The Optimal Rubbling Number of Ladders, Prisms and Möbius-ladders

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Abstract

A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed. In this new move, one pebble each is removed at vertices v and w adjacent to a vertex u, and an extra pebble is added at vertex u. A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using rubbling moves. The optimal rubbling number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable. We determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möblus-ladders.

1 Introduction

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. One of them is: How can we place the smallest number of pebbles such that every vertex is reachable (*optimal pebbling number*)? For a comprehensive list of references for the extensive literature see the survey papers [6, 7, 8].

Graph rubbling is an extension of graph pebbling. In this version, we also allow a move that removes a pebble each from the vertices v and w that are adjacent to a vertex u, and adds a pebble at vertex u. The basic theory of rubbling and optimal rubbling is developed in [2]. The rubbling number of complete m-ary trees are studied in [5], while the rubbling number of caterpillars are determined in [13]. In [11] the authors gives upper and lower bounds for the rubbling number of diameter 2 graphs.

In the present paper we determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möblus-ladders.

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2 Definitions

Throughout the paper, let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \to \mathbb{Z}$ where $p(v)$ is the number of pebbles placed at v. A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v \in V(G)} p(v)$. We say that a vertex v is *occupied* if $p(v) > 1$, else it is *unoccupied*.

Consider a pebble function p on the graph G. If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \to u)$ removes two pebbles at vertex v, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 2$ and $p'(u) = p(u) + 1$. If $\{w, u\} \in E(G)$ and $v \neq w$, then the *strict rubbling move* $(v, w \to u)$ removes one pebble each at vertices v and w, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 1$, $p'(w) = p(w) - 1$ and $p'(u) = p(u) + 1$.

A *rubbling move is* either a pebbling move or a strict rubbling move. A *rubbling sequence* is a finite sequence $T = (t_1, \ldots, t_k)$ of rubbling moves. The pebble function gotten from the pebble function p after applying the moves in T is denoted by p_T. The concatenation of the rubbling sequences $R = (r_1, \ldots, r_k)$ and $S = (s_1, \ldots, s_l)$ is denoted by $RS = (r_1, \ldots, r_k, s_1, \ldots, s_l)$.

A rubbling sequence T is *executable* from the pebble distribution p if $p_{(t_1, \ldots, t_i)}$ is nonnegative for all i. A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbling sequence T such that $p_T(v) \geq 1$. p is a solvable distribution when each vertex is reachable. Correspondingly, v is *k-reachable* under p if there is an executable T, that $p_T(v) \geq k$, and p is *k-solvable* when every vertex is k-reachable. A H subgraph is *k-reachable* if there is an executable rubbling sequence T such that $p_T(H) \geq k$. We say that vertices u and v are *independently reachable* if there is an executable rubbling sequence T such that $p_T(u) = 1$ and $p_T(v) = 1$.

The *optimal rubbling number* $\varrho_{\text{opt}}(G)$ of a graph G is the size of a distribution with the least number of pebbles from which every vertex is reachable.

Let G and H be simple graphs. Then the *Cartesian product* of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $g = g'$ and $(h, h') \in E(H)$ or if $h = h'$ and $(g, g') \in E(G)$. This graph is denoted by $G \Box H$.

P_n and C_n denotes the path and the cycle containing n distinct vertices, respectively. We call $P_n \Box P_2$ a *ladder* and $C_n \Box P_2$ a *prism*. It is clear that the prism can be obtained from the ladder by joining the 4 endvertices by two edges to form two vertex disjoint C_n subgraphs. If the four endvertices are joined by two new edges in a switched way to get a C_{2n} subgraph, then a *Möbius-ladder* is obtained.

We imagine the $P_n \Box P_2$ ladder laid horizontally, so there is an upper P_n path, and a lower P_n path, which are connected by "parallel" edges, called *rungs* of the ladder. Vertices on the upper path will be usually denoted by v_i, while vertices of the lower path by w_i. Also, if A is a rung (a vertical edge of the graph), then $\overline{A}$ denotes the upper, and $\underline{A}$ the lower endvertex of this rung. This arrangement also defines a natural left and right direction on the horizontal paths, and between the rungs.

3 Optimal rubbling number of the ladder

In this section we give a formula for the optimal rubbling number of ladders:

Theorem 3.1 *Let $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$, so $k = \lfloor \frac{n}{3} \rfloor$.*

$$\varrho_{opt}(P_n \Box P_2) = \begin{cases} 1 + 2k & if\ r = 0, \\ 2 + 2k & if\ r = 1, \\ 2 + 2k & if\ r = 2. \end{cases}$$

PROOF: We prove by induction on n. First we give a summary of the proof, then the necessary definitions and proofs of several Lemmas will be given.

Consider a p optimal distribution on $P_n \Box P_2$. Choose an appropriate $R = P_3 \Box P_2$ subgraph which contains maximum number of pebbles, delete the vertices of R and reconnect the remaining two parts to obtain $G^R = P_{n-3} \Box P_2$, called the *reduced graph*, see Fig. 1.

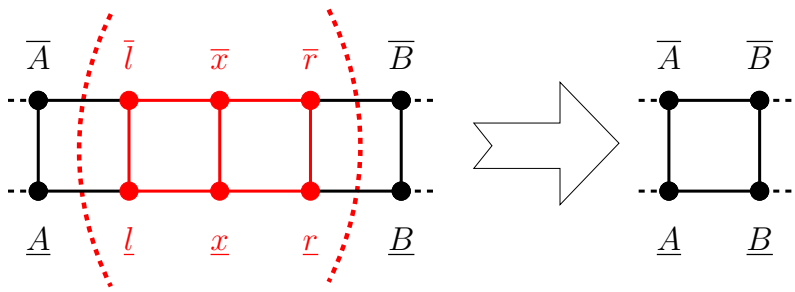

Figure 1: Deleting a $P_3 \Box P_2$ subgraph.

Now construct a solvable p' distribution for the new $P_{n-3} \Box P_2$ graph in the following way: p induces a distribution on the vertices which we have not deleted. In most of the cases we simply place $p(v)$ pebbles to all $v \in V(G) \backslash V(R)$, (i.e. do not change the original distribution), in some other cases we apply a simple operation on the original distribution. Finally, distribute and place $R_p - 2$ pebbles at vertices $\overline{A}$, $\underline{A}$, $\overline{B}$ and $\underline{B}$ in an appropriate way so that the new distribution on $P_{n-3} \Box P_2$ is solvable. Our aim is to show that it is always possible to find such a new distribution. This will be proved in several lemmas. These will imply

$$\varrho_{opt}(P_n \Box P_2) \geq \varrho_{opt}(P_{n-3} \Box P_2) + 2.$$

It is easy to see that this implies the theorem if we show that the theorem holds for $n = 1, 2, 3$.

Lemma 3.2

$$\varrho_{opt}(P_2) = 2$$
$$\varrho_{opt}(P_2 \Box P_2) = 2$$
$$\varrho_{opt}(P_3 \Box P_2) = 3$$

PROOF: The optimal distributions are shown in Fig. 3. It is an easy exercise to check that these distributions are optimal.

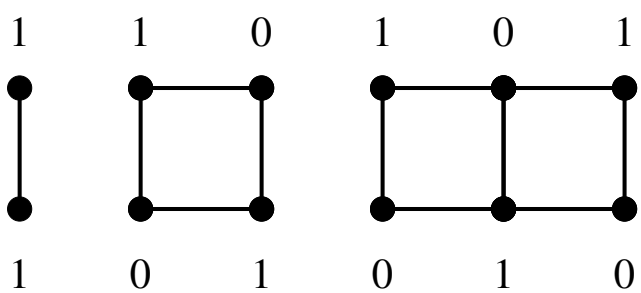

Figure 2: Optimal distributions of P_2, $P_2 \Box P_2$ and $P_3 \Box P_2$.

$\Box$

Lemma 3.3 *Let $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$, so $k = \lfloor \frac{n}{3} \rfloor$.*

$$\varrho_{opt}(P_n \square P_2) \leq \begin{cases} 1 + 2k & if\ r = 0, \\ 2 + 2k & if\ r = 1, \\ 2 + 2k & if\ r = 2. \end{cases}$$

PROOF: A solvable distribution with adequate size is shown in Fig. 3 for each case. $\square$

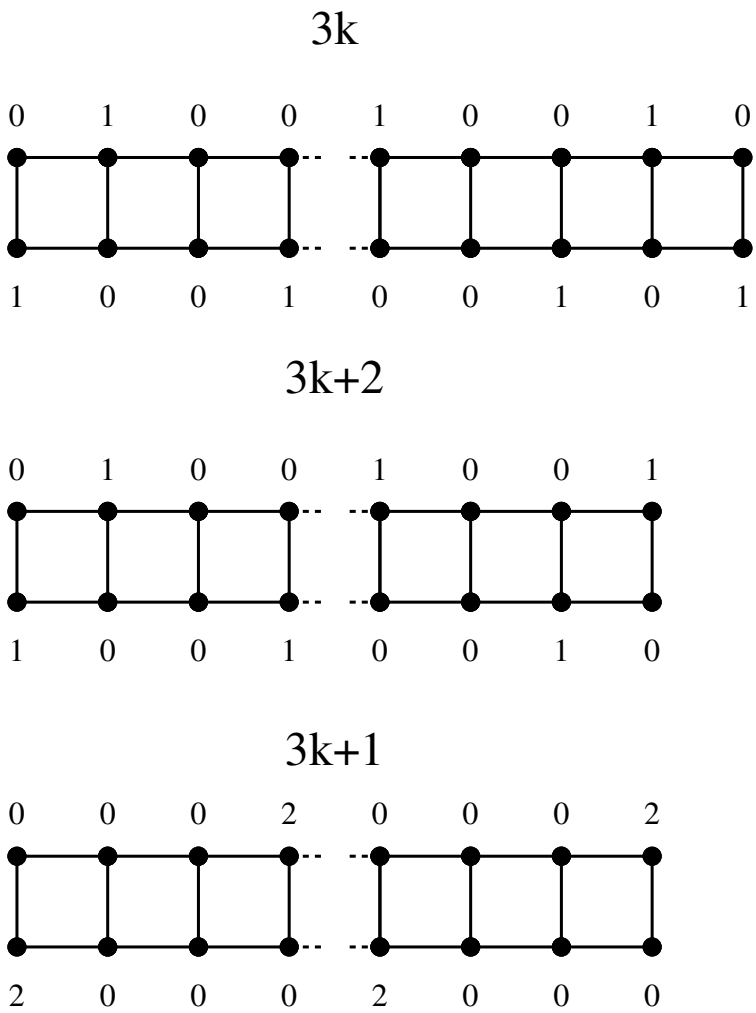

Figure 3: Optimal distributions.

Definition 3.4 *The above mentioned distribution constructed on G^R is denoted by p^R, called* reduced distribution. *It satisfies the following conditions:*

- *$p^R(v) = p(v)$ or $p^R(v) = p^R(w)$, if v is not contained by rung A or B and v and w contained in the same rung,*
- *$p^R(R) = p(R)$ if R is a rung other than A and B.*
- *$p^R(w) \geq p(w)$, if w is contained by rung A or B,*
- *$p^R(\overline{A}) + p^R(\underline{A}) + p^R(\overline{B}) + p^R(\underline{B}) = p(\overline{A}) + p(\underline{A}) + p(\overline{B}) + p(\underline{B}) + p(R) - 2$.*

One of the tools that is used in the proof is the "weight argument". This was introduced by Moews in [12], now it is extended for our situation.

Definition 3.5 *Let $d(x,v)$ denote the distance between vertices x and v, i.e. the length of the shortest path which connects them. The* weight-function *of a vertex x with respect to pebble distribution p is:*

$$w_p(x) = \sum_{v \in V(G)} \left(\frac{1}{2}\right)^{d(x,v)} p(v).$$

The left weight-function*, denoted by $Lw_p(x)$, is similar function, the difference is that the summation is taken only for vertices that do not lie right from x (i.e. for vertices lying left and the other vertex of the rung containing x). The* right weight-function*, denoted by $Rw_p(x)$ can be defined similarly.*

Definition 3.6 *Let p be a distribution on the graph $G = P_n \Box P_2$. Fix a vertex v and delete every vertex located right from it. We get a shorter G' graph which does not contain vertices located right from v. Let p' be a pebble distribution on G' such that $p'(v) = p(v)$ for each vertex of G'. We say that v is left k-reachable in G if it is k-reachable in G' under the distribution p'.* Right k-reachability *is defined similarly.*

Let $L_p(v)$ (and $R_p(v)$) denote the maximum k for which v is left-k-reachable (right-k-reachable).

Lemma 3.7 *$L_p(v) \leq Lw_p(v)$ and $R_p(v) \leq Rw_p(v)$ hold for any vertex v.*

PROOF: It is clear that a rubbling step cannot increase the value of the left (right) weight-function at v. However, if a sequence T of rubbling steps moved k pebbles to v from the left, then

$$k \leq Lw_{p_T}(v) \leq Lw_p(v)$$

holds, proving the first claim. The second claim can be proved similarly. $\Box$

In fact, a stronger statement can be proved.

Lemma 3.8 *$L_p(v) = \lfloor Lw_p(v) \rfloor$ and $R_p(v) = \lfloor Rw_p(v) \rfloor$ hold for any vertex v.*

PROOF: It is enough to show the first claim, the other can be shown similarly. In the following we only consider pebbles that are not right from v.

There are at most two vertices w and w' in the graph whose distance from v is d. w and w' has a common neighbour towards v. Move as many pebbles from w and w' to this neighbour by rubbling moves as possible. Use the same moves for every d in decreasing order, to obtain a distribution p'. Let us call this *greedy rubbling*. As a result, in p', the two vertices at distance $d > 0$ from v contains at most one pebble together. It is easy to see that $Lw_{p'}(v) = Lw_p(v)$. Therefore

$$Lw_p(v) = Lw_{p'}(v) = \sum_{x \in V(G)} \frac{1}{2^{d(x,v)}} p'(x) =$$

$$= p'(v) + \sum_{x \neq v} \frac{1}{2^{d(x,v)}} p'(x) \leq p'(v) + \sum_{d=1}^{k} \frac{1}{2^d} < L_{p'}(v) + 1 = L_p(v) + 1.$$

By Lemma 3.7 and the fact that $L_p(v)$ is an integer, the claim is proved. $\Box$

Definition 3.9 *Let p be a distribution on $P_n \Box K_2$ and let A be a rung. An executable rubbling sequence S is called A-biased if each rubbling move that takes a pebble from A to another rung only use pebbles from the same vertex of A. So when S is A-biased and $(\underline{A}, v \to w) \in S$ where $w \notin V(A)$, then $(\overline{A}, v' \to w') \notin S$ except in the case when $w' = \underline{A}$.*

The reason why we invent this notion is quite simple. Assume that a vertex v located left from A is reachable by an A-biased sequence S under distribution p. Furthermore, assume that all moves of S taking a pebble from A to another rung use only pebbles from $\underline{A}$. Let q be a modification of p such $q(u) = p(u)$ where $u \neq \underline{A}$ and $q(\underline{A}) = R_p(\underline{A})$. We can make a new sequence which acts only on $\underline{A}$ and vertices left from A and still reaches v under q. Finally, if we modify the graph or the distribution right from A, then v remains reachable if $\underline{A}$ remains right $R_p(\underline{A})$-reachable. This makes the rest of the proof substantially easier.

Lemma 3.10

i) When S is an A-biased sequence, T is a sequence which does not contain a move acting on rung A, and ST is executable, then ST is A-biased.

ii) The greedy rubbling sequence is A-biased.

These statements are direct consequences of the definition of A-biased sequences.

Lemma 3.11 *Let p be a solvable distribution of $P_n \square K_2$. Let A be an arbitrary rung in the graph, and let S_s denote an A-biased sequence that reaches vertex s. Such an S_s exists for all but one vertices located left from A. Furthermore, if there is an exception then we have to see the distribution shown in Fig. 4 during the execution of the rubbling sequence reaching s. (We have assumed on the figure that the exception vertex is v_i. This exception vertex can be different for different rungs.)*

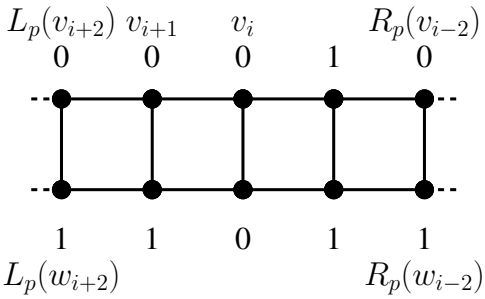

Figure 4: The only possible exception.

PROOF: Assume that for each s which is located right from v_i and located left from rung A, there exist a suitable S_s A-biased sequence. Now we show that either some S_{v_i} exists or v_i is the single exception. Let S be a rubbling sequence such $p_S(v_i) = 1$. If $R_p(v_i) \geq 1$ then the greedy rubbling sequence towards v_i from the right is executable and A-biased. $L_p(v_i) \geq 1$ means that v_i is reachable without any pebble of rung A, hence the statement holds trivially in this case. Thus we have to check cases where $R_p(v_i) = 0$ and $L_p(v_i) = 0$.

Our assumption implies that v_{i-1} is reachable with an A-biased sequence. Let T be a subsequence of S such that T contains only rubbling moves which act on vertices located right from v_i, and T is maximal. $R_p(v_i) = 0$ implies that $R_{p_T}(v_i) = 0$ which means that one of the following cases holds:

- **Case 1.** $p_T(v_{i-1}) = 1$, $p_T(w_{i-1}) = 1$
- **Case 2.** $p_T(v_{i-1}) = 1$, $p_T(w_{i-1}) = 0$
- **Case 3.** $p_T(v_{i-1}) = 0$, $p_T(w_{i-1}) = 1$
- **Case 4.** $p_T(v_{i-1}) = 0$, $p_T(w_{i-1}) = 2$
- **Case 5.** $p_T(v_{i-1}) = 0$, $p_T(w_{i-1}) = 3$

In cases from 3 to 5 we can replace T with a greedy sequence towards w_{i-1}, denoted by Z, its moves also act only on vertices located right from v_i, so $p_T(w_{i-1}) \leq p_Z(w_{i-1})$. In case 2 we replace T with a similar greedy Z that reaches v_{i-1}. v_i is reachable by the executable sequence $Z(S \backslash T)$, hence by Lemma 3.10 this sequence is A-biased. So we completed the proof for cases from 2 to 5.

Now let us prove case 1.

If $L_p(v_{i+1}) \geq 1$ then $(v_{i-1}, v_{i+1} \rightarrow v_i)$ can move a pebble to v_i after we apply T and some moves which act only on vertices not right from v_{i+1}. Thus we do not need a pebble at w_{i-1}, so T can be replaced again by a greedy sequence towards v_{i-1}. Now we show that if $L_p(v_{i+1}) = 0$ we see the distribution during the execution of S shown in Fig. 4.

$R_p(v_i) = 0$ implies $R_p(v_{i-1}) \leq 1$. The reachability of v_i, $L_p(v_{i+1}) = 0$ and $R_p(v_{i-1}) \leq 1$ implies that we can move a pebble to w_{i-1}, and for the same reasons it can be done only by the execution of a $(w_{i+1}, w_{i-1} \rightarrow w_i)$ move. Thus $Lp(w_{i+1}) = 1$. The conditions $R_p(v_i) = 0$, $L_p(v_{i+1}) = 0$, $L_p(w_{i+1}) = 1$, $p(w_i) = 0$ imply that the distribution shown in Fig. 4 have to be seen during the reach of v_i.

Finally we prove that at most one exception may exist. Assume that v_i is an exception. It is easy to see that if v_i is an exception then w_i can not be. Also, no vertex located right from v_i can be exception. We can reach v_{i+1} with the $\{(w_i, w_{i+2} \rightarrow w_{i+1}), (w_{i+1}, w_{i+1} \rightarrow v_{i+1})\}$ sequence after we reach w_i with an A-biased sequence. Any other vertex located left from v_i can not use a pebble at v_i or w_i, hence we do not need to use any pebbles of rung A to reach them. $\square$

Lemma 3.12 *Let p be a solvable distribution of $P_n \square K_2$, and let A be an arbitrary rung in the graph. Then exists a solvable distribution q satisfying the following conditions:*

1. $|q| = |p|$
2. $q(v) = p(v)$ *for all vertices v not located left from A.*
3. *There exist a sequence S_s for all vertices s located left from A which is A-biased and reaches s under q.*
4. *If T is an executable sequence under p then there is an executable sequence T' under q such $p_T(\overline{A}) = q_{T'}(\overline{A})$ and $p_T(\underline{A}) = q_{T'}(\underline{A})$.*

PROOF: If we do not get an exception while applying Lemma 3.11 then $q \equiv p$ trivially satisfies all conditions, so we are done. Otherwise, we have to change p. Assume that v_i is the exceptional vertex. Let q be the following distribution: $q(s) = p(s)$ for all vertices not located left from v_i and $q(w_j) = p(v_j)$, $q(v_j) = p(w_j)$ when $j > i$. (In other words, we just reflect the vertices located left from v_i on a horizontal axis.) Conditions 1 and 2 trivially hold again, as well as condition 3 for all vertices except $v_i, v_{i+1}, w_i, w_{i+1}$. $Lq(v_{i+1}) \geq 1$ so v_i is not an exception under q. $R_p(w_i) \geq 1$ and nothing has changed not left from w_i, so $R_q(w_i) \geq 1$ so condition 3 holds for these vertices, too.

The last condition is trivial if $p \equiv q$, otherwise none of the pebbles placed on the reflected vertices can be moved to rung A, because of the definition of the exception ($L_p(v_i) = L_p(w_i) = 0$). So let T' be the part of T which acts only on vertices located right from v_i. The fact that p and q are the same on these vertices implies that T' is executable. Thus the vertices of rung A and the other vertices located right from A can be reached from q. $\square$

Naturally, the "right-sided" version of the above "left-sided" lemma can be proved similarly. Now, since the distribution in the left-sided version is not changed on the right side, and in the right-sided version it is not changed in the left side, we can apply both versions simultaneously.

Corollary 3.13 *Fix rungs A and B such B is right from A. Then there is a q which fulfills Lemma 3.12 for A on the left side and for B on the right side.*

Corollary 3.14 *Let p be a solvable pebble distribution on the graph $G = P_n \Box K_2$. If a distribution p^R in the graph G^R satisfies*

- $R_p(\overline{A}) \leq R_{p^R}(\overline{A})$,
- $R_p(\underline{A}) \leq R_{p^R}(\underline{A})$,
- $L_p(\overline{B}) \leq L_{p^R}(\overline{B})$ *and*
- $L_p(\underline{B}) \leq L_{p^R}(\underline{B})$

then all vertices located left from rung A and located right from B are reachable from p^R.

PROOF: By Corollary 3.13 we can replace p with q which has same size, and all vertices located left from A (right from B) are reachable with an A-biased (B-biased) sequence. It is easy to see that $R_p(\overline{A}) = R_q(\overline{A})$, $R_p(\overline{A}) = R_q(\overline{A})$, $L_p(\overline{B}) = L_q(\overline{B})$ and $L_p(\underline{B}) = L_q(\underline{B})$ hold.

Let p^R be a distribution such the conditions hold and $p^R(v) = q(v)$ for all v located left from A, or right from B. Fix a v left from A. It is enough to show the statement just for such a v, the other case when v is right from B is similar.

An A-biased sequence S reaches v under q in G. Assume that S does not contain a $(\underline{A}, *\rightarrow w)$ move where $w \neq \overline{A}$. Let T be a subsequence of S such T contains only moves that act on vertices located left from rung A and on $\overline{A}$. T uses only $R_p(\overline{A})$ pebbles at $\overline{A}$ and reaches v under $q_{S\setminus T}$. $S\setminus T$ moves at most $R_q(\overline{A}) = R_p(\overline{A})$ pebbles to $\overline{A}$. Let Z be an executable sequence under p^R, such that Z does not act on any vertex left from A and moves $R_{p^R}(\overline{A})$ pebbles on $\overline{A}$. ZT is executable under p^R and reaches v.

The proof is the same in all other cases. $\Box$

The combination of Lemma 3.8 and Corollary 3 shows that it is enough to find a distribution p^R on G^R that satisfies the following inequalities:

$$\lfloor Rw_{p^R}(\overline{A}) \rfloor - \lfloor Rw_p(\overline{A}) \rfloor \geq 0$$

$$\lfloor Rw_{p^R}(\underline{A}) \rfloor - \lfloor Rw_p(\underline{A}) \rfloor \geq 0$$

$$\lfloor Lw_{p^R}(\overline{B}) \rfloor - \lfloor Lw_p(\overline{B}) \rfloor \geq 0$$

$$\lfloor Lw_{p^R}(\underline{B}) \rfloor - \lfloor Lw_p(\underline{B}) \rfloor \geq 0$$

For the sake of simplicity we call the set of these inequalities *original*. The original inequalities contain floor functions. Calculating without floor functions is much easier, hence we prefer to calculate with the following *modified* inequalities:

$$Rw_{p^R}(\overline{A}) - Rw_p(\overline{A}) \geq 0$$

$$Rw_{p^R}(\underline{A}) - Rw_p(\underline{A}) \geq 0$$

$$Lw_{p^R}(\overline{B}) - Lw_p(\overline{B}) \geq 0$$

$$Lw_{p^R}(\underline{B}) - Lw_p(\underline{B}) \geq 0$$

It is clear that if the modified inequalities hold then the original inequalities hold as well. On the other hand, the following lemma shows that the modified inequalities with a weak additional property imply that p^R is solvable.

Lemma 3.15 *If $p(R) \geq 4$ and the modified inequalities are satisfied then p^R is solvable.*

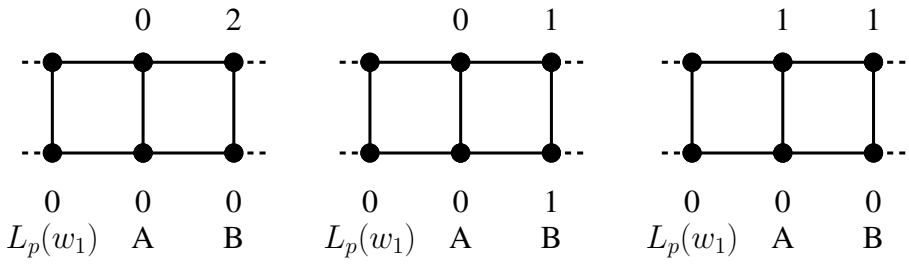

Figure 5: Three cases.

PROOF: By the above results we only need to check that $\overline{A}$, $\underline{A}$, $\overline{B}$, $\underline{B}$ are all reachable. By symmetry it is enough to do it for $\underline{A}$.

Assume that $p^R(\underline{A}) = 0$ and $p^R(\overline{A}) < 2$, otherwise $\underline{A}$ is trivially reachable under p^R. After the reduction, at least two pebbles will be placed somehow on the subgraph induced by rung A and B. Now it is easy to verify that if $L_p(w_1) \geq 1$ then $\underline{A}$ is reachable under p^R.

If $L_p(w_1) = 0$ then there are three remaining ways to distribute the two pebbles on A and B, these are shown on Fig. 5.

$R_p(A) = 0$, $L_p(w_1) = 0$ and the fact that $\underline{A}$ is reachable under p implies that $L_p(v_1) = 1$. The reduction leaves v_1 reachable from the left. It is easy to see that $\underline{A}$ is reachable with the help of this pebble in the second and the third case. In the first case, first we show that $R_p(\bar{l}) \geq 3$ which will imply that $R_{p^R}(\overline{B}) \geq 3$ and $\underline{A}$ is reachable again.

$R_p(\overline{A}) = 1$, otherwise w_1 can not be reachable under p. This implies that $R_p(\bar{l}) \geq 2$. The reachability of $\underline{A}$ under p requires that $R_p(l) \geq 3$, but the fact that $R_p(\underline{A}) = 0$ and $R_p(l) \geq 3$ excludes that $R_p(\underline{l}) > 0$, thus $R_p(\bar{l}) \geq 3$.

Assume that $R_{p^R}(\overline{B}) = 2$ and use the condition of $\overline{A}$ and Lemma 3.8. This results in the following contradiction:

$$\frac{3}{2} \leq Rw_p(\overline{A}) \leq Rw_{p^R}(\overline{A}) = \frac{Rw_{p^R}(\overline{B})}{2} < \frac{R_{p^R}(\overline{B}) + 1}{2} = \frac{3}{2}$$

□

Corollary 3.16 *p^R is solvable if one of the following statements holds:*

1. *The modified inequalities hold and $p(R) \geq 4$.*
2. *The original inequalities hold and the vertices of rung A and B are reachable from p^R.*

The elements of the graph family $P_n \square K_2$ have got several symmetries. Hence we can assume without loss of generality that $p(\bar{l}) + p(\underline{l}) \geq p(\overline{r}) + p(\underline{r})$ and $p(\bar{l}) + p(\overline{x}) \geq p(\underline{l}) + p(\underline{x})$.

3.1 The difference between the old and the new reachability

In this section we prove that we can find a reduction method for all graphs and for each pebbling distribution if the graph contains a $P_3 \square K_2$ subgraph which has the maximum number of pebbles, contains at least four pebbles. First we show an example how can we prove the solvability of a reduced distribution by calculation.

Let p be a solvable pebbling distribution which fulfills that $p(\bar{l}) + p(\overline{x}) \geq 4$. A proper reduction method in this case is the following: Take the pebbles from vertices $\bar{l}$ and $\overline{x}$, throw away two of these pebbles and place the remaining ones to $\overline{A}$. Place the other pebbles of R at vertices of rung A and B as shown in Fig. 6.

p^R is not uniquely defined, but we can show that p^R will be solvable in any way. The proof of this made by some calculations:

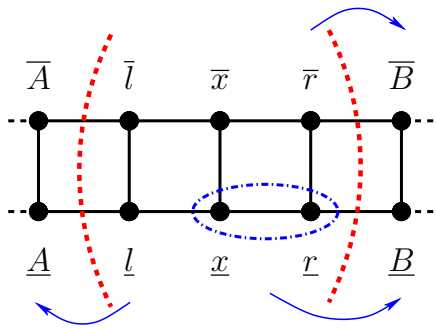

Figure 6: An example for a proper reduction method

$$
\begin{aligned}
Rw_{p^R}(\overline{A}) - Rw_p(\overline{A}) = {} & (p(\overline{A}) + p(\bar{l}) + p(\overline{x}) - 2 + \frac{1}{2}(p(\underline{A}) + p(\overline{r}) + p(\underline{l})) + \frac{1}{4}(p(\underline{x}) + p(\underline{r}))+ \\
& \underbrace{+\frac{1}{2}p(\overline{B}) + \frac{1}{4}p(\underline{B}) + \ldots)}_{\Delta_1}- \\
& - (p(\overline{A}) + \frac{1}{2}(p(\underline{B}) + p(\bar{l})) + \frac{1}{4}(p(\underline{l}) + p(\overline{x})) + \frac{1}{8}(p(\underline{x}) + p(\overline{r})) + \frac{1}{16}p(\underline{r})+ \\
& \underbrace{+\frac{1}{16}p(\overline{B}) + \frac{1}{32}p(\underline{B}) + \ldots)}_{\Delta_2} = \\
& = \underbrace{\frac{1}{2}p(\bar{l}) + \frac{3}{4}p(\overline{x}) - 2}_{\geq 0} + \underbrace{\frac{3}{8}p(\overline{r}) + \frac{1}{4}p(\underline{l}) + \frac{1}{8}p(\underline{x}) + \frac{3}{16}p(\underline{r})}_{\text{nonnegative}} + \underbrace{\Delta_1 - \Delta_2}_{\Delta} \geq 0
\end{aligned}
$$

We assumed that $p(\bar{l}) + p(\overline{x}) \geq 4$, which implies $\frac{1}{2}p(\bar{l}) + \frac{3}{4}p(\overline{x}) - 2 \geq \frac{1}{2}p(\bar{l}) + \frac{1}{2}p(\overline{x}) - 2 \geq 0$. $\Delta \geq 0$ because each vertex and it's pebbles right from B come closer to $\overline{A}$, even if it's rung have been reflected.

Similar calculations are need for $\overline{B}$, $\underline{A}$ and $\underline{B}$. The details are left to the reader, we only give here the crucial parts:

$$
Lw_{p^R}(\overline{B}) - Lw_p(\overline{B}) = \underbrace{\frac{3}{8}p(\bar{l}) + \frac{1}{4}p(\overline{x})}_{\geq 1} - 1 + \text{nonnegative} + \Delta \geq 0,
$$

$$
Rw_{p^R}(\underline{A}) - Rw_p(\underline{A}) = \underbrace{\frac{1}{4}p(\bar{l}) + \frac{3}{8}p(\overline{x})}_{\geq 1} - 1 + \text{nonnegative} + \Delta \geq 0,
$$

$$
Lw_{p^R}(\underline{B}) - Lw_p(\underline{B}) = \underbrace{\frac{3}{16}p(\bar{l}) + \frac{1}{8}p(\overline{x})}_{\geq \frac{1}{2}} - \frac{1}{2} + \text{nonnegative} + \Delta \geq 0.
$$

This implies that this is a proper reduction method.

There are 20 essentially different ways to place at least 4 pebbles to R. In most of these cases one can find a proper reduction method and a similar argument to verify it. For completeness these are listed in [1]. Unfortunately, an universal reducing method which is proper for every pebbling distribution does not exist. The cases where a proper reducing method does not exist can be seen in Fig. 7.

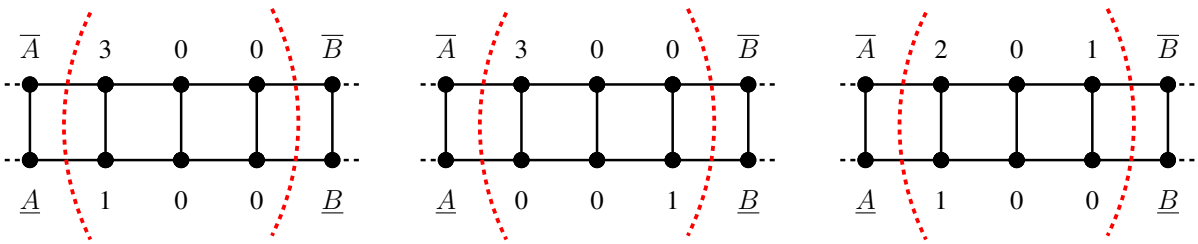

Figure 7: Exceptions when the modified inequalities do not hold

The first case can be solved easily. If R is at the left end of the graph then put two pebbles at the vertices of rung B. Else we can choose another subgraph R' which contains rung A, l and x and it is not an exception, so we have got a proper reduction method for it. We call this idea as *shifting technique*.

The second case, there is no reduction method for this R, unless we use the assumption that R contains the maximum number of pebbles from the set of $P_3\Box K_2$ subgraphs. So assume that each of the $P_3\Box K_2$ subgraphs contain at most four pebbles. The reducing method is the following: Put one pebble to $\overline{A}$ and the another to $\underline{B}$. It is clear that the vertices of rung A and B are reachable. The modified inequalities can be shown to hold for vertices $\underline{A}$, $\overline{B}$, $\underline{B}$, hence the original inequalities hold for these vertices, too. Now we need to show that $R_p(\overline{A}) = 1$, which implies that the original inequality holds for vertex $\overline{A}$.

Partition G to disjoint $P_3\Box K_2$ subgraphs. The maximality of R means that these subgraphs contains at most four pebbles. This gives an upper bound for $R_p(\overline{A})$.

$$R_p(\overline{A}) \leq \left\lfloor \frac{3}{2} + \frac{1}{16} + \frac{4}{16}\sum_{i=0}^{\infty}\left(\frac{1}{8}\right)^i \right\rfloor = \left\lfloor 1 + \frac{9}{16} + \frac{2}{7} \right\rfloor = 1$$

Therefore the original inequalities hold, hence this reduction method is proper.

Finally, consider third case and put a pebble to $\underline{A}$ and $\overline{B}$. The modified inequalities hold for $\underline{A}$, $\overline{B}$ and $\underline{B}$. We do what we have done in the second case. We can make an estimation again:

$$R_p(\overline{A}) \leq \left\lfloor 1 + \frac{1}{4} + \frac{1}{8} + \frac{4}{16}\sum_{i=0}^{\infty}\left(\frac{1}{8}\right)^i \right\rfloor = \left\lfloor 1 + \frac{3}{8} + \frac{2}{7} \right\rfloor = 1$$

So the original inequalities also hold.

We have shown that exist a solvable reduced distribution for every distribution which has a $P_3\Box K_2$ subgraph that contains at least four vertices.

3.2 Distributions where the maximal number of pebbles on a $P_3\Box K_2$ subgraph is three

Lemma 3.17 *Let p be a distribution and fix a subgraph R. Let T be an executable rubbling sequence which uses vertices located left from R. If every $P_3\Box K_2$ subgraph has at most three pebbles then $p_T(\overline{A}) + p_T(\underline{A}) \leq 3$. Furthermore, if equality holds then $p(A) = 3$.*

PROOF: The proof is based on a similar idea that we used in the previous subsection. Partition the graph to disjoint $P_3\Box K_2$ subgraphs. By the assumption all of these disjoint subgraphs may contain at most 3 pebbles. When $p(A) = 3$ we obtain the following estimate which completes the proof of this case:

$$p_T(\overline{A}) + p_T(\underline{A}) \leq \left\lfloor 3 + \frac{3}{8}\sum_{i=0}^{\infty}\left(\frac{1}{8}\right)^i \right\rfloor = \left\lfloor 3 + \frac{3}{7} \right\rfloor = 3$$

When $p(A) \leq 2$, then the third pebble of the subgraph is not on A, therefore its contribution is at most $\frac{1}{2}$:

$$p_T(\overline{A}) + p_T(\underline{A}) \leq \left\lfloor 2 + \frac{1}{2} + \frac{3}{8}\sum_{i=0}^{\infty}\left(\frac{1}{8}\right)^i \right\rfloor = \left\lfloor 2 + \frac{1}{2} + \frac{3}{7} \right\rfloor = 2$$

□

To continue the proof we have to find reducing methods for the cases when R contains three pebbles. There are sixteen different cases, these are shown on Fig. 8 with proper reduction methods.

We prove the solvability of the reduced pebbling distribution for one case, and leave the remaining ones for the reader to check. It is enough to check that the right (left) reachability of the vertices of rung A (B) are not decreasing, and all of them remain reachable.

Consider the case $p(\bar{l}) = p(\underline{x}) = p(\overline{r}) = 1$ and $p(\underline{l}) = p(\overline{x}) = p(\underline{r}) = 0$. A proper reducing method is the following: Place a pebble at $\overline{A}$. Since any $P_3 \square K_2$ contains at most 3 pebbles, we have $p(A) \leq 1$ and $p(B) \leq 1$. Lemma 3.17 shows that A is not left 3-reachable under p (it can not have 3 pebbles under p). This result shows that rung r is not left 2-reachable under p. Hence a vertex of B can get only one pebble from r by a strict rubbling move and not left 2-reachable. Similar statement holds for rungs l and A when we swap left and right directions. Now we can show that either there is a pebble on B or its right neighbours are right reachable.

To show this, assume the contrary, so $p(B) = 0$ and one of its right neighbour, u, is not right reachable. Then the other right neighbour cannot be right 2-reachable. This means that we can not reach the vertex of B which is adjacent to u without the use of the other vertex of B. But to move there a vertex, we consume all the pebbles which can be moved to the neighbourhood of B. So this vertex of B cannot be reached.

Moreover, the vertices of B can be left reachable if and only if $p(B) = 1$. There is also a similar fact for A. Now we need to check the reachability of the four vertices.

- $\overline{A}$: We place a pebble at this vertex, so it is right reachable.
- $\underline{A}$: l is not right 2-reachable, hence the extra pebble at $\overline{A}$ can act a pebble of $\underline{l}$ when we want to reach $\underline{A}$ from p^R.
- $\overline{B}$: If it is left reachable under p then it is also left reachable under p^R with the help of $\overline{A}$'s pebble. Otherwise this pebble assures reachability of $\overline{B}$ under p^R.
- $\underline{B}$: $R_p(\underline{A}) = 0$, thus the addition of an extra pebble at $\overline{A}$ makes $\underline{A}$ left reachable. So if B has a pebble then $\underline{B}$ is left reachable under p^R, otherwise simply reachable.

The proof of the solvability of the reduced distributions in the first 14 cases uses same tools and ideas. We can reduce case 15 and case 16 to case 10 and case 14 with the shifting technique.

3.3 Distributions where none of the $P_3 \square K_2$ subgraphs contains more than two pebbles

In this subsection we show that if p is solvable and none of the $P_3 \square K_2$ subgraphs contains more than two pebbles then all of them contain exactly two.

Lemma 3.18 *Let p be a distribution which satisfies that every $P_3 \square K_2$ subgraph contains at most 2 pebbles. If an $R = P_3 \square K_2$ subgraph satisfies that $p(R) < 2$ then p is not solvable.*

Statement 3.19 *Let p be a distribution which satisfies that every $P_3 \square K_2$ subgraph contains at most 2 pebbles. A rung g is 2-reachable from p if and only if $p(g) = 2$, and if it is 2-reachable then it can not get a pebble by a rubbling move.*

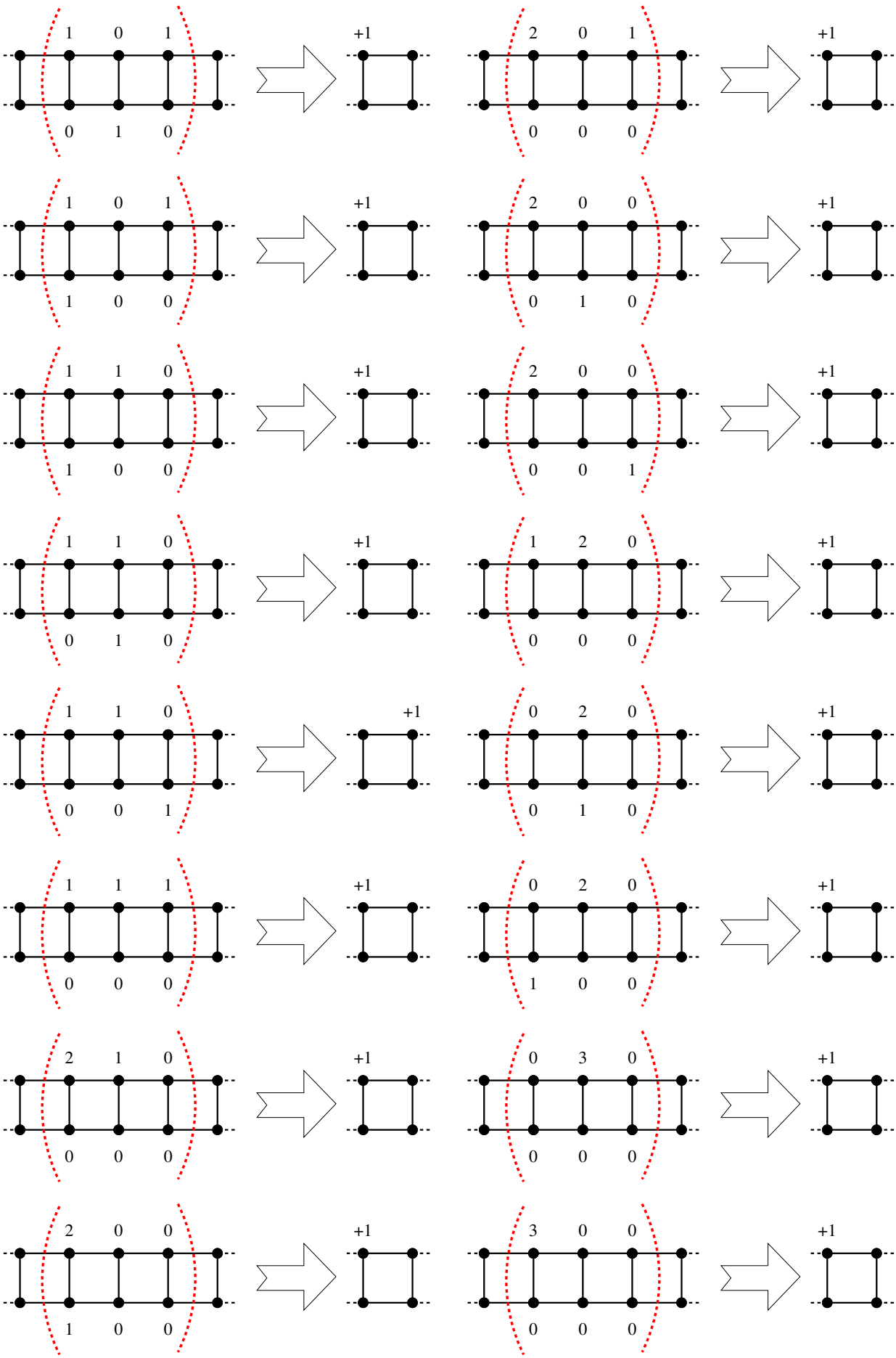

Figure 8: Reduction methods when every $P_3 \square K_2$ contains at most three pebbles.

It is easy to show this with the partition method described in the proof of Lemma 3.17.

PROOF:(Lemma 3.18) Let R be a $P_3 \Box K_2$ subgraph satisfying $p(R) \leq 1$. There are three essentially different cases (considering symmetry):

- $p(R) = 0$: Rung A and rung B contains at most two pebbles, hence one of the vertices of rung x is not reachable.
- $p(x) = 1$: The upper bound on $p(P_3 \Box K_2)$ implies that $p(A)$ and $p(B)$ is less than or equal to one. We can not move an extra pebble to rung A and B due to the previous statement. Hence the vertex of rung x which does not have a pebble is not reachable.
- $p(l) = 1$: Clearly B neither contains 3 pebbles nor can get an additional pebble. Furthermore A is not 2-reachable, thus one of the vertices of rung x is not reachable again.

□

□

4 Optimal rubbling number of the ladder

Now we are prepared to complete the proof of our main result.

Lemma 4.1 *If $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$ then:*

$$\varrho_{opt}(P_n \Box K_2) \geq \begin{cases} 2 + 2k & if\ r = 1 \\ 2 + 2k & if\ r = 2 \\ 1 + 2k & if\ r = 0 \end{cases}$$

PROOF: Let G be a counterexample where $V(G)$ is minimal, and let p be the optimal distribution on G. If there exists $R = P_3 \Box K_2$ subgraph with $p(R) \geq 3$ then we can apply one of the reduction methods described in the previous section and get a solvable distribution p^R on graph G^R. $V(G^R) = V(G) - 2$ and $\varrho_{opt}(G^R) \leq p^R(G^R) = p(G) - 2 = \varrho_{opt}(G) - 2$. Thus G^R is also a counterexample, which contradicts the minimality of G. So we can assume that every $P_2 \Box K_2$ subgraph contains at most two pebbles. By Lemma 3.18 every $P_3 \Box K_2$ subgraph contains exactly two pebbles and by Statement 3.19 that the solvability of p requires that the pattern has to start and end with two 1s or one 2. Thus number of pebbles on the rungs must have the following pattern:

$2\,0\,0\,2\,0\,0\,2\,0\,0\,2\,0 \ldots 0\,2$ or $1\,1\,0\,1\,1\,0\,1 \ldots 1\,0\,1\,1$

However, this means that G is not a counterexample. □

The combination of Lemmas 3.3 and 4.1 completes the proof of Theorem 3.1.

5 The 2-optimal rubbling number of the circle

In this section the 2-optimal rubbling number of the cycle is determined. It is interesting on its own, but it will be needed in the next section.

Theorem 5.1

$$\varrho_{2-opt}(C_n) = n$$

Definition 5.2 *Let p be a pebbling distribution on graph G. Let v be a vertex of degree two such that $p(v) \geq 3$. A 2-smoothing move from v removes two pebbles from v and adds one pebble at both neighbours of v.*

Clearly, if $p(v) \geq 3$, $d(v) = 2$ and u is 2-reachable under p, then u is 2-reachable under q which we obtained from p by making a 2-smoothing move from v. When no 2-smoothing move is available, we say that the distribution is 2-smooth.

PROOF:(Theorem 5.1) When every vertex on the path between vertices v and w are occupied, we say that v and w are *friends*. Assume that $\varrho_{2-opt}(C_n) < n$, so there is a 2-solvable distribution p with size $n-1$. Apply 2-smoothing moves recursively for every vertex which contain at least three pebbles. p has less than n vertices, thus this process ends in finitely many steps. Now we have a 2-solvable pebbling distribution q such $q(v) \leq 2$ for every vertex v. Denote the number of vertices containing i pebbles with x_i. We have the following two equalities:

$$x_2 + x_1 + x_0 = n$$

$$2x_2 + x_1 = n - 1$$

Which implies that $x_0 = x_2 + 1$. Is is easy to see that a 2-solvable 2-smooth distribution q on the circle has the following properties:

- An unoccupied vertex must have an occupied neighbour.
- An unoccupied vertex must have two different friends such that each of them contains two pebbles.
- When v and u are unoccupied neighbours and there are more unoccupied vertices, then v and u can not have a common friend.
- Every vertex which contains two pebbles has at most two unoccupied friends.

The pigeonhole principle with these statements show us that $x_2 \leq x_0$ which contradicts $x_2 = x_0 + 1$. Hence every 2-solvable pebbling distribution contains at least n pebbles.

On the other hand, $\varrho_{2-opt}(C_n) \leq n$. Place one pebble to each vertex to obtain a 2-solvable distribution. $\square$

6 Optimal rubbling number of the n-prism

Statement 6.1 ($k \geq 2$)

$$\varrho_{opt}(C_{3k-1}\square K_2) \leq \varrho_{opt}(P_{3k-2}\square K_2) \leq 2k$$

$$\varrho_{opt}(C_{3k}\square K_2) \leq \varrho_{opt}(P_{3k-1}\square K_2) \leq 2k$$

$$\varrho_{opt}(C_{3k+1}\square K_2) \leq \varrho_{opt}(P_{3k}\square K_2) \leq 2k+1$$

Notice that if an arbitrary rung A of $C_n\square K_2$ is deleted then we obtain $P_{n-1}\square K_2$. It is easy to see that if $k \geq 2$ then opposite ends of the ladder are reachable "in parallel" from the distributions shown in Fig. 3 such that every pebble contributes to only one end. Since A is adjacent to both ends of the ladder, both vertex of A can be reached, too. In one case this idea works even if we delete two adjacent rungs from the prism. So the optimal distributions of $P_{n-1}\square K_2$ or $P_{n-2}\square K_2$ gives a solvable distributions of the circle in the following way:

- When $n \equiv 0 \bmod 6$ use the distribution of case $3k+2$.

- When $n \equiv 3 \bmod 6$ use the distribution of case $3k+1$.
- When $n \equiv 1 \bmod 3$ use the distribution of case $3k$.
- When $n \equiv 2 \bmod 3$ use the distribution of case $3k+1$.

For the optimal distributions when $k = 1$ see Fig. 9.

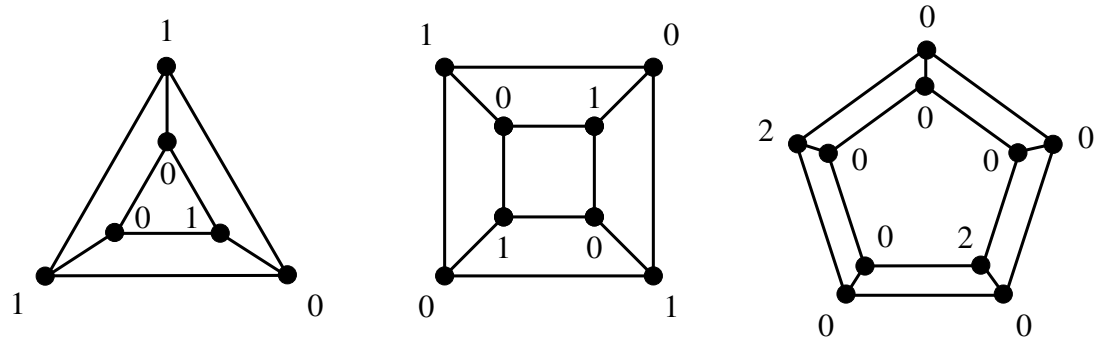

Figure 9: Optimal distributions of the n-prism when $n \leq 5$.

Lemma 6.2 *Let p be an optimal distribution of $C_n \Box K_2$ ($n \geq 5$). Then there exists a rung which is not 2-reachable.*

We are using the "collapsing technique", which is the following method. Let S be a subset of $V(G)$. Then the operation *collapsing* [3] S creates a new graph H. The vertex set of H is $u \cup V(G) \backslash S$, where u is a vertex which plays the role of S. More precisely, u is connected with a vertex v if v is neighbour one of the vertices of S. Moreover, the subgraph induced by $V(G) \backslash S$ is the same in both graphs. Let p be a pebbling distribution on G. Then we also define a *collapsed* distribution q on the collapsed graph H. The definition is simple: $q(u) = p(S)$ and $q(v) = p(v)$ for all $v \notin S$.

PROOF: Let q be a pebble distribution of C_n obtained from a solvable pebble distribution of $C_n \Box K_2$ by applying collapsing operations for each rungs independently. It is easy to see, that the 2-reachability of a rung implies that the vertex produced by its collapse is 2-reachable from q. Assume that each rung is 2-reachable from p. Then each vertex is 2-reachable from q and Lemma 5.1 implies that $q(C_n) \geq n$. However, Statement 6.1 implies that $p(C_n \Box K_2) < n$ and the definition of the collapsed distribution shows us that $q(C_n) = p(C_n \Box K_2)$, which is a contradiction. $\Box$

Definition 6.3 *Let p be a distribution of $P_n \Box K_2$. Let l and r be different vertices, such that l is located left from r. If l is k_l right reachable and r is k_r left reachable, but not independently, then we say that l and r are p-dependent.*

Lemma 6.4 *Let p be a distribution of $P_n \Box K_2$. Let l and r be two different vertices, such they belong to different rungs. If l and r are p-dependent, then all vertices located between their rungs are reachable.*

PROOF: Without loss of generality we may assume that l is located left from r. The condition implies the following facts: There is a rubbling sequence T_r acting only on vertices not located right from R (the rung containing r), such $p_{T_r}(r) = k_r$. If we consider a proper subsequence of T'_r (i. e. we delete at least one move from T_r) then $p_{T'_r}(r) < k_r$ or T'_r is not executable. We also have a T_l with the same properties for l. Finally, there is a vertex u, such T_r and T_l both acts on this vertex.

Let v be a vertex between the rungs of l and r, we show that v is reachable. Let V be the rung which contains v. It is trivial that at least one of T_r and T_l is acting on V. Assume that it is T_r. If T_r

acts on v then it has to receive a pebble at some point, so it is reachable. Otherwise, the other vertex of V (denote it with v') is reachable. T_r moves the pebble of v' towards r, there are two possibilities. First, when T_r uses a pebbling move to move this pebble from v'. In this case, we can change its destination vertex to v and so it is reachable. The second case is when T_r is using a strict rubbling move to move the pebble of v' towards a neighbour. The structure of the ladder implies that this strict rubbling move removes the other pebble from an other neighbour of v. So we can reach v by this strict rubbling move if we change its destination to v. $\square$

Lemma 6.5 *Let p be an optimal distribution of $C_n \square K_2$ ($n \geq 5$), and let C be a rung of this graph such that it is not 2-reachable under p. Consider the following properties:*

- $p(C) = 0$
- $L_p(\overline{L}) = 0$
- $L_p(\underline{L}) = 1$
- $R_p(\overline{R} \geq 2)$ *or* $R_p(\underline{R} \geq 2)$

Let $\text{ref}_c()$ be a reflecting operator, which reflects the whole p distribution across rung C. Furthermore, let $\text{ref}_h()$ be a similar operator which reflects the whole p distribution horizontally. If none of the $\{p, \text{ref}_c(p), \text{ref}_v(p), \text{ref}_c(\text{ref}_v(p))\}$ distributions fulfills all of the properties then $|p| \geq \varrho(P_{n-1} \square K_2)$. Otherwise $|p| \geq \varrho(P_{n-2} \square K_2)$.

PROOF: To prove our claim we construct solvable pebbling distributions for the $P_{n-1} \square K_2$ and $P_{n-2} \square K_2$ ladders from the optimal distribution of $C_n \square K_2$. Consider the solvable distribution p on $C_n \square K_2$ and the rung C which is not 2-reachable. It is trivial that each of the distributions $\{\text{ref}_c(p)$, $\text{ref}_v(p)$, $\text{ref}_c(\text{ref}_v(p))\}$ are solvable. If we delete the rung L or rung C from $C_n \square K_2$, then the remaining graph is $P_{n-1} \square K_2$. If we delete both L and C then the remaining graph is $P_{n-2} \square K_2$. We show that a slight modification of the induced distribution is a solvable distribution of these graphs.

This modification is the following: Let L_2 be the left neighbour of L. If we delete L, then we place all pebbles of $\underline{L}$ to $\overline{L_2}$, and place all the pebbles of $\overline{L}$ to $\underline{L_2}$.

C is not 2-reachable, thus no pebbling move can move a pebble from C to L. Thus only a strict rubbling move can move a pebble from C to L. The modification does not increase the distance of the pebbles from the vertices located left from L. Furthermore, the strict rubbling move from C is not needed, because the result of this move is the same as if we swap the pebbles of the vertices of L, but in the modified distribution these pebbles are placed closer to the remaining vertices. Hence if T is an executable rubbling sequence acting on L_2, L and C then we can construct T', such it is also executable, $p'_T(\overline{L_2}) = p_T(\overline{L_2})$ and $p'_T(\underline{L_2}) = p_T(\underline{L_2})$. The construction is easy: delete the strict rubbling move which uses a pebble from C, replace each occurrence of $\overline{L}$ with $\underline{L_2}$, and do the same with the $\underline{L}$, $\overline{L_2}$ pair.

Using this we can modify every executable rubbling sequence which acts on L.

Notice that if we delete the edges between L and C, and we have a vertex l of L and an r of R such they are p-dependent, then by Lemma 6.4 all vertices between L and R are still reachable.

Case 1:

First we handle the case when there is a rubbling sequence from the optimal distribution to each vertex of C that non of them use either the two edges between L, C or the two edges between C, R. We can assume that we have the first case, otherwise apply $\text{ref}_c()$. Now we delete L and show that the remaining part is solvable under a modified version of p.

If we delete the edges between L and C and a vertex of L and a vertex of R is p-dependent, then all vertices between L and R are reachable. Furthermore C is reachable our assumption, hence R is also. Therefore, the modified distribution is a solvable distribution in the graph obtained by deleting these edges, a $P_{n-1}\Box K_2$. Thus we may assume that the vertices of L and R are not p-dependent.

First we show, that there is no need to move a pebble from L across an edge between L and C to reach a vertex located right from C. A vertex of L cannot get two pebbles from the left, because it would violate the condition that C is not 2-reachable (C remains reachable from R since $l \in V(L)$ and $r \in V(R)$ are not p-dependent). So there is only a strict rubbling move available which requires a pebble at C.

If C does not have a pebble then first we need to move one there. This requires two pebbles at R and after two other moves we are able to move it back to R with the help of a pebble of L. Both vertices of R were reachable, we consumed two pebbles of R and moved one to it, so we just wasted the pebbles. Hence it is pointless to move a pebble from L to R.

If C has a pebble then we may assume that $\overline{C}$ has it. We can apply $\text{ref}_h()$ to achieve this. Now we can use only the pebble of $\underline{L}$. Our assumption that both vertices of C are reachable without the use of the edges between L and C implies that $\underline{R}$ is right-reachable. Hence $\underline{\text{L}}$ can not be left-reachable.

This completes the proof of this case.

Case 2:

Now we assume the opposite, that there are moves through the edges between L, C and R to reach the vertices of C.

When C has a pebble, then assume that it is placed on $\overline{C}$. The reachability of $\underline{C}$ implies that $\underline{R}$ is right reachable or $\underline{L}$ is left reachable. However these are the cases we covered in Case 1. Hence C can not contain a pebble.

If there is a left 2-reachable l of L and a right 2-reachable r of R then we have two possibilities. The first is that they are p-dependent. This means that if we delete C then p remains solvable on the graph because of Lemma 6.4. The second case, when they are not p-dependent, contradicts with the fact that C is not 2-reachable.

If both vertices of L are left reachable and both vertices of R are right reachable but none of them are left or right 2-reachable, then it is easy to see that we can not move a pebble from L to R through C and the same is true for the opposite direction. So in this case and everything remains reachable after the deletion of C.

It is possible that one of them can get more pebbles, so assume that $\overline{R}$ is right 2-reachable. In this case $\underline{R}$ and $\overline{R}$ are not reachable independently. The same is true for $\overline{L}$ and $\underline{L}$. Hence after we have used the two pebbles of $\overline{R}$ and have moved it to $\overline{C}$, we can not use the pebble at $\overline{C}$ by a strict rubbling move to increase the number of pebbles at $\overline{L}$ to 2. So we can delete C without any problem and the distribution remains solvable.

Now we have checked almost all the cases except the ones which gives bound with $P_{n-2}\Box K_2$. In this case at most 3 of the vertices of L and R are reachable from the proper direction. Otherwise, we can not reach both vertices of C or we have a case which we have already covered above.

Applying $\text{ref}_c()$ and $\text{ref}_h()$ we can guarantee that $\overline{L}$ is not left-reachable. Then $\underline{L}$ is left reachable, and both vertices of R are right reachable. The fact that $\overline{L}$ is not left reachable implies that we can move a pebble from R to $\overline{C}$ or to $\underline{L}$. C does not have a pebble so $R_p(R) \geq 2$. Now if we delete L, C and modify the distribution in the above way then we obtain a solvable distribution of $P_{n-2}\Box K_2$ which gives us the desired bound.

It remains to show that $R_p(\overline{R}) = R_p(\underline{R}) = 1$ leads to a contradiction. The only possible way to move a pebble with the use of R's pebbles to the neighbourhood of $\overline{L}$ is to consume a pebble from $\underline{L}$. On the other hand, $\underline{L}$ is a neighbour of $\overline{L}$, so we are not able to increase the number of pebbles at $\overline{L}$'s neighbourhood. So $\overline{L}$ is not reachable under p in the n-prism, which contradicts that p is an optimal

distribution of that graph. $\square$

Theorem 6.6

$$\varrho_{opt}(C_{3k-1}\square K_2) = \varrho_{opt}(P_{3k-2}\square K_2) = 2k$$
$$\varrho_{opt}(C_{3k}\square K_2) = \varrho_{opt}(P_{3k-1}\square K_2) = 2k$$
$$\varrho_{opt}(C_{3k+1}\square K_2) = \varrho_{opt}(P_{3k}\square K_2) = 2k+1$$

Except:

$$\varrho_{opt}(C_3\square K_2) = 3$$
$$\varrho_{opt}(C_4\square K_2) = 4$$

PROOF: Let p be an optimal pebbling distribution of $C_n\square K_2$, and let C be a rung which is not 2-reachable under p. Denote the neighbouring rungs of C with L and R.

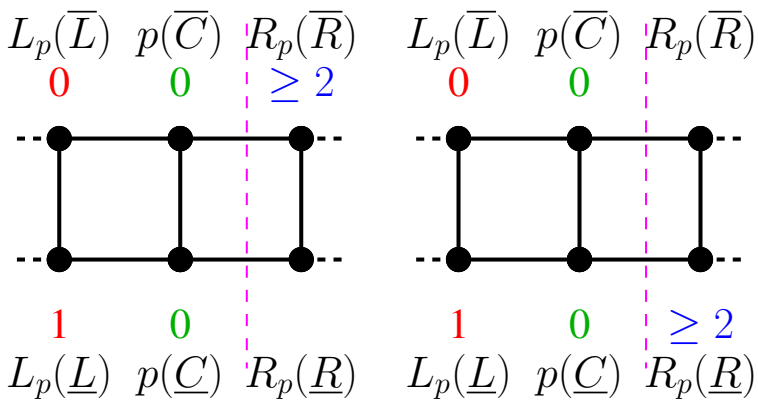

Figure 10: The red numbers denote $L_p()$, the blue ones $R_p()$.

Let $n = 3k + r$ where $n, k, r \in \mathbb{N}, 0 \leq r < 3$. Now we show for each r that the theorem holds. We apply Lemma 6.5 in each case.

When $r = 0$ then we use the general bound of Lemma 6.5, which always holds:

$$\varrho_{opt}(C_n\square K_2) \geq \varrho_{opt}(P_{n-2}\square K_2) = 2(k-1) + 2 = 2k.$$

When $r = 1$ we show that in Lemma 6.5 the better bound holds, since at least one of the conditions fail to hold. Indirectly, assume that $\varrho_{opt}(C_n\square K_2) = \varrho_{opt}(P_{n-2}\square K_2)$. This implies that all conditions in the Lemma hold, so we have one of the cases shown on Fig. 10. Delete the edges between R and C to obtain $P_n\square K_2$, place one more pebble at $\overline{C}$. This modification of p is clearly solvable on $P_n\square K_2$. This implies:

$$2k + 2 = \varrho_{opt}(P_n\square K_2) \leq \varrho_{opt}(C_n\square K_2) + 1 = \varrho_{opt}(P_{n-2}\square K_2) + 1 = 2(k-1) + 2 + 1 = 2k + 1,$$

which is a contradiction, hence:

$$\varrho_{opt}(C_n\square K_2) \geq \varrho_{opt}(P_{n-1}\square K_2) = 2k + 1.$$

When $r = 2$ assume indirectly again that $\varrho_{opt}(C_n\square K_2) = \varrho_{opt}(P_{n-2}\square K_2)$, then p fulfills the all properties. Delete the edges between C and R again, but now augment the graph as shown see Fig.

11. We place an extra pebble at $\overline{C}$ and another at the end of the new part. The augmented $P_{n+2}\Box K_2$ graph is solvable under this distribution. Using Theorem 3.1 these imply:

$$2k+4 = 2(k+1)+2 = \varrho_{opt}(P_{n+2}\Box K_2) \leq \varrho_{opt}(C_n\Box K_2)+2 = \varrho_{opt}(P_{n-2}\Box K_2)+2 = 2k+3,$$

which is a contradiction, hence

$$\varrho_{opt}(C_n\Box K_2) \geq \varrho_{opt}(P_{n-1}\Box K_2) = 2k.$$

□

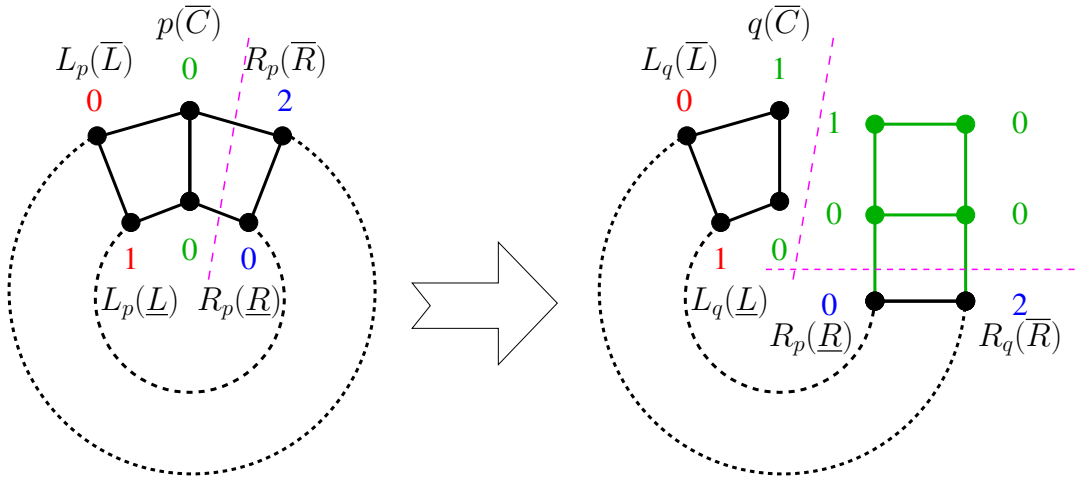

Figure 11: The augmented part is shown in green.

7 Optimal rubbling number of the Möbius-ladder

Theorem 7.1 *The Möbius-ladder with length n and the n-prism has the same optimal rubbling number, except when $n = 3$.*

For the optimal distributions of small graphs see Fig. 12. The proof given for the lower bound of Theorem 6.6 is works for this theorem, too. The upper bound also came from the optimal pebbling distributions of $P_n\Box K_2$.

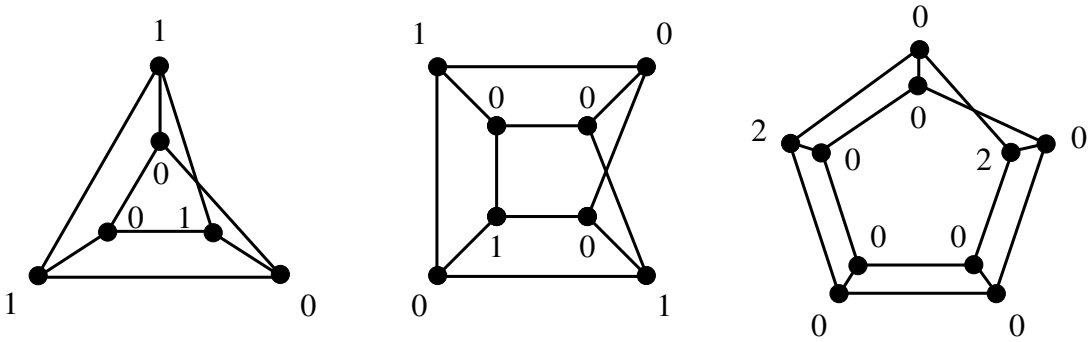

Figure 12: Optimal pebbling distributions of the Möbius-ladder when $n < 6$.

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References

[1] List of proper reduction methods for the $p(R) \geq 4$ case, `http://ww.cs.bme.hu/~kiskat/ladder_appendix.pdf`

[2] Ch. Belford and N. Sieben, *Rubbling and optimal rubbling of graphs*, Discrete Math. **309** no. 10 (2009) pp. 3436–3446.

[3] D.P. Bunde, E. W. Chambers, D. Cranston, K. Milans, D. B. West, *Pebbling and optimal pebbling in graphs* J. Graph Theory 57 no. 3. (2008) pp. 215–238.

[4] F. Chung, *Pebbling in hypercubes* SIAM J. Disc. Math. **2** (1989) pp. 467–472.

[5] L. Danz, *Optimal t-rubbling of complete m-ary trees*, REU project report, University of Minnesota Duluth, Department of Mathematics and Statistics, (2010).

[6] G. Hurlbert, *A survey of graph pebbling*, Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999), vol. 139 (1999), pp. 41–64.

[7] G. Hurlbert, *Recent progress in graph pebbling*, Graph Theory Notes N. Y. **49** (2005), pp. 25–37.

[8] G. Hurlbert, *Graph pebbling*, In: Handbook of Graph Theory, Ed: J. L. Gross, J. Yellen, P. Zhang, Chapman and Hall/CRC, Kalamazoo (2013), pp. 1428– 1449.

[9] P. Lemke, D. Kleitman, *An addition theorem on the integers modulo n* J. Number Theory **31** (1989) pp. 335–345.

[10] K. Milans, B. Clark, , *The complexity of graph pebbling.* SIAM J. Discrete Math. **20** no. 3 (2006) pp. 769–798.

[11] G. Y. Katona, N. Sieben, *Bounds on the Rubbling and Optimal Rubbling Numbers of Graphs* Electronic Notes in Discrete Mathematics, Volume 38 (2011), pp. 487-492.

[12] D. Moews, *Pebbling graphs*, J. Combin. Theory (B), 55 (1992) pp. 244–252.

[13] L. Papp, *Optimal rubbling numbers of graphs (in Hungarian)*, thesis, Budapest University of Technology and Economics, Department of Computer Science and Information Theory, (2010).