \cdots & 0 & 0 & 0 \\ 0 & K & K & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K & K & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & K & K & \cdots & K & 0 & 0 \\ 0 & 0 & K & K & \cdots & K & K & 0 \\ 0 & 0 & K & K & \cdots & K & 0 & K \end{pmatrix}$$

In addition we determine the dimension vectors of the indecomposable projective and injective representations, as seen in [1].

$$\mathbf{dim}\, P(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{dim}\, P(2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{dim}\, P(3) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{dim}\, P(4) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{dim}\, P(n) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{dim}\, P(n+1) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

If we write these into a single matrix, we get the so-called Cartan matrix:

$$\mathbf{C}_A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Using this matrix, we can calculate the dimension vectors of the indecomposable preprojective and preinjective representations:

$$\mathbf{\Phi}_A = -\mathbf{C}_A^t \mathbf{C}_A^{-1}$$

$$\mathbf{dim}\, \tau^{-m} P(j) = \mathbf{\Phi}_A^{-m}\, \mathbf{dim}\, P(j)$$

$$\mathbf{dim}\, \tau^{m} I(j) = \mathbf{\Phi}_A^{m}\, \mathbf{dim}\, I(j),$$

where $i \in \{1, \ldots, n+1\}$, $m \in \mathbb{N}$ and τ is the Auslander–Reiten translation. The key observation is the following lemma.

LEMMA 1. *Let $\mathbf{x} = (x_1, x_2 \ldots, x_{n+1})$ be a dimension vector of an indecomposable preprojective or preinjective $\widetilde{\mathbb{D}}_n$ representation. Then $\#\{x_i \neq x_{i+1} | i \in \{3, 4, \ldots, n-1\}\} \leq 2$.*

Proof. Using a well known theorem due to Kac–Moody we know that the roots of the quadratic form of a quiver are precisely the dimension vectors of the indecomposable representations of that quiver.

We can obtain these roots by solving the following equation (see page 267 of [1]):

$$4q_Q(\mathbf{x}) = (2x_1 - x_3)^2 + (2x_2 - x_3)^2 + (x_{n-1} - 2x_n)^2 + (x_{n-1} - 2x_{n+1})^2 + 2\sum_{i=3}^{n-2} (x_i - x_{i+1})^2 = 4$$

Since the components of the dimension vectors are integers, we can conclude that the $\sum_{i=3}^{n-2}(x_i - x_{i+1})^2$ component is equal to either $0, 1$ or 2, which is equivalent to $\#\{x_i \neq x_{i+1} | i \in \{3, 4, \dots, n-1\}\} \leq 2$. $\square$

The previous lemma tells us that the vector spaces on the main axis of an indecomposable preprojective or preinjective $\widetilde{\mathbb{D}}_n$ representation have at most 3 distinct dimensions, such that each dimension is repeated along the axis. This is very useful, since in this way we can get a $\widetilde{\mathbb{D}}_n$ representation from a $\widetilde{\mathbb{D}}_6$ representation by simply putting the identity matrix as morphism between the vector spaces with the same dimension. This representation remains indecomposable (which can be easily verified by substituting the dimension vector into the previous equation) and it is also a tree representation. Moreover, if the number of distinct dimensions of the vector spaces is less than 3, then we can get a $\widetilde{\mathbb{D}}_n$ representation from $\widetilde{\mathbb{D}}_4$ or $\widetilde{\mathbb{D}}_5$.

- 1st case: $\sum_{i=3}^{n-2}(x_i - x_{i+1})^2 = 0$
  It follows that $x_i = x_{i+1}$ for all $i \in \{3, 4, \dots, n-1\}$ i.e.
  $\#\{x_i \neq x_{i+1} | i \in \{3, 4, \dots, n-1\}\} = 0$
  This is equivalent to the $\widetilde{\mathbb{D}}_4$ case, see [5].
- 2nd case: $\sum_{i=3}^{n-2}(x_i - x_{i+1})^2 = 1$
  It follows that there exists a $j \in \mathbb{Z}$ such that $x_j \neq x_{j+1}$ and $x_i = x_{i+1}$ for all $i \in \{3, \dots, j-1, j+1, \dots, n-1\}$ i.e.
  $\#\{x_i \neq x_{i+1} | i \in \{3, 4, \dots, n-1\}\} = 1$
  This is equivalent to the $\widetilde{\mathbb{D}}_5$ case, see [8].
- 3rd case: $\sum_{i=3}^{n-2}(x_i - x_{i+1})^2 = 2$
  It follows that there exist $k, l \in \mathbb{Z}$ such that $k < l$ $x_k \neq x_{k+1}$ and $x_l \neq x_{l+1}$ for all $i \in \{3, \dots, k-1, k+1, \dots, l-1, l+1, \dots, n-1\}$ i.e.
  $\#\{x_i \neq x_{i+1} | i \in \{3, 4, \dots, n-1\}\} = 2$
  This is equivalent to the $\widetilde{\mathbb{D}}_6$ case, which is the subject of this paper.

3. REPRESENTATIONS OF THE QUIVER $\widetilde{\mathbb{D}}_6$

Applying the 3rd case from the previous lemma to $\widetilde{\mathbb{D}}_6$ we get that $(2x_1 - x_3)^2 + (2x_2 - x_3)^2 + (x_5 - 2x_4)^2 + (x_5 - 2x_7)^2 = 0$ which is equivalent to the next system of equations: $\begin{cases} 2x_1 = x_3 \\ 2x_2 = x_3 \\ x_5 = 2x_6 \\ x_5 = 2x_7 \end{cases}$

This means that every dimension vector of an indecomposable $\widetilde{\mathbb{D}}_6$ representation has the form $[x_1, x_1, 2x_1, x_4, 2x_6, x_6, x_6]$.

Now, using the previous lemma we deduce that there are only 4 possible dimension vectors, and these are the following:

$$[x_1, x_1, 2x_1, 2x_1+1, 2x_1+2, x_1+1, x_1+1]$$

$$[x_1, x_1, 2x_1, 2x_1+1, 2x_1, x_1, x_1]$$

$$[x_1, x_1, 2x_1, 2x_1-1, 2x_1, x_1, x_1]$$

$$[x_1, x_1, 2x_1, 2x_1-1, 2x_1-2, x_1-1, x_1-1].$$

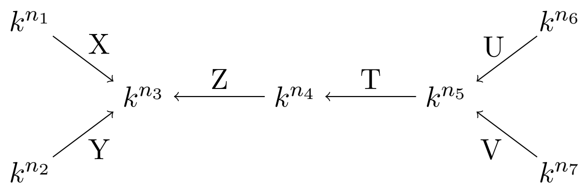
Since there are two possible orientations for each one, using the defect of the quiver, we determine whether they are the dimension vectors of a preprojective, preinjective or a regular representation.

For $n, m \geq 0$ we denote by $0_{n\times m} \in \mathbb{M}_{n\times m}$ and $0_n \in \mathbb{M}_{n\times n}$ the zero matrix and by $I_n \in \mathbb{M}_{n\times n}$ the identity matrix. In addition we use the following notations:

$$^{\circ}\Pi_{n,m} = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{M}_{n\times m}, \quad \Pi^{\circ}_{n,m} = \begin{bmatrix} 0 & \cdots & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \in \mathbb{M}_{n\times m},$$

$$\overline{I}_n = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{M}_{n\times n}, \quad \Sigma_{n,n+1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 \;\; 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \in \mathbb{M}_{n\times(n+1)}$$

The representations of our specially oriented $\widetilde{\mathbb{D}}_6$ quiver have the following form:



Here the matrices X, Y, Z, T, U, V correspond to the linear applications corresponding to the arrows (relative to the canonical bases). Note that the representation is uniquely determined by the matrix list (X, Y, Z, T, U, V) and its dimension vector is $[n_1, n_2, n_3, n_4, n_5, n_6, n_7]$.

Using these notations we now list the indecomposable preprojective $\widetilde{\mathbb{D}}_6$ representations. The first row contains the dimension vector of the representation, while the second one the morphism family of the representation. In every case $n \geq 0$.

$[2n+1, 2n, 4n+1, 4n+1, 4n, 2n, 2n]$

$$\left(\begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} {}^{\circ}\Pi_{2n,4n} \\ 0_{1,4n} \\ \Pi^{\circ}_{2n,4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}\right)$$

$[2n, 2n+1, 4n+1, 4n+1, 4n, 2n, 2n]$

$$\left(\begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} {}^{\circ}\Pi_{2n,4n} \\ 0_{1,4n} \\ \Pi^{\circ}_{2n,4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}\right)$$

$[2n, 2n, 4n+1, 4n+1, 4n+12n+1, 2n]$

$$\left(\begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}\right)$$

$[2n, 2n, 4n+1, 4n+1, 4n+1, 2n, 2n+1]$

$$\left(\begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}\right)$$

$[2n, 2n+1, 4n+1, 4n+1, 4n+1, 2n, 2n]$

$$\left(\begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix}\right)$$

$[2n+1, 2n, 4n+1, 4n+1, 4n+1, 2n, 2n]$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix}\right)$$

$[2n+1, 2n+1, 4n+2, 4n+2, 4n+1, 2n, 2n+1]$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+1} \\ \Pi^{\circ}_{2n+1,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}\right)$$

$[2n+1, 2n+1, 4n+2, 4n+2, 4n+1, 2n+1, 2n]$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+1} \\ \Pi^{\circ}_{2n+1,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}\right)$$

$[2n+1, 2n, 4n+2, 4n+2, 4n+2, 2n+1, 2n+1]$

$$\left(\begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, [I_{4n+1}], [I_{4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$[2n, 2n+1, 4n+2, 4n+2, 4n+2, 2n+1, 2n+1]$,

$$\left(\begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+1}], [I_{4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$[2n+1, 2n+1, 4n+2, 4n+2, 4n+2, 2n+1, 2n]$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], [I_{4n+2}], \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}\right)$$

$[2n+1, 2n+1, 4n+2, 4n+2, 4n+2, 2n, 2n+1]$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], [I_{4n+2}], \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n+1,2n+2,4n+3,4n+3,4n+2,2n+1,2n+1]$$

$$\left(\begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ {}^\circ\Pi_{2n+1,2n+2}\end{bmatrix}, [I_{4n+3}], \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+2}\\ 0_{1,4n+2}\\ \Pi^\circ_{2n+1,4n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{2n+1}\\ I_{2n+1}\end{bmatrix}\right)$$

$$[2n+2,2n+1,4n+3,4n+3,4n+2,2n+1,2n+1]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ {}^\circ\Pi_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, [I_{4n+3}], \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+2}\\ 0_{1,4n+2}\\ \Pi^\circ_{2n+1,4n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{2n+1}\\ I_{2n+1}\end{bmatrix}\right)$$

$$[2n+1,2n+1,4n+3,4n+3,4n+3,2n+1,2n+2]$$

$$\left(\begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}\right)$$

$$[2n+1,2n+1,4n+3,4n+3,4n+3,2n+2,2n+1]$$

$$\left(\begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}\right)$$

$$[2n+2,2n+1,4n+3,4n+3,4n+3,2n+1,2n+1]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}\right)$$

$$[2n+1,2n+2,4n+3,4n+3,4n+3,2n+1,2n+1]$$

$$\left(\begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}\right)$$

$$[2n+2,2n+2,4n+4,4n+4,4n+3,2n+2,2n+1]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, [I_{4n+4}], \begin{bmatrix} {}^\circ\Pi_{2n+2,4n+3}\\ \Pi^\circ_{2n+2,4n+3}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ {}^\circ\Pi_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}\right)$$

$$[2n+2,2n+2,4n+4,4n+4,4n+3,2n+1,2n+2]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, [I_{4n+4}], \begin{bmatrix} {}^\circ\Pi_{2n+2,4n+3}\\ \Pi^\circ_{2n+2,4n+3}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix} \begin{bmatrix} I_{2n+2}\\ {}^\circ\Pi_{2n+1,2n+2}\end{bmatrix}\right)$$

$$[2n+1,2n+2,4n+4,4n+4,4n+4,2n+2,2n+2]$$

$$\left(\begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, [I_{4n+4}], [I_{4n+4}], \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ I_{2n+2}\end{bmatrix}\right)$$

$$[2n+2,2n+1,4n+4,4n+4,4n+4,2n+2,2n+2]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, [I_{4n+4}], [I_{4n+4}], \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ I_{2n+2}\end{bmatrix}\right)$$

$$[2n+2,2n+2,4n+4,4n+4,4n+4,2n+1,2n+2]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, [I_{4n+4}], [I_{4n+4}], \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ I_{2n+2}\end{bmatrix}\right)$$

$$[2n+2,2n+2,4n+4,4n+4,4n+4,2n+2,2n+1]$$

$$\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, [I_{4n+4}], [I_{4n+4}], \begin{bmatrix} I_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}\right)$$

$[2n, 2n, 4n+1, 4n+1, 4n, 2n, 2n]$
$\left(\begin{bmatrix} I_{2n} \\ \overline{I}_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ \overline{I}_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,4n} \\ 0_{1,4n} \\ \Pi^\circ_{2n,4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix} \right)$
$[2n, 2n, 4n+1, 4n+1, 4n+1, 2n, 2n]$
$\left(\begin{bmatrix} I_{2n} \\ 0_{2n+1,2n} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ \overline{I}_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ \overline{I}_{2n} \end{bmatrix} \right)$
$[2n+1, 2n+1, 4n+2, 4n+2, 4n+1, 2n, 2n]$
$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+2} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+1} \\ \Pi^\circ_{2n+1,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ \overline{I}_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ \overline{I}_{2n} \end{bmatrix} \right)$
$[2n+1, 2n+1, 4n+3, 4n+3, 4n+2, 2n+1, 2n+1]$
$\left(\begin{bmatrix} I_{2n+1} \\ \overline{I}_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+2} \\ 0_{1,4n+2} \\ \Pi^\circ_{2n+1,4n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$[2n+1, 2n+1, 4n+3, 4n+3, 4n+3, 2n+1, 2n+1]$
$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+2,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \overline{I}_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ \overline{I}_{2n+1} \end{bmatrix} \right)$
$[2n+2, 2n+2, 4n+4, 4n+4, 4n+3, 2n+1, 2n+1]$
$\left(\begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}, \begin{bmatrix} I_{4n+4} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+2,4n+3} \\ \Pi^\circ_{2n+2,4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \overline{I}_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ \overline{I}_{2n+1} \end{bmatrix} \right)$
$[n+1, n+1, 2n+2, 2n+1, 2n, n, n]$
$\left(\begin{bmatrix} {}^\circ\Pi_{n,n+1} \\ \overline{I}_{n+1} \\ 0_{1,n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,n+1} \\ {}^\circ\Pi_{n,n+1} \\ \overline{I}_{n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{n+1,2n+1} \\ \Pi^\circ_{n+1,2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{n,2n} \\ 0_{1,2n} \\ \Pi^\circ_{n,2n} \end{bmatrix}, \begin{bmatrix} I_{n} \\ 0_{n} \end{bmatrix}, \begin{bmatrix} 0_{n} \\ I_{n} \end{bmatrix} \right)$

Now we list the preinjective tree representations. The first row is the dimension vector, the second one is the morphism family associated to the representation.

$[2n+1, 2n, 4n, 4n, 4n, 2n, 2n]$
$\left(\begin{bmatrix} {}^\circ\Pi_{2n,2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix} \right)$
$[2n, 2n+1, 4n, 4n, 4n, 2n, 2n]$
$\left(\begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix} \right)$
$[2n, 2n, 4n, 4n, 4n, 2n+1, 2n]$
$\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix} \right)$

$$[2n, 2n, 4n, 4n, 4n, 2n, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix}, [I_{4n}], [I_{4n}], \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}\right)$$

$$[2n, 2n+1, 4n+1, 4n+1, 4n+1, 2n+1, 2n+1]$$
$$\left(\begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} I_{2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n, 4n+1, 4n+1, 4n+1, 2n+1, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} I_{2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}\right)$$

$$[2n, 2n, 4n, 4n, 4n+1, 2n, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, [I_{4n}], \begin{bmatrix} {}^\circ\Pi_{2n,4n+1} \\ \Pi^\circ_{2n,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n, 2n, 4n, 4n, 4n+1, 2n+1, 2n]$$
$$\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, [I_{4n}], \begin{bmatrix} {}^\circ\Pi_{2n,4n+1} \\ \Pi^\circ_{2n,4n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}\right)$$

$$[2n+1, 2n, 4n+1, 4n+1, 4n+2, 2n+1, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], [\Sigma_{4n+1,4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n, 2n+1, 4n+1, 4n+1, 4n+2, 2n+1, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], [\Sigma_{4n+1,4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+1, 4n+1, 4n+1, 2n+1, 2n]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} I_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+1, 4n+1, 4n+1, 2n, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^\circ_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n+2, 4n+2, 4n+2, 4n+2, 2n+1, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, [I_{4n+2}], [I_{4n+2}], \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}\right)$$

$$[2n+2, 2n+1, 4n+2, 4n+2, 4n+2, 2n+1, 2n+1]$$
$$\left(\begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], [I_{4n+2}], \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+2, 4n+2, 4n+2, 2n+1, 2n+2]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], [I_{4n+2}], \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+2, 4n+2, 4n+2, 2n+2, 2n+1]$$
$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], [I_{4n+2}], \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n+2, 2n+1, 4n+3, 4n+3, 4n+3, 2n+2, 2n+2]$$
$$\left(\begin{bmatrix} I_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}\right)$$

$$[2n+1, 2n+2, 4n+3, 4n+3, 4n+3, 2n+2, 2n+2]$$

$$\left(\begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ {}^{\circ}\Pi_{2n+1,2n+2} \end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+2, 4n+2, 4n+3, 2n+2, 2n+1]$$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix} \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+3} \\ \Pi^{\circ}_{2n+1,4n+3} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,2n+2} \\ I_{2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+2, 4n+2, 4n+3, 2n+1, 2n+2]$$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix} \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+3} \\ \Pi^{\circ}_{2n+1,4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,2n+2} \\ I_{2n+2} \end{bmatrix}\right)$$

$$[2n+1, 2n+2, 4n+3, 4n+3, 4n+4, 2n+2, 2n+2]$$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}, [I_{4n+3}], [\Sigma_{4n+3,4n+4}], \begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}\right)$$

$$[2n+2, 2n+1, 4n+3, 4n+3, 4n+4, 2n+2, 2n+2]$$

$$\left(\begin{bmatrix} I_{2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+3}], [\Sigma_{4n+3,4n+4}], \begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}\right)$$

$$[2n+2, 2n+2, 4n+3, 4n+3, 4n+3, 2n+1, 2n+2]$$

$$\left(\begin{bmatrix} I_{2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ {}^{\circ}\Pi_{2n+1,2n+2} \end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}\right)$$

$$[2n+2, 2n+2, 4n+3, 4n+3, 4n+3, 2n+2, 2n+1]$$

$$\left(\begin{bmatrix} I_{2n+2} \\ {}^{\circ}\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+1, 4n+1, 4n+1, 2n+1, 2n+1]$$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n,2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+1}], [I_{4n+1}], \begin{bmatrix} \overline{I}_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{2n,2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}\right)$$

$$[2n, 2n, 4n, 4n, 4n+1, 2n+1, 2n+1]$$

$$\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, [I_{4n}], \begin{bmatrix} {}^{\circ}\Pi_{2n,4n+1} \\ \Pi^{\circ}_{2n,4n+1} \end{bmatrix}, \begin{bmatrix} \overline{I}_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{2n,2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+2, 4n+2, 4n+3, 2n+2, 2n+2]$$

$$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+3} \\ \Pi^{\circ}_{2n+1,4n+3} \end{bmatrix}, \begin{bmatrix} \overline{I}_{2n+2} \\ {}^{\circ}\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{2n+1,2n+2} \\ \overline{I}_{2n+2} \end{bmatrix}\right)$$

$$[2n+1, 2n+1, 4n+1, 4n+1, 4n+2, 2n+1, 2n+1]$$

$$\left(\begin{bmatrix} \overline{I}_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{2n,2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}, [I_{4n+1}], [\Sigma_{4n+1,4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$$

$$[2n+2, 2n+2, 4n+3, 4n+3, 4n+4, 2n+2, 2n+2]$$

$$\left(\begin{bmatrix} \overline{I}_{2n+2} \\ {}^{\circ}\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{2n+1,2n+2} \\ \overline{I}_{2n+2} \end{bmatrix}, [I_{4n+3}], [\Sigma_{4n+3,4n+4}], \begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}\right)$$

$$[2n+2, 2n+2, 4n+3, 4n+3, 4n+3, 2n+2, 2n+2]$$

$$\left(\begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n+2} \\ I_{2n+2} \end{bmatrix}, [I_{4n+3}], [I_{4n+3}], \begin{bmatrix} \overline{I}_{2n+2} \\ {}^{\circ}\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{2n+1,2n+2} \\ \overline{I}_{2n+2} \end{bmatrix}\right)$$

$$[n, n, 2n, 2n+1, 2n+2, n+1, n+1]$$

$$\left(\begin{bmatrix} I_{n} \\ 0_{n} \end{bmatrix}, \begin{bmatrix} 0_{n} \\ I_{n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{n,2n+1} \\ \Pi^{\circ}_{n,2n+1} \end{bmatrix}, [\Sigma_{2n+1,2n+2}], \begin{bmatrix} \overline{I}_{n+1} \\ 0_{1,n+1} \\ {}^{\circ}\Pi_{n,n+1} \end{bmatrix}, \begin{bmatrix} \Pi^{\circ}_{n,n+1} \\ 0_{1,n+1} \\ \overline{I}_{n+1} \end{bmatrix}\right)$$

To check the correctness of the representations listed above, we checked the following three things: indecomposability, the defect and the fact that these presentations are indeed tree presentations.

For every preprojective (or preinjective) indecomposable M representation we have that $\dim_k \mathrm{End}(M) = 1$ and conversely $\dim_k \mathrm{End}(M) = 1$ implies indecomposability. Using this fact we checked if the endomorphism ring of the representations is one dimensional.

The defect of a representations can be calculated using the Euler form.

Finally we verified if the presentations are tree presentations. We know due to Ringel in [7] that if M is indecomposable, then its coefficient tree associated to any basis is connected. We checked that the number of 1's in each presentation is the global dimension minus one, which implies that in our connected coefficient quiver the number of edges equals the number of vertices minus one, which implies that the coefficient quiver is indeed a tree.

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