



COLLECTIVE DECISION-MAKING UNDER THE INFLUENCE OF BRIBERS AND TEMPORAL CONSTRAINTS

Marina Bannikova

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UNIVERSITAT ROVIRA I VIRGILI

Collective decision-making under the influence of bribers and temporal constraints

PhD dissertation by
Marina Bannikova

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UNIVERSITAT ROVIRA I VIRGILI

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COLLECTIVE DECISION-MAKING
UNDER THE INFLUENCE OF
BRIBERS AND TEMPORAL CONSTRAINTS

PHD DISSERTATION BY
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Introduction

It is always difficult to make decisions, even when a single person has to decide between several alternatives. Some decisions are easy to make: what to have for lunch, which book to read in the evening, or which wine to buy. A simple visit to a supermarket creates a lot of questions for which answers must be decided upon (how many packages, which brand, which flavour, etc.). Usually, people do not consider these situations as a matter of a decision, although in formal sense, before undertaking a certain action (cook spaghetti or pick up Dostoevsky from the book shelf) a person performs a fast decision process in his¹ head.

These small decisions do not affect much the life of a person, but there are crucial decisions that may change his life forever. For instance, a recent graduate from a high school who needs to decide which college to go to and which future speciality to enrol in; or a young girl standing in front of the altar, who is asked to choose whether she wants to spend the rest of her life with a young man standing next to her.

When the order of the preferences is clear (“I prefer Russian literature to English one”, “My dream is to be a teacher”, or “I love this man more than anyone in the whole world”), the decision seems to be an easy operation to undertake. Though, in most cases, people have to sacrifice their most preferred alternatives in order to achieve more utility: “I would read Tolstoy, but tomorrow I have a test on Microeconomics, so I would be better off taking a look at Mas Colell one more time”; “...but lawyers earn more”, and so on.

¹We use the pronoun “he” in a gender-neutral sense: “he” can be equally taken to refer to a man or a woman. The reader is free to choose who stands behind “he”.

Additionally, some decisions only affect the person who decides and his utility, but usually it may affect more people and therefore, their utilities, too. In such a case it would be fair that such decisions would be the product of a collective will. But what if a decision affects every citizen in a city, or even in the whole country? It would be fair to ask an opinion of everyone, but undertaking this feat would likely not be possible to perform. When there are many people who have to decide, the decision-making process would take a lot of time, hence it would seem to be more convenient to reduce the group of decision-makers, as illustrated in [Figure 1](#). In modern democratic societies, the solution is to elect representatives, to whom the power to decide would be delegated. The most typical example is a parliament, formed by representatives who are elected by citizens: parliament is charged with the power of making decisions that affect the whole country. A court of jurors can also serve as an illustration: a small group of people (jurors) is randomly selected to make a decision on behalf of the people.

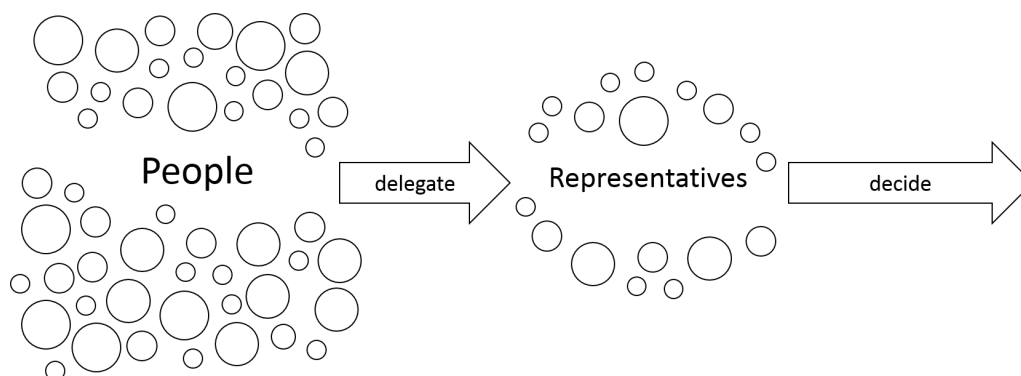


Figure 1. Decision-making by reducing the number of individuals involved.

Consider the most prominent example: parliament. The process of electing the representatives is studied by scientists from different fields: political science, social choice theory, public economics. Political science considers the political component of the election process: the rule of law and government, the role of parliament in implied polities. Electoral competition theory treats the elections from the point of view of candidates who are competing for the votes of citizens. Symmetrically, social choice theory concentrates on the voters' behaviour and their preferences. Firstly, social choice theorists are interested in how to aggregate the preferences of voters. We cannot refuse the importance of the rule for dividing the population

into representative districts, since history provides certain examples of gerrymandering – a practice of manipulating district boundaries in order to establish a political advantage for a particular party or candidate. But probably, the most desirable feature of voting procedures is that voters reveal their opinions sincerely. The sincerity of the voters is important when these voters are the citizens electing representatives, but it is even more important when these voters are the elected representatives, who are in charge of deciding the future of the whole country.

Manipulation

Every citizen of a democratic country votes for a certain candidate or a party in order to delegate them his decision power. He trusts in him, based on his statements during the electoral campaign. It is then not surprising that he wants to be sure that politicians are honest in their promises, and they act as they promised they would act during their electoral campaign. But the manipulability of smaller groups is easier than in larger groups: it is easier to influence the decision of a smaller group (of elected representatives) than the decision of the whole group of citizens. Consequently, in solving one problem (the difficulty to make a decision in a large group) by reducing the number of decision-makers, we might create another problem: manipulability, since if it is found to be easy to strategically influence voting outcomes, one cannot be confident about the reliability and legitimacy of political decisions.

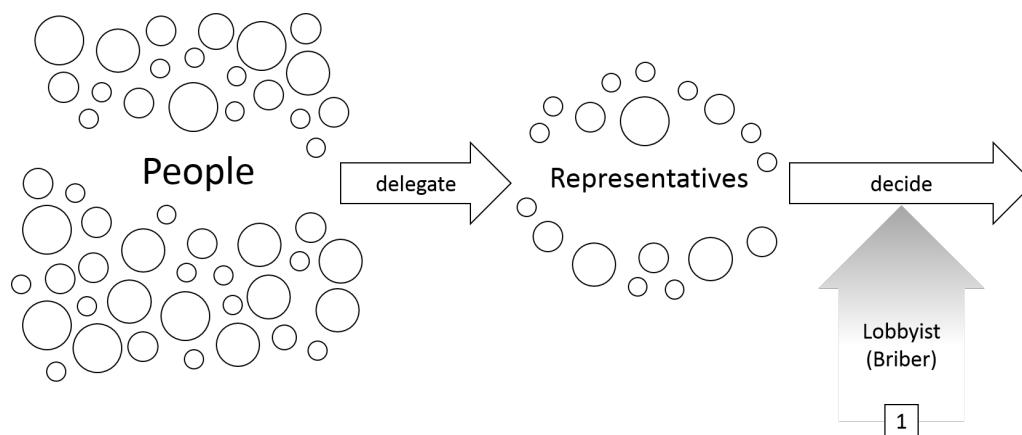


Figure 2. Manipulation in decision-making.

The pressure on representatives in order to influence their decision – process [1] in [Figure 2](#), – can be legal (lobbying in the United States) or illegal (bribery and corruption). The body of lobbying literature attributes both a positive and a negative effect to lobbying activities; see, for instance, [Grossman and Helpman \(1996\)](#) and [Dal Bó \(2007\)](#). The positive view of lobbying relies on the idea that lobbying groups provide information to decision-makers, although there is a certain problem of incentives to reveal only the information that favours the interest of the lobby, hence, the decision by parliament may be biased. The negative side is based on the fact that lobbying groups may corrupt decision-makers and, therefore, persuade them to decide in favour of the groups.

Usually, corruption is considered at the stage of electing the representatives. Scholars analyse how different electoral systems are connected with corruption; see, for instance [Myerson \(1993\)](#), [Myerson \(1999\)](#), [Persson et al. \(2003\)](#), [Rose-Ackerman \(2005\)](#), [Birch \(2007\)](#). Less attention seems to have been paid to what happens inside the parliament once it is elected (see, [Diermeier et al. 2003](#) and [Diermeier and Vlaicu 2011](#)). This issue is mostly studied empirically: see, for example, [Charron \(2011\)](#) and [Pelizzo \(2006\)](#). Much less attention seems to have been devoted to the connection between corruption and the structural characteristics of the parliaments, such as the number of seats (or size) of the parliament, the number of parties with representation, and the decision rules that are adopted by the parliament. These characteristics might have an influence on the existence of political corruption inside the parliament. Motivated by this possibility, in [Chapter 1](#) we develop a model of a parliament formed by political parties, that must decide whether to accept or reject a proposal in the presence of a briber who is interested in having the proposal approved. We calculate an average cost for the briber, expressed in terms of the number of seats in the parliament that on average have to be bribed to ensure that the voting outcome is the one which the briber wants.

Timing

Another problem that may worry a voter is that elected politicians actually will make a particular decision once they are elected. A citizen votes for a member of a parliament in order that he decides in the name of this citizen. But in some cases it appears that politicians

are indecisive. The most recent example of the Spanish government shows that sometimes the elected representatives are indecisive. Elected in December 2015, the lower house of the Spanish parliament had two months to define the government of Spain. Some candidate had accepted the invitation of the Head of the State to try to obtain the support of the parliament and become Prime Minister, but due to the lack of consensus, the four main party leaders failed to come to an agreement within the allotted time. For this reason, new general elections are held in Spain in June 2016. Public opinion polls show that the voters are not happy about the waste of resources and time which happened owing to the inability of the parliament to reach a decision on who is going to be the Prime Minister. The parties were involved in a bargaining process of trying to form coalitions. There were two parliamentary sessions aimed at making a final decision about the election of a Prime Minister. This situation can serve as a perfect example of the importance of temporal constraints in an iterative collective decision process.

Any voting procedure, and especially an iterative voting procedure, involves the time factor, which imposes definite restrictions on the collective decision process.

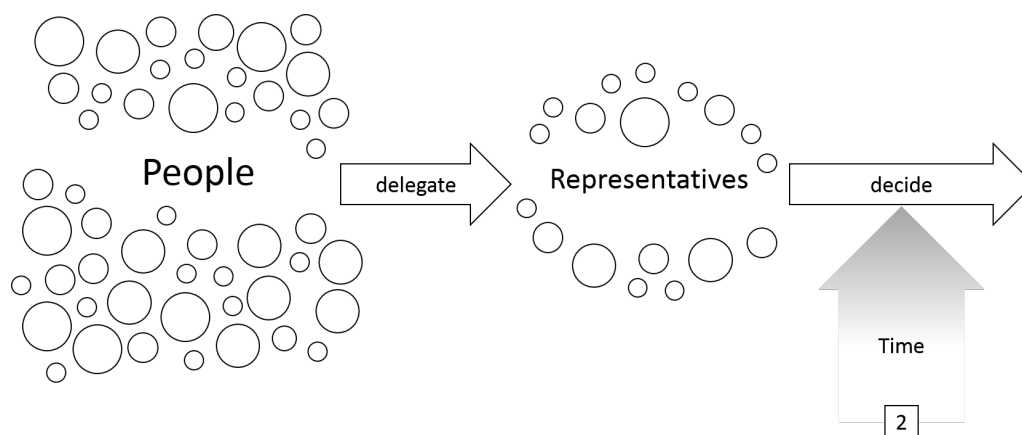


Figure 3. Temporal constraints in decision-making.

The significance of time for a decision-making process is motivated by at least two considerations. First, time may induce some costs: inflation, discount factors, interest rates if some money issue is under voting; costs of voting procedure, as in the case of a parliamentary session; costs of delay can be very large if fast developing contemporary technologies are the matter to be decided. Second, there might be a certain limit whereby the decision must be

taken; for instance, the two months restriction for a Spanish parliament to elect a Prime Minister. Or in the case of a jury, the deadline imposed by the judge, so that if the jurors do not decide before the deadline, a mistrial is declared. Motivated by the importance of time restrictions in [Chapter 2](#) we develop two models of iterative voting, one with explicit time costs and another with a deadline.

Usually, the situation where several individuals try to reach an agreement is considered by scholars as a bargaining process with delay costs, see, for instance, [Rubinstein \(1982\)](#). The presence of a deadline and its effects is also widely studied in bargaining models, as in, for instance, [Ma and Manove \(1993\)](#) or more recently in [Roth et al. \(2016\)](#). But decision-makers are not always engaged in a bargaining process: sometimes, they participate in a formal voting procedure, which can include several stages, or iterations. [Kwiek \(2014\)](#) presents a model of a group of people (a conclave) who must choose one out of two options. He also assumes delay costs, which affect the utilities of every voter. Commonly, in an iterative voting process there is no finite stage, that is, the voting continues unless there is an outcome obtained or it is clear that there is no possible outcome to be obtained; see, [Meir et al. \(2010\)](#), [Reijngoud and Endriss \(2012\)](#), [Reyhani and Wilson \(2012\)](#), [Obraztsova et al. \(2015\)](#). Contrariwise, we assume that the deadline is defined as a strict limit of the number of voting stages. Such a scenario is motivated by real-life examples, such as the two months limit for the Spanish parliament to elect a Prime Minister, or as a time limit for jurors to reveal their verdict.

This thesis consists of two chapters, devoted to the two problems of collective decision-making described above: manipulation and time restrictions.

Brief summary of [Chapter 1](#)

First, [Chapter 1](#) presents a general model of a parliament formed by parties, which must decide whether to accept or reject a proposal (a bill, for instance) in the presence of a briber, who is interested in having the proposal passed.

As a preliminary step we collected data on 172 countries: their parliaments, level of corruption, perceptions of corruption of parliament, and perceptions of corruption of political parties. We find weak empirical evidence that corruption increases as the number of parties

rises. To provide a theoretical explanation of this finding, we present a simple theoretical model. We compute the number of deputies a briber needs to persuade, on average, in parliaments with different structures as described by the number of parties, the voting quota, and the allocation of seats among parties. Applying analytical and computational approaches, and running simulations of the model, we find that the average number of seats needed to be bribed decreases as the number of parties increases (and a lower number of seats required to be bribed appears to encourage bribery).

Due to the mathematical complexity of the problem, the analytical approach provides a solution for just two specific cases: parliaments with two parties (the minimum number of parties) and parliaments with single-member parties (which is equivalent to having in parliament the maximum number of parties). The computational approach allows us to deal with parliaments consisting of up to ten parties (according to our empirical data, 77% of countries have no more than 10 parties, and, therefore, the restriction to ten parties seems reasonable). Restricting the number of seats a party may have allows us to consider four different cases: (i) the general case, when each party can occupy any number of seats; (ii) parliaments formed by similarly-sized parties; (iii) parliaments without small parties, and (iv) parliaments without big parties. We show that the average number of seats to be bribed is smaller in parliaments without small parties; under a simple majority, the average number of seats needed to be bribed is smaller for parliaments in which one party has a majority, but under a qualified majority it hardly changes.

Next, we adapt the bribing cost definition in order to characterise the ease of manipulation, by including the likelihood of parties' acceptance of a bribe. We show that the ease of manipulating a legislature decision by the briber increases with the number of parties. A high electoral threshold leads to fewer parties represented and, consequently, decreases the ease of changing a legislature decision by the lobbyist. On the other hand, a high electoral threshold may cause a misrepresentation of voters. We show that if the threshold is higher than 6%, then the impact of the misrepresentation effect becomes significant.

Brief summary of [Chapter 2](#).

Even without the presence of a briber, for a group of people it might be difficult and fast

to make a decision, especially when the process involves certain time restrictions. [Chapter 2](#) provides theoretical models of iterative voting procedures by considering time restrictions in two different ways: as a delay cost and as a deadline for making a decision.

The first model suggests an iterative voting procedure in which several voters try to reach a unanimous decision. The time restriction is introduced as *a delay cost*: personal utilities are decreasing with time, so that the later the decision is taken, the lower the utility that the individuals obtain. This model can be seen as representing members of a parliament (or a political party, or any committee, or jurors) who are deciding whether to accept or to reject a proposal (or choosing one of them to be a speaker or a head of committee). Until they do not agree on a certain option, the voting does not end, no decision is taken, no utility is obtained and, moreover, the passage of time is costly.

As a first step, we consider two voters who are choosing between two alternatives via a sequential voting procedure. Both voters have complete information about each other's preference. At each voting stage, individuals vote in a fixed order for one of the two alternatives. If they both vote for the same alternative, the voting procedure stops and the alternative is implemented. If they vote for different alternatives, then the voting passes to the next stage and the procedure repeats in the same way. At every stage, each voter prefers the same alternative to the other and has utilities decreasing with stages. Each voter has a degree of impatience, indicating when it is better voting for the non-preferred alternative now, rather than waiting for the next stage and obtaining the preferred alternative. Since both voters know their own degree of impatience degree and that of the other voter, intuition suggests that the more patient voter will manage to get his preferred alternative. Contrary to this intuition, in the unique solution of the sequential voting procedure obtained by backward induction, the first voter gets his preferred alternative at the first stage.

Subsequently, we assume that the voting order is reversed at some point, so that the voter, who previously had voted first, begins to vote in the second place. The result obtained under a fixed order holds only under the condition that the impatience degree of the second voter lies before the moment when the voting order is changed. Otherwise, the unique solution of the sequential voting procedure obtained by backward induction is that the voter who votes second at the first stage gets his preferred alternative immediately at the first stage.

Further, the model is extended up to n voters and m alternatives, assuming the fixed voting order. When there are more than two voters, intuition may suggest that not only the patience of the voters matters, but also the number of voters supporting the same alternative. At first glance, it seems that if the alternative is supported by the majority of the voters, this alternative will be implemented. Yet, our result tells that, in the unique solution of the sequential voting procedure obtained by backward induction, the first voter gets his preferred alternative at the first stage.

The second model introduces a time restriction as a temporal limit for voters to make a decision: *a deadline*. This feature is motivated by the fact that in many situations there is a fixed final stage when the decision must be taken, otherwise the worst outcome arises. We consider a model of a group of individuals seeking a consensus under a strict deadline.

The most prominent example of seeking a consensus under a deadline is, perhaps, a jury trial, when the judge places a deadline by which the jury must reach a unanimous decision, otherwise the judge declares a mistrial. A mistrial is commonly perceived to be worse than any decision the jury might render. As a result, while each juror has his own idea about the fairness of each possible trial outcome, he may eventually choose to vote for a less fair outcome, rather than cause a mistrial by breaking unanimity.

Motivated by this possibility, we propose a model for the above scenario — that we call Consensus Under a Deadline — based on a time-bounded iterative voting process. We provide theoretical features of Consensus Under a Deadline, particularly focusing on convergence conditions and the quality of the final decision. We prove that, if there is enough time for all the voters to change their vote, convergence always occurs.

We propose a time-bounded iterative voting algorithm that checks if and when the process converges to a unanimous decision, for two simple types of voter behaviour, lazy vs. proactive. We perform an experimental evaluation of the algorithm on four real-world datasets. The experiments reveal where the lazy and the proactive voters differ, and that a tradeoff exists between the fairness of the result and the effort required of the voters during the process.

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UNIVERSITAT ROVIRA I VIRGILI

COLLECTIVE DECISION-MAKING UNDER THE INFLUENCE OF BRIBERS AND TEMPORAL CONSTRAINTS

Marina Bannikova

CHAPTER 1

CORRUPTION IN REPRESENTATIVE DEMOCRACIES

Overview. We present a simple theoretical model of parliaments formed by parties, which must decide whether to accept or reject a proposal in the presence of a briber, who is interested in having the bill passed.

We find that the average number of seats needed to be bribed decreases as the number of parties increases. We investigate the cases of restricting the minimum or the maximum number of seats a party may have.

A high electoral threshold leads to fewer parties being represented, and, consequently, decreases the ease of changing a parliament decision by the lobbyist. On the other hand, a high threshold may cause a misrepresentation of voters. We show that if the threshold is higher than 6%, the impact of the misrepresentation effect becomes significant.

Keywords: parliament; briber; parties; electoral threshold.

Preface

This chapter goes back to the beginning of my endeavours as a student researcher, during my time as a masters student in 2011. My masters thesis resulted in several papers and, currently, in my writing the present chapter of my PhD thesis. I often wondered whether “possible extensions” and “future research” were ever actually carried out and now I can tell that at least some of them are. This chapter aims to answer some open questions that were presented in my masters thesis 4 year ago.

My interest in Social Choice in corrupt environments were generated by my past experience at auditor in the Chamber of Control and Accounts of Moscow. In this line, my masters thesis was aimed at finding a connection between parliament structures and proneness to bribing from an analytical perspective. The results can be found in [Section 1.5](#).

*Immediately after my masters thesis I aimed at generalising the solution and changing the model in a manageable way but I did not completely succeed; hence, as other ideas and projects arose, I gave up trying. However, in 2013, I presented the original results at the workshop *Mathematics of Electoral Systems: Voting, Apportioning and Districting (MES-VAD)*, in Budapest. During that workshop I had the privilege of meeting Attila Tasnádi, who generously invited me for a research visit at the Department of Mathematics, Corvinus University of Budapest. During my stay in Spring 2015,¹ I found out that the problem I was trying to solve is an NP-hard problem; the analytical approach was therefore not suitable and different methods were required. Attila Tasnádi honoured me by being my coauthor and suggested that we apply a computational approach. The results are presented in [Section 1.6](#).*

The research continued and resulted in a joint paper with Artyom Jelnov from Ariel University in Israel, whom I had met for the first time at the Summer school of COST Action IC1205 in Caen, France. We met again during my research visit to the Center for the Study of Rationality in Jerusalem in Autumn 2015. There, interesting discussions on the matter of bribing and lobbying of parliaments led to a joint collaboration, and Artyom Jelnov honoured me by being the coauthor of the paper which is partially presented in [Section 1.7](#).

¹I would like to express my gratitude to the COST Action IC1205 “Computational Social choice” programme for the corresponding financial support.

1.1 Introduction

Voting is the most common procedure by the means of which decisions of a group of people are taken, whether it is a nation, a director's board or a committee. The nation makes a decision about what is the common goal and how to achieve it, and a parliament is trusted to represent the people's vision of the joint future of the society. In a modern democracy, parties represent groups of people in a parliament.

While the representatives are acting for the sake of social welfare, no problem arises. But an exogenous agent, a briber, can provide incentives to the representatives to forget their duty and act for the sake of personal profits, which happens less or more frequently depending on the country and may be related to certain relevant characteristics of a parliament. Hence, the question to be studied is which characteristics of the parliament favour the occurrence of bribing in the parliament.

Illegal lobbying procedures take place when interest groups influence a decision through bribes, favours, or promises of future benefits. In this chapter, we suggest a model to deal with the bribing of deputies in order to persuade them to vote for a decision preferred by the briber.

The analysis in this chapter is not only motivated by the empirical and theoretical considerations indicated above, showing that multipartism is associated with high corruption, but also by the data we collected concerning 172 countries. A simple descriptive statistical analysis of the data reveals that parliaments with more parties correspond to higher levels of corruption. These findings are presented in [Section 1.3](#).

The study of parliamentary structures helped us to define the framework of our theoretical model. And more importantly, this empirical exercise raised interesting questions to be answered. What favours parliamentary bribing? Having more or less parties? A greater or a smaller election threshold? An equal or an unequal allocation of seats between parties? The presence or the absence of a majority party?

To answer these questions, we propose a simple model to calculate the average bribing cost (measured in the number of members of the parliament) of an exogenous briber, interested in having a certain decision approved by the parliament. The briber tries to manipulate the

parliament decision by offering a payment to parties for the change of their votes on some proposed bill. Therefore, the briber needs to solve an optimisation problem: to obtain the support for the bill with minimal payments to parties.

As a first attempt in [Section 1.5](#), a precise formula of the average bribing cost in a parliament formed by two parties or by single-member parties is presented. Due to the complexity of the formulae, it is not possible to compare the results directly. Although intuition suggests that in a parliament with single member parties the briber never needs to bribe an excessive number of seats, it does not guarantee that the average bribing cost is lower, since there might be more cases when a briber needs to bribe some parties in the case of single-member parties, than in a parliament with two parties.

Since the optimisation problem of the briber is NP-hard² for more than two parties, analytically it seems extremely difficult to provide a general result, but it becomes possible if a computational approach is applied. [Section 1.6](#) provides a computational result, which tells that having more parties makes bribing less costly and, consequently, more bribing is likely to occur. Assuming different restrictions on the allocation of seats among parties, we also find that the egalitarian allocation of seats between parties is not always the most preferred method; that the election threshold reduces the number of seats to be bribed on average; and that the presence of a big party in parliament only affects bribing under a simple majority.

To be more realistic, the parties should differ not only in the number of seats in parliament, but also in their political views. Assume that the “Yes” answer favours the left wing. A strong right wing party is likely to be never bribed for a “Yes” decision, or the price to be paid for each seat will be much higher than for a central party or a right wing party with more liberal views. Therefore, for a specific question we can define a bribing propensity – a measure of the likelihood to accept a bribe. And, even if there are two parties that are equal in their political views, one could be more corruptible than the other. In [Section 1.7](#) we introduce the randomness of parties’ opinions about the bill and drop the assumption that all the parties are equally likely to accept the bribe. Taking this into account, the model is extended by introducing citizens who voted for the parties. We apply a restriction on the total amount

²Non-deterministic polynomial-time hard (NP-hard) is the complexity class of decision problems that are intrinsically harder than those that can be solved by a nondeterministic Turing machine in polynomial time.

of payments the briber can make. As distinguished from the previous sections, parties are assumed to be not equally likely to accept the bribe, an assumption that can be motivated simply by the existence of different levels of parties' corruption or by their political views. If the total amount is below the maximal total amount the briber agrees to pay, he obtains his desired decision. Otherwise, parties vote according to their initial standings.

Given the obtained results, it would be logical to apply a policy which reduces the number of parties. One of the institutional tools which can be applied is an electoral threshold. Theoretically, the higher the threshold the fewer the parties in the parliament. But, on the other hand, an electoral threshold can create a misrepresentation, as occurred in the 2015 Catalan (i.e., Catalonia, Spain) general elections. Each of 11 parties which participated in the elections were implicitly requested to declare their opinion on the matter of Catalonia's political independence: two of the parties strictly supported it and five of the parties did not (other parties were not clear about their views). In the election, 48.05% of the voters voted for the 'yes' parties and 50.82% voted for the 'no' parties. Yet, since one 'no' party and indifferent parties did not pass the electoral threshold of 3%, it follows that 72 MPs (53.33%) belong to 'yes' parties, and 63 MPs (46.66%) to 'no' parties. Therefore, the electoral threshold caused a significant difference between the opinions of voters and representatives. At the same time, the voters cannot be sure that the parties will vote for or against independence, as they promised. Their vote can change due to the variations of their initial position or because of the (il)legal pressure from outside of the parliament.

Inspired by this example, we analyse how the probability that the final decision by the parliament coincides with the popular will depends on the electoral threshold, taking into account both misrepresentation and ease to manipulate by the briber effects. The simulations show that, for thresholds below 6%, this probability is almost unaffected by the changes in the threshold. As for a threshold higher than 6%, this probability is decreasing with the threshold.

[Section 1.8](#) concludes the chapter. The tables summarising many of the numerical results are gathered in Appendix.

1.2 Related literature

Usually, corruption, and bribing as a part of corruption, are studied in relation to their negative influence on the social welfare of the country, on its economic growth (Mauro 2008, Shleifer and Vishny 1993), public investment (Tanzi and Davoodi 1998), foreign direct investments (Wei 2000), inflation (Myles and Yousefi 2015), or business regulation (Breen and Gillanders 2012, Banerjee 1997, Guriev 2004). Some research is focused on the factors favouring corruption and the ways to fight it. The key factors eliminating corruption are media freedom and political rights, and democratisation (Bhattacharyya and Hodler 2015), rule of law (for instance, Croix and Delavallade 2011 show that “if Zimbabwe had Denmark’s rule of law and democracy levels, its annual income growth would double and the level of corruption would decrease from 3.2 to 0.2, inferior to the Norwegian level”), . Recent research by Jetter et al. (2015), shows that democratisation reduces corruption only in economies that have already crossed a GDP per capita level of approximately US\$ 2,000 (in 2005), but for poorer nations democratisation seems to increase corruption. Not only the quality of democracy, but also the age of democracy plays a role. Treisman (2000) shows that older democracies foster significantly lower levels of corruption compared to younger democracies. Later empirical work by Persson et al. (2003) reinforced this conclusion.

The fundamental components of representative democracy are political parties, a parliament, elections, and electoral systems. In this field, there is a number of studies exploring the elections, voting rules, electoral districts and their vulnerability to corruption and bribing. Myerson (1993) finds that approval voting and proportional representation are fully effective in excluding corrupt parties, plurality voting is partly effective, and the Borda Count method of voting is ineffective. His findings were contested later by some empirical studies (Persson et al. 2003, Rose-Ackerman 2005, and Birch 2007, for instance) and by the work of Myerson (1999). Specifically, Kunicová and Rose-Ackerman (2005) show that proportional representation electoral systems are associated with higher levels of corruption than single-member district systems. Other studies do not find this relationship significant: for instance, Persson et al. (2003) find that switching from strictly proportional to strictly majoritarian elections only has a small negative effect on corruption.

Once the parliament is elected, the influence on its decision by exogenous pressure groups is widely studied in the lobbying literature (see, for instance, [Grossman and Helpman 1996](#), [Bennedsen and Feldmann, Bennedsen and Feldmann 2006](#)), and [Helpman and Persson 1998](#)). The positive side of lobbying is that lobbying groups provide information to policy makers (though there is a clear incentive to furnish only the information that favours the interest of the lobby, so that the decision by policy makers is likely to be biased). The negative side is just based on the fact that lobbying groups may bribe policy makers to decide in favour of their groups. This chapter is motivated by this possibility.

The illegal side of lobbying, such as direct bribing, is usually studied empirically. For instance, [Charron \(2011\)](#) investigates empirically the connection between party systems and corruption. He finds that multipartism in countries with dominance of single-member districts is associated with higher levels of corruption, while the party system's influence on corruption plays no role in countries with proportional representation. [Chang and Golden \(2007\)](#) show that a higher number of political parties makes it difficult for the public to monitor the behaviour of politicians and, therefore, corruption increases in multi-party systems. [Lederman et al. \(2005\)](#) demonstrate that democracies, parliamentary systems, political stability, and freedom of the press are all associated with lower corruption. Further, [Pelizzo \(2006\)](#) shows that the potential for corruption is inversely related to the parties' levels of institutionalisation, that is, the more a party is institutionalised, the less likely it is to become involved in corrupt practices.³ [Dal Bó \(2007\)](#) presents an interesting game-theoretical model on the exogenous influence over collective decisions. In particular, he proves that collusion

³According to [Pelizzo \(2006\)](#) the institutionalisation of political parties occurs when parties develop the following characteristics: adaptability, complexity, autonomy, and coherence. The level of adaptivity of a political party "reflects its age (how long the party has been in existence), its generational age (whether and how many times the party has been able to transfer power from one generation to the next one), and whether it has been able to adapt to environmental changes." The complexity of political parties reflects the combination of two sets of characteristics: the number of organisational levels (national, regional, provincial, local) and the number of units at each level of organisation. The level of autonomy is defined by whether the party is able to finance most of its activities with the revenue generated by the membership fees and dues; and whether it has a fairly developed bureaucratic apparatus and selects its leaders from within the party organisation.

among voters, such as political parties, and the existence of “disciplinary devices”, such as party discipline, might help to avoid lobbying (or bribing) or “setting its price”.

1.3 Stylised facts⁴

Our first aim is to investigate whether the empirical evidence suggests a connection between parliament structure and level of corruption of the country and political parties and parliament itself. First, we collected data for 172 countries concerning their parliaments and level of corruption. Our attention is restricted to year 2011, since this is the most recent year for which all the sources provide the most complete information.

It is difficult to define corruption and bribing. Moreover, as corruption and bribing generally are illegal, violators try to keep them secret. Furthermore, based on culture and traditions, what in one country is considered to be illegal bribing, in another country might be seen as a social norm. There are some debates on how corruption should be measured (see, for instance, [Svensson 2005](#)). As the general indicator of corruption we decided to take the World Governance Indicator of Control of Corruption (WGICC), as provided by the World Bank.⁵ It is an aggregate indicator that reflects “perceptions of the extent to which public power is exercised for private gain, including both petty and grand forms of corruption, as well as ‘capture’ of the state by elites and private interests”. The range goes from -2.5 (low control of corruption) to 2.5 (high control of corruption). Since we are interested in bribing

⁴This section is written in collaboration with Attila Tasnádi from the Department of Mathematics, Corvinus University of Budapest, Hungary.

⁵The World Governance Indicators (WGI) compile and summarise information from over 30 existing data sources that report the views and experiences of citizens, entrepreneurs, and experts in the public, private and NGO sectors from around the world, on the quality of various aspects of governance. The WGI draw on four different types of source data: surveys of households and firms (9 data sources including the Afrobarometer surveys, Gallup World Poll, and Global Competitiveness Report survey), commercial business information providers (4 data sources including the Economist Intelligence Unit, Global Insight, Political Risk Services), non-governmental organisations (11 data sources including Global Integrity, Freedom House, Reporters Without Borders), and public sector organisations (8 data sources including the CPIA assessments of World Bank and regional development banks, the EBRD Transition Report, French Ministry of Finance Institutional Profiles Database). For more, see [Kaufmann et al. \(2010\)](#) and the official Web page of World Governance indicators: <http://info.worldbank.org/governance/wgi/index.aspx>.

and corruption inside of the parliament, we selected two indicators, provided in the Global Corruption Barometer by Transparency International: perceptions of corruption of political parties (PP) and perceptions of corruption of parliament (PL).⁶ These indexes show people's perceptions of the extent to which some constituents of their democratic system are affected by corruption. For year 2011 these indexes are available for 98 countries of our sample. The indexes range from -5 (extremely corrupt) to -1 (not at all corrupt). For simplicity, since "the greater the index is the less there is corruption" we had to invert these indicators, which originally ranged from 1 (not at all corrupt) to 5 (extremely corrupt).

If the parliament is bicameral, only the lower house is taken into account. The size of parliament equals the number of representatives forming the parliament. In almost 70% of the 116 countries there are no independent representatives. In 56 countries (32.5 %) there are independent deputies. In [Table 1.1](#) the basic descriptive statistics are presented for the total sample and separately for these categories.

On average, a parliament consists of around 216 seats. It is noteworthy that the Chinese parliament, consisting of 2987 deputies, has an extremely high number of representatives. Excluding China from the sample gives almost 200 seats on average and 182 seats on average among those parliaments without independent deputies (without significant change in other variables). It may be inferred from the data that, generally, big parliaments tend to have independent deputies, or that allowing independent deputies creates big parliaments.

On average, the number of parties is 8.73, half of the countries have no more than 6 parties (5 and 8 for parliaments without and with independent deputies, respectively). In the whole sample, India has the maximum number of parties with 37 parties. In parliaments without independent deputies, the maximum number of parties can be found in Chad with 31 parties.

In 56 countries of the second group of countries having independent deputies, on average there are almost 20 independents. Half of these countries have no more than 8 independent deputies. There are some extreme cases, such as Afghanistan with 164 independents and

⁶The Global Corruption Barometer is a public opinion survey that offers views of the general public on corruption and its impact on their lives, including personal experiences with bribes, see more at <https://www.transparency.org/gcb201011>.

Table 1.1. Descriptive statistics of parliament characteristics and corruption level in 172 countries in 2011.

Variable	Total		No independent deputies		Independent deputies	
	Number of countries	Mean	Number of countries	Mean	Number of countries	Mean
Size of the parliament	172	216.10 (*)	116	206.30 (**)	56	236.30
Number of parties	172	8.73	116	7.27	56	10.77
Number of independent deputies	172	6.50	116	0	56	19.9
World Governance Indicator (WGICC) -2.5 - low control of corruption 2.5 - high control of corruption	172	-0.077	116	-0.060	56	-0.364
Perceptions about corruption of political parties (PP): -5 - extremely corrupt -1 - not at all corrupt	99	-1.26	60	-1.23	39	-1.32
Perceptions about corruption of parliament (PL): -5 - extremely corrupt -1 - not at all corrupt	99	-1.50	60	-1.52	39	-1.46

Without the parliament of China, the average size would be 200 (*) and 182 (**)

SOURCE: World Bank Data, Transparency International.

20 parties, Belarus with 125 independents and 3 parties in a parliament of 125 seats, and Syria with 81 independents and 11 parties. There is the problematic issue of how to treat

independent deputies: should each be treated as a separate party or as a party called “others” with(out) party discipline? We have decided to restrict our attention to 116 parliaments, formed only by parties.

Taking as a fact that corruption and bribing are negatively correlated with income (see, for instance, [Mauro 2008](#), [Shleifer and Vishny 1993](#), [Tanzi and Davoodi 1998](#)), we divide these countries into four groups by income level using the classification provided by the World Bank.⁷ [Table 1.2](#) shows the average values for these four groups of countries.

Higher income countries on average have fewer parties than lower income countries. The size of parliament in the upper-middle-income group is biased due to the parliament of China: without it, the average size of the parliament would be 181.54 (with almost no change in the other variables), and the conclusion is that on average the parliament is bigger in higher income countries. This conclusion can be explained by the fact that higher income countries (with some exceptions) have been enjoying democracy for longer, so that parliaments may be bigger in order to better represent the more diverse views of the citizens. At the same time, the average number of parties is smaller in higher income countries, which can be explained by the fact that, in the long run, the small parties do not eventually survive or they form coalitions in order to be more effective or just to survive.

On average, the political parties and the parliament are perceived to be highly corrupt in upper-middle-income countries (-3.98 and -3.85, respectively) and are perceived as the least corrupt in low-income countries (-3.37 and -3.13, respectively), but the difference is not as crucial as for WGICC.

Stylised fact 1. *Parliaments in high-income countries have more seats and fewer parties than parliaments in low-income countries.*

Stylised fact 2. *Parties in the parliaments of low-income countries are not perceived as corrupt as parties in the parliaments of high- and middle-income countries.*

⁷Economies are currently divided into four income groupings: low, lower-middle, upper-middle, and high. Income is measured using gross national income (GNI) per capita, in U.S. dollars, converted from local currency using the World Bank Atlas method. Estimates of GNI are obtained from economists in World Bank country units; and the size of the population is estimated by World Bank demographers from a variety of sources.

Table 1.2. Parliament structure and corruption: averages by income level groups in 2011.

Variable	High-income	Upper- middle- income	Lower- middle- income	Low-income
Size of the parliament (number of seats)	213.61	260.57*	184.100	147.727
Number of parties in the parliament	6.58	6.89	7.82	8.45
World Governance Indicator (WGICC) -2.5 - <i>low control of corruption</i> 2.5 - <i>high control of corruption</i>	1.210	-0.097	-0.527	-0.807
Perceptions of corruption of political parties -5 - <i>extremely corrupt</i> -1 - <i>not at all corrupt</i>	-3.686	-3.982	-3.8274	-3.367
Perceptions of corruption of the parliament -5 - <i>extremely corrupt</i> -1 - <i>not at all corrupt</i>	-3.231	-3.847	-3.553	-3.133
Number of countries	33	37	28	18

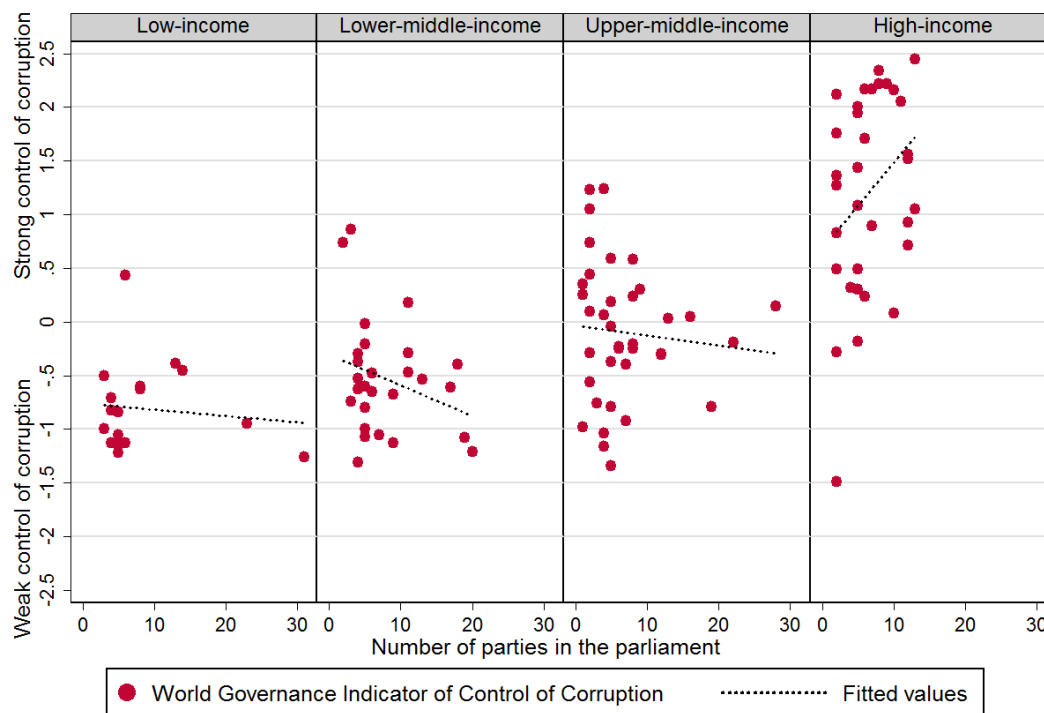
* Without the parliament of China the average size would be 184.83.

SOURCE: World Bank Data, Transparency International.

High-income countries enjoy the strongest control of corruption (1.21 on average) while the low-income countries experience weak control of corruption (-0.87 on average). This empirical evidence is confirmed by the aforementioned studies.

Stylised fact 3. *There appears to be a strong positive correlation between control of corruption and income of a country.*

Figure 1.1 illustrates the stylised fact 3 and addresses the question of whether the control of corruption decreases with the increase in the number of parties.



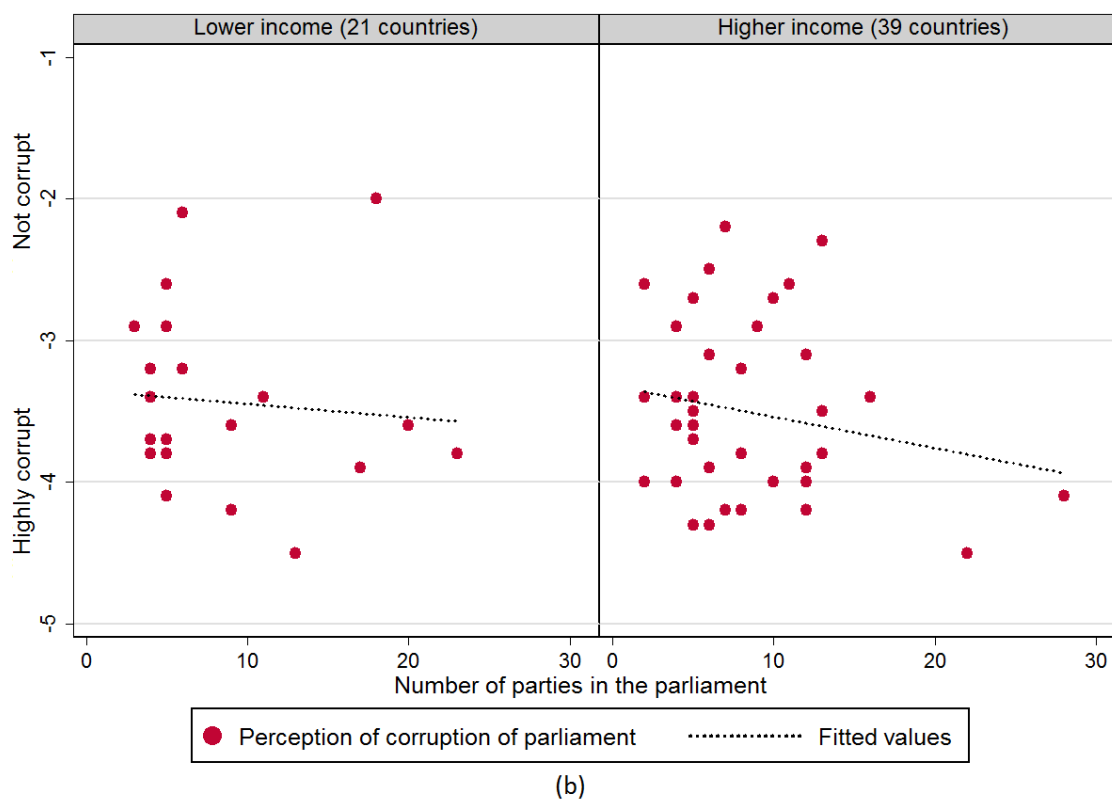
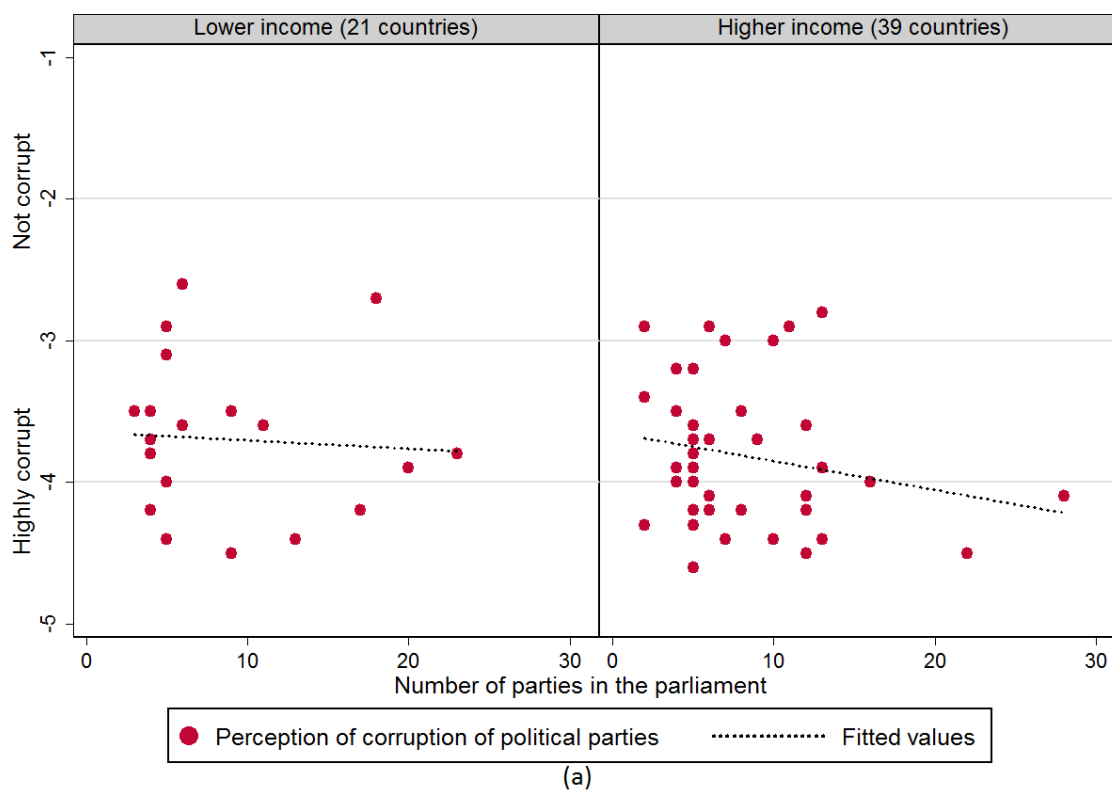
Source: World Bank Data, year 2011.

Figure 1.1. World Governance Indicator of Control of Corruption correlated with the number of parties for 116 countries in year 2011.

The linear fit indicates a negative relationship between the control of corruption and the number of parties: with more parties there seems to be associated a lower level of control of corruption. In all groups except the high-income group there is a decrease in the control of corruption shown by linear fitted values.

In high-income countries democracy was established a long time ago and has become more mature. With the development of democracy, parliamentary parties become more efficient, stable, and started to form coalitions, but after some time, more parties with more focus on specific issues begin to appear, such as the Party for the Animals in the Dutch Parliament, Animal Justice Party in the Legislative Council of New South Wales (Australia), and green parties. Such parties are hardly known in low-income countries. The conjecture is then that

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Source: Transparency International, Global Corruption Barometer, year 2011.

Figure 1.2. Perception of corruption of (a) political parties and (b) parliaments correlated with the number of parties for 116 countries in year 2011.

a more developed democracy tends to produce a small rise in the number of parties, because in mature democracies most big issues have been resolved (or there is an agreement on them among most parties) and citizens may start worrying about issues or problems which are perceived as secondary (like animal rights), or viewed as less important in younger democracies.

Due to the low number of observations for indexes PP and PL, we merged the four income groups into two groups: “lower income” (21 countries) and “higher income” (39 countries). Both groups show that the higher the number of parties in the parliament, the more the parliament and political parties are perceived as corrupt.

Stylised fact 4. *In countries with more parties in the parliament people seem to perceive the parliament and the political parties to be more corrupt, especially in high-income countries.*

1.4 The model

We consider a parliament given. We shall denote by m the number of parties in the parliament, by n the number of seats in the parliament, and by $q \leq n$ the minimum number of favourable votes required to pass a bill. Furthermore, we shall denote by $M = \{1, \dots, m\}$ the set of parties and by n_1, \dots, n_m their respective number of representatives. Hence, $\sum_{i=1}^m n_i = n$ and $n_i \geq 1$ for all $i \in M$.

We assume that there is party discipline, that is, concerning any bill all representatives of a party vote in the same way. Besides simplifying our calculations, our assumption of party discipline can be partly justified by the observation (see, for instance, [Bowler et al. 1999](#)) that in many countries, especially in the case of important issues, after party internal discussions, all representatives of a party vote in the same way. Hence, in our model parties are the decision makers and vote with either yes or no, henceforth, denoted by Y and N , respectively. The parties' initial decisions will be denoted by $d_1, \dots, d_m \in \{Y, N\}$, which might be determined by the parties' ideological standpoints, while the set of parties initially supporting a bill will be denoted by $D = \{i \in M \mid d_i = Y\}$ and the respective number of their representatives by $n_D = \sum_{i \in D} n_i$.

Since we aim at establishing a connection between the number of parties and the vulnerability with respect to corruption, we will admit several sets of possible allocations of seats

and all possible initial decisions for given m , n , and q . We may restrict the set of all possible allocations of seats by a lower bound L and by an upper bound U on party sizes to obtain a possibly restricted set of admissible seat allocations, where we will denote the respective set of admissible seat allocations and all possible initial decisions by

$$\Omega_n^m(L, U) = \left\{ (n_1, \dots, n_m, d_1, \dots, d_m) \in \{1, \dots, n\}^m \times \{Y, N\}^m \mid \sum_{i=1}^m n_i = n, L \leq n_i \leq U \right\}$$

and we will refer to any $\omega \in \Omega_n^m(L, U)$ as a possible *state* of the parliament. In most cases, the set of states will be relatively large because we do not want to restrict ourselves to already observed seat distributions of the past since in the future any admissible seat distribution might be possible. The set of all possible allocations of seats, which we also call the unrestricted case, can be obtained by setting $L = 1$ and $U = n - m + 1$, meaning that each party at least has 1 seat and can occupy a maximum of $n - m + 1$ seats, which guarantees that each party has at least one seat.

For a given allocation n_1, \dots, n_m of seats among parties, and initial decisions d_1, \dots, d_m , a briber may consider bribing parties in order to get a bill passed if it did not already obtain sufficient support, that is, if $n_D < q$. We assume for simplicity that the cost of bribing a party (i.e., to turn its initial N to Y) is proportional to its number of representatives, and therefore we measure the associated cost of bribing by its size. Assuming bribing costs proportional to the sizes of parties is a natural starting assumption in the case of a lack of further information. Clearly, factors such as the parties' initial commitments to their decisions or party size may have an effect on bribing costs in a more general setting. However, trying to incorporate these additional factors would increase the number of free parameters in our model, which also would critically slow down the computing of the bribing costs. Consequently, for a given allocation of seats and for a given state $\omega \in \Omega_n^m(L, U)$, the cost of bribing, which we shall denote by $\beta(\omega)$, is defined as

$$\beta(\omega) = \min \left\{ \sum_{i \in I} n_i \mid I \subseteq M \setminus D(\omega) \text{ and } \sum_{i \in I} n_i \geq \max\{q - n_D(\omega), 0\} \right\}. \quad (1.1)$$

There are several links to the literature on power indexes and the way how we formulated

the briber's problem (1.1). While function β is single-valued, it would have been possible to define values for each parties by the minimisation problem (1.1), that is, to introduce a new power index by determining the number of cases in which a party collects a bribe or the amount it receives by taking the average values of the respective solutions of problem (1.1). Therefore, it would be possible to obtain a modified, restricted, and weighted Deegan and Packel (1978) power index. The obtained power index would be modified in the sense that minimal winning coalitions would be replaced with minimum winning coalitions, restricted in the sense that parties from set D would be excluded from the calculations, and weighted in the sense that seats would serve as weights. Though it would be interesting to consider the respective power indexes, we do not, since we are only interested in the briber's costs.

Instead of assuming that the cost of bribing a party is proportional to its number of representatives, another approach would have been to assume that the cost of bribing a party was proportional to its power index. However, this would have raised two problems: the choice of the appropriate power index and an extreme increase in computation time since determining power indexes is a computationally complex task, as has been shown by Matsui and Matsui (2001).

In a spirit similar to Laruelle and Valenciano (2002), in the case of power indexes, we allow for all possible decisions of parties, and we assume that any admissible state emerges with equal probability, which is a particular case in Laruelle and Valenciano (2002). Hence, we define the average bribing cost by

$$A_n^m(L, U) = \frac{\sum_{\omega \in \Omega_n^m(L, U)} \beta(\omega)}{\#\Omega_n^m(L, U)}. \quad (1.2)$$

The aim of Example 1 is to illustrate the model.

Example 1. Consider a parliament with six members $n = 6$ distributed among $m = 2$ parties and a simple majority voting rule, so that $q = 4$. There are no restrictions on the number of seats that a party can occupy, hence, $L = 1$ and $U = 5$.

If $\#D = 2$, that is, both parties vote Y, there are five ways of dividing six seats between two parties, so there are 5 states of the parliament.

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States, ω	n_D	Need to bribe	Seats to bribe, $\beta(\omega)$
(1, 5, Y, Y)	5	No	0
(2, 4, Y, Y)	5	No	0
(3, 3, Y, Y)	5	No	0
(4, 2, Y, Y)	5	No	0
(5, 1, Y, Y)	5	No	0

For clarity we state that $\omega = (1, 5, Y, Y)$ stands for a parliament where party 1 has 1 seat and party 2 has 5 seats, and both parties vote positively. In none of these states is there a need to bribe: $\beta(\omega) = 0$ for every state.

If $\#D = 1$, that is, one party votes Y, this party includes n_D members. As a result, n_D can take values from 1 to 5. Accordingly, there are 10 possible states of the parliament.

States, ω	n_D	Need to bribe	Seats to bribe, $\beta(\omega)$
(1, 5, Y, N)	1	Yes	1
(2, 4, Y, N)	2	Yes	2
(3, 3, Y, N)	3	Yes	3
(4, 2, Y, N)	4	No	0
(5, 1, Y, N)	5	No	0
(1, 5, N, Y)	5	No	0
(2, 4, N, Y)	4	No	0
(3, 3, N, Y)	3	Yes	3
(4, 2, N, Y)	2	Yes	4
(5, 1, N, Y)	1	Yes	5

It is obvious that states of the parliament where party 1 votes Y and where party 2 votes Y are symmetric. In states where $n_D \geq 4$, there is no need to bribe; if $n_D < 4$, there is a need to bribe the other party, that is, $\beta(\omega) = n - n_D$ members.

If $\#D = 0$, that is, no party voted Y, there are 5 states of the parliament, and in each of them there is a need to bribe.

States, ω	n_D	Need to bribe	Seats to bribe, $\beta(\omega)$
(1, 5, N , N)	0	Yes	5
(2, 4, N , N)	0	Yes	4
(3, 3, N , N)	0	Yes	3
(4, 2, N , N)	0	Yes	4
(5, 1, N , N)	0	Yes	5

If party 1 has fewer seats than the voting quota of 4 and party 2 is greater than 4 or equal to 4, there is a need to bribe party 2. That is, if $n_1 = 1$ or 2, then there is a need to bribe $n_2 = n - 1 = 5$ or 4 members, respectively. If party 1 has at least four seats and party 2 has less than four seats, then there is a need to bribe party 1. That is if $n_1 = 4$ or 5, then there is a need to bribe 4 and 5 members, respectively. If both parties are smaller than 4, then there is a need to bribe them both, that is, 6 members.

By 1.2, the average bribing cost is the sum of the minimum number of members of the parliament that have to be bribed, divided by the total number of states:

$$A_n^m(1, 5) = \frac{0 + 5 + 4 + 3 + 3 + 4 + 5 + 5 + 4 + 4 + 5 + 6}{5 + 5 + 5 + 5} = 2.4.$$

Therefore, in the parliament with 6 members, having a simple majority rule, and the presence of party discipline, the briber needs to bribe on average almost 2.5 members.

Replacing the uniform distribution above, the set of admissible states would require further empirical analysis and would result in a dramatic increase in the number of parameters in a modified version of expression (1.2), as well as a critical computational slowdown in its evaluation. Looking at expressions (1.1) and (1.2), we can see that parties do not act strategically since we take all possible profiles of party decisions into consideration. Thus, our problem is of a combinatorial nature. It can be verified that the cardinality of $\Omega_n^m(1, n - m + 1)$ in the unrestricted case equals $\binom{n-1}{m-1} 2^m$,⁸ which makes a brute force algorithm quite time-consuming. Luckily, the brute force algorithm enabled us to determine average bribing costs

⁸There are 2^m ways of assigning to m parties one of two possible decision: Yes or No. For each possible assignment there are $\binom{n-1}{m-1}$ ways of dividing n seats among m parties.

for up to 10 parties within a few hours. The algorithm consists of generating all possible allocations of seats (we have taken symmetries into account in order to speed up the procedure), generating all possible initial decisions and then solving problem (1.1) for each possible state of the parliament.

To determine the computational complexity of calculating the average bribing cost, we will need the recognition version of the famous knapsack problem, which was shown by Karp (1972) to be NP-complete. The knapsack problem is specified in the following way: consider a knapsack that has a limit weight of $W \in \mathbb{Z}_+$, which can be filled with m objects of weights $w_1, \dots, w_m \in \mathbb{Z}_+$ and respective values $v_1, \dots, v_m \in \mathbb{Z}_+$. The target goal is to pick a subset of objects $I \subseteq M$ such that their total value is at least $V \in \mathbb{Z}_+$. The knapsack problem asks whether there exists a set $I \subseteq M$ such that $\sum_{i \in I} w_i \leq W$ and $\sum_{i \in I} v_i \geq V$. From Papadimitriou (1994, p. 202) we also know that the problem is NP-complete if $v_i = w_i$ for all $i \in M$ and $V = W$. Now, we can check that the recognition version of problem (1.1) is NP-complete. In the recognition version of problem (1.1), minimising bribes is replaced with the question of whether passing the bill is feasible by bribing at most a given number of b representatives, i.e., there exists an $I \subseteq M \setminus D$ such that

$$\sum_{i \in I} n_i \leq b \text{ and } \sum_{i \in I} n_i \geq \max\{q - n_D, 0\}. \quad (1.3)$$

Hence, in order to obtain a polynomial time reduction from the restricted version of the knapsack problem to problem (1.3), let $D = \emptyset$, $n_i = v_i$ for all $i \in M$, $q = V$ and $b = V$. Therefore, if $P \neq NP$, then there does not exist a polynomial time algorithm for solving problem (1.1), and nor does one exist for (1.2).

Nehama (2015) investigated the computational complexity of a problem related to (1.1) (the so-called optimal lobbying problem), which is on the one hand more general than problem (1.1), since it allows to vote on multiple issues, but on the other hand is more restrictive than (1.1), since it gives each agent (i.e., voter) the same weight, while in our problem (1.1) each agent (i.e., party) can receive a different weight.

1.5 Analytical approach

As we have already noted, the problem of choosing parties to be bribed is NP-hard, which makes it hard to solve analytically. As a first attempt we propose a solution for the particular cases of two parties and n parties.

1.5.1 Parliament with two parties

The case of two parties is a significant one. Firstly, as the empirical data tell, 10% of studied countries have exactly two parties in parliament. A second reason is motivated by the fact that in voting for specific motions, opinions may easily become polarised or dichotomic, so that it is as if only two parties existed. In fact, voting for a particular question could split the parliament into two parts, one representing the government party and the other representing the opposition parties, or one representing left-wing politics and the other representing right-wing politics.

The case of n parties (or n single member parties) can be considered as an example of voting by the citizenship on a certain proposal. In real life there are certain questions that cannot be expected to be answered sincerely by the members of the parliament. For instance, voting for reducing the number of members of the parliament. Besides, there are some crucial decisions which perhaps should be voted for by all the inhabitants of a country. For example, should the form of government change from monarchy to republic? Is part of the nation (Scotland, Québec, Catalonia) to be allowed to secede from the whole nation (United Kingdom, Canada, Spain, respectively)? Why should such a crucial question be under the voting of only a few hundred of representatives? Constitutions have selected certain issues to be voted by citizens themselves directly (referendum) - such as reforms of the constitution itself or the final approval of international treaties.

As the last point of the analysis in this section, we provide an intuition for comparison of the two results obtained, [Proposition 1](#) and [2](#).

First consider a parliament with two parties, the members of which are obliged to cast the vote of the party. We assume that each party can occupy at least 1 seat and at most $n - 1$ seats, that is, $L = 1$ and $U = n - 1$. The next proposition tells us what is the average

bribing cost in this case.

Proposition 1. *For a given voting quota $q \in [1, n]$, $L = 1$ and $U = n - 1$, a parliament with n members and two parties, the average bribing cost A_n^2 is*

$$A_n^2 = \frac{(q-1)(2n-q)}{2(n-1)}, \text{ if } q > n/2, \quad (1.4)$$

$$A_n^2 = \frac{(q-1)(2n-q) + 3/4n^2 - n}{4(n-1)}, \text{ if } q = n/2, \quad (1.5)$$

$$A_n^2 = \frac{(q-1)(4n-3q) + 1/4n^2}{4(n-1)}, \text{ if } q < n/2 \text{ and } n \text{ is even}, \quad (1.6)$$

$$A_n^2 = \frac{(q-1)(4n-3q) + 1/4(n^2-1)}{4(n-1)}, \text{ if } q < n/2 \text{ and } n \text{ is odd}. \quad (1.7)$$

Proof. There are four possible assignments of the Y and N vote to two parties. And, for each assignment there are $n - 1$ ways of dividing n seats among parties with each party getting at least one seat, but not all seats. Hence, the total number of states is $4(n - 1)$. Consider each possible assignment separately.

Case 1. $\#D = 2$, in other words, both parties vote positively. There is no need to bribe: $\beta = 0$ for any ω .

Case 2. $\#D = 1$, parties vote differently: $d_1 \neq d_2$. Since there are two symmetric cases, assume, without loss of generality, that party 1 votes Y, that is, $d_1 = Y$. When $n_1 \in [q, n - 1]$, there is no need to bribe. When $n_1 \in [1, q - 1]$, then in each state it is necessary to bribe the other party, $n_2 = n - n_1$. Hence, the aggregate bribe for such cases is $\beta = \sum_{i=1}^{q-1} (n - i) = n(q - 1) - \frac{q(q-1)}{2}$. For the whole case the bribe is

$$n(q-1) - \frac{q(q-1)}{2} + n(q-1) - \frac{q(q-1)}{2} = (2n-q)(q-1).$$

Case 3. $\#D = 0$, that is, both parties vote N. In all possible states in this case it is necessary to bribe one party or both parties, depending on q .

Case 3a. $q > n/2$. When party 1 has less than $n - q$ seats, that is, $n_1 \in [1, \dots, n - q]$, there is a need to bribe the other party: $n_2 = n - n_1$, therefore, for such states the aggregate bribe is equal to $\sum_{i=1}^{n-q} (n - i) = \frac{n(n-1)}{2} - \frac{q(q-1)}{2}$. When $n_1 \in (n - q, \dots, q - 1]$, there is a need to bribe both parties (n members) in each of $2q - n - 1$ states, so, $\beta = n(2q - n - 1)$.

When $n_1 \in [q, n-1]$, then it is necessary to bribe this party, hence the aggregate bribe is $\sum_{i=q}^{n-1} i = \frac{n(n-1)}{2} - \frac{q(q-1)}{2}$. The aggregate bribe for this subcase is:

$$n(n-1) - q(q-1) + n(2q-n-1) = (q-1)(2n-q).$$

Case 3b. Next, consider $q = n/2$, which implies that n is even. While $n_i \in [1, \dots, q-1]$, there is a need to bribe the other party, for these states the aggregate bribe is $\sum_{i=1}^{n/2-1} (n-i) = n(q-1) - \frac{q(q-1)}{2}$. If $n_1 = n_2 = q = n/2$, it is necessary to bribe one of the two parties: $\beta = n/2$. When $n_1 \in (q, \dots, n-1]$, there is a need to bribe party 1; the aggregate bribe for these case equals to $\sum_{i=q+1}^{n-1} i = \frac{n(n-1)}{2} - \frac{t(t+1)}{2}$. Aggregating the bribe for this subcase gives:

$$n(q-1) - \frac{q(q-1)}{2} + n/2 + \frac{n(n-1)}{2} - \frac{t(t+1)}{2} = \frac{3n^2}{4} - n.$$

Case 3c. n is even and $q < n/2$. When $n_1 \in [1, q-1]$, there is a need to bribe the other party, the aggregate bribing cost is $\sum_{i=1}^{q-1} (n-i) = n(q-1) - \frac{q(q-1)}{2}$. When $n_1 \in [q, n/2-1]$, there is a need to bribe this party, $\beta = \sum_{i=q}^{n/2-1} i = \frac{n(n-2)}{8} - \frac{q(q-1)}{2}$. When $n_1 = n_2 = n/2$, $\beta = n/2$. When $n_1 \in [n/2+1, n-q]$, $\beta = \sum_{i=n/2+1}^{n-q} (n-i) = \frac{n(n-2)}{8} - \frac{q(q-1)}{2}$. When $n_1 \in [n-q+1, n-1]$, then $\beta = \sum_{i=1}^{q-1} (n-i) = n(q-1) - \frac{q(q-1)}{2}$. Aggregating gives total seats to be bribed in this subcase:

$$2n(q-1) - 2q(q-1) + \frac{n(n-2)}{4} + \frac{n}{2} = 2(n-q)(q-1) + \frac{n^2}{4}.$$

Case 3d. n is odd and $q < n/2$. When $n_1 \in [1, \dots, q-1]$, there is a need to bribe the other party, $\beta = \sum_{i=1}^{q-1} (n-i) = n(q-1) - \frac{q(q-1)}{2}$. When $n_1 \in [q, (n-1)/2]$, there is a need to bribe this party: $\beta = \sum_{i=q}^{(n-1)/2} i = \frac{(n-1)(n+1)}{8} - \frac{q(q-1)}{2}$. When $n_1 \in [(n+1)/2, n-q]$, there is a need to bribe n_2 : $\beta = \sum_{i=(n+1)/2}^{n-q} (n-i) = \frac{(n-1)(n+1)}{8} - \frac{q(q-1)}{2}$. When $n_1 \in (n-1, n)$, then there is a need to bribe this party: $\beta = \sum_{i=1}^{q-1} (n-i) = n(q-1) - \frac{q(q-1)}{2}$. Aggregating total seats to be bribed gives:

$$2n(q-1) - 2q(q-1) + \frac{(n-1)(n+1)}{4} = 2(n-q)(q-1) + \frac{n^2-1}{4}.$$

Aggregating all these cases together and taking into account the total number of states $4(n-1)$ proves [Proposition 1](#).

□

It is easy to verify that, for relatively big n , the difference between (1.6) and (1.7) is $\frac{1}{16(n-1)}$ which is less than $1/16$, so there is practically no difference between A_n^2 for even and odd parliaments if $q < n/2$.

Example 2. Consider an odd parliament, and simple majority, that is, $q = \frac{n+1}{2}$. By (1.4), the average bribing cost is $A_n^2 = \frac{3n-1}{8} \approx \frac{3}{8}n$. In other words, under a simple majority the briber may need to bribe, on average, three eighths of the parliament, which is a little bit more than one third. Consider a qualified majority, with two-thirds rule: $q = 2n/3$. For simplicity, assume that n is divisible by 3 and let n be odd. By 1.4, the average bribing cost is $A_n^2 = \frac{4n(n-3/2)}{9(n-1)} \approx \frac{4}{9}n$. Increasing the voting quota from a simple majority to two-thirds makes the average bribing cost increase by up to four ninths of the parliament, which is less than half of the parliament.

1.5.2 Parliament with n parties

We consider next a parliament where each member represents a party. It can be seen as relaxing the assumption of party discipline. Proposition 2 indicates what the average bribing cost is in this case.

Proposition 2. *For a given voting rule $q \in [1, n]$ and a parliament with n parties consisting of a single member, the average bribing cost is*

$$A_n^n = \frac{1}{2^n} \sum_{k=0}^{q-1} (q-k) \binom{n}{k} \quad (1.8)$$

Proof. There is only one way of dividing n seats among m parties, since $m = n$. The state of the parliament is defined by the members of set D – the parties which voted Y. Hence, there are 2^n states of the parliament.

The number of states in which k parties vote Y is equal to the binomial coefficient $\binom{n}{k}$. If $k \in [q, \dots, n]$, there is no need to bribe. If $k \in (0, \dots, q-1]$, there is a need to bribe $q-k$ members for each state out of $\binom{n}{k}$ states. Summing up and dividing by the total number of states gives the average bribing cost for $m = n$ parties $A_n^n = \frac{1}{2^n} \sum_{k=0}^{q-1} (q-k) \binom{n}{k}$.

□

The comparison of parliaments with two and n parties suggests the following intuition: in the case of two parties, under the assumption that members of the party are voting according to the party line, on average a briber needs to bribe more votes than it is exactly needed to achieve the desired outcome; in the case of n parties, there is no state where it is necessary to bribe seats in excess. Although the number of states with the respect to which the average bribing cost is calculated is greater for the case of n parties for $n > 3$, and at the same time the bribe is presumed to be smaller, these facts do not allow a precise comparison between these cases. This is because it is not clear whether there are more or less states of the parliament when bribing occurs, which affects the number of components of the sum in the numerator of A .

Due to the complexity of the problem, the analytical approach involves dealing with too many possibilities to determine the precise formulae for more than 2 parties. Hence in the next session the solution is investigated using a computational approach.

1.6 Computational approach⁹

We provide the results of the calculations for a parliament of 100 seats formed by a number of parties ranging from 2 to 10. We restrict the number of parties for computational reasons. For instance, for 10 parties there are more than a quadrillion possible states of the parliament, and in each of them an optimal decision has to be chosen. The data from [Section 1.3](#) show that on average there are 7 parties in parliaments without independent deputies, with half of the countries having no more than 5 parties and 77% having no more than 10 parties, and because of this the restriction to ten parties seems reasonable and justifiable.

We provide the results for each possible voting quota $1 \leq q \leq n$, but mostly we concentrate our attention on majority rules and, specifically, the simply majority, two-thirds, three-quarters, four-fifths majority, and unanimity.

There are four cases to be considered, depending on the restrictions on the party size:

1. All party sizes are possible, which we call the unrestricted case,
2. Similarly sized parties,

⁹This section is written in collaboration with Attila Tasnádi from the Department of Mathematics, Corvinus University of Budapest, Hungary.

3. Without small parties,
4. Without big parties.

The first case is the most general, as any allocation of seats between parties is possible.

The second case is motivated by the analysis of the allocation of seats among parties in 172 countries, which indicates that in some parliaments seats are divided almost equally between parties. The two-party parliament of Malta provides a perfect example: After general elections in 2008, out of 69 seats, the Nationalist Party acquired 35 seats and the Malta Labour Party acquired 34 seats. We might compare political parties that are competing for seats with firms that are competing for a market share. On many product markets we can observe an increasing level of concentration, i.e., having increasingly fewer ‘non-negligible’ firms on the market. The same logic can be applied to the “political market”, where the firms are the parties and the consumers are the voters. Besides, some voting procedures favour the emergence of big parties, hence the small parties have incentives to form federations and run elections together; for example, the Catalan party CiU, which was a federation of two constituent parties for more than thirty years, the Democratic Convergence of Catalonia and the Democratic Union of Catalonia.

Case 3 is motivated by the existence of the restriction on parties (or candidates from a party) to run in the elections. For instance, in the United Kingdom a candidate for the parliamentary election is required to present a signed assent of ten registered electors plus a deposit of £500 which is forfeited if the candidate wins less than 5% of the vote. A typical restriction for entering parliament is to obtain a certain proportion of votes (election threshold). Usually, the election threshold is 5-7%. In some countries a threshold of 10% is implemented. In a purely proportional electoral system in a parliament of 100 seats one seat represents 1% of the votes. Furthermore, an election threshold of 5% means that no party occupies less than 5 seats.

Case 4 is motivated both by empirical evidence and by the symmetry of Case 3. In 55 countries out of 116 (47%) there is a party which occupies the majority of seats. In 24 countries there is a party with more than 70% of the seats. In 13 of the countries there is a party with more than 80% of the seats. Further, in 8 countries there is a party which occupies more than 90% of the seats. Another motivation is as follows. If it is relevant to

impose a lower limit on parties' vote share ratio (or have another rule which eliminates small parties), why not impose an upper limit? Although there are no such limits in practice, it would be interesting to see what effects there would be.

1.6.1 Unrestricted case

The average bribing cost decreases with the increase in the number of parties, as [Figure 1.3](#) illustrates. The maximum average bribing cost (50 seats) is achieved under unanimity for any number of parties, which is obvious, since positive and negative decisions are equally likely. The decrease is less in proportion to the increase in the number of parties and to the increase in the required level of support. The two-party parliament appears to be the least vulnerable to bribing, in the sense that in this case the bribing cost is the highest.

It is noteworthy that the average bribing cost of a parliament with less parties can be replicated in a parliament with more parties by changing the voting rule. For instance, under a simple majority the average bribing cost of the 2-party parliament is 37.2 seats, while for the 3-party parliament it is 29 seats. By changing the voting rule from a simple majority to two-thirds, the average bribing cost becomes 37.6 seats for the 3-party parliament. Such a change seems to be an extremely large one. But for more parties such a large change is not required. For instance, under a simple majority the average bribing cost for a parliament with 7 parties is 13.5 seats. To replicate this cost in a parliament with 8 parties, it is enough to increase the voting threshold from 51 to 53, where the average bribing cost would be 13.7 for a parliament with 8 parties. All these observations and [Figure 1.3](#) give us the following result.

Result 1. *If in a parliament with at most ten parties all possible seat allocations are equally likely, then for any voting quota necessary to reach a decision the average bribing cost decreases as the number of parties increases.*

Generally, the policy makers (or “designers” of the parliament) might be interested in restricting access of parties to the parliament, or in forcing parties to form coalitions and, therefore, decreasing the number of parties in order to increase the cost of bribing, and thus discourage corruption.

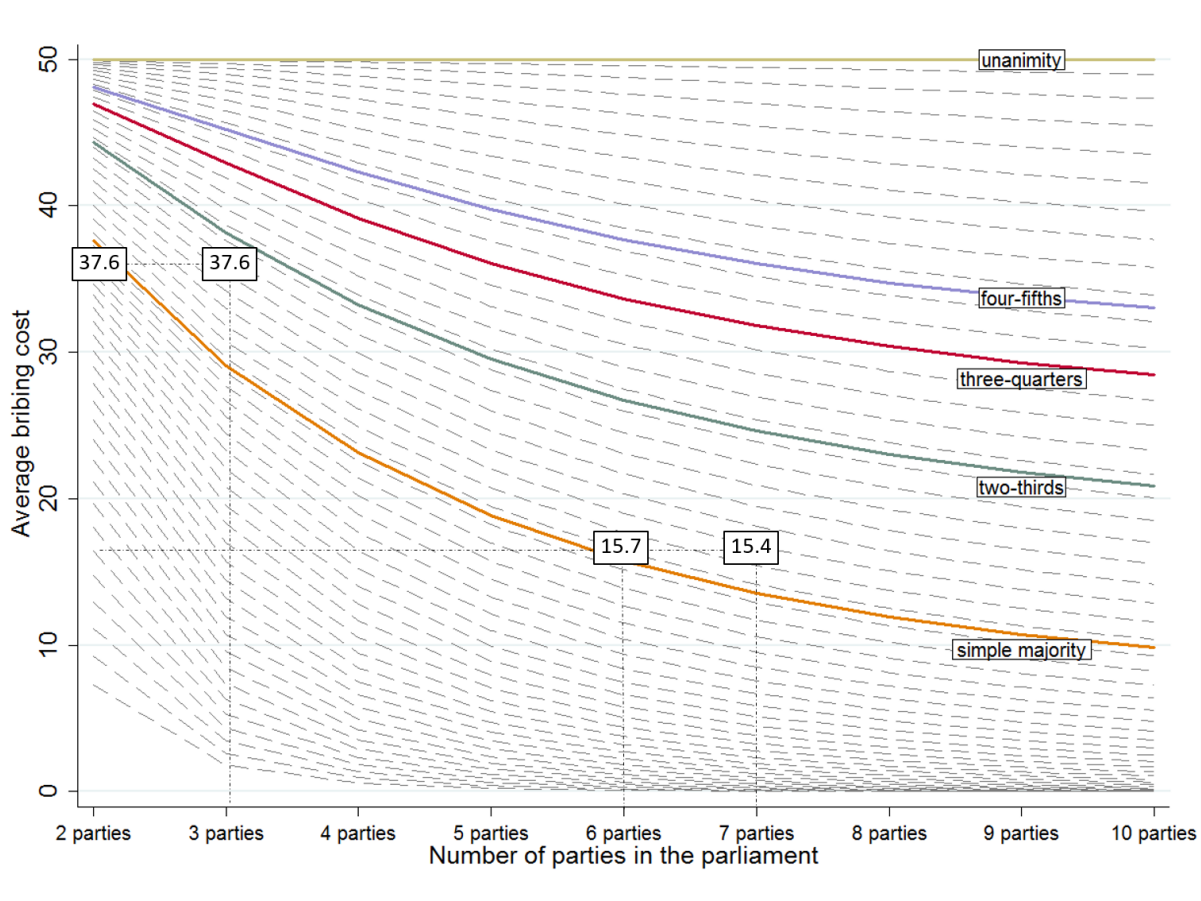


Figure 1.3. Average number of members of parliament of 100 seats needed to be bribed in order to achieve a positive decision (average bribing cost) for unrestricted allocations of seats among 2-10 parties for different voting quotas.

1.6.2 Parliaments with similarly sized parties

We now restrict our attention to parliaments with similarly sized parties. To do so, we impose different lower and upper bounds on the party size by allowing a different degree γ of variation from the egalitarian allocation $n_i = n/m$:

$$L = \left\lfloor \frac{n}{m} - \gamma \frac{n}{m} \right\rfloor, U = \left\lceil \frac{n}{m} + \gamma \frac{n}{m} \right\rceil \quad (1.9)$$

$$\left\lfloor \frac{n}{m} - \gamma \frac{n}{m} \right\rfloor \leq n_i \leq \left\lceil \frac{n}{m} + \gamma \frac{n}{m} \right\rceil. \quad (1.10)$$

The special case of $\gamma = 0\%$ will also be called the equally sized case.

In Figure 1.4 the result for $\gamma = 20\%$ is presented along with the unrestricted allocation. Results for different degrees of size variations are presented in Table 1.4.

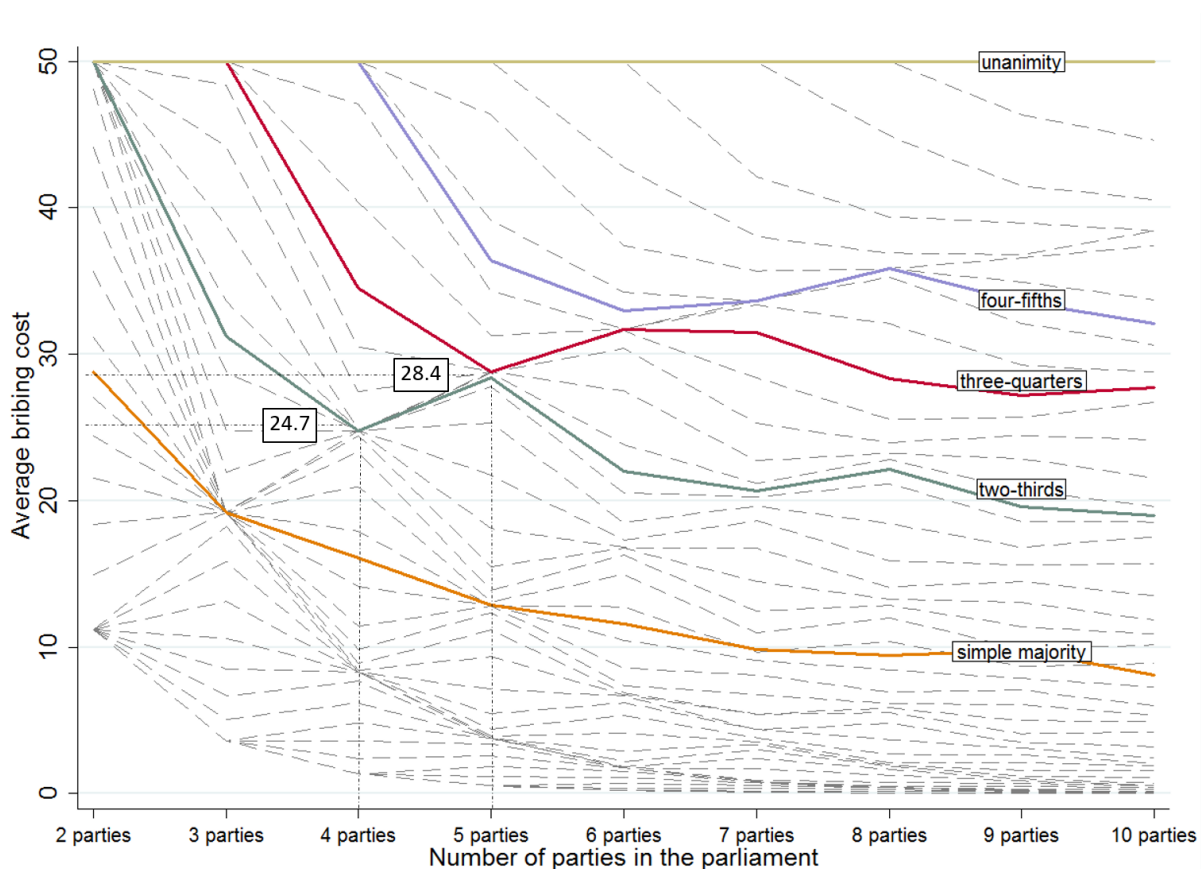


Figure 1.4. Average number of members of parliament of 100 seats (average bribing cost) needed to be bribed in order to achieve a positive decision for similarly sized 2-10 parties for different voting quotas.

Results for unanimity remain the same as under unrestricted allocation. For a simple majority it is true that the average bribing cost decreases with the increase in the number of parties. Under a simple majority the average bribing cost in parliaments with 3-9 similarly sized parties is smaller than the average bribing cost in parliaments with more parties for the unrestricted case. For a parliament with two parties it holds for weak ($\gamma > 5\%$) restrictions on the equality of parties, and does not hold for strong ($\gamma \leq 5\%$) restrictions.

But for the qualified majority rules, the average bribing cost does not always decrease with the increase in the number of parties. For instance, under a two-thirds rule, the average

bribing cost increases when we pass from a parliament with 4 parties to a parliament with 5 parties. In this case, it might be preferred to encourage more parties to appear in order to pass from 4 to 5, and therefore to increase the average bribing cost. This example for 5 parties and the obtained results (see [Table 1.4](#)) give us the next result.

Result 2. *If the allowed seat allocations of a parliament with at most ten parties are those in which parties do not differ “greatly” in size from each other, then:*

- (a) *the average bribing cost still monotonically decreases in the number of parties when decisions are taken by a simple majority;*
- (b) *when some form of a super majority is required to make a decision, the average seems to be eventually decreasing, but may increase depending on the number of parties.*

Consider a particular example of 5 parties and the most frequently applied rules: a simple majority and a two-thirds rule. Different γ s produce different restrictions of parties' sizes. The ideal equal allocation would be that each party occupies exactly 20 seats: $\gamma = 0\%$. By increasing γ , we relax the restriction up to $10 \leq n_i \leq 30$, when $\gamma = 50\%$. Kyrgyzstan and Latvia provide good examples of similarly sized allocation of seats between parties in 2011: $\gamma = 41\%$ (28, 26, 25, 23, 18), and $\gamma = 52\%$ (31, 22, 20, 14, 13), respectively. Austria can be considered as an example of an extremely relaxed similarly sized allocation: $\gamma = 186\%$ (57, 51, 34, 21, 20).

[Figure 1.5](#) shows that under a simple majority the unrestricted allocation is more costly than all the possible similarly sized allocations, but under the two-thirds rule it performs worse than a 10% size variation as well as an equal allocation. As a justification, the six-party parliament with unrestricted allocation is presented. For a simple majority, it is more costly than all the similarly sized allocations between five parties. Hence, it might be the case that under a simple majority policy makers would rather prefer to encourage a new party to come in (or to lower the election threshold in order to allow one more party, albeit a small one, to enter parliament). But if the two-thirds rule is used, this would decrease the average bribing cost instead of increasing it.

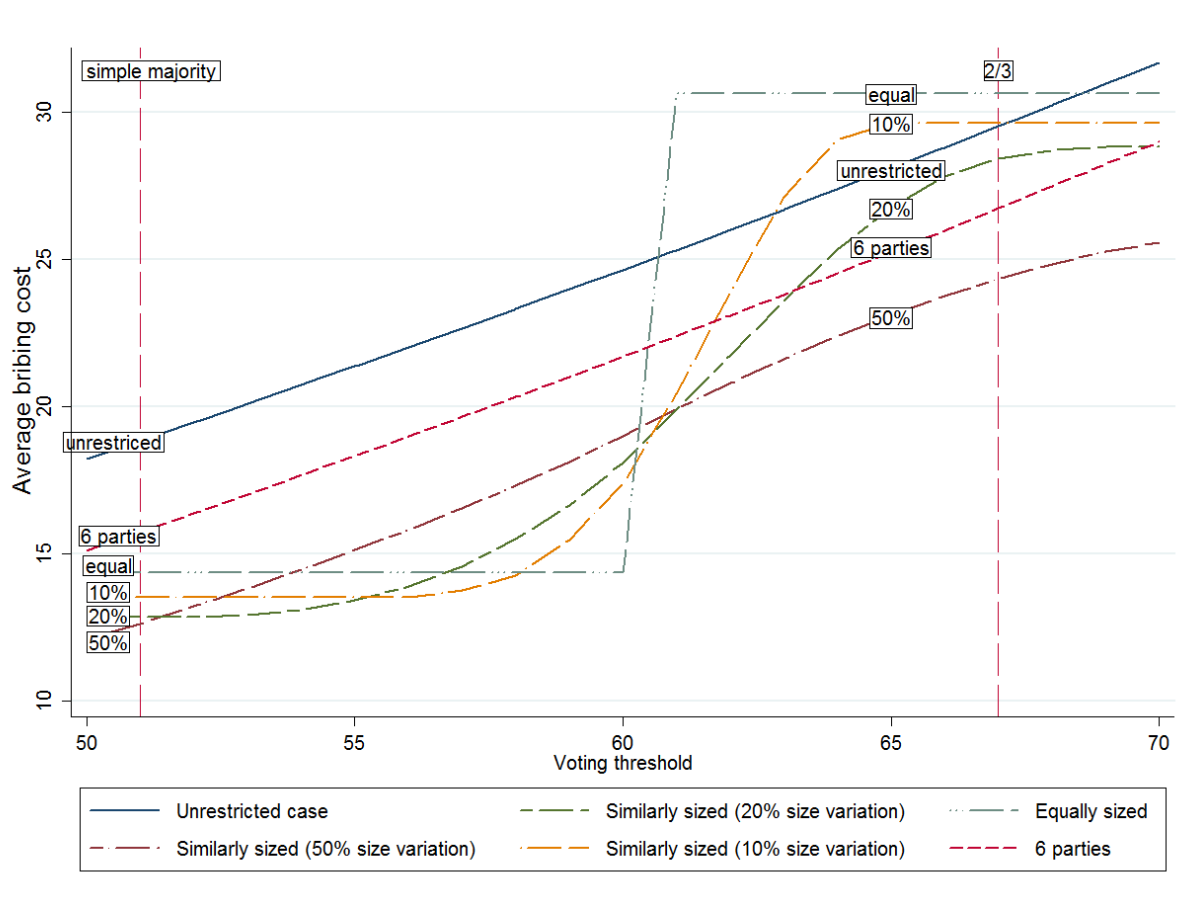


Figure 1.5. Average number of members of parliament of 100 seats needed to be bribed in order to achieve a positive decision (average bribing cost) for 5 similarly sized parties.

1.6.3 Parliaments without small parties

In this subsection we compute the average bribing cost for parliaments without small parties. We restrict the minimal number of seats which a party needs to occupy: $L = 5$ seats, $L = 10$ seats. The lower bound can be seen as an election threshold: A certain percentage of votes a party has to have in order to enter the parliament.

At first glance, it seems that eliminating small parties would increase the average bribing cost. First, the number of states $\#\Omega_n^m$ decreases. Second, no small parties to be bribed means that the briber will be bribing more seats in excessive of what he needs: $\beta(\omega)$ increases. But it appears that there are fewer states where the briber needs to act. Therefore, although some of the components of the sum $\sum_{\omega \in \bar{\Omega}_n^m} \beta(\omega)$ are greater, the sum itself gets smaller with

the elimination of small parties.

In [Figure 1.6](#) average bribing costs for different election thresholds (lower bounds) in a parliament with 5 parties are presented.

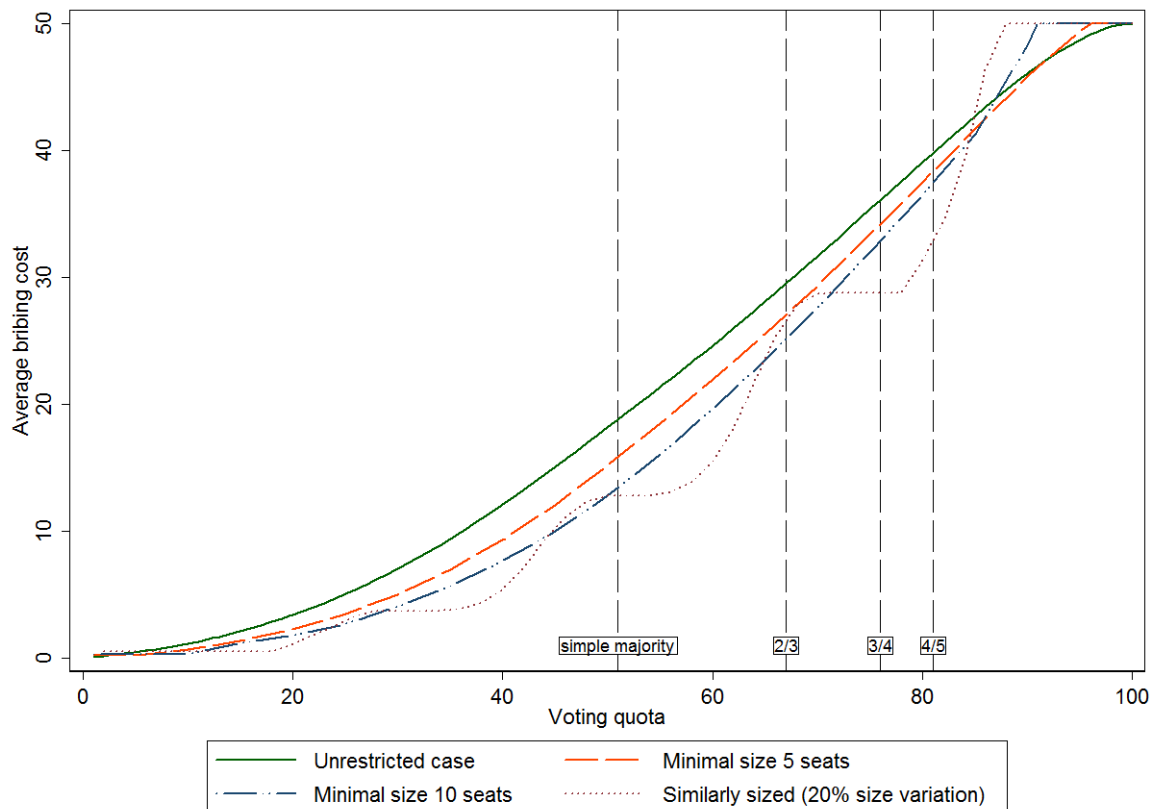


Figure 1.6. Average number of members of parliament of 100 seats needed to be bribed in order to achieve a positive decision (average bribing cost) for 5 parties occupying not less than 5 (10) seats.

It is easy to see that for all majority rules, the average bribing cost decreases with the increase of the lower bound. Note that in this case, under a simple majority for a parliament with 5 parties and a lower bound of 10 seats, the average bribing cost is 13.3 seats, which is smaller than for a parliament of 6 or 7 parties and with unrestricted allocation, where the average bribing cost is 15.7 and 13.5 seats, respectively (see [Figure 1.3](#) and [Table 1.5](#)). For all parliaments with more than 5 parties the same conclusion holds: the average bribing cost in a parliament without small parties is greater than in parliaments with one more party in the unrestricted case. Therefore, it might be the case that for a parliament with more than

5 similarly sized parties the corruption cost can be increased if one more party enters the parliament in order to deter the briber. For parliaments with less than 5 parties, this does not hold for qualified majority rules.

Result 3. *If the admissible seat allocations of a parliament with at most ten parties are those in which parties cannot occupy less than a certain number of seats (5 or 10 seats, for instance), then the average bribing cost decreases in the minimally required number of seats.*

The restriction $L = 10$ means that if there are 10 parties in the parliament, all the parties have exactly 10 seats. And, this makes the average bribing cost greater than for a parliament with 9 or 8 parties. More details can be seen in [Table 1.5](#) in Appendix to this chapter.

1.6.4 Parliaments without big parties

If there is a lower limit, then why not consider upper limits? Of course, generally, if there is a big party in a parliament, it reflects the views of the majority of the voters, hence, such a restriction is not fair with respect to the views of the citizens. But the question is: Why do we care about the representation of the views of the voters when we allow a big ‘dictatorial’ party and at the same time neglect the full representation of the views when we establish the election threshold? This might be if society considers it necessary to restrict the entrance of small parties, but it might consider also to restrict the ‘dictatorship’ in parliaments.

In this subsection we compute the average bribing cost that restricts the upper bound. The softest restriction is that no party has more than 70% of the seats, $U = 70$. The strongest restriction is that no party has a majority of the seats, $U = 50$. The results for 5 parties are presented in [Figure 1.7](#).

Under a simple majority the average bribing cost decreases, that is, the presence of a ‘dictatorial’ party makes bribing easier. But, under qualified majorities, even the strictest demand of “no party has more than 50% of seats” hardly makes a difference.

In a parliament with several parties, the requirement that no party has a majority does not eliminate a big part of the states of the parliament. But in a 2-party parliament it means that all the allocations except $(n_1, n_2) = (50, 50)$ are eliminated. The condition $U = 60$ actually refers to the case of similarly sized parties in the parliament: no party occupies less

Chapter 1

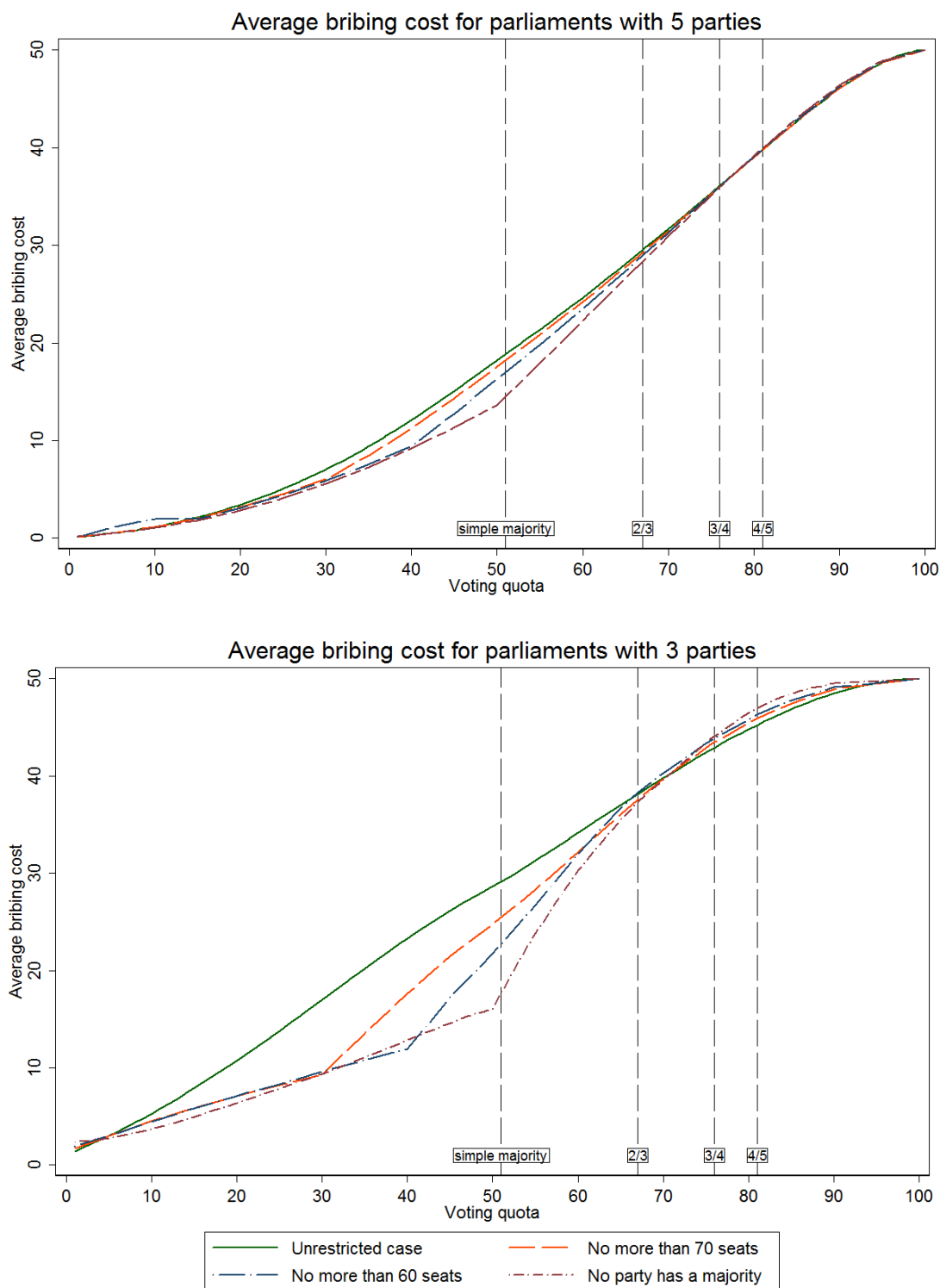


Figure 1.7. Average number of members of parliament of 100 seats needed to be bribed in order to achieve a positive decision (average bribing cost) for 5 and 3 parties occupying not more than 70 (60 and 50) seats.

than 40 or more than 60 seats. Hence, consider a parliament with 3 parties. The requirement $U = 50$ eliminates quite a substantial number of the states of the parliament.

The average bribing costs for a 3-party parliament under upper bound restrictions are shown in [Figure 1.7](#). Under a simple majority, the average bribing cost gets smaller as the upper bound restriction diminishes. The strictest case of “no party has a majority” decreases the bribing cost from 29 seats to 17.8 seats. For the two-thirds rule there is almost no difference. And, for the other qualified majorities, the cost slightly increases the smaller the upper bound is.

Example 3. For instance, the restriction $U = 50$ means that either there is no ‘small party’ (with at most 20 seats) or there is one ‘small party’. If four-fifths is required, in the case of no small party, it means that if there is one party voting positively, $\#D = 1$, then to reach the quota, the briber has to bribe both parties. If $\#D = 2$, the briber still has to bribe 1 more party. In the worst case $\#D = 0$, the briber has to bribe the 3 parties in order to reach the quota. If there is a small party and one of the non-small parties votes Y, the briber has to bribe the other non-small party, because with bribing the small party he will not reach the quota.

Our numerical calculations (also see [Table 1.6](#)) suggest the following conclusion.

Result 4. *If the admissible seat allocations of a parliament with at most ten parties are those in which parties do not occupy more than a certain number of seats (e.g., 50, 60, or 70 seats), then:*

- (a) *the average bribing cost decreases with the rise in the maximum size when decisions are taken by a simple majority;*
- (b) *when some form of a super majority is required to make a decision, the average bribing cost does not vary when different restrictions on maximum party size are applied.*

It is obvious that the parliament structure is not a constant, but is determined by citizens, their views, and electoral rules. In [Subsection 1.6.3](#), we present a first attempt to analyse the effect of the absence of small parties, which can be caused by the existence of an election

threshold. The next section studies what is the effect that the electoral threshold may have on the ease of influence on the voting outcome by a briber through its impact on the number of parties in the parliament.

1.7 Simulations¹⁰

Next, we aim at capturing a bigger picture, taking into account not only parties which are represented in the parliament, but also the citizens who voted for them.

First, we assume that the number of parties is given, as in [Section 1.5](#) and [Section 1.6](#). We focus only on the simple majority rule, since it is the most used rule in the parliamentary voting. The briber stands for or against the bill, and the maximal total amount of payment he agrees to make is random. Contrary to the previous sections, the minimal demands of parties to change their position are random variables. If the briber does not have enough resources to obtain the majority of the votes in favour of his opinion, the parties vote according to their initial standings. We calculate the probability that the final decision of the parliament coincides with the popular will. We call this probability the *fairness* of the political system. As the next step, we introduce the electoral threshold, which allows to analyse how the probability depends on the electoral threshold, taking into account both misrepresentation (which increases with the electoral threshold) and ease of manipulation (which increases with the electoral threshold) by the briber effects.

In order to introduce these extensions the model has to be changed as described below.

1.7.1 Number of parties given

There are m given parties and n seats. We allow different degrees of parties' opinions about the bill by letting $d_i \in [-1, 1]$ be a position of party i on the bill. Nonpositive d_i means that party i supports the bill, but will vote against it for payment $|d_i|$ or higher. Positive d_i means that party i opposes the bill but will vote for it for payment d_i or more. Let the number of seats allocated to parties be random according to a Poisson probability distribution: $n_i \sim$

¹⁰This section is written in collaboration with Artyom Jelnov from the Department of Economics and Business Administration, Ariel University, Israel.

$Poiss(\lambda)$ for each $i \in \{1, \dots, m\}$. Assume that parties' positions are random, too. For each party $i \in \{1, \dots, m\}$

$$d_i = 2X_i - 1 \quad (1.11)$$

where $X_i \in [0, 1]$, $X_i \sim Beta(\alpha, \alpha)$, α and λ are given parameters. Note that as $\alpha \rightarrow \infty$, the distribution $Beta(\alpha, \alpha)$ converges to the Bernoulli distribution with parameter $\frac{1}{2}$. Namely, as α increases, d_i with increasing probability are closer to 0, which means a less polarised ideologically political system.

Without loss of generality, we assume that the briber is interested in bill passing. If the briber might be for or against the bill with equal probability, the results are the same by symmetry. The briber offers to each party a nonnegative payment for voting for the bill, and every party may accept or reject the offer. Denote by $p_i \in [0, 1]$ the payment offer to party i , $i \in \{1, \dots, m\}$. Party i votes for the bill iff $p_i - d_i \geq 0$

The briber minimises the total amount of payment to parties. Namely, the briber solves

$$T(m) = \min \left\{ \sum_{i=1}^m p_i \right\} \text{ s.t. } \sum_{p_i - d_i \geq 0} n_i > \frac{\sum_{j=1}^m n_j}{2}. \quad (1.12)$$

$T(m)$ is a minimal cost that the briber has to pay in order to persuade the members of parliament to vote in favour of the bill. The expected $T(m)$ will be denoted as $E_T(m)$, which characterises the mathematical expected value of the ease of manipulation of the decision of the parliament (i.e., bribing cost). We introduce a concept termed “ease of manipulation”, which is slightly different from “average bribing cost”, used in previous sections in order to avoid confusion between the two terms. Ease of manipulation takes into account simulated degrees of parties' support of the bill, which in previous sections was considered to be equal for every party (equal to 1/2, to be precise), and simulated distribution of seats among parties, which in previous sections was considered to be equally likely to happen.

Figure 1.8 presents the results of a Monte Carlo simulation for $E_T(m)$ depending on m (the number of trials is 1,000,000). It shows that the ease of manipulate increases with the number of parties, namely, the expected total payment required to bribe is decreasing with the number of parties.

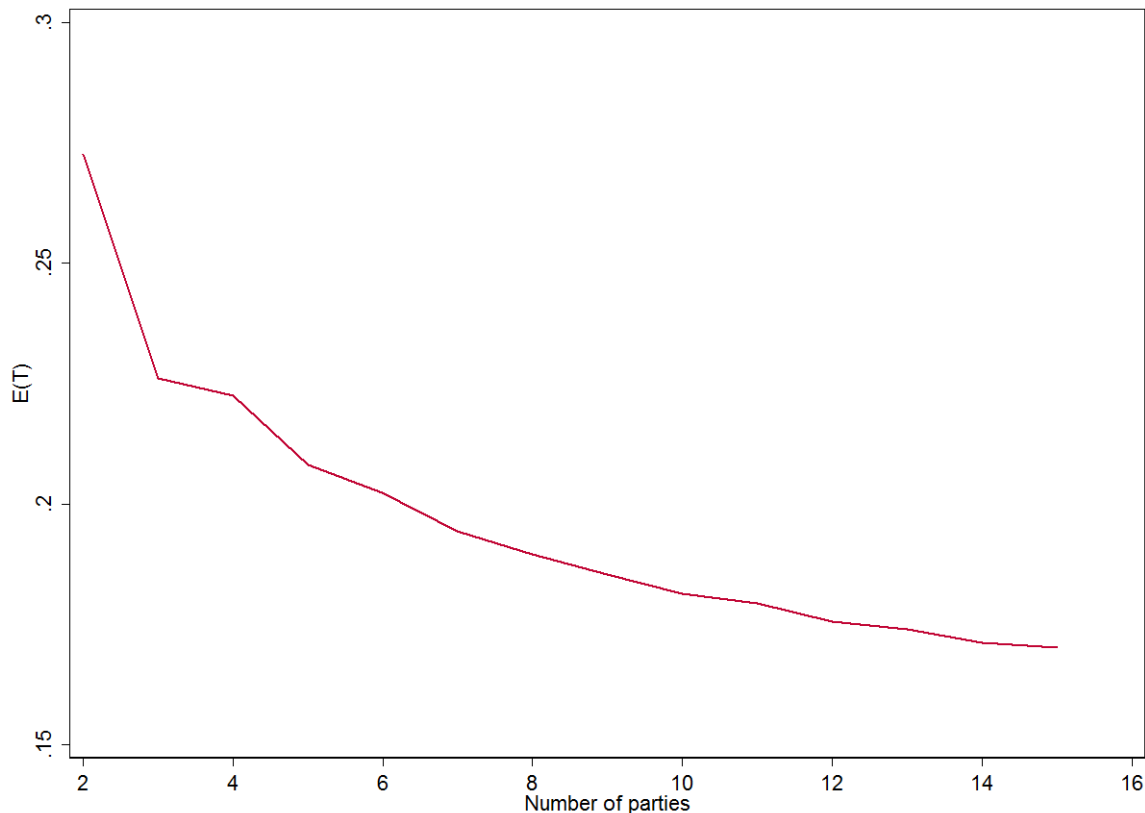


Figure 1.8. Ease of manipulation, results for 1,000,000 trials.

1.7.2 Given electoral threshold

In the next step, we do not assume that the number of parties is given, but that it is the result of citizens' voting for parties and implementation of the election threshold. Let a random number of parties m' compete in general elections for a given number of seats in a parliament: $m' \sim Poiss(\lambda_p)$, λ_p is a given parameter. Assume that after the election takes place, the new elected parliament has to vote for or against an important bill/issue. Therefore, even before the elections, all the parties declare their position on the bill: $d'_i \in [-1, 1]$, $i \in \{1, \dots, m'\}$, and d'_i is randomly picked as in (1.11).

We introduce new elements into the model in order to represent the citizens who participate in general elections. For $i \in \{1, \dots, m'\}$, let $v_i \in \mathbb{N}$ be the number of popular votes received by party i . Assuming that each party position coincides with its voters' position, we define *popular preference* $\Pi \in \{A, R, I\}$ concerning the bill as “Accept”, “Reject”, or

“Indifferent”:

$$\Pi = \begin{cases} A, & \text{if } \sum_{d'_i \leq 0} v_i > \sum_{d'_i > 0} v_i \\ R, & \text{if } \sum_{d'_i \leq 0} v_i < \sum_{d'_i > 0} v_i \\ I, & \text{if } \sum_{d'_i \leq 0} v_i = \sum_{d'_i > 0} v_i \end{cases}$$

A widely-used institutional rule which is implemented in general elections is the electoral threshold. Parties obtaining a share of the popular vote that is lower than the predefined threshold do not receive any seat in the parliament. Let $0 \leq e < 1$ be the electoral threshold. Party i passes the threshold t iff $v_i > e \sum_{j=1}^{m'} v_j$. Let $m \in \mathbb{N}$ be the number of parties that passes the threshold:

$$m = \# \left\{ i \in m' \mid v_i > e \sum_{j=1}^{m'} v_j \right\}.$$

Seats are allocated to m parties proportionally to their popular vote share, where surplus votes are allocated according to the widely-used d’Hont method (see Farrell 2011). Given $v_{\pi(1)}, \dots, v_{\pi(n)}$, let n_1, \dots, n_m be the number of seats allocated to m parties.¹¹

Hence, d_i is an initial position on the bill of party i , that passes the threshold. That is, $|d_i|$ means the minimal demand for payment to party i to vote for the bill. Note that $d_i \leq 0$ means that party i supports the bill if there is no payment.

The briber has his standing d_B concerning the bill, which is randomly chosen as in (1.11). Negative d_B means that he supports the bill, positive d_B means that he opposes it, and $d_B = 0$ means indifference. The briber agrees to pay to parties for changing their standing about the bill in an amount less than $|d_B|$. If this sum is not sufficient to assure the majority in the parliament desired by the briber, he prefers to not make any payment. Let

$$d_i = \begin{cases} d'_{\pi(i)}, & \text{if } d_B \leq 0 \\ -d'_{\pi(i)}, & \text{if } d_B > 0 \end{cases}$$

$i \in \{1, \dots, m\}$.

The briber offers non-negative payments p_1, \dots, p_n to parties, such that the total payment $T(m)$ minimises (1.12). If $T(m) < |d_B|$, the parliament adopts the decision $\Pi_l(e)$ about the

¹¹We define a mapping $\pi : \{1, \dots, m\} \rightarrow \{1, \dots, m'\}$ which is interpreted as follows: if party $i \in \{1, \dots, m'\}$ passes the threshold e , its index among other parties which passes the threshold is $\pi^{-1}(i)$.

bill as desired by the briber, namely,

$$\Pi_l(e) = \begin{cases} A & , d_B < 0 \\ R & , d_B > 0 \\ I & , d_B = 0 \end{cases}$$

In the case $T(m) \geq |d_B|$, the lobbyist does not make any payment and parties in the parliament vote according to their initial standings. Namely, for $T(m) \geq |d_B|$

$$\Pi_l(e) = \begin{cases} A & , \sum_{d_i < 0} w_i > \sum_{d_i > 0} w_i \\ R & , \sum_{d_i < 0} w_i < \sum_{d_i > 0} w_i \\ I & , \sum_{d_i < 0} w_i = \sum_{d_i > 0} w_i \end{cases}$$

We define the *fairness* of the threshold q as the probability that $\Pi = \Pi_l(e)$.

Figure 1.9 presents results for 100 seats, and 100,000 trials. It follows from Figure 1.9, that for thresholds below 6%, fairness does not suffer; as for thresholds higher than 6%, the ‘fairness’ is decreasing with the threshold.

Remark 1. Results for different values of α are similar for models in sections 1.7.1 and 1.7.2. These models should not be sensitive to changes in the number of seats, in λ and λ^* , since results depend on a fraction of seats/popular votes of each party out of a total number of seats/popular votes.

1.8 Conclusions

The literature has been mostly concerned with finding evidence of correlation between income level of the country with its democratic development, including parliaments and with levels of corruption and bribing. The connection between bribing and parliaments is usually considered at the time of general elections, from the point of view of simple voters and bribing their votes. However, in this chapter we aimed at analysing the bribing once the parliament is already elected. First, we collected some empirical data for 172 countries. The data showed that in high-income countries parliaments are bigger, with fewer parties; in low-income countries parliaments are smaller, with more parties. Higher income countries show stronger control of corruption, but the parliament and political parties are perceived to be corrupt as much as in lower income countries, which have weaker control of corruption.

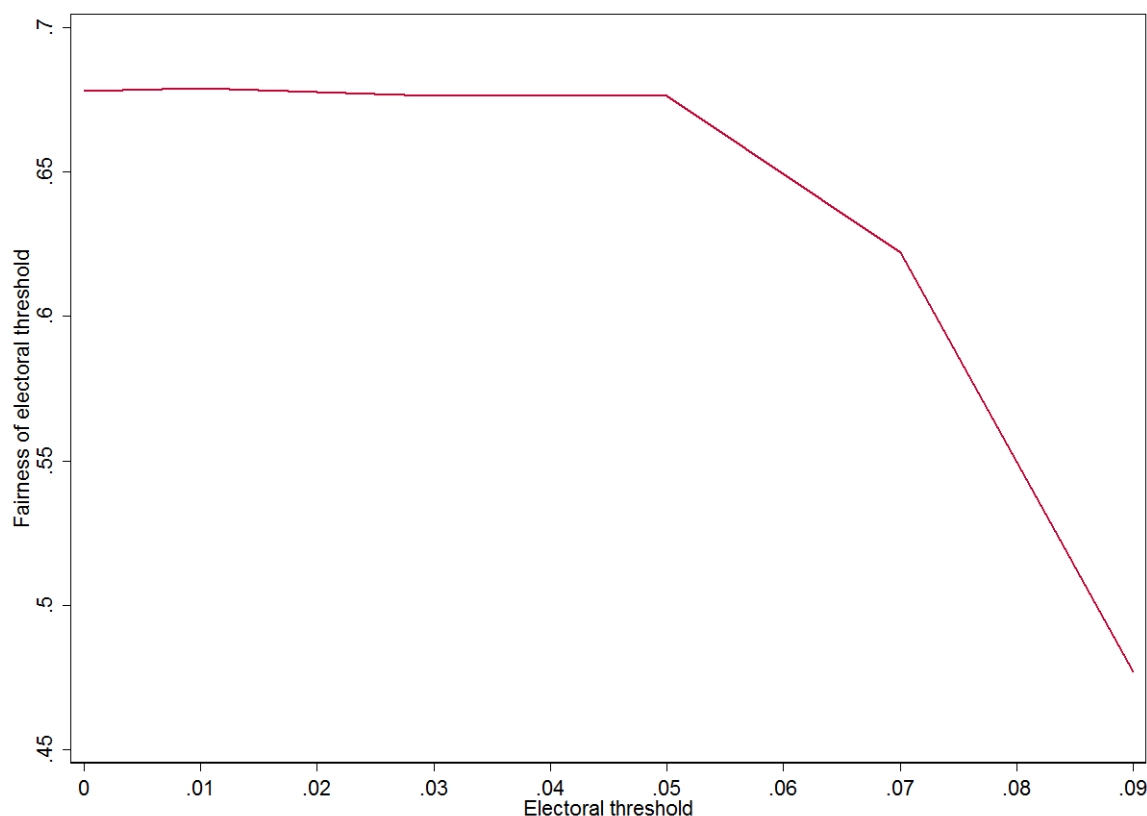


Figure 1.9. Fairness of electoral threshold, results for 100 seats and 100,000 trials.

Though analytically it was not possible to provide a general result, a computational approach allowed to provide a solution for less than 10 parties. Under the assumptions that each party votes “yes” or “no” with equal probabilities and that all possible allocations of seats between parties are equally likely, for both simple and qualified majority rules, the number of seats to be bribed decreases with the number of parties in the parliament. The decrease is smaller the larger the number of parties. The average bribing cost of a parliament with less parties can be reproduced in a parliament with more parties by increasing the voting threshold. And the greater the number of parties, the smaller the required change.

In a parliament with similarly sized parties, the average bribing cost increases in the number of parties only under the simple majority rule. For qualified majorities there is no general conclusion, though we can define average bribing cost for different parliaments. Comparing the unrestricted case with similarly sized parties it appears that there might be

the case when encouraging one more party to enter parliament increases significantly the average bribing cost.

Under the simple majority, the average bribing cost in parliaments with 3-9 similarly sized parties is smaller than the average bribing cost in parliaments with more parties in the unrestricted case. For a parliament with two parties, it holds for weak ($\gamma > 5\%$) restrictions of equality of parties, and does not hold for strong ($\gamma \leq 5\%$) restrictions.

In parliaments without small parties, the average bribing cost gets smaller with the increase of the lower bound of party size. In a purely proportional system, this result can serve as a proxy for an election threshold to enter parliament. Hence, the conclusion is that the election threshold makes bribing less costly for both simple and qualified majority rules.

Under the simple majority, the average bribing cost is smaller for parliaments without a big ‘dictatorial’ party: the greater the restriction for the maximum size of a party, the smaller is the average bribing cost. Under a qualified majority, for parliaments with more than 3 parties, the average bribing cost almost does not change if the maximum party size is restricted. And, in parliaments with 2 or 3 parties, it even gets greater for strict qualified majority rules.

To be closer to reality, we introduce the randomness of opinions of parties and briber about the motion to be voted for. Besides, we relax an assumption that all the parties are equally likely to accept the bribe, and the cost directly depends on the size of the party. Performed simulations show that the ease of manipulation is inversely related to the number of parties.

The number of parties that enter into a parliament is conditioned by the electoral threshold. The higher the threshold, the less parties succeed in obtaining seats in the parliament. One may conjecture that an extremely high threshold, such that only two party will pass over it, is the optimal threshold. However, in many democracies this is not the case. In parliaments elected according to the proportional representation system, the electoral threshold is, with few exceptions, between 1% to 5%, and the number of parties is normally between 3 and 10. Obviously, the main reason why many democracies do not impose a severe constraint on the number of parties is their desire to avoid an extreme misrepresentation of voters. In this sense, a lower threshold improves the representativeness of the parliament. But on the

other hand, as we have claimed above, it makes it easier to manipulate the parliament decision. We find that for thresholds that exist in most parliamentary democracies with the proportional representation system (1-5 %), the ‘fairness’ in our sense is close to be optimal. As for thresholds higher than 6%, ‘fairness’ decreases with the threshold.

For future research, the proposed model can be extended to answer several more questions. It would be interesting to see if relaxing party discipline changes the result. The conjecture is that the result will hold, since each party would be treated as a parliament with n parties as in [Section 1.5](#).

The most difficult extension seems to be relaxing the assumption on uniform distribution above the set of admissible seat allocations among parties, which we somehow tried to introduce in [Section 1.7](#). Additional empirical studies would help to identify exactly the underlying distribution or put relevant constraints on the admissible or relevant distributions.

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Appendix

Table 1.3. Average bribing cost in the unrestricted case.

Number of parties	Simple majority ($q = 51$)	Two-thirds ($q = 67$)	Three-quarters ($q = 76$)	Four-fifths ($q = 81$)	Unanimity ($q = 100$)
2 parties (1-99 seats)	37.6263	44.3333	46.9697	48.0808	50
3 parties (1-98 seats)	29.0864	38.1541	42.9001	45.2342	50
4 parties (1-97 seats)	23.1213	33.2305	39.15	42.3316	50
5 parties (1-96 seats)	18.8375	29.5129	36.0822	39.784	50
6 parties (1-95 seats)	15.7425	26.7247	33.675	37.694	50
7 parties (1-94 seats)	13.5083	24.6222	31.8154	36.0356	50
8 parties (1-93 seats)	11.891	23.0253	30.3895	34.7452	50
9 parties (1-92 seats)	10.7078	21.8062	29.3031	33.7562	50
10 parties (1-91 seats)	9.82603	20.8735	28.4815	33.009	50

q denotes the voting quota.

Table 1.4. Average bribing cost in parliaments with similarly sized parties

Number of parties	Simple majority ($q = 51$)	Two-thirds ($q = 67$)	Three-quarters ($q = 76$)	Four-fifths ($q = 81$)	Unanimity ($q = 100$)
Similarly sized parties $\gamma = 20\%$					
2 parties (40-60 seats)	28.8095	50	50	50	50
3 parties (26-40 seats)	19.2098	31.2232	50	50	50
4 parties (20-30 seats)	16.1199	24.7941	34.5297	50	50
5 parties (16-24 seats)	12.8413	28.4224	28.8007	36.4211	50
6 parties (13-20 seats)	11.5675	22.0271	31.7242	32.9582	50
7 parties (11-18 seats)	9.86938	20.6706	31.4796	33.6585	50
8 parties (10-15 seats)	9.41272	22.1754	28.3424	35.8307	50
9 parties (8-14 seats)	8.25432	19.6143	27.2125	33.595	50
10 parties (8-12 seats)	8.07644	19.0056	27.7276	32.1036	50
Similarly sized parties $\gamma = 10\%$					
2 parties (45-55 seats)	28.6364	50	50	50	50
3 parties (30-37 seats)	19.9609	29.1406	50	50	50
4 parties (22-28 seats)	16.6166	25.4445	34.5563	50	50
5 parties (18-22 seats)	13.5367	29.6309	29.6309	36.6339	50
6 parties (15-19 seats)	11.8884	21.7073	32.4011	32.4011	50
7 parties (12-16 seats)	10.3089	20.9326	34.1929	34.6124	50
8 parties (11-14 seats)	9.78004	23.8548	28.5392	36.4417	50
9 parties (10-13 seats)	8.77813	19.3543	26.9587	37.6636	50
10 parties (9-11 seats)	8.5203	19.3537	28.7816	32.4175	50
Similarly sized parties $\gamma = 5\%$					
2 parties (47-53 seats)	29.4286	50	50	50	50
3 parties (31-35 seats)	20.3125	29.875	50	50	50
4 parties (23-27 seats)	17.2316	25.7728	35.0176	50	50
5 parties (19-21 seats)	13.8909	30.0551	30.0551	37.8922	50
6 parties (15-18 seats)	12.31	21.7421	32.7208	32.7208	50
7 parties (13-15 seats)	10.78	21.6277	35.2329	35.2329	50
8 parties (11-14 seats)	9.78004	23.8548	28.5392	36.4417	50
9 parties (10-12 seats)	8.88799	19.6184	26.8467	38.1372	50
10 parties (9-11 seats)	8.5203	19.3537	28.7816	32.4175	50

Table 1.5. Average bribing cost in parliaments without small parties.

Number of parties	Simple majority ($q = 51$)	Two-thirds ($q = 67$)	Three-quarters ($q = 76$)	Four-fifths ($q = 81$)	Unanimity ($q = 100$)
All parties have at least 5 seats					
2 parties (5-95 seats)	36.6484	43.9451	46.8132	48.022	50
3 parties (5-90 seats)	27.0958	36.7797	42.0966	44.7704	50
4 parties (5-85 seats)	20.4281	31.1244	37.6559	41.2997	50
5 parties (5-80 seats)	15.8401	27.0654	34.184	38.3344	50
6 parties (5-75 seats)	12.8541	24.2351	31.6769	36.0924	50
7 parties (5-70 seats)	10.9797	22.2661	29.9251	34.4999	50
8 parties (5-65 seats)	9.75383	20.8996	28.7172	33.3971	50
9 parties (5-60 seats)	8.88106	19.9592	27.8899	32.6389	50
10 parties (5-55 seats)	8.21115	19.3099	27.3258	32.1205	50
All parties have at least 10 seats					
2 parties (10-90 seats)	35.4321	43.6296	46.8519	48.2099	50
3 parties (10-80 seats)	24.61	35.2485	41.4988	44.7667	50
4 parties (10-70 seats)	17.4809	29.0347	36.3618	40.6857	50
5 parties (10-60 seats)	13.3932	25.1299	32.8744	37.3922	50
6 parties (10-50 seats)	11.3317	22.7091	30.7821	35.54	50
7 parties (10-40 seats)	10.0116	21.3342	29.3397	34.8836	50
8 parties (10-30 seats)	9.27403	20.7983	28.0235	35.262	50
9 parties (10-20 seats)	8.76792	19.4428	27.3474	37.1779	50
10 parties (10 seats)	12.3828	20.6641	30.1172	40.0098	50

q denotes the voting quota.

Table 1.6. Average bribing cost in parliaments without big parties.

Number of parties	Simple majority ($q = 51$)	Two-thirds ($q = 67$)	Three-quarters ($q = 76$)	Four-fifths ($q = 81$)	Unanimity ($q = 100$)
All parties have at most $U = 70$ seats					
2 parties (30-70 seats)	30.7371	46.9268	50	50	50
3 parties (1-70 seats)	25.4275	37.5353	43.4775	45.9011	50
4 parties (1-70 seats)	21.5153	32.6633	39.1304	42.472	50
5 parties (1-70 seats)	18.2133	29.2364	36.009	39.7919	50
6 parties (1-70 seats)	15.5236	26.6162	33.6345	37.6845	50
7 parties (1-70 seats)	13.438	24.585	31.668	36.0297	50
8 parties (1-70 seats)	11.8701	23.0137	30.3839	34.7428	50
9 parties (1-70 seats)	10.7019	21.8028	29.3014	33.7554	50
10 parties (1-70 seats)	9.82449	20.8726	28.481	33.0088	50
All parties have at most $U = 60$ seats					
2 parties (40-60 seats)	28.8095	50	50	50	50
3 parties (1-60 seats)	22.6744	38.256	43.8613	46.3128	50
4 parties (1-60 seats)	19.5115	32.4879	39.2759	42.6737	50
5 parties (1-60 seats)	16.9778	28.9168	36.0005	39.8569	50
6 parties (1-60 seats)	14.8498	26.3778	33.5854	37.6926	50
7 parties (1-60 seats)	13.1058	24.4478	31.7584	36.0227	50
8 parties (1-60 seats)	11.7193	22.9455	30.36	34.7358	50
9 parties (1-60 seats)	10.638	21.7722	29.2896	33.7511	50
10 parties (1-60 seats)	9.79885	20.8598	28.4758	33.0067	50
All parties have at most $U = 50$ seats					
2 parties (50 seats)	50	50	50	50	50
3 parties (1-50 seats)	17.7613	37.3067	44.1043	46.9841	50
4 parties (1-50 seats)	15.9171	31.8911	39.261	42.9766	50
5 parties (1-50 seats)	14.4658	28.3709	35.9116	39.9645	50
6 parties (1-50 seats)	13.1862	25.9196	33.4819	37.7124	50
7 parties (1-50 seats)	12.0665	24.1115	31.668	36.0115	50
8 parties (1-50 seats)	11.1054	33.7241	30.2933	34.7188	50
9 parties (1-50 seats)	10.2933	21.6383	29.246	33.7373	50
10 parties (1-50 seats)	9.61375	20.7842	28.45	32.9976	50

q denotes the voting quota.

CHAPTER 2

GATHERING SUPPORT FROM RIVALS

Overview. A group of individuals has to agree on a certain alternative out of several alternatives. Consider iterative voting procedures, where voters cast their vote repeatedly. At each stage of the voting procedure, voters observe the result of the previous stage and vote again according to a certain protocol. Iterative voting involves timing, which can be presented as a cost of delay or a presence of a deadline. This chapter studies iterative voting procedures with delay costs and with a deadline.

First, we assume that there is a delay cost, so that voters' utilities are decreasing with time. We study whether the patience of the voters, determined by the utilities, favours winning.

Second, we consider that there is a certain deadline for the decision to be taken. We model a process of decision-making of a group of individuals, such as a trial jury, for instance. We study theoretical features of an iterative voting process and conduct an experimental study.

Keywords: iterative voting; subgame perfect equilibrium; decision deadline; delay cost; decision process convergence; price of anarchy.

Preface

This chapter originated upon an idea that I first presented at the X Encuentro de la REES (Red Española de Elección Social – Spanish Network of Social Choice) in 2013, in Malaga where I was fortunate enough to receive useful and generous feedback from some leading experts in Social Choice in Spain, such as Salvador Barberà, Jordi Massó, Bernardo Moreno, and Dolors Berga. Their comments, suggestions and interest in the model encouraged to me to work further on the project.

This effort initially resulted in a small note published in the proceedings of the 2015 Group Decision and Negotiation Conference. A more advanced version was shortly after presented by me in a workshop at the Department of Quantitative Methods and Economic Theory at Universitat d'Alacant and it gave rise to my active discussion with Carmen Beviá, José Alcalde and Josep E. Peris, whose sharp observations lead to some significant improvements in the paper.

The transit from the initial idea to the version included in this chapter was effected in collaboration with José-Manuel Giménez-Gómez and Peio Zuazo-Garin, who honoured me by being my co-authors. The model would not be so mathematically elegant without their effort and contribution.

During my research visit to the Federmann Center for the Study of Rationality at Hebrew University of Jerusalem, I was most warmly welcomed by Lihi Naamani-Deri, whom I first met at Summer School of Computational Social Choice in Caen, France, in 2014. We were strongly motivated to do a joint paper; soon, we were given the opportunity of working together within the most remarkable team of researchers.

During my stay in Jerusalem, where I had the privilege of meeting Svetlana Obraztsova, Zinovi Rabinovich, and Jeffrey S. Rosenschein. Most pleasant conversations and discussions with Svetlana Obraztsova about iterative voting procedure gave a birth to the model of searching for a consensus under imposed deadline, which is included in this chapter. The model was significantly improved, thanks to brilliant comments and suggestions by our coauthor Jeffrey S. Rosenschein. The model would not exist in its present form without great inspiration and sharp observations by Zinovi Rabinovich, and the hard work and management skills of Lihi Naamani-Deri.

2.1 Introduction

In iterative voting settings individuals cast their vote repeatedly, starting from some initial profile. At each stage, the individuals observe the outcome and one or more of them may wish to change their vote. Usually, the voting does not stop until a majority, or all the voters in the case of unanimity, agree on the decision. For instance, in jury trials jurors are voting until they reach a verdict that is supported by the majority or unanimously; or in the the Catholic Pope's election, when voting must continue until a super majority of cardinals agree on an alternative.

Since there might be several stages until the decision is taken, time plays a crucial role in any iterative voting procedure. Hence, two questions arise: what changes with time, and will the time run out or not. The first question is motivated by the fact that certain delay costs can be applied to voters, so that their utility might decrease with time. The second question is important because there are some cases where there exists a certain deadline, that is, a moment in time before which the decision must be taken. For instance, the cardinals have exactly one year to decide who will be the new Catholic Pope; or parliaments of some countries (e.g., Spain, Israel) have a certain deadline for choosing a new prime minister. Hence, either time constraints affect individuals directly through their utilities, or time constrains affect the procedure itself, implying that there is a certain time window within which a decision must be taken. Inspired by these two effects of timing, we propose two models in order to capture these effects.

First, consider a group of individuals that need to agree on a unique alternative out of several options. Consider unanimity, so that the chosen alternative should be supported by every individual. Let the individuals vote sequentially, one by one. Assume perfect information, so that they see the votes of the individuals who had voted before. If they do not agree on the same alternative, let them vote again, and so on, until a decision is taken. And finally, introduce timing in the form of time costs, so that the later they agree, the less they get. In this situation several questions can be interesting: (i) whether voters would be willing to change their vote and, (ii) and if so, then when should they change their vote? (iii) when the decision is taken? (iv) is the decision is taken at all?

[Section 2.2](#) answers these questions. We propose a simple model, which represents the situation in which individuals must agree on choosing one alternative out of the set of alternatives, by voting sequentially. Whenever voters disagree, the procedure is repeated. Voters' utilities decrease with time and for each voter there exists a critical stage beyond which he prefers accepting his least preferred alternative rather than continuing the process. Both voters have complete information about each other's preference, the voting results of previous stages and the votes declared at the current stage by the preceding individuals. Although we may intuitively predict that the most patient voter wins, under a fixed voting order the voter who votes first obtains his preferred option at the first stage of the procedure.

Hence, the result indicates that the voting order gives certain advantage to the voter who votes first. But, what if this advantage has its limit? Next, we consider a particular case with two individuals and two alternatives, assuming that there is a stage when the voting is reversed. That is, starting from this stage, the voter who voted first cedes his position to the voter who voted second. It may be seen as a deadline for the first voter to exploit his advantage. It appears that, under certain conditions, the voter does not have even a possibility to use his advantage, and the alternative that is preferred by the other voter (who votes in second place at the beginning) is chosen.

As the next step, consider time effects not as a discount factor for personal utilities, but as a time limit for the voting process itself: introduce a deadline, so that the decision must be reached before a certain time limit. Before the voting starts, let the voters reveal their most preferred alternatives. If no decision is taken, ask the voters whether somebody changed his mind and wishes to recast his vote. Let one voter change his vote and then question all the voters again, in order to ascertain if somebody wants to change his vote. And so on, until the deadline comes. If they do not agree before the time is up, then the worst outcome for all voters will arise. Such a scenario can be easily imagined in a court of jurors, where if no decision is reached before a certain deadline, a mistrial is declared. And a mistrial is commonly perceived to be worse than any decision, the jury might render. This model suggests questions such as: (i) How often will the group reach a consensus? (ii) What effort will be required from the voters? (iii) What attributes will the decision have?

With the inspiration of this scenario, in [Section 2.3](#) we define a strictly formalised time-

bounded iterative voting process — *Consensus Under a Deadline (CUD)* — based on a time-bounded iterative voting process. We provide some theoretical features of CUD scenarios, particularly focusing on convergence guarantees and the quality of the final decision. An extensive experimental study demonstrates the more subtle features of CUDs.

The chapter is organised as follows. [Section 2.2](#) presents a model of iterative voting with sequential voting at each stage, [Section 2.3](#) presents a model of iterative voting with a deadline, and [Section 2.4](#) recapitulates the main findings and suggests future research questions.

2.2 A model of iterative voting without a deadline¹

In this section we propose a simple model of iterative voting, where at each stage individuals are arranged in a fixed linear order, and at each stage they cast their votes in that order. We consider the most strict voting rule, unanimity: if all the voters unanimously support the same alternative, then the voting stops with this alternative chosen; otherwise, the procedure is repeated at the next stage.

Voters have strict preferences over alternatives, which are persistent with stages: at each stage a voter prefers the same alternative as the other voters. For each voter a utility function is defined which represents the preferences. Both utility functions and preferences are known by all the voters; in other words, it is assumed perfect information. Due to the time costs, we assume that the utilities decrease with the passing of the stages. There is an impatience degree for each voter: a moment in time (a stage) when it is worth voting for the non-preferred alternative now, rather than waiting for the next stage and voting for the preferred alternative.

Intuition suggests that the more patient voter will manage to get his preferred alternative. It is shown that in the unique solution of the sequential voting procedure obtained by backward induction, the first voter gets his preferred alternative at the first stage.

First, in [Subsection 2.2.2](#) a basic case of two voters and two alternatives is described. This

¹This section is written in collaboration with José-Manuel Giménez-Gómez and Peio Zuazo-Garin from the Department of Economics at Universitat Rovira i Virgili, Spain.

case is of particular interest: the presence of two voters makes it impossible to apply other rules rather than dictatorship or unanimity; besides, two alternatives imply that voting for one alternative means vetoing the other alternative. Intuition suggests that the most patient voter wins independently of the voting order, since he can wait until the rival loses his patience and, therefore, agrees with any alternative in order to stop the procedure. Yet, the result predicts that in the unique subgame perfect equilibrium the first voter gets his preferred alternative at the first stage, independently of his degree of impatience. Due to the aforementioned particularity of the 2 agents – 2 alternatives case, it is not clear whether the result whereby the first voter obtains his preferred alternative holds for a general case. Hence, as the next step in [Subsection 2.2.4](#) we consider the general case of n individuals and m alternatives. Having individuals with different degrees of patience suggests that the more patient voter will have a certain advantage over the less patient voter. Yet, the general result again states that patience does not matter and only the order determines the outcome. Formally, we obtain that in the unique subgame perfect equilibrium an alternative that is proposed by the first voter wins, even if this voter is the least patient of the voters.

2.2.1 Related literature

The study of models where individuals try to agree on a question originates from the [Rubinstein \(1982\)](#) bargaining model, and involves a vast number of different contexts. [Baron and Ferejohn \(1989\)](#) extends the model from 2 to n agents and presented a dynamic model of bargaining in legislatures, when at each round a randomly selected voter makes a proposal to vote by a committee. The member who makes a proposal is shown to have agenda power; hence, with a closed rule, the first proposal is passed.

The Baron-Ferejohn model produced a huge interest from researchers; see, [Yang \(1992\)](#), [Ma and Manove \(1993\)](#), [Krishna and Serrano \(1996\)](#), [Ponsatí and Sákovics \(1996\)](#), [Banks and Duggan \(2000\)](#), [Eraslan \(2002\)](#), [Kalandrakis \(2004\)](#), [Frechette et al. \(2005\)](#), [Banks and Duggan \(2006\)](#), [Cho and Duggan \(2009\)](#), [Diermeier and Fong \(2011\)](#), [Battaglini et al. \(2012\)](#), [Bowen and Zahran \(2012\)](#), [Diermeier and Fong \(2012\)](#), [Fuchs and Skrzypacz \(2013\)](#). It did not lose its actuality nowadays. For instance, [Eraslan and McLennan \(2013\)](#) show, in generalising [Eraslan \(2002\)](#), the uniqueness of equilibria, assuming linear utilities, heterogeneous

discount factors and any voting quota in the Baron-Ferejohn model. The most recent work by [Anesi and Seidmann \(2015\)](#) proposes a dynamic bargaining model with endogenous default option as an extension of the Baron-Ferejohn model. In contrast to the original model, it is permissible for players to have different discount factors and different probabilities of being selected to act first. At each period a player is randomly selected to make a proposal, if the proposal is accepted by the committee, then it is implemented and considered to be a new default option for the next period. If the committee does not accept the proposal, then the current default option is implemented. One of the question this section studies is how the endogeneity of the evolving default affects the equilibria.

Other scholars introduce incomplete information. First, [Rubinstein \(1985\)](#) develops his initial model by dropping the assumption of perfect information. Later, [Ponsatí and Sákovic \(1996\)](#) and [Hörner and Lovo \(2009\)](#) study incomplete information. Some other researchers (see for instance, [Schmidt 1993](#), [Peški 2014](#)) assume asymmetric information. For a broad overview of the theoretical and empirical literature on bargaining with incomplete information, one should refer to [Ausubel et al. \(2002\)](#).

Searching for a decision by a group of people can be seen not only as bargaining, but as sequential voting, which is usually treated as a repeated game (see, for instance, [Benoit and Krishna 1985](#), [Dekel and Piccione 1997](#), [Morton and Williams 1999](#), [Dekel and Piccione 2000](#), [Eliaz et al. 2007](#)). Recently, [Compte and Jehiel \(2010\)](#) construct a model in which at each stage the committee is asked to accept or to decline a certain proposal, and in the case of rejection the procedure passes to the next stage and a new proposal is considered. Those authors study which members have more impact on the decision under different majority rules. One of the interesting results is that under unanimity, when proposals vary along a single dimension, the extremists determine the final decision. In their framework the extremists are the voters with more intense preferences and therefore with the highest degree of patience. Adapting the results of [Compte and Jehiel](#) to the model that is considered in this section it is expected that the voters with a higher degree of patience are those who define the result.

The closest model to the one presented here is a model by [Kwiek \(2014\)](#). He considers a decision-making conclave, choosing between two alternatives under a super-majority rule

(including unanimity). If a decision is not reached in the first round of voting, then the procedure repeats in the next round, and so on, until the required super-majority is reached. The delay in time is increasingly costly to each player. The question that is asked is: which rule offers higher utilitarian welfare? In answering this question Kwiek finds that there is a subgame perfect Nash equilibrium that leads to a unique voting outcome in the first round. This outcome coincides with the alternative that is preferred by the pivotal voter with the greater indifference time (or, in other words, a higher degree of patience).

The current approach studies how agents make a unanimous decision over a set of alternatives, assuming a potentially infinite number of voting stages, such that the delay of the decision implies some costs for the voters. We assume, contrary to [Kwiek \(2014\)](#), that there can be more than two alternatives. Unlike in the work of [Kwiek \(2014\)](#), it is assumed that the ordering of the voters is fixed and the agreement is met by unanimity. Each voter has a degree of impatience, indicating when it is worth voting for the non-preferred alternative now, rather than for the preferred alternative later. We do not restrict our attention to the cases when voters must be different in their degree of patience, as it is applied by Kwiek. Since both voters know their own degree of impatience and that of the other voter, intuition suggests that the more patient voter will manage to get his preferred alternative. We show that the subgame perfect equilibrium is unique, such that the first voter gets his preferred alternative at the first stage, independently of his degree of impatience. The result contradicts the result obtained by [Kwiek \(2014\)](#).

The result of the uniqueness of the subgame perfect equilibrium is similar to the result of [Rubinstein \(1982\)](#), where the proposal of the first individual is accepted by the other individual at the beginning. Besides, there is a huge body of literature on voting by conformity that shows that people are likely to accept a proposal immediately. For instance, [Bernheim \(1994\)](#) states that the voters are willing to conform because they recognise that even small departures from the social norm will seriously impair their status. Despite this penalty, agents with sufficiently extreme preferences refuse to conform. Applying this idea to the model considered here suggests that the voters with a higher degree of patience (extreme voters) are not likely to conform with the first voter. Some researchers suggest that observing the actions of the other agents would induce individuals to believe that these agents are

better informed and, therefore, these individuals are likely to imitate their behaviour (see, for instance, Banerjee 1989). Herrera and Martinelli (2006) develop a model based on the idea that voters follow a leader and attract other voters to also follow that leader. Rivas and Rodríguez-Álvarez (2012) also study the effect of the presence of leaders between the voters on the information transmission among themselves. In the model studied in this section leadership can be presented as taking the initiative and voting first.

2.2.2 Analysis of the two-voter case, fixed order

We consider a sequential voting procedure with complete information and fixed voting ordering. Let $V = \{1, 2\}$ denote the set of individuals (voters). A decision is made via sequential unanimity voting, and the voters have to choose between two alternatives $C = \{x, y\}$.

The voting procedure unfolds in a series of (potentially unending) stages, measured in discrete time $T = \{1, 2, \dots, t, t + 1, \dots\}$. Let $t = \infty$ denote the situation, when no decision is taken at any stage. The alternative implied in this case is denoted by θ . Formally, time is defined as $T = \{1, 2, \dots, t, t + 1, \dots\}$.

Preferences. Each individual $i \in V$ is endowed with a utility function u_i defined over the set of pairs (t, c) , where $c \in C$ and t is a stage. Formally, the utility function for every $i \in V$ is defined as follows: $u_i : T \times C \cup \{(\infty, \theta)\} \rightarrow \mathbb{R}$.

Voting schedule. The individuals are supposed to be engaged in a sequential voting procedure to collectively select one of the two alternatives. The voting procedure is based on a fixed ordering of the two individuals. For the sake of comprehension and without loss of generality, we restrict our attention first to the case in which player 1 votes first at every stage.

Stage 1. Individual 1 votes for x or votes for y . Knowing 1's choice, individual 2 next votes for x or for y . If both individuals vote for the same alternative c , then the procedure ends and, for $i \in V$, individual i gets utility $u_i(1, c)$. If the individuals do not vote for the same alternative, then the procedure moves to stage 2.

Stage t . If stage t is reached, then again individual 1 votes for x or votes for y . Knowing 1's choice, individual 2 next votes for x or votes for y . If both individuals vote for the same

alternative c , then the procedure ends and, for $i \in V$, individual i gets utility $u_i(t, c)$. If the individuals do not vote for the same alternative, then the procedure moves to stage $t + 1$.

Stage ∞ . Potentially it can happen that voters never vote for the same alternative, then we assume that no decision is taken and θ is implied. In this case, each voter i obtains $u_i(\infty, \theta)$.

The individuals' utility functions are assumed to satisfy conditions PER, IMP, REV, and TER, stated below.

Firstly, each individual has an alternative that is always more preferred than the other: the individual either prefers x over y at each time t or prefers y over x at each time t . In other words, each voter is persistent in his preferences. Let $\alpha_i \in C$ designate voter i 's most preferred alternative and $\beta_i \in C$ the least preferred alternative, i.e., $\alpha_i \neq \beta_i$, for each $i \in V$.

Axiom PER (Persistence). For $\alpha_i, \beta_i \in C$, such that $\alpha_i \neq \beta_i$, each voter $i \in V$, and each stage $t \in T$, $u_i(t, \alpha_i) > u_i(t, \beta_i)$.

Secondly, each voter is impatient, i.e., the time delay induces losses for the voters' utility. Axiom IMP asserts that the more the time passes to obtain an alternative, the smaller the corresponding utility.

Axiom IMP (Impatience). For each voter $i \in V$, each alternative $c \in C$, and each stage $t \in T$, $u_i(t, c) > u_i(t + 1, c)$.

Thirdly, for each individual, there is a time t_i making the individual always prefer to obtain the least preferred alternative now, rather than to obtain the most preferred alternative a moment immediately later (before t_i the opposite happens: it is better to have the most preferred alternative a moment later than to have the least preferred alternative immediately). Intuitively, t_i represents the moment at which i loses patience: it no longer pays to wait for the possibility of obtaining in the future the most preferred alternative by disregarding the possibility of obtaining now the least preferred alternative. This moment t_i shall be called the reversal stage of voter i . Note that we allow reversal stages to be equal.

Axiom REV (Reversion). For each voter $i \in V$, there is a reversal stage $t_i \in T$, such that, for each $t' < t_i$, $u_i(t', \beta_i) < u_i(t' + 1, \alpha_i)$, and for each $t \geq t_i$, $u_i(t, \beta_i) > u_i(t + 1, \alpha_i)$.

Finally, both voters prefer to stop the procedure at any stage, rather than never stopping. Hence, the utility obtained in this case should be smaller than the utility of obtaining any other outcome. The situation of no decision taken is denoted by θ , and the stage in this case is denoted by $t = \infty$.

Axiom TER (Termination). For each voter $i \in V$, each alternative $c \in C$, and each stage t , $u_i(t, c) > u_i(\infty, \theta)$.

We assume complete information, so that each voter knows the preferences, utilities and, therefore, reversal times of himself and the other voter.

To illustrate the voting procedure, consider [Example 4](#).

Example 4. There are two voters, both voting sequentially for two alternatives $\{x, y\}$. At each stage, voter 1 votes first either for x or for y , then voter 2 votes either for x or for y . Voter 1 prefers x and voter 2 prefers y : $\alpha_1 = \beta_2 = x$ and $\alpha_2 = \beta_1 = y$. The utilities are defined according to the four axioms: PER, IMP, REV, and TER. [Figure 2.1](#) shows the first five stages and the utilities of possible outcomes at these stages (we assume, that, starting from the sixth stage, utility functions for both voters fulfil the four axioms).

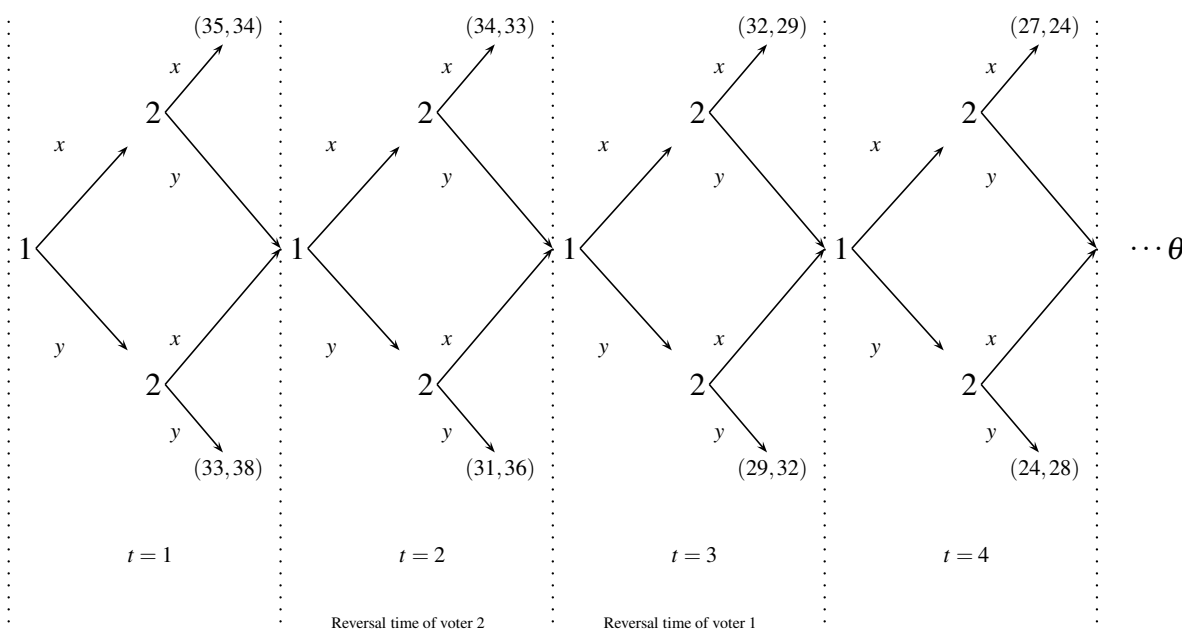


Figure 2.1. Example of voting procedure

For instance, assume that the procedure arrives at the second stage, $t = 2$, and individual 1 votes for y , while individual 2 votes for y , too. Then, the voting stops and the alternative y is implemented, so that individuals 1 and 2 obtain 31 and 36, accordingly. If individual 2 votes for x , then the voting passes to the next stage, $t = 3$, and the individual 1 is asked to vote again. If voters never vote for the same alternative, then the outcome is considered to be θ implemented at stage ∞ .

It is easy to see that the reversal time t_1 of voter 1 equals to 3: if at stage 3 he obtains his non-preferred alternative, y , it gives him a higher utility then waiting for the next stage and obtaining his most preferred alternative, x . That is, $u_2(2, y) > u_2(3, x)$. And, the reversal time of voter 2 equals to 2: $u_1(3, x) > u_2(4, y)$.

Subgame perfect equilibria

Since there is a fixed order, so that voter 1 always votes first, and voter 2 votes second, we can describe the procedure in terms of an offer by voter 1 to be accepted or rejected. That is, at each stage voter 1 offers some alternative and voter 2 either accepts it by voting for the same alternative or rejects it by voting for another alternative. Acceptance or rejection are denoted by a and r , respectively.

The described procedure represents the collective decision mechanism, by means of which voters carry on voting until they reach a unanimous decision. It can be seen as a game. We define a normal-form game $G = \langle V, (S_i)_{i \in V}, (U_i)_{i \in V} \rangle$ as follows:

Players. There is a set of players $v = \{1, 2\}$.

Strategies. A strategy for player 1 is a description of which alternative to offer at each stage, and a strategy for player 2 consists of a description of which alternatives he accepts or rejects at each stage. This is formalised by a set of strategies $S_1 = \{s_1 : T \rightarrow C\}$ for player 1, and a set of strategies $S_2 = \{s_2 : T \times C \rightarrow \{a, r\}\}$ for player 2.

Since T is an infinite set, each strategy is an infinite vector: $s_1 = (s_1(1), s_1(2), \dots, s_1(t - 1), s_1(t), \dots)$, where for each element $s_1 \in S_1$ and for each stage $t \in T$, $s_1(t)$ is interpreted as the alternative offered by player 1 at stage t . For instance, the strategy in which 1 always offers alternative x is $s_1 = (x, x, \dots, x, \dots)$.

By analogy, for player 2, for each $s_2 \in S_2$, each $t \in T$ and each $c \in C$, $s_2(t, c) = a$ is

interpreted as player 2 accepting alternative c at stage t , and $s_2(t, c) = r$ is interpreted as player 2 rejecting alternative c at stage t .

We denote $S = S_1 \times S_2$, and we refer to each $s \in S$ as a *strategy profile*.

Payoff functions. If at some stage player 2 accepted the offered alternative, then this alternative is implemented at this stage, which can be called the termination stage. Hence, the outcome is defined as the pair – (implemented alternative, termination stage). The termination stage represents the time when there is an agreement. It occurs if player 1 offers an alternative such that player 2 accepts it. Formally, the termination stage is given by $t(s) = \min \{t \in T | s_2(t, s_1(t)) = a\}$ if, under strategy profile s , the agreement is met at some stage t , and $t(s) = \infty$, otherwise. The outcome $o(s)$, induced by s_1, s_2 , is the alternative offered by player 1 at the termination stage. It is given by $o(s) = s_1(t(s))$, if $t(s) \neq \infty$, and $o(s) = \theta$, otherwise.

It is noteworthy that each strategy profile $s \in S$ induces a termination stage and an outcome in an endogenous way. Outcomes induce payoffs. Consequently, each voter gets a utility corresponding to the chosen alternative at the termination stage: $u_i(t(s), s_1(t(s)))$, whereby $u_i(\infty, \theta)$ in case of non-acceptance of any alternative at every stage.

Taking this into account, preferences on a set of outcomes $T \times C$, which are given by functions $(u_i)_{i \in V}$, are easily extended to the set of strategy profiles S , as $U_i(s) = u_i(t(s), o(s))$ for any player i and any strategy profile s . In this way, we obtain a well-defined pair of payoff functions $U_i : S \rightarrow \mathbb{R}$.

The main result tells us that under PER, IMP, REV and TER, every subgame perfect equilibrium induces the same outcome and that an agreement is achieved at the initial stage. Nonetheless, it remains valid independently of the degree of patience of each player. If at some stage t a player prefers to stop the procedure and, at the same stage, the other player prefers to continue, then the first voter's preferred alternative will be selected at this stage, and henceforth, at all the previous stages.

Subgame perfect equilibria. The proposed model allows for voters' asymmetries in patience, hence it seems natural to expect that such asymmetries, which we assume to be known, can impact the outcome of the voting procedure. Specifically, we may expect that the most

patient player could try to delay the agreement until the stage at which the least patient voter prefers to agree on any alternative, rather than to keep voting. In order to explore this hypothetical strategic advantage we recall first a standard solution concept, subgame perfect equilibria, and study next which outcomes are consistent with such behaviour. To this end, let us introduce first some auxiliary notions that allow to capture the dynamic nature of the voting procedure.

Conditional termination stage. For each strategy profile $s \in S$ and stage $t \in T$, we define the *conditional termination* stage $\vec{t}(s|t)$ of s at t as $\min\{t' \geq t | s_2(t', s_1(t')) = a\}$, whenever the players meet an agreement; and ∞ , otherwise.

Thus, $\vec{t}(s|t)$ simply represents at which stage an agreement would be met under strategy s , if no agreement had been met before stage t . In other words, the termination stage tells us when is the next stage after stage t (and including stage t) when players reach an agreement.

Conditional outcome. For each strategy profile $s \in S$ and stage t , we define the *conditional outcome* $o(s|t)$ of s at t as $s_1(\vec{t}(s|t))$, if $\vec{t}(s|t) \neq \infty$; and θ , otherwise.

A conditional outcome is the outcome at termination stage; in other words, conditional outcome tells us which alternative will be chosen if the voting started at t .

Player 1's conditional payoff. For each strategy profile $s \in S$, each stage $t \in T$ and each alternative $c \in C$, let $U_1(s|t) = u_1(\vec{t}(s|t), o(s|t))$ represent player 1's *conditional payoff* at t under s .

Player 2's conditional payoff. For each strategy profile $s \in S$ and each alternative $c \in C$, let $U_2(s|t, c)$ represent player 2's *conditional payoff* at stage t , after c is offered under the strategy profile s as $u_2(t, c)$ if $s_2(t, c) = a$; and $u_2(\vec{t}(s|t+1), o(s|t+1))$, otherwise.

The conditional payoffs of both voters are defined for a time t . Based on the strategies of both voters, the conditional payoff is a utility that voters get if they meet an agreement at the current stage

A subgame perfect equilibrium is defined as a profile of mutual best responses in terms of conditional payoffs.

Definition 1. Subgame perfect equilibrium (SPE). We say that strategy profile $s^* \in S$

is a subgame perfect equilibrium ($SPE(G)$) of G if:

- (i) $s_1^* \in \operatorname{argmax}_{s_1 \in S_1} U_1(s_2^*, s_1 | t)$ for all $t \in T$.
- (ii) $s_2^* \in \operatorname{argmax}_{s_2 \in S_2} U_2(s_1^*, s_2 | t, c)$ for all $(t, c) \in T \times C$.

The described game is potentially infinite, which makes it difficult to apply backward induction in order to find SPE. Hence, we need to eliminate the possibility of an infinity of the game and a possibility of the θ outcome. [Lemma 1](#) tells us that the game always stops the most at the reversal time of player 2, t_2 .

Lemma 1. *For any $s^* \in S$, if $s_2^* \in \operatorname{argmax}_{s_2 \in S_2} U_2(s_1^*, s_2 | t_2, c)$, then $s_2^*(t_2, c) = a$ for any $c \in C$.*

Proof. We show that at stage t_2 , the best reply of player 2 is to accept any alternative which player 1 suggests. This implies that t_2 is a conditional termination stage and the corresponding conditional outcome is $o(s^*, t_2) = s_1^*(t_2)$.

To check it, we assume that the best reply for player 2 at t_2 is to reject any alternative that 1 offers. We compare payoffs of this strategy and a strategy which differs from that one only at t_2 . Showing that the latter gives a better payoff for player 2 induces that rejection is not the best strategy at t_2 . Consequently, at t_2 the best strategy for 2 is to accept.

Suppose that c is such an alternative that the best reply of player 2 at stage t_2 is to reject it: $s_2^*(t_2, c) = r$. Let $s'_2 \in S_2$ be the strategy of player 2, defined as the best reply at all the stages, except the stage t_2 , where player 2 accepts alternative c . Formally, $s'_2(t_2, c) = a$ and $s'_2(t, c') = s_2^*(t, c')$ for any $(t, c') \neq (t_2, c)$.

Then, note that since under strategy s'_2 player 2 accepts alternative c , he obtains a payoff $U_2(s_1^*, s'_2 | t_2, c) = u_2(t_2, c)$. At the same time, as we supposed, the best reply of player 2 is to reject c ; that is $s_2^*(t_2, c) = r$. Therefore, under this strategy player 2 obtains a payoff in some future: $U_2(s_1^*, s_2^* | t_2, c) = u_2(t', c')$ for some $t' > t_2$ and some c' .

Comparing these two possible outcomes tell that, by REV and IMP, strategy s'_2 gives better a payoff than the supposed best reply strategy s_2^* . Formally, $U_2(s_1^*, s'_2 | t_2, c) > U_2(s_1^*, s_2^* | t_2, c)$, by TER and IMP if $t' = \infty$, and by REV and IMP if $t' \neq \infty$.

□

Lemma 2 shows that at t_2 the best strategy of player 1 is to propose his preferred alternative.

Lemma 2. *For any $s^* \in S$, if $s_1^* \in \operatorname{argmax}_{s_1 \in S_1} U_1(s_1, s_2^* | t_2)$, then $s_1^*(t_2) = \alpha_1$.*

Proof. **Lemma 1** demonstrated that at t_2 the best reply of player 2 is to accept any alternative that player 1 suggests. Given this, we need to show that the best strategy for 1 is to offer his preferred alternative, α_1 .

By **Lemma 1**, if player 1 gets either (i) $u_1(t_2, \alpha_1)$ if he offers α_1 , or (ii) $u_1(t_2, \beta_1)$ if he offers β_1 . By PER $u_1(t_2, \alpha_1) > u_1(t_2, \beta_1)$, therefore, the best strategy of player 1 is to suggest his preferred alternative, since he knows that the best strategy of player 2 is to accept. □

Lemma 3 tells that at any stage before the reversal time of player 2 the best strategy of 2 is to accept every proposal of player 1, and of 1 is to offer his most preferred alternative.

Lemma 3. *For any $s^* \in SPE(G)$ and any $t \in [1, t_2)$ we have (i) $s_2^*(t, c) = a$ for any $c \in C$, and (ii) $s_1^*(t) = \alpha_1$.*

Proof. We proceed by induction:

Base step. By **Lemma 1** and **Lemma 2**, (i) and (ii) hold for $t = t_2$. That is, at stage t_2 player 1 offers α_1 and player 2 accepts it. This leads to outcome α_1 at time t_2 .

Inductive step. Suppose that $\hat{t} < t_2$ is such that the claim holds for $\hat{t} + 1$. Let us check that it also holds for \hat{t} . We begin with (i).

Suppose, by contradiction, that alternative $c \in C$ is such that player 2 is better off by rejecting it: $s_2^*(\hat{t}, c) = r$. Define a strategy $s'_2 \in S_2$ as the best strategy at all the stages except the current stage: $s'_2(\hat{t}, c) = a$ and $s'_2(t, c') = s_2^*(t, c')$ for any $(t, c') \neq (\hat{t}, c)$.

By the induction hypothesis, at the next stage the best strategy of 1 is to offer α_1 and the best reply of 2 is to accept it: $s_1^*(\hat{t} + 1) = \alpha_1$ and $s_2^*(\hat{t} + 1, c) = a$. Then, since player 2 rejects the offer at the current stage, i.e., $s_2^*(\hat{t}, c) = r$, he would get the conditional payoff $U_2(s^* | \hat{t}, c) = u_2(\hat{t} + 1, \alpha_1)$. At the same time under strategy $s'_2(\hat{t}, c)$ player 2 accepts an offer and, consequently he obtains $U_2(s_1^*, s'_2 | \hat{t}, c) = u_2(\hat{t}, \alpha_1)$. Therefore, $U_2(s_1^*, s'_2 | \hat{t}, c) > U_2(s_1^*, s_2^* | \hat{t}, c)$, by IMP, if $c = \alpha_1$, and by PER, if $c = \beta_1$. Thus, we conclude that $s_2^* \notin$

$\operatorname{argmax}_{s_2 \in S_2} U_2(s_1^*, s_2 | \hat{t}, c)$, which is a contradiction. To sum up, the best strategy for player 2 at \hat{t} step is to accept.

Next we show that the best reply of player 1 at the inductive stage is to offer his most preferred alternative. Suppose, by contradiction, that the best strategy for 1 is to offer his nonpreferred alternative: $s_1^*(\hat{t}) = \beta_1$. Define then a strategy $s'_1 \in S_1$ as follows: $s'_1(\hat{t}) = \alpha_1$; and $s'_1(t) = s_1^*(t)$, if $t \neq \hat{t}$. By (i), the best strategy of 2 is to accept any alternative offered: $s_2^*(\hat{t}, c) = a$ for any $c \in C$; hence $U_1(s_2^*, s_1^* | \hat{t}) = u_1(\hat{t}, \beta_1)$, and $U_1(s_2^*, s'_1 | \hat{t}) = u_1(\hat{t}, \alpha_1)$. Therefore, by PER, we conclude that $s_1^* \notin \operatorname{argmax}_{s_1 \in S_1} U_1(s_2^*, s_1 | \hat{t})$, which is a contradiction.

To sum up, it has been shown that at the inductive stage \hat{t} , player 1 offers his most preferred alternative and player 2 accepts it under the best strategy profile. \square

Lemma 1, Lemma 2, and Lemma 3 allowed us to prove that under the four axioms PER, IMP, REV, and TER, in the unique SPE, the first voter votes for his preferred alternative, and the second voter always agrees to vote for the same alternative, and therefore, a unanimous decision is obtained at the first stage. This result is formulated in the following proposition:

Proposition 3. *Given PER, IMP, REV, and TER, every subgame perfect equilibrium strategy profile of G agrees with the first voter's preferred alternative at the first stage; i.e., $(t(s^*), c(s^*)) = (1, \alpha_1)$ for any $s^* \in SPE(G)$.*

Proof. The proposition follows from applying the inductive step from Lemma 3 to $\hat{t} = 1$. \square

Remark 2. It is important to note that the axiom REV was applied only regarding player 2. In other words, only the reversal time of player 2 affects the outcome. Hence, the conjecture that patience matters is not confirmed: even an impatient voter, if he votes first, obtains his preferred alternative immediately at the first stage.

2.2.3 Analysis of the two-voter case, voting order with a reverse

Assume that there is a time, that we shall call it $\tau \in T$, $\tau > 1$, such that from τ on the order is reversed. That is, starting from τ voter 2 votes first, and voter 1 votes second. It can be seen as the moment when voter 1 stops having an advantage of voting first, as happens

with a fixed order. Intuition suggests that voter 1 will not always win, since he has a certain deadline of his advantage. On the other hand, since Proposition 3 tells us that he obtains his preferred alternative at the first stage, it may happen that the change of order does not affect the outcome at the first stage.

Assume, that both voters know τ . And now the question is whether the Proposition 3 holds.

To simplify the notation, let $i_1(t) \in V$ denote the individual which votes first at $t \in T$, let and $i_2(t) \in V$ denote the individual who votes second. Hence, for each stage $t < \tau$, $i_1(t) = 1$ and $i_2(t) = 2$; and for each stage $t < \tau$, $i_1(t) = 2$ and $i_2(t) = 1$. The voting process is illustrated in Figure 2.2.

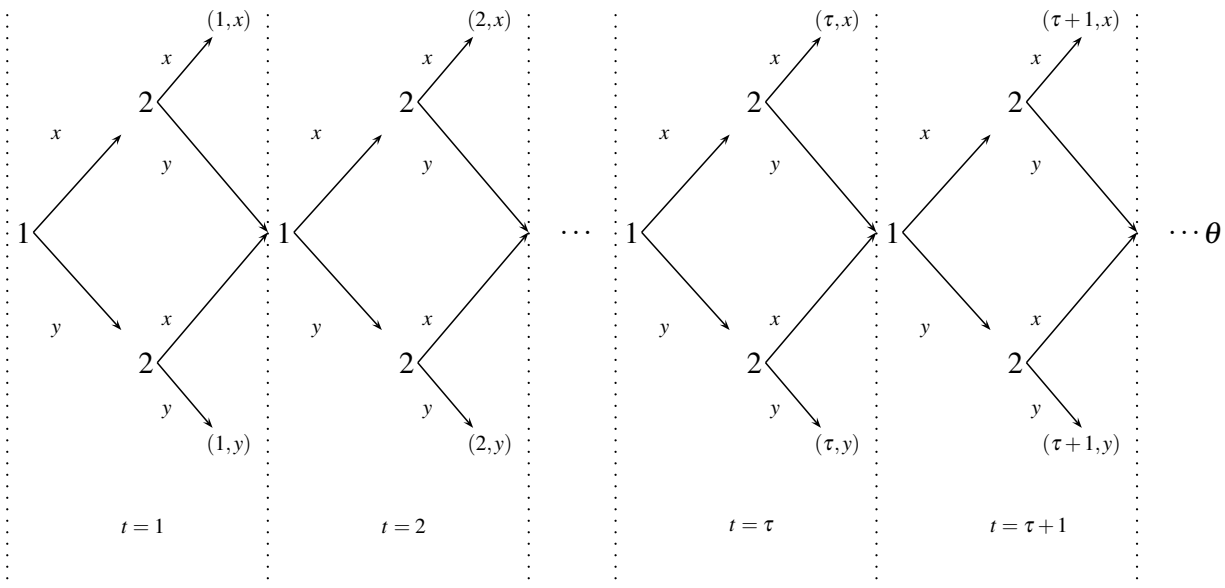


Figure 2.2. Example of voting procedure with reversed voting order.

Strategies. Before stage τ , the strategies remain the same as in the fixed-order case and after stage τ the strategies are mirrored for the players. Thus, a strategy for player $i \in V$ is a description of what alternative to offer at each stage t when $i = i_1(t)$, and what alternatives he accepts or rejects at each stage t when $i = i_2(t)$. Hence, for each voter a strategy consists of two elements, formally $s_i = (s_{i=i_1}, s_{i=i_2})$, where $s_{i=i_1}$ denotes a strategy of player i at the stages when he votes first, and $s_{i=i_2}$ denotes a strategy of player i at the stages when he votes second. Formally, for voter 1 the strategy s_1 consists of

$s_{1=i_1} : \{1, \dots, \tau - 1\} \rightarrow C$ and $s_{1=i_2} : \{\tau, \dots\} \times C \rightarrow \{a, r\}$; and for voter 2 the strategy s_2 consists of $s_{1=i_1} : \{1, \dots, \tau - 1\} \times C \rightarrow \{a, r\}$ and $s_{1=i_2} : \{\tau, \dots\} \rightarrow C$.

As previously, we denote $S = S_1 \times S_2$, and we refer to each $s \in S$ as a *strategy profile*.

The termination stage is given by $t(s) = \min \{t \in T | s_{i_2}(t, s_{i_1}(t)) = a\}$ if, under strategy profile s the agreement is met at some stage t , and $t(s) = \infty$, otherwise. The outcome in this case is the alternative offered by player i_1 at the termination stage, which is given by $o(s) = s_{i_1}(t(s))$, if $t(s) \neq \infty$, and $o(s) = \theta$, otherwise. Utility corresponding to the chosen alternative at the termination stage is $u_i(t(s), s_{i_1}(t(s)))$ ($u_i(\infty, \theta)$), in the case of non-acceptance of any alternative at every stage). Thus, payoff functions are defined as in the fixed-order case.

The concepts of conditional termination stage, conditional outcome, and conditional payoffs remain the same:

Conditional termination stage. For each strategy profile $s \in S$ and stage $t \in T$, we define the *conditional termination stage* $\vec{t}(s|t)$ of s at t as $\min\{t' \geq t | s_{i_2}(t', s_{i_1}(t')) = a\}$, whenever the players meet an agreement; or ∞ , otherwise.

Conditional outcome. For each strategy profile $s \in S$ and stage t , we define the *conditional outcome* $o(s|t)$ of s at t as $s_{i_1}(\vec{t}(s|t))$, if $\vec{t}(s|t) \neq \infty$; or θ , otherwise.

Player i_1 's conditional payoff. For each strategy profile $s \in S$, each stage $t \in T$ and each alternative $c \in C$, let $U_{i_1}(s|t) = u_{i_1}(\vec{t}(s|t), o(s|t))$ represent player i_1 's *conditional payoff* at t under s .

Player i_2 's conditional payoff. For each strategy profile $s \in S$ and each alternative $c \in C$, let $U_{i_2}(s|t, c)$ represent player 2's *conditional payoff* at stage t , after c is offered under the strategy profile s as $u_{i_2}(t, c)$ if $s_{i_2}(t, c) = a$; or $u_{i_2}(\vec{t}(s|t+1), o(s|t+1))$, otherwise.

First, consider that $t_2 < \tau$. Hence, at all stages including t_2 the voting order remains the same as in the fixed-order case. The next proposition shows that the result of [Proposition 3](#) holds if $t_2 < \tau$.

Proposition 4. *If $t_2 < \tau$, then every subgame perfect equilibrium strategy profile agrees with the first voter's preferred alternative at the first stage; i.e., $(t(s^*), c(s^*)) = (1, \alpha_1)$ for any*

$s^* \in SPE(G)$.

Proof. When $t_2 < \tau$, then by [Lemma 1](#) at stage t_2 player 2 accepts any offer, hence the procedure never arrives to the stage τ . By [Lemma 2](#), the best strategy for player 1 is to offer his most preferred alternative, $\alpha_1 = x$. Next, applying the inductive step from [Lemma 3](#) to $\hat{t} = 1$, we obtain that [Proposition 3](#) holds even if there is such a stage τ , when the order starts to be reversed. \square

Next, consider the opposite situation: the reversal time of player 2 is greater than, or equal to, τ . We aim to show that in the unique subgame perfect equilibrium, the alternative preferred by player 2 is obtained at the first stage. To prove this, we formulate several lemmas. [Lemma 4](#) tells us that at stage τ , player 2 suggests his preferred alternative, and player 1 accepts it.

Lemma 4. *If $t_2 \geq \tau$, then for any $s^* \in SPE(G)$ we have $s_1^*(\tau, \alpha_2) = a$ and $s_2^*(\tau) = \alpha_2$.*

Proof. Note that $i_1(t) = 2$ for all $t \geq \tau$. Taking this into account, we replicate the results of [Lemma 1](#), [Lemma 2](#), and [3](#).

There are two possible cases, depending on whether t_1 is greater or smaller than τ . First, consider $t_1 > \tau$. Replicating [Lemma 1](#) for t_1 , we obtain that $s_2^*(t_1) = \alpha_2$ and $s_1^*(t_1, c) = a$. We replicate the inductive step from [Lemma 3](#) for any t , such that $\tau \leq t_1$. Next, we apply it to $t = \tau$. Thus, we obtain $s_2^*(\tau) = \alpha_2$ and $s_1^*(\tau, c) = a$.

Second, consider $t_1 \leq \tau$. Replicating [Lemma 1](#) for $\tau \geq t_1$ gives that $s_2^*(\tau) = \alpha_2$ and $s_1^*(\tau, c) = a$. \square

Next, we show that at the stage before τ the best strategy of player 2 is to accept the alternative α_2 and to reject the alternative β_2 . Given this, the best strategy of player 1 is to offer his non-preferred alternative, $\alpha_2 = \beta_1$.

Lemma 5. *If $t_2 \geq \tau$, then for any $s^* \in SPE(G)$ we have $s_1^*(\tau-1) = \alpha_2$ and $s_2^*(\tau-1, \alpha_2) = a$.*

Proof. [Lemma 4](#) implies that τ is a conditional termination stage and the conditional outcome is $o(s^*, \tau) = s_2(\tau, s_2(\tau, c))$. Accordingly, player 1 gets conditional payoff $u_1(\tau, \alpha_2)$ and player 2 gets conditional payoff $u_2(\tau, \alpha_2)$.

It is obvious that by PER and IMP, the best strategy for player 2 is to always accept α_2 at any t . Hence, we need to show that if player 1 offers β_2 , then the best strategy for player 2 is to reject it.

Suppose that at stage $\tau - 1$ player 1 votes for $\beta_2 = \alpha_1$. If player 2 rejects the alternative β_2 , he obtains a conditional payoff $u_2(\tau, \alpha_2)$. At the same time, if he accepts the alternative, that would give him $u_2(\tau - 1, \beta_2)$. Since $\tau - 1 < t_2$, by REV, $u_2(\tau, \alpha_2) > u_2(\tau - 1, \beta_2)$, then the best strategy for player 2 is to reject β_2 .

Next, we show that at stage $\tau - 1$ the best reply of voter 1 is to offer his non-preferred alternative, $\alpha_2 = \beta_1$.

If player 1 offers $\alpha_1 = \beta_2$, as shown, player 2 rejects it, hence player 1 obtains payoff $u_1(\tau, \alpha_2)$. If player 1 offers $\beta_1 = \alpha_2$, then player 2 accepts it, hence player 1 gets $u_1(\tau - 1, \alpha_2)$. By IMP, $u_1(\tau, \alpha_2) > u_1(\tau - 1, \alpha_2)$, and the best strategy for player 1 is to offer $\alpha_2 = \beta_1$. \square

Lemma 6 states that at all the stages before τ , the best strategy of player 1 is to offer his non-preferred alternative, which player 2 accepts.

Lemma 6. *If $t_2 \geq \tau$, then for any $s^* \in SPE(G)$ and any $t \in [1, \tau)$ we have $s_1^*(t) = \alpha_2$ and $s_2^*(t, \alpha_2) = a$.*

Proof. Next, we show that at any $t < \tau$ the best strategy for player 1 is to suggest α_2 , and the best reply of player 2 is to accept it.

Base step. **Lemma 5** showed that at stage $\tau - 1$, player 1 offers α_2 and player 2 accepts it.

Inductive step. Suppose that $\hat{t} < \tau - 1$ is such that the claim holds for $\hat{t} + 1$. Let us check that it also holds for \hat{t} .

By the induction hypothesis, at the next stage the best strategy of 1 is to offer α_2 and the best reply of 2 is to accept it: $s_1^*(\hat{t} + 1) = \alpha_2$ and $s_2^*(\hat{t} + 1, \alpha_2) = a$.

Assume that at \hat{t} player 1 offers $\alpha_1 = \beta_2$. Then, player 2 obtains (i) $u_2(\hat{t}, \beta_2)$ if he accepts it, and (ii) $u_2(\hat{t} + 1, \alpha_2)$ if he rejects it. Since $\hat{t} < t_2$, by REV the best strategy for player 2 is to reject $\alpha_1 = \beta_2$.

Assume that at \hat{t} player 1 offers $\beta_1 = \alpha_2$. Then, player 2 obtains (i) $u_2(\hat{t}, \alpha_2)$ if he accepts it or (ii) $u_2(\hat{t} + 1, \alpha_2)$ if he rejects it. Since $\hat{t} < t_2$, by REV the best strategy for player 2 is

to accept α_2 .

Knowing this, player 1 obtains (i) $u_1(\beta_1, \hat{t} + 1)$ if he offers α_1 or (ii) $u_1(\beta_1, \hat{t})$ if he offers β_1 . By IMP, the best strategy for player 1 is to offer his non-preferred alternative. \square

[Lemma 4](#), [Lemma 5](#), and [Lemma 6](#) allow us to prove that if $t_2 \geq \tau$, then [Proposition 6](#) does not hold.

Proposition 5. *If $t_2 \geq \tau$, then every subgame perfect equilibrium strategy profile agrees with the second voter's preferred alternative at the first stage; i.e., $(t(s^*), c(s^*)) = (1, \alpha_2)$ for any $s^* \in SPE(G)$.*

Proof. Applying the inductive step from [Lemma 6](#) to $\hat{t} = 1$ gives that the alternative preferred by player 2 is obtained at the first stage. \square

2.2.4 Analysis of the n -voter case

Next, we extend the model with fixed voting order to more than 2 voters and 2 alternatives, and check whether [Proposition 3](#) holds. While patience might be not the crucial parameter, as happened in the case of two agents as described above, it is interesting to study whether the alternative preferred by the plurality rule wins in the result of sequential voting.

In order to solve the general case some more notation is needed. We extend the set of voters from two to n voters, that is, let $V = \{1, 2, \dots, n\}$ denote the set of voters, with typical elements i and j , where $n > 2$. Additionally, we extend the set of alternatives from 2 to m , that is, let C denote a set of alternatives, such that $\#C = m \geq 2$. We assume that the preferences of the voters are dicotomic. That is, although there are more than two alternatives, for each voter there is a unique most preferred alternative, and the preference order among other alternatives can be undefined. Hence, let $\alpha_i \in C$ designate voter i 's most preferred alternative, and $\beta_i \in C$ designate any other alternative.

Consequently, the formal definition of REV (described in [Subsection 2.2.2](#)) is slightly modified, taking into account that there are several non-preferred alternatives β_i for each voter. That is, REV defines a reversal time as a stage when the voter can be better off by choosing now any non top-preferred alternative, rather than wait until next stage and obtain his most preferred alternative.

Axiom REV (Reversion). For each voter $i \in V$, there is a reversal stage $t_i \in T$, such that for each $t' < t_i$, $u_i(t', \beta_i) < u_i(t' + 1, \alpha_i)$, and for each $t \geq t_i$, $u_i(t, \beta_i) > u_i(t + 1, \alpha_i)$ for any $\beta_i \neq \alpha_i$.

As in [Subsection 2.2.2](#), time is discrete: $t \in T$; the ordering of voters is fixed and remains the same at each stage; and we assume that voter 1 votes first at every stage, voter 2 votes second, and so on.

Stage 1. Voter 1 votes (offers) one alternative out of the set of alternatives. Let $c \in C$ denote this alternative. In knowing the choice of voter 1, voter 2 next votes either for the same alternative c , or for some other alternative. It is supposed that 2 accepts the offer of voter 1, if 2 votes for the same alternative, and rejects it otherwise. Further, in knowing the votes of voters 1 and 2, voter 3 also votes either for the alternative offered by 1 alternative (voter 3 accepts 1's offer), or for some other alternative (3 rejects 1's offer). And so on, until voter n votes in the same way. As in [Subsection 2.2.2](#), let a denote a voter's acceptance and let r denote a voter's rejection of the offered alternative. If all the individuals $i \geq 2$ accepted the alternative offered by 1, then the procedure ends and, for $i \in V$, individual i gets utility $u_i(1, c)$. If some voter does not vote for this alternative, then the procedure moves to stage 2.

Stage t . If stage t is reached, then again individual 1 votes for some alternative c . In knowing 1's choice, voter 2 next votes for c or for some other alternative. In knowing the choices of voters 1 and 2, voter 3 also votes for c or for some other alternative. And so on, until voter n acts in the same way. If all the voters $i \geq 2$ accepted the alternative offered by 1, then the procedure ends and, for $i \in V$, voter i gets utility $u_i(t, c)$. If these voters do not vote for this alternative, then the procedure moves to stage $t + 1$.

Stage ∞ . If no decision is taken at any stage, then each voter i obtains $u_i(\infty, \theta)$, which by TER is the worst for every voter.

Therefore, two possible outcomes may appear: $(t, c) \in V \times C$, where t represents the first stage in which every voter agreed with the vote of voter 1, and $(t, c) = (\infty, \theta)$, where at every stage, some voter $i \geq 2$ voted differently, except for c .

The normal-form game $G = \langle V, (S_i)_{i \in V}, (U_i)_{i \in V} \rangle$ identifies the proposed procedure as

follows:

Players. We have a set of players $V = \{1, \dots, n\}$. At each stage, player 1 proposes alternative $c \in C$, and then the rest of players sequentially accept or reject 1's proposal, following the fixed order.

Information. Available information is formalised for each player i as follows

$$H_i = \begin{cases} T & \text{if } i = 1, \\ T \times C & \text{if } i = 2, \\ T \times C \times \{a, r\}^{i-2} & \text{if } i \geq 3. \end{cases}$$

Each element $h_i \in H_i$ describes player i 's available *information*:

- The stage in which the choice being made is: $t(h_i) = \text{Proj}_T(h_i)$ for any $h_i \in H_i$ and any $i \in V$.
- The first voter's proposal: $c_1(h_i) = \text{Proj}_C(h_i)$ for any $h_i \in H_i$ and any $i \geq 2$.
- Whether each of the precedent voters accepted or rejected the proposal: $d(h_i) = \text{Proj}_{\{a, r\}^{\{2, \dots, i-1\}}}(h_i)$ for any $h_i \in H_i$ and any $i \geq 3$, where $d_j(h_i)$ denotes the decision corresponding to j according to h_i for each $j = 2, \dots, i-1$.

Strategies. A strategy is a description of actions that are contingent on available information. A strategy for player 1 consists of a description of the alternative he offers at each stage. Hence, voter 1's set of *strategies* is given by $S_1 = A^{H_1}$. Furthermore, for each voter $i \geq 2$, a strategy describes whether he accepts or rejects each alternative at each stage, conditional on what precedent voters decided, so that, for each voter $i \geq 2$, $S_i = \{a, r\}^{H_i}$. Let $S = \prod_{i \in V} S_i$ denote the set of strategy *profiles*, and for each player i , let $S_{-i} = \prod_{j \neq i} S_j$ denote the set of *partial profiles* of all the voters, except the voter i .

Payoff functions. Note that each strategy profile $s \in S$ determines a sequence of actions at each stage t ; let:

$$\begin{aligned} s_1(t, s_{-1}) &= s_1(t), \text{ and,} \\ s_i(t, s_{-i}) &= s_i(t, s_1(t, s_{-1}), \dots, s_{i-1}(t, s_{-(i-1)})) \end{aligned}$$

for any $i \geq 2$, recursively. Then, $(s_i(t, s_{-i}))_{i \in V}$ specifies player 1's offer at stage t , and the remaining players' response at stage t to that offer. Obviously, for each player $i \geq 2$, $h_i(t, s_{-i}) = (t, (s_j(t, s_{-j}))_{j=1}^{i-1})$ is a well-defined element of H_i . Based on the latter, it is easy to see that each strategy profile $s \in S$ induces a termination stage (the moment in which the agreement takes place) and an outcome in an endogenous way. Specifically, the termination stage is given by,

$$t : S \longrightarrow T \cup \{\infty\}$$

$$s \longmapsto \begin{cases} \min \bigcap_{i \geq 2} \{t \in T \mid s_i(t, h_i(t, s_{-i})) = a\} & \text{if such a minimum exists,} \\ \infty & \text{otherwise,} \end{cases}$$

and the outcome is given by $o(s) = s_1(t(s))$ when $t(s) \neq \infty$ and $o(s) = \theta$, otherwise.

Taking this into account, preferences on the set of outcomes $(T \times C) \cup \{(\infty, \theta)\}$, which are given by functions $(u_i)_{i \in V}$ described in 2.2.2, are straightforwardly extended to the set of strategies S as follows: let $U_i(s) = u_i(t(s), o(s))$ for any player i and any strategy profile s . In this way, we obtain well-defined payoff functions $U_i : S \rightarrow \mathbb{R}$.

Subgame perfect equilibria. The proposed model allows for voters' asymmetries in patience, hence it seems natural to expect that such asymmetries, which we assume to be known, can impact on the outcome of the voting procedure. We recall first a standard solution concept, subgame perfect equilibria, and study next which outcomes are consistent with such behaviour. But, beforehand let us first introduce some auxiliary notions.

Conditional termination stage. For each strategy profile $s \in S$, each player $i \neq j \in V$ and $h_i \in H_i$, let $s_j(h_i, s_{-i})$ represent player j 's *conditional choice* under strategy profile s in the case when h_i was reached. If $j < i$, then j 's choice is described by h_i , while, otherwise, it needs to be computed recursively. Now, let $h_1(h_i, s) = t(h_i)$, and $h_j(h_i, s) = (t(h_i), (s_k(h_i), s_{-k}))_{k=1}^{j-1}$ for any $j \geq 2$. Such $h_j(h_i, s)$ represents the information that player j would have in the case where h_i had been reached and s was played from that moment on. Obviously, $h_j(h_i, s) \in H_j$ and we refer to it as player j 's *conditional information* under strategy profile s in the case h_i was reached.

For each strategy profile $s \in S$ and stage $t \in T$, $\vec{t}(s|t)$ denotes the conditional termination

stage under s in the case t was reached, that is, $\vec{t}(s|t) = \min \bigcap_{i \geq 2} \{t' \geq t | s_i(t', s_i(t', s_{-i})) = a\}$, if such minimum exists, or $\vec{t}(s|t) = \infty$, otherwise.

Additionally, for each player $i \in V$ and $h_i \in H_i$, $\vec{t}(s|h_i)$ denotes the *conditional termination stage* under s at h_i , i.e., $\vec{t}(s|h_i) = t(h_i)$ if $s_j(h_i, s_{-j}) = a$ for any $j \geq 2$, or $\vec{t}(s|h_i) = \vec{t}(s|t(h_i) + 1)$, otherwise.

Thus, $\vec{t}(s|t)$ ($\vec{t}(s|h_i)$) simply represents which stage would lead to an agreement, if an agreement was reached at t (h_i). Hence, the difference between the stages is that $\vec{t}(s|t)$ imposes that stage t has been reached, while $\vec{t}(s|h_i)$ imposes that stage $t(h_i)$ has been reached and that alternative $c(h_i)$ has been offered at $t(h_i)$, and for $j = 2, \dots, i - 1$, decision $d_j(h_i)$ has been made at $t(h_i)$.

Conditional outcome. For each strategy profile $s \in S$, each voter $i \in V$, and each $h_i \in H_i$, we define the *conditional outcome* $o(s|h_i)$ of s at h_i as $s_1(\vec{t}(s|h_i))$ if $\vec{t}(s|h_i) \neq \infty$, or θ , otherwise. That is, $o(s|h_i)$ represents the outcome that voter i would have obtained, if h_i has been reached and s was played from there on.

Conditional payoff. For each strategy profile $s \in S$, each voter $i \in V$ and each $h_i \in H_i$, voter i 's *conditional payoff* at h_i under strategy profile s is given by $U_i(s|h_i) = u_i(\vec{t}(s|h_i), o(s|h_i))$. That is, $U_i(s|h_i)$ represents the payoff that voter i would obtain, if h_i has been reached and s was played from there on.

Then, as usual, subgame perfect equilibrium is defined as profiles of mutual best responses in terms of conditional payoffs.

Definition 2 (Subgame perfect equilibrium (SPE)). We say that strategy profile $s^* \in S$ is a subgame perfect equilibrium of G if for any voter $i \in V$,

$$s_i^* \in \bigcap_{h_i \in H_i} \operatorname{argmax}_{s_i \in S_i} U_i(s_{-i}^*, s_i | h_i),$$

Hereinafter, $SPE(G)$ denotes the set of subgame perfect equilibria of G .

The main result goes in line with the result obtained for 2 individuals: under PER, IMP, REV and TER, every subgame perfect equilibria induces the same outcome and that an

agreement is achieved at the initial stage. Nonetheless, it remains valid independently from the degree of patience of each individual.

Let us introduce some notation first. Let t^* denote the greatest reversal time of all the voters: $t^* = \max\{t_1, \dots, t_n\}$. Recall that voter 2's available information is of type (t, c) , and for $i \in \{3, \dots, n\}$, (t, c, d_{-i}) . Hence, for the sake of consistency and clarity, we consider that $a_{-2} = d_{-2}$ are empty sets, and for any $i \geq 3$, $a_{-i} = d_{-i} = (d_j)_{j=2}^{i-1}$, where $d_j = a$ for any $j = 2, \dots, i-1$. Furthermore, we denote (t, c, a_{-2}) by (t, c) for any $t \in T$ and any $c \in C$.

Similarly to [Lemma 1](#), [Lemma 7](#) next shows that voting will never continue after the greatest reversal time is reached and at this stage the best reply of all the voters $i \geq 2$ is to accept whatever voter 1 suggested.

Lemma 7. *For any $s^* \in SPE(G)$, any $c \in C$, and any $i \geq 2$, $s_i(t^*, c, a_{-i}) = a$.*

Proof. We check it by induction. For the initial step, suppose that the best reply of voter n at stage t^* is to reject the alternative proposed by player 1 and accepted by everyone, before the player n . Formally, $s^* \in SPE(G)$ and $c \in C$ are such that $s_n^*(t^*, c, a_{-n}) = r$. Obviously, this implies that $\vec{t}(s^*|t^*, c, a_{-n}) > t^*$. Define s'_n as a strategy, such that at stage t^* it is not the best reply and at all the other stages this strategy is the best reply. Formally, $s'_n(t^*, c, a_{-n}) = a$ and $s'_n(h_n) = s_n^*(h_n)$ if $h_n \neq (t^*, c, a_{-n})$. Then, either by TER, if $\vec{t}(s^*|t^*, c, a_{-n}) = \infty$, or by REV, if $\vec{t}(s^*|t^*, c, a_{-n}) < \infty$, $U_n(s_{-n}^*, s'_n|t^*, c, a_{-n}) = u_n(t^*, c) > U_n(s^*|t^*, c, a_{-n})$. Thus, rejection is not the best reply of voter n : $s_n^* \notin \operatorname{argmax}_{s_n \in S_n} U_n(s_{-n}^*, s_n|t^*, c, a_{-n})$, and this is a contradiction, because we assumed that $s^* \in SPE(G)$. Therefore, it is shown that the best reply of voter n is to accept the proposed alternative.

For the inductive step, suppose that $i \in \{2, \dots, n-1\}$ is such that the lemma holds for any $j > i$, therefore it suffices to check that it also holds for i . Suppose that $s^* \in SPE(G)$ and $c \in C$ are such that $s_i^*(t^*, c, a_{-i}) = r$. Obviously, this implies that $\vec{t}(s^*|t^*, c, a_{-i}) > t^*$. Define, then, s'_i as follows: $s'_i(t^*, c, a_{-i}) = a$ and $s'_i(h_i) = s_i^*(h_i)$ if $h_i \neq (t^*, c, a_{-i})$. Then, either by TER, if $\vec{t}(s^*|t^*, c, a_{-n}) = \infty$, or by REV, if $\vec{t}(s^*|t^*, c, a_{-n}) < \infty$, $U_i(s_{-i}^*, s'_i|t^*, c, a_{-i}) = u_i(t^*, c) > U_i(s^*|t^*, c, a_{-i})$. Thus, $s_i^* \notin \operatorname{argmax}_{s_i \in S_i} U_i(s_{-i}^*, s_i|t^*, c, a_{-i})$, and this is a contradiction, because we assumed that $s^* \in SPE(G)$. \square

[Lemma 8](#) next states that, at the greatest reversal time the best reply for player 1 is to

propose his most preferred alternative, since by [Lemma 7](#) all other players will accept it.

Lemma 8. *For any $s^* \in SPE(G)$, $s_1(t^*) = \alpha_1$.*

Proof. We proceed by contradiction. Suppose that the best strategy for player 1 is to propose some alternative that is different from the most preferred alternative, formally: $s^* \in SPE(G)$ is such that $s_1(t^*) = \beta_1$. Under strategy s^* by [Lemma 7](#), $U_1(s_{-1}^*, s_1^*|t^*) = u_1(t^*, \beta_1)$.

At the same time, let s'_1 be a strategy such that at t^* , player 1 proposes his preferred alternative, and at all other stages this strategy is the best one: $s'_1(t^*) = \alpha_1$ and $s'_1(t) = s_1^*(t)$ for any $t \neq t^*$. Under strategy s'_1 by [Lemma 7](#), $U_1(s_{-1}^*, s'_1|t^*) = u_1(t^*, \alpha_1)$.

By IMP, $U_1(s_{-1}^*, s'_1|t^*) > U_1(s_{-1}^*, s_1^*|t^*)$, hence, $s_1^* \notin \operatorname{argmax}_{s_1 \in S_1} U_1(s_{-1}^*, s_1|t^*)$, and this is contradiction, because we assumed that $s^* \in SPE(G)$.

Therefore, given [Lemma 7](#), the best reply of player 1 at the greatest reversal stage is to propose his most preferred alternative. □

Next, [Lemma 9](#) states that for any stage before the greatest reversal stage, the best reply for player 1 is to propose his most preferred alternative, and for all the other players the best reply is to accept whatever he suggested.

Lemma 9. *For any $s^* \in SPE(G)$ and any $t \leq t^* \in T$ we have (i) $s_i(t, c, a_{-i}) = a$ for any $i \geq 2$, and (ii) $s_1(t) = \alpha_1$.*

Proof. We prove both cases by induction on t . We use [Lemma 7](#) and [Lemma 8](#) as the starting point for (i) and (ii), respectively. Suppose that $t < t^*$ is such that both claims hold for any $t' \in \{t+1, \dots, t^*\}$, we check that it also holds for t .

- (i) We proceed by induction on i . For the initial step ($i = n$), suppose that $s^* \in SPE(G)$ and $c \in C$ are such that $s_n^*(t, c, a_{-n}) = r$. Then, $\vec{t}(s^*|t, c, a_{-n}) > t$, since if there is some rejection at stage t , then *a fortiori* an agreement, if reached, must arrive later than t . Define s'_n as follows: $s'_n(t, c, a_{-n}) = a$ and $s'_n(h_n) = s_n^*(h_n)$ if $h_n \neq (t, c, a_{-n})$. By construction of s'_n , $U_n(s_{-n}^*, s'_n|t, c, a_{-n})$. In addition, given the inductive hypothesis (which states that at $t+1$, voter 1 offers c , and that every voter $i \geq 2$ agrees) and by $s^* \in SPE(G)$ and $\vec{t}(s^*|t, c, a_{-n}) > t$, $U_n(s^*|t, \alpha, a_{-n}) =$

$u_n(t+1, c)$. Consequently, by IMP, $U_n(s_{-n}^*, s'_n|t, c, a_{-n}) > U_n(s^*|t, c, a_{-n})$. Therefore, $s_n^* \notin \operatorname{argmax}_{s_n \in S_n} U_n(s_{-n}^*, s_n|t, c, a_{-n})$, which is a contradiction to $s^* \in SPE(G)$.

For the inductive step, suppose that $i \in \{2, \dots, n-1\}$ is such that Lemma 9 holds for any $j > i$. We check that it holds i . Suppose that $s^* \in SPE(G)$ and $c \in C$ are such that $s_i^*(t, c, a_{-i}) = r$. Obviously, this implies that $\vec{t}(s^*|t, c, a_{-i}) > t$. Define s'_n as follows: $s'_n(t, c, a_{-i}) = a$ and $s'_n(h_n) = s_n^*(h_n)$ if $h_n \neq (t, c, a_{-i})$. By construction of s'_i , $U_i(s_{-i}^*, s'_i|t, c, a_{-i})$. In addition, given the inductive hypothesis and by $s^* \in SPE(G)$ and $\vec{t}(s^*|t, c, a_{-i}) > t$, $U_i(s^*|t, c, a_{-i}) = u_i(t+1, c)$. Consequently, by IMP, $U_i(s_{-i}^*, s'_i|t, c, a_{-i}) > U_i(s^*|t, c, a_{-i})$. Therefore, $s_i^* \notin \operatorname{argmax}_{s_i \in S_i} U_n(s_{-i}^*, s_i|t, c, a_{-i})$, which is a contradiction, since $s^* \in SPE(G)$.

(ii) We proceed by contradiction. Suppose that $s^* \in SPE(G)$ is such that $s_1(t) \neq \alpha_1$. Define s'_1 as follows: $s'_1(t) = \alpha_1$ and $s'_1(t') = s_1^*(t')$ for any $t' \neq t$. By $s^* \in SPE(G)$ and (i), $U_1(s_{-1}^*, s'_1|t) = u_1(t, \alpha_1)$. If $\vec{t}(s^*|t) = t_i$, then, $\vec{t}(s^*|t) \geq t$. Hence, by PER, $U_1(s_{-1}^*, s'_1|t) > U_1(s_{-1}^*, s_1^*|t)$. In addition, if $\vec{t}(s^*|t) > t$, by REV, $U_1(s_{-1}^*, s'_1|t) > U_1(s_{-1}^*, s_1^*|t)$. Therefore, $s_1^* \notin \operatorname{argmax}_{s_1 \in S_1} U_1(s_{-1}^*, s_1|t)$, which is a contradiction to $s^* \in SPE(G)$. □

Based on Lemma 7 and Lemma 9, we obtain the main result, which tells that in the unique subgame perfect equilibrium the voter who votes first, obtains his most preferred alternative immediately at the first stage.

Proposition 6. *Every subgame perfect equilibrium strategy profile agrees with the first voter's preferred alternative at the first stage; i.e., $(t(s), o(s)) = (1, \alpha_1)$ for any $s \in SPE(G)$.*

Proof. The proposition follows from Lemma 9 by applying (i) and (ii) when $t = 1$. □

2.3 A model of iterative voting with a deadline²

Consider, for instance, a jury trial. There are a few alternatives (e.g., “guilty”, “not guilty”, “not proven”). A civil trial jury is asked to reach a decision by majority agreement, while a criminal case decision requires unanimity, i.e., all the jurors must agree on the verdict. If no decision is reached before a certain deadline, a mistrial is declared. A mistrial is possibly the worst alternative for every juror, since it means that the case is not decided. During the actual trial the jurors do not communicate with one another, and during deliberations they discuss the case only inside the jury room. The intent may be that (at first) they should independently decide which verdict they consider to be the most fair, and potentially have an order of preferences over possible verdicts. Jury deliberation consists of a discussion about the trial case, and searching for agreement about a verdict. In the discussion jurors reveal some information about which verdict they consider to be fair. Using a simple voting process (i.e., “raise your hands”), jurors vote to find the most preferred outcome. The process is iterative, whereby at every stage some discussion is held, followed by another voting process. It ends either when the jury reaches a consensus, or when the deadline is reached, whichever is the sooner of the two.

A jury trial is not the only scenario where consensus under a deadline may be required. Other examples include a group of friends discussing their next vacation destination; members of a university department choosing a day for a weekly seminar; a scientific committee deciding where to hold next year’s conference; and even a casual problem such as a family dinner can cause some discussions.

All of the situations above have several common features. First, there is a strict deadline for reaching an agreement: a judge’s requirement, national holidays, the start of an academic year, the budget approval deadline, or even dinner time. Second, assuming that individuals are, at least, somewhat different in their preferences, it is unlikely that a unanimous consensus will be reached immediately by a simultaneous vote. A consensus is reached, if at all, only

²This section is written in collaboration with Lihi Naamani-Deri from the Industrial Engineering and Management Department at Ariel University, Israel; Svetlana Obraztsova and Jeffrey S. Rosenschein from the Department of Computer Science at Hebrew University of Jerusalem, Israel; and Zinovi Rabinovich from Mobileeye Vision Technologies Ltd., Israel.

after several rounds of a sequential voting process. Commonly used methods include emails, phone calls, messages, or Online platforms such as Doodle.

Inspired by these scenarios, we define a strict, formal, time-bounded iterative voting process. The process begins when each voter reveals his most preferred alternative. The preferences are aggregated using majority voting with a threshold (unanimity being the most extreme threshold). If a consensus is not instantly reached, a voting process begins. Voters may chose to change their vote. For instance, a voter that realises that his most preferred alternative has no chance of being elected might decide to change his mind and vote for another alternative. At each stage, all voters that wish to vote apply for a voting slot. One voter is chosen randomly; the chosen voter states his currently most-preferred alternative. Each stage is defined as one second. The process ends when a consensus is reached, or when the deadline is reached, whichever is the sooner of the two. The voters' ultimate goal is to reach a consensual decision, and do so before the deadline.

Assuming rational voter behaviour, we define two types of voters: a *proactive voter*, who likes to participate actively in the procedure and will not miss any chance to state his preferences, and a *lazy voter*, who prefers not to act until it is really necessary. We assume a sequential, iterative voting process, and that each voter has a strict order of preferences over the alternatives, that he keeps strictly to himself. Bargaining is not permitted. The voter is allowed to apply for a “voting slot”, i.e., an opportunity to vote. When the voter is given permission to vote, he reveals the alternative for which he wants to vote. The voter is allowed to strategically change his vote as often as he likes.

We provide what is, to the best of our knowledge, the first model for iterative voting processes with a deadline ([Subsection 2.3.2](#)). The model can easily be generalised to other voting rules, but for the ease of exposition, we initiate this line of research with a specific model, namely, *Consensus Under a Deadline (CUD)*, based on plurality with a threshold (also known as majority). Then, in [Subsection 2.3.3](#), we prove the theoretical properties of CUDs, such as stopping, guarantees of runs with no mistrial runs, and the (additive) Price of Anarchy bounds. Since the progress of CUDs has a rigorous algorithmic description, it is possible to effectively simulate such games. We resort to such a simulation to investigate statistical properties and tradeoffs of CUDs. In more detail, [Subsection 2.3.4](#) concentrates on

Unanimous CUDs, and provides an encompassing experimental analysis of CUD trade-offs. In particular, we measure the effects of voter behavioural types (lazy vs. proactive) on the number of voting steps and the Price of Anarchy. We find an indication of a trade-off between the fairness of the voting outcome and the effort required from the voters during the process.

2.3.1 Related literature

Deadline scenarios are abundant in bargaining models; “[f]inal deadlines are fixed time limits that end a negotiation” (Ma and Manove 1993, Moore 2004). One of the first experimental studies on deadlines in bargaining demonstrated the deadline effect: a majority of agreements are obtained in the final seconds before the deadline; even complete information does not speed agreement much, contrary to what might be expected (Roth et al. 2016).

The concession rate increases as the deadline approaches (Cramton and Tracy 1992, Lim and Murnighan 1994). However, there is a probability that no agreement will be reached before the deadline (Ma and Manove 1993); and more than half of negotiations may end without agreement (Cramton and Tracy 1992). Perhaps surprisingly, moderate deadlines were found to have a positive effect on the outcome of negotiation, though the participants expected that the deadline would hurt their negotiation (Moore 2004).

While in a bargaining procedure the outcome may be a compromise among the most preferred outcomes by the rivals, in CUDs a strict alternative must be chosen. For example, in a jury trial there is no compromise between “guilty” and “not-guilty”. We focus on iterative voting processes under a deadline and do not allow negotiation or bargaining. Informally, a deadline is a fixed time limit that ends the voting process. In general, CUDs closely follow the way other iterative voting games progress, e.g., Meir et al. (2010), Reijngoud and Endriss (2012), Naamani-Dery et al. (2015), Obraztsova et al. (2015). However, CUDs have several unique features. First, although CUDs do build up from a known voting rule, they work directly with the set of possible winners (i.e., alternatives that might be chosen by the majority), and behave much like non-myopic games based on local-dominance (see, e.g., Meir et al. 2014, Lev et al. 2015, Meir 2015). On the other hand, the distinction between lazy and proactive voter behaviour links CUDs with biased voting (see, e.g., Elkind et al. 2015). The concept of a *default alternative* (e.g., a mistrial) is also similar to the way lazy-biased voters

act: an alternative outside the alternative set is introduced, namely an abstention. However, unlike the abstention of lazy-biased voters, a default alternative is enforced on all voters in circumstances that encompass the entire voter set.

Another feature that distinguishes CUDs from other iterative voting processes is a *deadline timeout*. The standard assumption in the iterative voting literature is that voting processes do not stop, unless none of the voters has a way to further manipulate the outcome. Convergence is the subject of an extensive research effort, both to determine when these processes stop and with what ballot profile (Meir et al. 2010, Reijngoud and Endriss 2012, Reyhani and Wilson 2012, Obraztsova et al. 2015). CUDs, on the other hand, always stop, as we will show in Proposition 7.

To the best of our knowledge, an iterative voting procedure, where voters must vote for certain well-defined alternatives, and where a deadline exists as a restriction on the number of voting stages, has not been examined. This model is a first attempt at analysing theoretical features of CUDs in iterative voting and demonstrating them experimentally, with a focus on the convergence rate and the quality of the final decision.

2.3.2 Model

In this section we formally model an attempt to reach a decision under a deadline based on a bounded iterative voting process. If the iterative process converges to an alternative, that alternative is called the *winner*. We concentrate on an extension of the majority rule, i.e., on the plurality voting rule with a threshold. However, the model can be further extended to more general voting rules. The threshold can be tight, requiring the decision to be unanimous, or relaxed, requiring a certain threshold of the voters to agree on the decision. The formal details of our model are as follows.

Let $V = \{1, 2, \dots, n\}$ be a set of n voters, $C = \{c_1, c_2, \dots, c_m\}$ a set of m alternatives, and τ the number of discrete time slices before a deadline. We assume that the process starts at time $t = \tau$ and finishes at deadline $t = 0$, that is, the sequence of the stages is as follows: $(\tau, \tau - 1, \dots, t + 1, t, t - 1, \dots, 1, 0)$.

Each voter is characterised by a *truthful preference* $a_i \in \mathcal{L}(C)$, where $\mathcal{L}(C)$ is the space of complete and non-reflexive orderings over the set of alternatives C . We write $a_i(c, c')$ if

voter i prefers c to c' .

At the beginning of the process, every voter $i \in V$ casts a ballot $b_i^t \in C$ that reveals his choice at time $t = \tau$. The ballots of all voters are collected, forming a *ballot profile*, $\mathbf{b}^t = (b_1^t, \dots, b_n^t)$. The *score* of each alternative within a given ballot profile is the sum of the voters that voted for this alternative: $\mathbf{sc}_c(\mathbf{b}^t) = |\{i | b_i^t = c\}|$. A *score vector* is a collection of scores of all alternatives $\mathbf{sc}(\mathbf{b}^t) = (\mathbf{sc}_1(\mathbf{b}^t), \dots, \mathbf{sc}_m(\mathbf{b}^t))$. For convenience, a shorthand $\mathbf{s}^t = (s_1^t, \dots, s_m^t) = \mathbf{sc}(\mathbf{b}^t)$ is used, and we omit the time superscript to denote an arbitrary score vector.

The collection of score vectors is public information, and is visible to all voters. Using the score vector, the possible winners at time t can be computed using a *Possible Winner Function (PWF)*. Our notion of PWF extends the classical concept of possible winners (that refers to expansions of partial preference ballots, see, e.g., [Xia and Conitzer \(2011\)](#)) to iterative ballot modifications. Specifically, PWFs capture the possibility that some sequence of ballot changes will make an alternative the final winner of the voting process. More formally, a Possible Winner Function, denoted \mathcal{F} , maps a score vector at time t to a set of *possible winners*, and has the form $\mathcal{F} : \Delta \rightarrow 2^C$, where Δ is the space of all possible score vectors.

We investigate two PWFs: *Iterative Majority (IMaj)*, and its special sub-case *Iterative Unanimity (IUn)*. The possible winner function of IMaj is defined by:

$$\mathcal{F}_\sigma^{IMaj}(\mathbf{s}, t) = \{c \in C | \sigma - s_c < t + 1\}.$$

That is, the difference between the alternative's score in the next step, and the score that the alternative needs in order to win, is bounded by time. Once the deadline is reached, $\mathcal{F}(\mathbf{s}, 0)$ is either (i) a singleton, containing the alternative with the highest score (the winner), or (ii) empty, signifying that the default alternative must be adopted.³

The majority threshold is $\sigma > \frac{n}{2}$ and the unanimous threshold is $\sigma = n$. Hence, the possible winner function for IUn is generated by setting $\sigma = n$, i.e., $\mathcal{F}^{IUn} = \mathcal{F}_n^{IMaj}$.

Once the set of possible winners is computed, at any given time $t \in [0, \tau]$, the voter decides whether to change his vote and produce a ballot $b_i^t \in C$, i.e., state his vote at time t . Note that the voter's decision is based on the best possible winner for that voter, and on

³With selfish utility optimisation (see below), this will entail time dependent decision-making.

the voter's utility function.

The voter's best possible winner: For any $W \subset C$ the best alternative in W w.r.t. the voter's truthful preferences a_i is denoted $\mathbf{top}_i(W) \in W$. That is, w is the best possible winner for voter i if voter i prefers it over any other alternative in the possible winner set: $w = \mathbf{top}_i(W)$ iff for all $c \in W \setminus \{w\}$ $a_i(w, c)$ holds.

The voter's utility function: Each voter has a utility function that matches a score vector at time t to the voter's utility: $u_i : \Delta \times [0, \tau] \rightarrow \mathbb{R}$. We assume that the utility function is consistent with the voter's truthful preferences a_i for any $t \in \tau$.

We will particularly emphasise the classes of *lazy consistent* and *proactive consistent* utility functions. Intuitively, a lazy consistent utility function drives the voter to change his ballot only if a better alternative can be injected into the set of possible winners, while a proactive consistent function would also induce a ballot change if the score of the best possible winner can be improved. Formally, these utility classes are defined as follows.

Definition 3. A utility function u_i is *lazy consistent* if for all $\mathbf{s}, \mathbf{s}' \in \Delta$ and $t \in [0, \tau]$ the following condition holds:

$$u_i(\mathbf{s}, t) > u_i(\mathbf{s}', t) \iff a_i(w, w'),$$

where $w = \mathbf{top}_i(\mathcal{F}(\mathbf{s}, t))$ and $w' = \mathbf{top}_i(\mathcal{F}(\mathbf{s}', t))$.

Definition 4. A utility function u_i is *proactive consistent* if for all $\mathbf{s}, \mathbf{s}' \in \Delta$ and $t \in [0, \tau]$ the following condition holds:

$$u_i(\mathbf{s}, t) > u_i(\mathbf{s}', t) \iff a_i(w, w') \vee ((w = w') \wedge (s_w > s'_w))$$

where $w = \mathbf{top}_i(\mathcal{F}(\mathbf{s}, t))$ and $w' = \mathbf{top}_i(\mathcal{F}(\mathbf{s}', t))$.

We assume that all voters are selfish and myopic, and seek to maximise their utility function $u_i(\mathbf{s}, t)$. Note that, although here we only use \mathcal{F}^{IMaj} and \mathcal{F}^{IU_n} , CUDs can be naturally extended to more general forms of PWFs.

For a better understanding of the voting mechanism, consider a simple example of three voters.

Example 5. There are three voters, $V = \{1, 2, 3\}$. There are three alternatives to be voted for $C = \{a, b, c\}$. Assume the strictest voting rule, unanimity, that is, $\sigma = n$. They need to reach a consensus in three time periods, $\tau = 2$, starting from time $t = \tau$ and finishing at deadline, when $t = 0$. Assume that 3 voters have the following preferences:

voter 1	voter 2	voter 3
a	b	c
b	c	a
c	a	b

Stage $t = \tau = 2$. Voters reveal their truthfully preferred alternative, $b^\tau = (b_1^\tau, b_2^\tau, b_3^\tau) = (a, b, c)$. The scores are $s^2 = (s_a^2, s_b^2, s_c^2) = (1, 1, 1)$.

In order to be a winner an alternative has to have σ votes. At the moment each alternative has only one vote. There are two stages ahead, hence any alternative can gather two alternatives more. That is why, all the three alternatives $\{a, b, c\}$ are possible winners at the current stage.

Stage $t = \tau - 1 = 1$. Each voter needs to decide whether he wishes to change his vote.

Here, a concern may appear that the voters consider their options before they are picked to vote. It can be seen as answering a question of “what I will vote if I am picked to vote?” The issue is that, based on this question, he decides whether he wishes to have, or not, a possibility to be picked up and asked to vote. In the current example all the voters prefer anyone but him to switch to his preferred alternative. Hence, nobody would raise a hand and wish to change. The other way to think is to pick up a voter and ask him whether he wants to change his vote. If he does not, then pick another voter and ask him the same question. Until some voter changes his vote or until every voter is asked, the time is stationary. Hence, in the considered example, even if the first two picked voters refuse to change their vote, the last picked voter will be forced to change his vote, otherwise the worst outcome occurs.

For instance, consider voter 1. He might raise his hand if he wishes to change his vote, or he might decline to change his vote. Hence, there are 3 possible votes: either he stays with a ($b_1^1 = a$), or he changes to b or c ($b_1^1 = b$ or $b_1^1 = c$). If he declares that he wishes to change, he

might be randomly picked to do so, hence, he will be the only voter who changes his vote at the current stage, therefore, he can predict the consequences of each of these possible votes, as [Table 2.1](#) illustrates.

Table 2.1. [Example 5](#). Possible decisions of voter 1 at stage $t = 1$.

Voters	Possible vote, b_i^1	Scores, (s_a^1, s_b^1, s_c^1)	Possible winners	
voter 1	a	$(1,1,1)$	$\{\}$	\rightarrow prefers to switch to b
	b	$(0,2,1)$	$\{b\}$	
	c	$(0,1,2)$	$\{c\}$	

If he decides to keep his vote for a (row 2 of [Table 2.1](#)), then the scores of the alternatives will be $sc(b^1) = (s_a^1, s_b^1, s_c^1) = (1, 1, 1)$ and there will be no possible winner, because $n - s_x^1 > 1 + 1$ for any alternative $x \in C$.

Assume that voter 1 decides to change his vote, and instead of voting a he votes b (row 3 of [Table 2.1](#)). Then, the scores are $sc(b^1) = (s_a^1, s_b^1, s_c^1) = (0, 2, 1)$. Since $n - s_b < \tau - 1 + 1$, alternative b is a possible winner.

If voter 1 decides to switch to c (row 4 of [Table 2.1](#)), then the scores would be $(s_a^{\tau-1}, s_b^{\tau-1}, s_c^{\tau-1}) = (0, 1, 2)$, and c would be a possible winner. Both for lazy consistent and proactive voters, since the utility function is consistent with truthful preference, voter 1 prefers to switch to b .

The same argumentation can be applied to voters 2 and 3. As the result, each voter raises his hand to revote. Possible changes are gathered in [Table 2.2](#).

Table 2.2. [Example 5](#). Possible decisions of voters 2 and 3 at stage $t = 1$.

Voters	Possible vote, b_i^1	Scores, (s_a^1, s_b^1, s_c^1)	Possible winners	
voter 2	a	$(2,0,1)$	$\{a\}$	\rightarrow prefers to switch to c
	b	$(1,1,1)$	$\{\}$	
	c	$(1,0,2)$	$\{c\}$	
voter 3	a	$(2,1,0)$	$\{a\}$	\rightarrow prefers to switch to a
	b	$(1,2,0)$	$\{b\}$	
	c	$(1,1,1)$	$\{\}$	

Therefore, all the three voters “raise their hands” and wish to change their votes. Assume that voter 1 is randomly picked to change his vote. Accordingly, $b^1 = (b_1^1, b_2^1, b) = (b, b, c)$ and $s^1 = (s_a^1, s_b^1, s_c^1) = (0, 2, 1)$, which makes b the only possible winner.

Stage $t = 0$. Voters 1 and 2 have no reason to change their vote since, as it is shown in Table 2.3, whatever they vote, they cannot be better off. Voter 3 can be better off by switching to b , as it is illustrated in Table 2.3 since, if he votes for b , the ballot would be $b^0 = (b_1^0, b_2^0, b_3^0) = (b, b, b)$ and the scores would be $s^0 = (s_a^0, s_b^0, s_c^0) = (0, 3, 0)$, which makes b the only possible winner. Consequently, voter 3 decides to change his vote from c to b . Since he is the only one who wishes to change (that is, he is the only possible voter to be randomly picked), he changes to b .

Table 2.3. Example 5. Possible decisions of voters 1 and 2 at stage $t = 0$.

Voters	Possible vote, b_i^0	Scores, (s_a^0, s_b^0, s_c^0)	Possible winners	
voter 1	a	(1,1,1)	{}	
	b	(0,2,1)	{}	
	c	(0,1,2)	{}	
voter 2	a	(1,1,1)	{}	
	b	(0,2,1)	{}	
	c	(0,1,2)	{}	
voter 3	a	(1,2,0)	{}	
	b	(0,3,0)	{ b }	→ prefers to switch to b
	c	(0,2,1)	{}	

After voter 3 changes his vote, the voting ballot is $b^1 = (b, b, b)$ and $s^1 = (0, 3, 0)$. Since $s_b = \sigma = n$, alternative b is chosen as a winner.

We assume that each utility function is homogeneously either lazy consistent or proactive consistent. A *lazy voter* prefers not to change his vote if it is not necessary. Hence, if his top choice is not in a set of possible winners and at the same time his vote does not change the set of possible winners, then he does not change his vote. A *proactive voter* prefers to act, because if he can, he tries to speed up the process and add points to the score of the

alternative he prefers the most out of the set of possible winners. To illustrate the difference between *lazy* and *proactive voters*, consider Example 6.

Example 6. There are six voters, $V = \{1, 2, 3, 4, 5, 6\}$. Let voters 1-5 be *lazy* and voter 6 be *proactive*. There are four alternatives to be voted for $C = \{a, b, c, d\}$. Assume unanimity, $\sigma = n = 6$. Voters need to reach a consensus in six time periods, $\tau = 5$. Assume that voters have the following preferences:

voter 1	voter 2	voter 3	voter 4	voter 5	voter 6
a	a	b	b	c	d
b	c	c	a	b	a
c	b	a	c	d	b
d	d	d	d	a	c

Stage $t = \tau = 5$. Voters reveal their truthfully preferred alternative, hence the ballot is $b^5 = (b_1^5, b_2^5, b_3^5, b_4^5, b_5^5, b_6^5) = (a, a, b, b, c, d)$ and, consequently, the scores are $s^5 = (s_a^5, s_b^5, s_c^5, s_d^5) = (2, 2, 1, 0)$. Possible winners are $\{a, b, c, d\}$.

Stage $t = \tau - 1 = 4$. Each voter needs to decide whether he wishes to change his vote. Voters 1-4 cannot be better off if they change their vote, as is illustrated in Table 2.4.

If any of voters 1-4 changes his vote from his top-choice to some other alternative, that would exclude his top-choice from the set of possible winners. Therefore, they do not “raise their hands” in order to change their votes.

Consider voter 5. Whatever he votes, the set of possible winners consists only of a and b as is illustrated in Table 2.4. Hence, by Definition 3 voter 5, being a *lazy voter*, prefers not to change his vote.

Consider voter 6. As is illustrated in Table 2.4, only if he votes for c will the set of possible winners be $\{a, b, c\}$, and in all other cases it will include only a and b .

Since he is a *proactive voter*, then, by Definition 4, given that he prefers a to b , he “raises his hand” because he can be better off by voting for a . Since he is the only voter who wants to change his vote at the current stage, therefore, he is picked to change it.

Table 2.4. Example 6. Possible decisions of voters 1-4 at stage $t = 4$.

Voters	Possible vote, b_i^4	Scores, s^4	Possible winners	
voter 1 (<i>lazy</i>)	a	(2,2,1,1)	$\{a, b\}$	→ prefers not to change
	b	(1,3,1,1)	$\{b\}$	
	c	(1,2,2,1)	$\{b, c\}$	
	d	(1,2,1,2)	$\{b, d\}$	
voter 2 (<i>lazy</i>)	a	(2,2,1,1)	$\{a, b\}$	→ prefers not to change
	b	(1,3,1,1)	$\{b\}$	
	c	(1,2,2,1)	$\{b, c\}$	
	d	(1,2,1,2)	$\{b, d\}$	
voter 3 (<i>lazy</i>)	a	(3,1,1,1)	$\{a\}$	→ prefers not to change
	b	(2,2,1,1)	$\{a, b\}$	
	c	(2,1,2,1)	$\{a, c\}$	
	d	(2,1,1,2)	$\{a, d\}$	
voter 4 (<i>lazy</i>)	a	(3,1,1,1)	$\{a\}$	→ prefers not to change
	b	(2,2,1,1)	$\{a, b\}$	
	c	(2,1,2,1)	$\{a, c\}$	
	d	(2,1,1,2)	$\{a, d\}$	
voter 5 (<i>lazy</i>)	a	(3,2,0,1)	$\{a, b\}$	← prefers not to change
	b	(2,3,0,1)	$\{a, b\}$	
	c	(2,2,1,1)	$\{a, b\}$	
	d	(2,2,0,1)	$\{a, b\}$	
voter 6 (<i>proactive</i>)	a	(3,2,1,0)	$\{a, b\}$	← prefers to switch to a
	b	(2,3,1,0)	$\{a, b\}$	
	c	(2,2,2,0)	$\{a, b, c\}$	
	d	(2,2,1,1)	$\{a, b\}$	

Consequently, at stage $t = 4$ the ballot is $b^3 = (b_1^3, b_2^3, b_3^3, b_4^3, b_5^3, b_6^3) = (a, a, b, b, c, a)$, and alternatives have scores $s^3 = (s_1^3, s_b^3, s_c^3, s_d^3) = (3, 2, 1, 0)$, which make alternatives a and b the only possible winners.

It is not clear whether CUD converges more often or faster for *lazy* or *proactive voters*. Intuition suggests that *proactive* voters start to switch their votes earlier, which will lead CUD to converge faster. At the same time, *lazy* voters seems to wait until the last moment, when they cannot avoid changing their vote. Hence, it might be the case that for *lazy* voters CUD does not converge as fast as for *proactive* voters. We further test in our experiments this conjecture .

The algorithm for a CUD iterative voting game is given in [Algorithm 1](#). Since score vectors simply accumulate ballots from \mathbf{b}^t , we use algebraic operations between a score vector and a ballot. That is, if $\mathbf{s}' = \mathbf{s} - c$ (respectively, $\mathbf{s} = \mathbf{s} + c$) then $s'_k = s_k$ for all $k \in C \setminus \{c\}$ and $s'_c = s_c - 1$ (respectively, $s'_c = s_c + 1$).

The algorithm receives as input a Possible Winner Function \mathcal{F} and the initial time until the deadline τ . Iteratively, as long as the deadline has not been reached, all voters calculate the current score vector (line 3). If there is only one possible winner, w , a decision has been reached and the game ends (lines 4-5). The game also ends if there are no possible winners (lines 7-8). If the game continues, every voter calculates what is his best possible winner, given the current score vector (line 11).

If there are ties (i.e., if a few alternatives receive the same score), the voter selects the alternative that is ranked highest in his truthful preferences a_i . Each voter decides if he wants to vote, the decision being based on his current possible winners and on his utility function. Voters who want to vote “raise their hands”, i.e., are collected into a set I (line 13). A random voter is chosen from set I (line 14). The chosen voter casts his ballot (lines 15-18) and we are now one step closer to the deadline (line 19).

In order to analyse the quality of the result of a CUD game, features of voting processes can be adapted. One such feature is the *Additive Price of Anarchy (PoA)* [Branzei et al. \(2013\)](#). We adapt the additive PoA to CUDs as follows.

Definition 5. Let \mathbf{a} be the truthful profile of voters participating in a CUD, \mathbf{b} a ballot profile consistent with \mathbf{a} (i.e., $b_i = \mathbf{top}_i(C)$), and $\mathbf{s} = \mathbf{sc}(\mathbf{b})$. Denote all alternatives that the CUD may converge to by \widehat{C} . Then the CUDs *additive Price of Anarchy* is

$$PoA^+(\mathbf{a}) = \max_{c \in C} s_c - \min_{c \in \widehat{C}} s_c$$

Algorithm 1 Consensus Under Deadline: Game Progress

Input: PWF \mathcal{F} , deadline timeout τ

Input: Set of voters V , set of alternatives C ,

Input: Truthful profile \mathbf{a} , and utilities u_i .

Initialise: Set $t \leftarrow \tau$, and $b_i^t \leftarrow \mathbf{top}_i(C)$ for all $i \in V$

- 1: **while** $t \geq 0$ **do**
- 2: Ballots b_i^t are declared
- 3: All voters calculate $\mathbf{s}^t = \mathbf{sc}(\mathbf{b}^t)$
- 4: **if** $\mathcal{F}(\mathbf{s}^t, t) = \{w\}$ for some $w \in C$ **then**
- 5: **return** w as the winner ▷ Game stops
- 6: **end if**
- 7: **if** $\mathcal{F}(\mathbf{s}^t, t) = \emptyset$ **then**
- 8: **return** *mistrial* ▷ Game stops
- 9: **end if**
- 10: **for** $i \in V$ **do**
- 11: $w_i \leftarrow \arg \max_{c \in C} u_i(\mathbf{s}^t - b_i^t + c, t - 1)$ ▷ ties are broken by a_i
- 12: **end for**
- 13: $I \leftarrow \{i \in V \mid w_i \neq b_i^t\}$
- 14: $j \leftarrow \text{Random}(I)$ ▷ random voter choice
- 15: **for** $i \in V$ **do**
- 16: $b_i^{t-1} \leftarrow b_i^t$
- 17: **end for**
- 18: $b_j^{t-1} = w_j$ ▷ Only j is allowed to revote
- 19: $t \leftarrow t - 1$
- 20: **end while**

Namely, Additive PoA is the score of the least preferred alternative that could become the winner of a CUD, subtracted from the score of the truthful winner.

Example 7. There are six proactive voters, $V = \{1, 2, 3, 4, 5, 6\}$, 3 alternatives, $C = \{a, b, c\}$, and four stages before the deadline, $\tau = 4$. Preferences are as follows:

voter 1	voter 2	voter 3	voter 4	voter 5	voter 6
a	a	a	b	b	c
b	c	b	a	c	b
c	b	c	c	a	a

At stage $t = 4$ voters reveal their truthful preferences, hence the ballot is $b^4 = (b_1^4, b_2^4, b_3^4, b_4^4, b_5^4, b_6^4) = (a, a, a, b, b, c)$. There are 2 possible winners: $\{a, b\}$.

Stage $t = 3$. Voters 1-3 do not want to change their votes since, whatever they vote, they cannot better off, as it is illustrated in [Table 2.5](#).

Table 2.5. [Example 7](#). Possible decisions of voters 1-3 at stage $t = 3$.

Voters	Possible vote, b_i^3	Scores, s^3	Possible winners	
voters 1-3	a	(3,2,1)	$\{a\}$	→ prefers not to change
	b	(2,3,1)	$\{b\}$	
	c	(2,2,2)	$\{\}$	

Consider voters 4 and 5. Since they are both *proactive* voters then, by [Definition 4](#), they would change their votes to a as is illustrated in [Table 2.6](#).

Table 2.6. [Example 7](#). Possible decisions of voters 4-5 at stage $t = 3$.

Voters	Possible vote, b_i^3	Scores, s^3	Possible winners	
voters 4-5	a	(4,1,1)	$\{a\}$	→ prefers to switch to a
	b	(3,2,1)	$\{a\}$	
	c	(3,1,2)	$\{a\}$	

Consider voter 6. Since he prefers b to a , he would switch to b , as is illustrated [Table 2.7](#).

Table 2.7. Example 7. Possible decisions of voters 4-5 at stage $t = 3$.

Voter	Possible vote, b_i^3	Scores, s^3	Possible winners	
voter 6	a	(4,2,0)	$\{a\}$	\rightarrow prefers to switch to b
	b	(3,3,0)	$\{a, b\}$	
	c	(3,2,1)	$\{a\}$	

At this stage, there three voters who “raise their hands” and wish to change their vote: voters 4, 5, and 6. Assume that voter 6 is randomly chosen. He changes his vote and now supports b . Hence, the ballot is $b^3 = (b_1^3, b_2^3, b_3^3, b_4^3, b_5^3, b_6^3) = (a, a, a, b, b, b)$. And, there are 2 possible winners: $\{a, b\}$.

Stage $t = 2$. All the voters wish to change their votes. Voters 1-3 would change their vote to b , and voters 4-6 would change to b . Assume that voter 1 is randomly chosen. He changes his vote in favour of b . Consequently, all the process converges to alternative b as a winner.

Additive Price of Anarchy equals to the score of the plurality winner in the truthful profile minus the score of the final winner in the truthful profile:

$$PoA^+(\mathbf{a}) = \max_{c \in \mathcal{C}} s_c - \min_{c \in \hat{\mathcal{C}}} s_c = s_a^T - s_b^T = 3 - 2 = 1$$

Hence, since PoA^+ is greater than 0, it means that the voting process provoked some anarchy, so that the deserving winner, that is, the winner by plurality at the beginning (alternative a), did not win due to the applied protocol CUD. The price for this anarchy is 1 vote, that is, the difference between the deserving winner and the declared winner is 1 vote.

The applied protocol CUD may cause some anarchy in certain cases, but it can be compensated for by the fact that it always converges to a certain chosen alternative if there is at least a chance to converge, as we will show in next section, Besides, further into the experimental part, [Subsection 2.3.4](#), we show that Price of Anarchy in CUDs is relatively small, a fact that might outweigh the disadvantage of having an anarchy.

2.3.3 Theoretical features

From the first look at [Algorithm 1](#), it may appear that it is incomplete, since there is no default decision after the main loop ends. The following proposition shows that this eventuality never occurs. That is, the loop never completes, but is always broken by one of the game's stopping conditions (Line 5 and Line 8). Note that [Proposition 7](#) applies for both types of voter, lazy and proactive. In fact, all of our theoretical results are applicable to both of these voter types.

Proposition 7. *For $\mathcal{F}_\sigma^{IMaj}$, for any $\sigma \in (\frac{n}{2}, n]$, and for all consistent utility functions, CUD either stops at $t = \tau$ with a mistrial, or at time $t \in [0, \tau]$ with a valid alternative that is declared as the winner.*

Proof. First, we prove by contradiction that if there are possible winners at the beginning of the process, then it converges with some valid alternative as a winner. It implies that at stage τ the set of possible winners is not empty, nor is it empty at the final stage 0, since there is a winner. Suppose that it does not hold: the process does not converge, even if there were possible winners at the beginning. In other words, at the final step $t = 0$ the set of possible winners is empty, although at the initial step $t = \tau$ it is not empty. This can happen if at some time between these two steps the set of possible winners becomes empty (including step 0). Consider time τ' , such that for all $\tau'' > \tau'$ the set of possible winners is not empty and for τ' it is empty.

Consider step $\tau' + 1$. There are two possible scenarios: (i) no voter changes the vote, and (ii) some voter changes the vote.

Suppose that at step $\tau' + 1$ there are no voters who wish to change their votes. Since at $\tau' + 1$ the set of possible winners is not empty, for all the voters the utility is not zero under the current strategy. Hence, the set of possible winners could not become empty at step τ' .

Suppose that at step $\tau' + 1$ there are some voters who change their vote. The only reason why a voter would wish to change his vote is, that he can be better off by changing the vote from an alternative which is no longer in a set of possible winners to some other alternative which remains in a set of possible winner. It implies that the utility for them is positive, therefore, the set of possible winners is not empty. The obtained contradiction proves that if

the set of possible winners is not empty at τ , it cannot become empty at 0, therefore, there is a valid alternative that is declared as a winner.

As second step, we prove that if at τ the set of possible winners is empty, it cannot be not empty at 0. The proof is obtained straightforwardly from the definition of a possible winner. Consider some alternative $c \in C$, which at time τ has the maximum score s_c out of other alternatives. Since the set of possible winners is empty at step τ , it implies that $\sigma - s_c \geq \tau + 1$. Even if at each step until the deadline there will be a voter that changes his vote for this alternative c , at time 0 which can get $\tau - 1$ more votes than at the beginning. The alternative c will be in the set of possible winners at 0 if the following condition holds: $\sigma - s_c - \tau + 1 < 0 + 1$, which contradicts the condition that c is not a possible winner at τ .

□

In light of [Proposition 7](#), we will slightly overload the concept of *convergence*. Convergence, in general iterative voting schemes, means that the game stops at some stable point. For a CUD, on the other hand, we will say that it converges if the game stops by declaring some $w \in C$ the winner. If the game stopped with a mistrial, we will say that the CUD did not converge.⁴

Now, having established that the algorithm always stops, we can place a condition on the game features that ensures that it stops with a valid alternative, rather than a mistrial.

Proposition 8. *Let $\mathbf{a} = (a_1, \dots, a_n)$ be the truthful profile of preferences, let τ be the deadline time, and let \mathbf{b} be the ballot profile induced by \mathbf{a} , i.e., $b_i = \mathbf{top}_i(C)$. CUD stops with some $w \in C$, if and only if there is an alternative $c \in C$ so that $\mathbf{sc}_c(\mathbf{b}) \geq \sigma - \tau$.*

Proof. The condition $\mathbf{sc}_c(\mathbf{b}) \geq \sigma - \tau$ implies that c is a possible winner by the definition of possible winner. Hence, at τ the set of possible winners at least contains one alternative c . Then, by [Proposition 7](#) the process converges to some alternative that is chosen as a winner.

□

[Proposition 8](#) essentially provides a finer bound on what the initial scores must look like, so that CUD converges. Intuitively, the proposition says that for an alternative to become the

⁴Intuitively, a mistrial represents a deadlock or a livelock, similar to the infinite running time of non-converging iterative voting models in the standard sense.

declared winner, there must be enough time for it to gather additional support to achieve the majority threshold. However, [Proposition 8](#) does not guarantee that a particular alternative will be declared a winner. For such a guarantee, a much more stringent condition must be required of τ and n , as the following proposition states.

Proposition 9. *Let $\mathbf{a} = (a_1, \dots, a_n)$ be the truthful profile, let τ be the deadline time, and let \mathbf{b} be the ballot profile induced by \mathbf{a} , i.e., $b_i = \mathbf{top}_i(C)$. If there is an alternative $c \in C$, so that $\mathbf{sc}_c(\mathbf{b}) \geq \max \left\{ \lfloor \frac{n}{2} \rfloor + 1, \sigma - \tau \right\}$, then CUD stops with c as the winner.*

Proof. Let c be a possible winner at step τ . Suppose that c is not a winner at step 0, i.e. it is not in a set of possible winners at 0. Let τ' be a stage, such that (i) at τ' one voter with c as his top-choice changes his vote and (ii) at all stages before ($\tau'' > \tau'$) no voter, whose top-choice is c , did not change the vote.

Since a voter with c as his top-choice changes his vote, it implies 2 conditions. First: even if he would keep his vote to c , then c would not be a possible winner. Consequently, since $s_c^\tau > \lfloor \frac{n}{2} \rfloor + 1$ at stage τ and did not decrease before stage τ' , then, by Definition of PW, $\tau' < \lfloor \frac{n}{2} \rfloor - 1$.

Second: there exists an alternative c' , such that the considered voter changes his vote in favour of this alternative, which implies that c' is in the set of possible winners.

Note that $s_{c'}^{\tau'} \leq n - s_c^{\tau'} \leq \lfloor \frac{n}{2} \rfloor - 1$. But there is a contradiction, because c' is not a possible winner, since $s_{c'}^{\tau'} < s_c^{\tau'}$. However, c stops to be a possible winner. Even if c' gets one vote from c , it implies that $s_{c'}^{\tau'-1} \leq s_c^{\tau'-1}$. That is, c is the winner at 0.

□

[Example 8](#) demonstrates that the bound of [Proposition 9](#) is tight. That is, if the score was any lower than [Proposition 9](#) suggests, then, for at least some truthful profiles, a CUD would have more than one alternative that can be declared the winner.

Example 8. Let the number of voters n be even, and the number of alternatives $m \geq 2$. Furthermore, assume that $\tau > \frac{n}{2}$. Let \mathbf{a} be such that for $i \in [1, \frac{n}{2}]$, $\mathbf{top}_i(C) = c_1$ holds, and for $i \in [\frac{n}{2} + 1, n]$, $\mathbf{top}_i(C) = c_2$ holds. All other alternatives may appear in a_i in any order. Then, both c_1 and c_2 can possibly be declared as the winner in a CUD.

Having dealt with the characterisation of alternatives that can be declared a winner, and with those that are guaranteed to become a winner, we can exploit this knowledge to place some bounds on the additive Price of Anarchy for CUDs. Firstly, it is obvious that in no case can the additive Price of Anarchy be greater than n , since the maximum and minimum scores that any alternative can have at time τ are n and 0, respectively. [Lemma 10](#) defines the conditions under which the additive Price of Anarchy equals to zero.

Lemma 10. *Let $\mathbf{a} = (a_1, \dots, a_n)$ be the truthful profile, and let τ be the deadline time. If $\tau \leq \sigma - \lfloor \frac{n}{2} \rfloor$, then $PoA^+(\mathbf{a}) = 0$.*

Proof. Assume that $\tau \leq \sigma - \lfloor \frac{n}{2} \rfloor$. Note that, even if at each step τ every voter would change his vote in favour of the same alternative c , this alternative c would get not more than τ points.

$$s_c^\tau + \tau \geq \sigma \Leftrightarrow s_c^\tau \geq \sigma - \tau \geq \lfloor \frac{n}{2} \rfloor$$

Note that, there can be a maximum two alternatives with score $\lfloor \frac{n}{2} \rfloor$. If c is the only one, then he is the winner, hence $PoA^+(\mathbf{a}) = 0$. Suppose there are two such alternatives: w and c . If they have equal scores at τ : $s_w^\tau = s_c^\tau = \lfloor \frac{n}{2} \rfloor$, then, whoever wins, $PoA^+(\mathbf{a}) = 0$. Another possibility is that there are two such alternatives: w and c , such that w has more points, i.e., $s_w^\tau = s_c^\tau + 1$, and all other alternative have 0 points. Alternative c would win only if every supporter of w would change to c , which would take all τ stages. But, from those who initially voted for w , no voter would change his vote to c , since they are better off by keeping their votes for w . Hence, w will win, and consequently, $PoA^+(\mathbf{a}) = 0$.

□

2.3.4 Experimental features

We evaluated the behaviour of lazy and proactive voters on four real world datasets: the Sushi dataset (5,000 voters, 10 alternatives) [Kamishima et al. \(2005\)](#), the T-shirt dataset (30 voters, 11 alternatives), the 2003 course dataset (146 voters, 8 alternatives) and the 2004 course dataset (153 voters, 7 alternatives). The three latter datasets are taken from the Preflib library [Mattei and Walsh \(2013\)](#). For a fixed number of n ($n = 10, 20$ or 30)

voters, we varied the time left until the deadline: $t \in [1, \tau]$. For each experimental setting, we created 20 random sets of voter profiles by sampling with returns from each dataset. For each set of voter profiles, the experiment was conducted 30,000 times.

We examined: (1) The convergence rate, i.e., the ratio of games that converge to the total number of games within a given sub-class (e.g., games with 10 voters, or 7 alternatives); (2) The number of changes in votes that are required for the process to converge; and (3) The Additive Price of Anarchy for a process that has converged ([Definition 5](#)).

Our theoretical results are relevant for all majority thresholds $\sigma \in (\frac{n}{2}, n]$. However, to investigate the finer features of CUDs, we fix this parameter as n , the number of voters. Even though it means that we experimentally study an extreme CUD case, i.e., unanimity, fixing σ allows us to exclude it as a free parameter and to better concentrate on studying complex game features, such as the Additive Price of Anarchy.

In order to conclude which voter type performs best over multiple datasets, we followed a robust non-parametric procedure proposed by [García et al. \(2010\)](#). This procedure allows us to drop the assumption that the differences between the voter types are normally distributed, and is thus more adequate than a t-test. We first used the Friedman Aligned Ranks test in order to reject the null hypothesis that all heuristics perform in the same way. This was followed by the Bonferroni-Dunn test to find whether one of the heuristics performs significantly better than other heuristics.

2.3.5 Results

Convergence: As [Proposition 8](#) indicates, convergence always occurs when there is enough time until the deadline to allow voters to change their vote (i.e., when the number of iterations is larger than the number of voters). The experiments reveal that there is no difference in the convergence rate between lazy and proactive voters. Interestingly, we find that in all experimental settings the process converges even when the time until deadline is somewhat smaller than the number of voters, i.e., the convergence rate (estimated from 30K processes for each experimental setting) reaches 1 with *less* time than is theoretically necessary. [Table 2.8](#) shows the time left until the deadline when the convergence rate equals 1 (i.e., when all experiments converge).

Table 2.8. Convergence time for all processes

Number of voters	2004 course	2003 course	Sushi	T-shirts
10	6	7	8	8
20	13	14	14	14
30	19	23	24	23

Each row represents a different number of voters, and each column represents a different dataset. The number of alternatives in the dataset is indicated in brackets. For example, for the 2004 course dataset (column 2), for 10 voters (row 2), all of the experiments converge when the initial time before the deadline is $\tau \geq 6$. The process seems to begin to converge faster when there are fewer alternatives. The exact impact of the alternative number on convergence should be examined on a wider variety of datasets; we leave this for future research. The more voters there are, the longer it takes the process to converge. This is illustrated in [Figure 2.3](#), on the 2004 course dataset (similar results were obtained for the other datasets).

Required number of vote changes: [Table 2.9](#) shows, for different datasets and a varying number of voters, the normalised average of vote changes required to reach a consensus.

Table 2.9. Number of vote changes

Datasets	10 voters		20 voters		30 voters	
	lazy	proactive	lazy	proactive	lazy	proactive
2003 course	3.98	5.17	6.79	7.86	9.05	10.24
2004 course	3.55	5.73	6.38	9.51	8.89	12.71
Sushi	4	4.73	6.66	7.45	8.41	9.35
T-shirts	3.96	4.84	6.19	6.88	7.87	8.57

According to the Friedman test, there is a significant difference between the number of vote changes performed when either all voters are lazy or all of them are proactive. According to the Bonferroni-Dunn test, proactive voters require a significantly higher number of vote

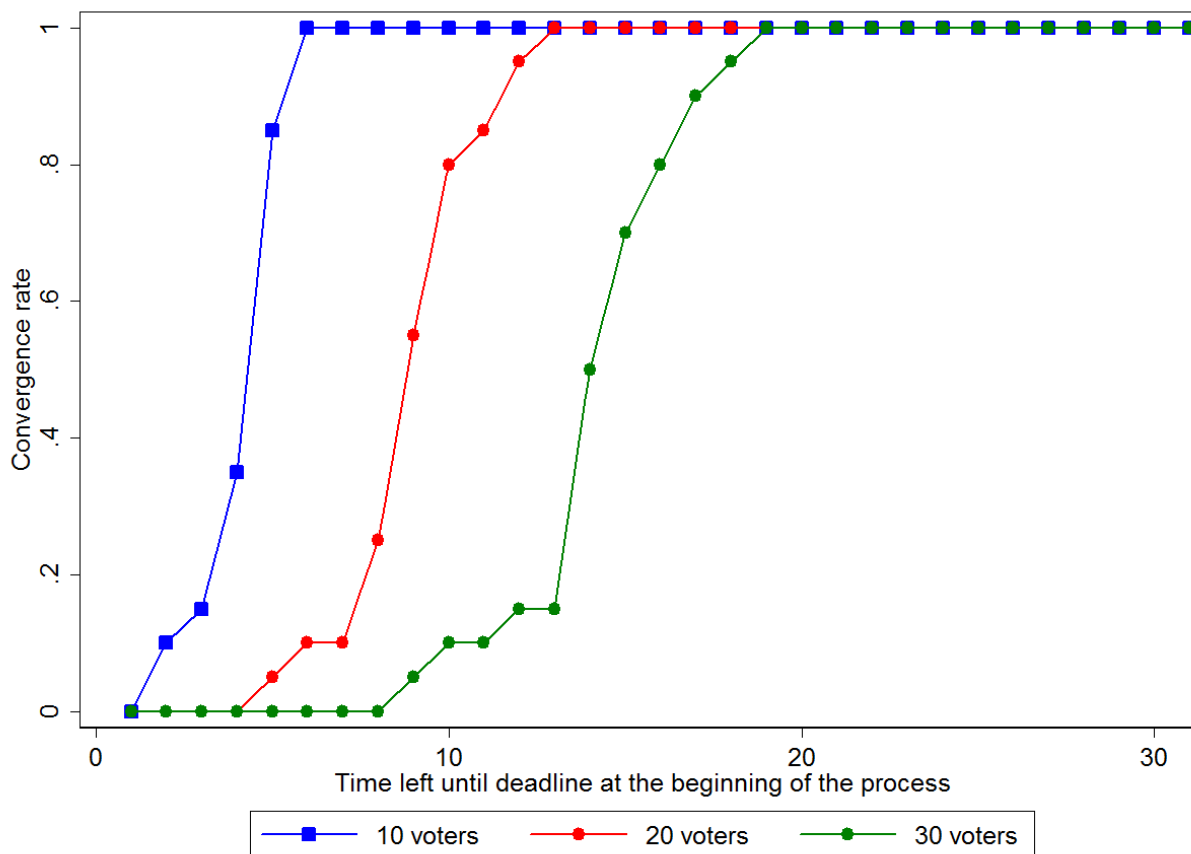


Figure 2.3. Voting process convergence rate.

changes in order to reach a consensus. Note that, regardless of the difference in the number of required vote changes, the converge rate for both voter types is the same. That is, proactive voters do not require more rounds to converge, but simply begin changing their votes sooner, more in advance of the deadline.

Additive price of anarchy: The Additive Price of Anarchy was computed as the plurality score of the least preferred alternative that was elected to be a unanimous winner in one of the 30,000 experiments, subtracted from the plurality score of the truthful winner. For example, consider 12 voters and 4 alternatives and the following scores at the beginning of the process: $c_1 = 2, c_2 = 6, c_3 = 1, c_4 = 3$. The truthful plurality winner is c_2 with a score of 6. If in one of the experiments c_3 is the unanimous winner, the additive price of anarchy is 5. If c_3 does not win in any of the experiments, but in some of them c_1 wins, the price is 4. Table 2.10 shows the normalised average of the additive price of anarchy. Although one can

notice a trend in the data, and see that for 20 and 30 voters the additive price of anarchy is somewhat higher for lazy voters, we could not confirm a significant difference using the Friedman test. Less accurate measures such as a simple t-test reveal that the additive price of anarchy is significantly higher for lazy voters, in the case of 30 voters. Unfortunately, overall, the statistical analysis was inconclusive, and no significance test could decide the issue completely.

Table 2.10. Additive Price of Anarchy

Datasets	10 voters		20 voters		30 voters	
	lazy	proactive	lazy	proactive	lazy	proactive
2003 course	0.21	0.23	0.52	0.38	0.6	0.33
2004 course	0.03	0.03	0.02	0.02	0.18	0.18
Sushi	0.15	0.15	0.29	0.27	0.12	0.12
T-shirts	0.69	0.27	0.38	0.35	0.51	0.3

To conclude, it is noteworthy that although the convergence rate is equal for both voter types, the required number of vote changes is higher for the proactive voters than for the lazy voters, whereas, the additive price of anarchy seems to be lower for the proactive voters than for the lazy voters. This may indicate that there is a tradeoff between the accuracy of the results and the required voter effort, at least when the number of voters is relatively large.

2.4 Concluding remarks

In this chapter we have considered iterative voting procedures when a group of individuals with different preferences is asked to agree on a certain alternative. Since the procedure is iterative, it involves time. The effect of time can be introduced in several ways. The most natural way is to introduce the cost of delay. It affects the voters directly, so that the later the decision is taken, the smaller the payoffs of individuals. The other way is inspired by the fact that, usually, for each decision there is a certain time limit, when the decision must be taken. The time restriction, or deadline, affects the procedure itself, and through it, affects

the individuals. The delay cost can be great or insignificant, strict or not, but both, delay cost and deadline, affect the individuals and their behaviour.

We have presented two models of iterative voting, with a focus on the two possible time effects: the cost of delay and a deadline.

Time is costly

The first model of iterative voting introduced a time effect through the cost of delay of the decision. We assume that the cost of delay may be different for individuals, so that it may affect their patience in different ways.

As the first step, we assumed that at each stage individuals cast their votes in the same fixed order. The result disproves the previous natural intuition that the most patient voter wins the voting procedure. Although at first glance, patience should have an impact on the outcome, it appears that the first voter's preferred alternative is chosen, even if this voter is the least patient. Surprisingly, the support of the plurality (or majority) of the voters does not guarantee that such an alternative wins. Everything depends on the preferences of the first voter.

It is easy to provide real-life examples of the fact that the first voter who takes the initiative wins. Consider a situation with two players: a firm and a labour union in the case of a strike. If the strike happens already, it is to tell that the union sets up some claims on the enterprise. Although it seems that the enterprise is more patient, with more resources at its disposal, in the case of a strike it is likely that the enterprise agrees with the claims of the union. In the case when the strike is about to happen the enterprise has the power of the first move: it can prevent a strike by making an offer to the workers that will not be rejected. This offer is less generous than the offer in the case when the strike has already happened.

If the order of the voters is not exogenous and is defined by the voters themselves, then it is likely that the most extreme voter would take the initiative and votes first. Therefore, in this case the result coincides, to some degree, with the results of [Compte and Jehiel \(2010\)](#) and [Rivas and Rodríguez-Álvarez \(2012\)](#).

In other words, being more patient does not guarantee the victory of one's preferred

alternative. In the present framework what really matters is the order of the voters.

Next, we restrict our attention to the case of two voters and introduce a stage when the voting order reverses. The previously obtained result that the first voter's alternative wins holds only if the reversal time (the stage when voter loses his patience) of the voter who initially votes second is smaller than the stage when the order reverses. Otherwise, the second voter's alternative is chosen at the first stage. In other words, the patience of the second voter is what defines the result.

The model can be extended along at least two directions. First, to modify existing elements of the model, for instance, to introduce explicitly a final stage. Second, to modify the procedure. The most natural extension is not to fix the order of the voters and make it random at each stage (as a continuation and generalisation of the considered case where the voting order is reversed only once). The other extension would be to allow coalitions or to imply other voting rules. The presence of bribing seems to be challenging and promising. The bribing can be presented in different ways: utility transfer between the voters, direct payments, or increasing the probability to vote first. Maybe, in this case the veto power of the individuals is reduced and the degree of impatience matters.

Time is limited

In the second model of iterative voting we introduce a time effect as a presence of a certain deadline – the limit in time when the decision must be taken. When a group of individuals with different preferences is asked to agree on a certain alternative, it would be natural to expect that in some cases the consensus will be not reached, even if the deadline is moderately far or very far. Surprisingly, our model of an iterative voting process with time restriction (CUD) predicts that if there is a possibility to converge, then a consensus will be reached.

If there is, at least, one alternative for which there is enough time to gain the missing votes (in other words, a possible winner), then the process converges with such an alternative chosen. Furthermore, if there is an alternative which, *a priori* is the top choice of the majority of voters, the process converges to that very alternative, although still being subject to a sufficiently long deadline timeout. These results remain valid, even for the strictest

special case of our model, Unanimity, and are confirmed (for the sake of sensibility) by our experiments.

It is obvious that not all individuals behave identically. We define two types of voters: proactive and lazy, according to their behaviour. Proactive voters are, in a sense, eager to change their vote, even if just to ensure that their preferred possible winner gets one more point. Lazy voters change their votes only when it is necessary to do so, i.e., when their vote is pivotal to keeping a particular alternative as a possible winner.

It would be natural to interpret proactive voters as those that actively seek consensus. This, however, is false. Our experiments show that the convergence rates of both proactive and lazy voter CUDs are the same. On the other hand, the number of vote changes until convergence is higher for proactive voters. In a way, they are inefficient in their behaviour. However, there is a benefit to the proactive voters' activism.

Our experiments looked deeper into the *Additive Price of Anarchy* (PoA) as a measure of winner quality. Theoretical results, while showing principal bounds of PoA, do not provide specific trade-offs. On the other hand, while re-confirming our bounds experimentally, our experiments indicate that there might be an interesting trade-off with regard to PoA. Namely, the final winner is closer to the truthful plurality winner (which has lower PoA) for proactive voters, than for lazy voters.

Currently, the model assumes that all voters have the same type; either all are lazy, or all are proactive. However, given the possible tradeoff with PoA found in experiments, it would be interesting to analyse the behaviour of mixed voter populations, e.g., how the ratio of proactive and lazy voters affects the balance between the number of revotes and PoA. This could also refine our dataset sufficiently for statistical tests to unequivocally determine the significance of this tradeoff. Furthermore, the two voter types are an initial foray. It is more feasible now to construct other varieties of voters.

We plan to expand our experimental base, and seek additional tradeoffs between various model parameters. For example, we conjecture that the convergence rate is affected by the number of alternatives, and we plan to investigate whether the convergence rate is subject to a tradeoff between the number of alternatives, the number of voters, and the deadline timeout.

A more challenging extension, however, would be to adapt our model to use general Positional Scoring Rules (PSRs), rather than a simple majority, e.g., veto, approval, and Borda. These rules would allow us to express complex semantic structures over the set of alternatives, still motivated by the jury trial example. For example, in some cases the jurors are asked not only to state “guilty” and “not-guilty”, but also to define the amount of damages or penalties in civil trials, or the recommendations for sentencing in criminal trials. There are many disagreements and discussions about the death penalty. Thus, some jurors may wish to chose guilty, but also wish to veto the death penalty, while some other jurors would approve of both life imprisonment and the death penalty. In these situations veto and approval voting are far more appropriate than majority voting.

Stepping even further away from our basic model, we need to investigate what information about the current vote affects a CUDs outcome, and in what way. For example, rather than using utilities that depend on an anonymous score vector, we can use weights to express the fact that the opinion of certain voters is more influential. The same technique would allow us to impose a price on the number of times a voter changes his mind, e.g., his vote loses influence, as being unstable. Simultaneous revoting will be another research direction, being a very non-trivial modification.

One more possible extension is to eliminate the discussion process before the voting, which implies totally imperfect information at the moment when voting starts. Then, the set of voters who “raised the hand” to vote will also include those who did not declare any vote. To guarantee the convergence, these voters have to have a higher probability to vote than those who voted and changed the vote couple of times. Lazy voters will likely be waiting till the last moment and casting their vote for the most supported alternative. Proactive voters will be likely to vote several times, and changing their vote. How will it affect the convergence rate? By how much will it increase the PoA? These questions are for further research.

One final remark regarding our model has to do with how faithful this representation is for a jury deliberation. The obvious questions concern time discretisation and the randomisation over the set I (all voters who wish to change their vote). First, time discretisation in CUDs can be achieved by setting each time-slide to be equal to the time necessary to calculate the modified ballot w_i (Lines 3-11 of Algorithm 1). Assuming that this time is the same

for all agents, synchronisation at Line 14 creates a race condition that naturally implements randomisation over I . Although these assumptions may appear stringent at first, they are common for many distributed decision systems, e.g., Dec-POMDPs (e.g., see an overview in [Oliehoek \(2012\)](#)). Naturally, in our future research, we will relax these limitations.

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