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Physics”

COSMOLOGICAL APPLICATIONS OF EXTENDED THEORIES OF GRAVITY

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- M. Paoella, S. Capozziello, *Hojman symmetry approach for scalar-tensor cosmology*, Phys. Lett. A **379**, 1304-1308 (2015).
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Foreword

This work investigates the cosmological applications of higher-order theories of gravity in four dimensions. In particular, we begin dealing with the possibility to obtain massive modes in the framework of effective field theories recovered by extending General Relativity and taking into account generic functions of the curvature invariants. In particular, adopting the minimal extension of $f(R)$ gravity, an effective field theory with massive modes is straightforwardly recovered. This approach allows to evade shortcomings like ghosts and discontinuities if a suitable choice of expansion parameters is performed. Next, we stress one of the most important problem related to Extended Theories of Gravity that is the lack of a definitive, unique theory able to address the different shortcomings of General Relativity. In fact, several models have been proposed in order to address the dark side problem in cosmology and these models should be constrained also at ultraviolet scales in order to achieve a correct fundamental interpretation. We proceed analyzing the possibility to constrain $f(R)$ theories at UV scales comparing quantum vacuum states in given cosmological backgrounds. Specifically, we compare Bogolubov transformations associated to different vacuum states for some $f(R)$ models. The procedure consists in fixing the $f(R)$ free parameters by requiring that the Bogolubov coefficients can be correspondingly minimized to be in agreement with both high redshift observations and quantum field theory predictions. In such a way, the particle production is related to the value of the Hubble parameter and then to the given $f(R)$ model. The approach is developed in both metric and Palatini formalism.

The second part of this thesis is devoted to the search for exact solutions for Extended Theories of Gravity that is very useful in order to control the physical meaning

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of these theories. To this goal, useful tools are Noether and Hojman approaches. The application of Hojman conservation theorem is presented in the framework of scalar-tensor cosmologies allowing to fix the form of the coupling $F(\phi)$, of the potential $V(\phi)$ and to find out exact solutions for related cosmological models. Afterwards, Noether point symmetries are applied to metric-Palatini hybrid gravity in order to select the $f(\mathcal{R})$ functional form, to find analytical solutions for the field equations and for the related Wheeler-DeWitt equation and finally to Gauss-Bonnet cosmological models, where F is a generic function of the curvature scalar R and the Gauss-Bonnet topological invariant \mathcal{G} , showing that the functional form of the $F(R, \mathcal{G})$ function can be determined by the presence of symmetries. Exact solutions for some specific cosmological models are found out. Finally, cosmological inflation is discussed in the framework of $F(R, \mathcal{G})$ gravity. In principle, this theory can exhaust all the curvature budget related to curvature invariants without considering derivatives of R , $R_{\mu\nu}$, $R_{\sigma\mu\nu}^\lambda$ etc. in the action. Cosmological dynamics is analysed resulting driven by two effective scalar fields, specifically a R scalaron and a \mathcal{G} scalaron, working respectively at early and very early epochs of cosmic evolution. In this sense, a double inflationary scenario naturally emerges.

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Chapter 1

Introduction

Gravity is the oldest and the most fascinating physical interaction that was investigated. Galileo Galilei (1564-1642) was the first who investigated terrestrial gravity discovering that bodies fall at a rate independent of their mass. He obtained his result with the introduction, at the end of 16th century, of an inclined plane to slow the fall, a water clock to measure its duration, and a pendulum to avoid rolling friction. It was in 1665 that Isaac Newton (1642-1727) related terrestrial gravity to celestial gravity. At the end of the *Principia*, Newton described gravitation as a cause that operates on the sun and planets “*according to the quantity of solid matter which they contain and propagates on all sides to immense distances, decreasing always as the inverse square of the distances*”.¹ Therefore he introduced the “inverse-square gravitational force law”.

The two key ideas of Newton’s theory of gravity were that of an absolute space, fixed, identical for all observers and not influenced by the mass and the idea that the inertial and the gravitational mass coincide, Weak Equivalence Principle (WEP). Despite the success of Newtonian model of gravitation in the prediction of a variety of phenomena, there were experimental and theoretical contradictions which undermined its foundations. In 1855 Le Verrier observed a 35 seconds of arc per century excess precession of Mercury’s orbit and later on, in 1882, Simon Newcomb measured this precession more accurately to be 43 seconds of arc per century. This experimental fact was not predicted by Newton’s theory.

¹Isaac Newton, *Philosophiae Naturalis Principia Mathematica*, trans. by Andrew Motte, revised and annotated by F. Cajori (University of California Press, 1966), p.546.

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From a theoretical point of view, in 1893 Ernst Mach stated a principle that was later called by Albert Einstein “*Mach's Principle*”. This is the first constructive attack to Newton’s idea of absolute space after the 18th century debate between Gottfried Wilhelm von Leibniz and Samuel Clarke (Clarke was acting as Newton’s spokesman) on the same subject, known as the Leibniz-Clarke Correspondence [6].

Mach’s idea was reformulated later on by Einstein who stated that inertia originates from interaction between masses. So the acceleration of a mass is relative to distant masses and not relative to absolute space. This is obviously in contradiction with Newton’s ideas, according to which inertia was always relative to the absolute frame of space. Later on, Dicke [7] gave a further interpretation of Mach’s Principle stating that the gravitational constant should be a function of the mass distribution in the Universe. This is different from Newton’s idea of the gravitational constant as being universal and unchanging. Therefore Newton’s basic axioms have to be reconsidered.

In 1905, when Albert Einstein completed Special Relativity, Newtonian gravity had to face a serious challenge. Special Relativity was introduced to explain a series of phenomena related to non-gravitational physics and this new theory is not consistent with Newtonian gravity. For example, the latter invokes notions of instantaneous influence of one body on another, whereas Special Relativity states that the velocity of an interaction cannot be greater than the speed of light.

Special Relativity incorporates Maxwell’s theory of electricity, magnetism and light, therefore one might expect that the next logical step would have been to develop a new theory of the other classical force, gravitation, which would generalize Newton’s theory and make it compatible with Special Relativity. However, Einstein chose a completely different path and instead developed General Relativity (GR), a new theory of space-time structure and gravitation. The Equivalence Principle and Mach’s principle provided the primary motivation for formulating a new theory.

In 1915, Einstein completed the theory of GR. Remarkably, the theory matched perfectly the experimental result for the precession of Mercurys orbit, as well as other experimental findings like the Lense-Thirring gravitomagnetic precession (1918) [8, 11] and the gravitational deflection of light by the Sun, as measured in 1919 during a Solar eclipse by Arthur Eddington [12].

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GR overthrew Newtonian gravity and continues to be up to now an extremely successful and well-accepted theory for gravitational phenomena. It is important to notice that GR reduces to Newtonian gravity in weak field limit of gravitational field strength and velocities. Hence, GR is a generalization of Newton's theory of gravity that maintain, although in a revisited form, some of its cornerstones, like the Equivalence Principle, but abandon some of its axioms, like the notion of an absolute frame.

From a mathematical point of view, it has been possible to formulate GR in particular thanks to the fact that, in the previous decades, Gauss, Riemann, Ricci Curbastro, Christoffel and Levi-Civita had formulated the so called *absolute differential calculus*, that is independent of the intrinsic structure of the geometric manifold in which is applied. The major value of this mathematical approach is that all of its notion can be applied to whatever manifold, such as curved space-time, and for whatever coordinates transformations. The advantage is remarkable because space-time is not necessarily flat.

1.1 The Equivalence Principle

The Equivalence Principle has played an important role in the development of gravitation theory. Newton regarded this principle as such a cornerstone of mechanics that he devoted the opening paragraphs of the *Principia* to a detailed discussion of it. He also reported there the results of pendulum experiments he performed to verify the principle. According to Newton, the Equivalence Principle demanded that the "mass" of any body, namely that property of a body (inertia) that regulates its response to an applied force, be equal to its "weight" that property that regulates its response to gravitation. Bondi (1957) coined the terms "inertial mass" m_I and "passive gravitational mass" m_G , to refer to these quantities, so that Newton's second law and the law of gravitation take the forms

$$\mathbf{F} = m_I \mathbf{a}, \quad \mathbf{F} = m_G \mathbf{g}, \quad (1.1)$$

where \mathbf{g} is the gravitational field. The Equivalence Principle can then be stated succinctly saying that for any body

$$m_I = m_G. \quad (1.2)$$

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An alternative statement of this principle is that all bodies fall in a gravitational field with the same acceleration regardless of their mass or internal structure. Newton's equivalence principle is now generally referred to as the "Weak Equivalence Principle" (WEP). It was Einstein who added the key element to WEP that revealed the path to GR. If all bodies fall with the same acceleration in an external gravitational field, then to an observer in a freely falling elevator in the same gravitational field, the bodies should be unaccelerated (except for possible tidal effects due to inhomogeneities in the gravitational field, which can be made as small as one pleases by working in a sufficiently small elevator). Thus insofar as their mechanical motions are concerned, the bodies will behave as if gravity were absent. Einstein went one step further. He proposed that not only should mechanical laws behave in such an elevator as if gravity were absent but so should all the laws of physics, including, for example, the laws of electrodynamics. This new principle led Einstein to GR. It is now called the "Einstein Equivalence Principle" (EEP).

The EEP then states: (i) WEP is valid, (ii) the outcome of any local nongravitational test experiment is independent of the velocity of the (freely falling) apparatus, and (iii) the outcome of any local nongravitational test experiment is independent of where and when in the universe it is performed. EEP is the foundation for all gravitation theories that describe gravity as a manifestation of curved spacetime, the so-called metric theories of gravity.

1.2 The requirement for a self-consistent theory of gravity

In considering relativistic theories of gravity it is necessary to impose some a priori conditions that have to be satisfied from the phenomenological point of view. First of all, every relativistic theory must explain astronomical observations mapping the orbits of planets and the potential well of self-gravitating structures such as galaxies and clusters. In other words the theory must reproduce the Newtonian dynamics in its weak-field, slow-motion limit. At the Post-Newtonian level, the theory must pass the classical Solar System tests, which have by now become very precise. A second

requirement is that the theory should reproduce correctly the observed galactic dynamics taking into account the observed baryonic constituents (*e.g.* luminous components as stars, sub-luminous components as planets, dust and gas), radiation and reproduce the Newtonian potential which is, by assumption, extrapolated to galactic scales. Besides, the theory must address the problem of the generation of large scale structures (galaxy clusters, superclusters, voids, and filaments) and finally, the cosmological dynamics must be reproduced. This means predicting in a self-consistent way the Hubble parameter H_0 , the deceleration parameter q_0 , the density parameters, etc. Astronomical observations and experiments probe directly standard baryonic matter, radiation, and indirectly the overall attraction of gravity acting at all scales and depending on distance.

1.3 The Einstein-Hilbert formulation of General Relativity

GR is the simplest theory which try to satisfies the above requirements[19]. It is based on the assumption that space and time have to be entangled into a single space-time structure and its properties are described by a space-time metric ($g_{\mu\nu}$), as in Special Relativity. However, the space-time metric does not need to have the flat form it has in Special Relativity. Indeed, the curvature, *i.e.* the deviation of the space-time metric from flatness, accounts for the physical effects usually ascribed to a gravitational field therefore we can see gravity as an aspect of space-time structure. Furthermore, the curvature of space-time is related to the stress-energy tensor of the matter in space-time via an equation postulated by Einstein. In this way the structure of space-time is related to the matter content of space-time.

The idea that the space-time should be a curved manifold came from the earlier ideas of Riemann who stated that the Universe should be a curved manifold and that its curvature should be established on the basis of astronomical observations [13]. It is important to underline that a requirement of GR is that, in the limit of no gravitational forces, the space-time structure has to reproduce the Minkowski space- time. Therefore, the equations of GR must reduce to the equation satisfied in Special Relativity in

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the case where the metric is flat.

There are three basic assumption on which GR is formulated

- The “Principle of Relativity” is the requirement that all observers are equally valid for describing physics. In particular, inertial frames (which do not exist globally) are not a priori preferred. This postulate addresses the main shortcoming of Special Relativity, being based on preferred inertial frames and Lorentz boosts between them.
- The “Principle of General Covariance”, that requires field equations to be the same in all coordinate systems, which is equivalent to the geometric view of physics, and states that all coordinate systems are in principle equivalent in the description of physics.
- The “Principle of Equivalence”, that amounts to require acceleration effects to be locally indistinguishable from gravitational effects (roughly speaking, the equivalence between the inertial and the gravitational mass).

In addition to these three principles, one imposes that causality is preserved (“Principle of Causality”, *i.e.* that each point of space-time should admit a universally valid notion of past, present and future) which is the same for all physical observers.

Let us also recall that the older Newtonian theory of space-time and gravitation, that Einstein wanted to reproduce at least in the limit of weak gravitational forces (what is called today the “post-Newtonian approximation”), required space and time to be absolute entities and required particles to move in a preferred inertial frame, following curved trajectories, the curvature of which (*i.e.* the acceleration) had to be determined as a function of the sources (*i.e.* the “forces”).

It was on these bases that Einstein was led to postulate that the gravitational forces have to be described by the curvature of a metric tensor field $g_{\mu\nu}$ related to the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

of a four-dimensional space-time manifold, having the same signature of Minkowski metric, *e.g.*, the so-called “Lorentzian signature”, herewith assumed to be $(-, +, +, +)$.

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As previously mentioned Einstein postulated that space-time is curved in itself and that its curvature is locally determined by the distribution of the sources, *e.g.*, being space-time a continuum, by the four-dimensional generalization of what in Continuum Mechanics is called the “matter stress-energy tensor”, *e.g.* a rank-two symmetric tensor $T_{\mu\nu}^{(m)}$.

Once a metric $g_{\mu\nu}$ is given, its curvature is expressed by the *Riemann* (or *curvature*) tensor

$$R_{\alpha\beta\mu}^{\nu} = \Gamma_{\alpha\mu,\beta}^{\nu} - \Gamma_{\beta\mu,\alpha}^{\nu} + \Gamma_{\alpha\beta}^{\sigma}\Gamma_{\sigma\mu}^{\nu} - \Gamma_{\mu\beta}^{\sigma}\Gamma_{\sigma\alpha}^{\nu} \quad (1.3)$$

where the commas denote partial differentiation. Its contraction

$$R_{\alpha\mu} \equiv R_{\alpha\beta\mu}^{\beta} \quad (1.4)$$

is the *Ricci tensor*, while

$$R \equiv R_{\mu}^{\mu} = g^{\mu\nu} R_{\mu\nu} \quad (1.5)$$

is the *scalar* (or *Ricci*) *curvature* of $g_{\mu\nu}$. Einstein initially contemplated the equations for the dynamics of gravity

$$R_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}^{(m)} \quad (1.6)$$

where $\kappa^2 = 8\pi G$ (in units in which $c = 1$) contains the gravitational coupling constant G . These equations turned out to be physically and mathematically inconsistent. As pointed out by Hilbert [14], they do not derive from an action principle; there is no action which reproduces them exactly through a variation¹. Einstein’s reply was that he knew that the equations were physically unsatisfactory, since they were incompatible with the continuity equation deemed to be satisfied by any reasonable form of matter. Assuming that the latter consists of a perfect fluid with stress-energy tensor

$$T_{\mu\nu}^{(m)} = (P + \rho)u_{\mu}u_{\nu} + P g_{\mu\nu} \quad (1.7)$$

where u^{μ} is the four-velocity of the fluid particles and P and ρ are the pressure and energy density of the fluid, respectively, the continuity equation requires $T_{\mu\nu}^{(m)}$ to be covariantly constant, *i.e.*, to satisfy the conservation law

$$\nabla^{\mu} T_{\mu\nu}^{(m)} = 0 \quad (1.8)$$

¹This is not entirely correct but this point is not essential.

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where ∇_μ denotes the covariant derivative operator of the metric $g_{\mu\nu}$. In fact, $\nabla^\mu R_{\mu\nu}$ does not vanish, except in the special case $R = 0$. Einstein and Hilbert independently concluded that the incorrect field equations (1.6) had to be replaced by the correct ones

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(m)} \quad (1.9)$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (1.10)$$

is now called the *Einstein tensor* of $g_{\mu\nu}$. These equations can be derived by minimizing an action and satisfy the conservation law (1.8) since the relation

$$\nabla^\mu G_{\mu\nu} = 0, \quad (1.11)$$

holds as a contraction of the Bianchi identities that the curvature tensor of $g_{\mu\nu}$ has to satisfy [20]. The Lagrangian that, when varied, produces the field equations (1.9) is the sum of a “matter” Lagrangian density $\mathcal{L}^{(m)}$, the variational derivative of which is

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}^{(m)})}{\delta g^{\mu\nu}} \quad (1.12)$$

and of the gravitational (*Hilbert – Einstein*) *Lagrangian*

$$\sqrt{-g}\mathcal{L}_{HE} = \sqrt{-g}R \quad (1.13)$$

where g is the determinant of the metric $g_{\mu\nu}$.

As it became clear a few years later, the choice of Hilbert and Einstein was completely arbitrary, but it was certainly the simplest one both from the mathematical and the physical points of view. As it was later clarified by Levi-Civita in 1919, curvature is not a “purely metric notion” but is also related to the linear connection defining parallel transport and covariant differentiation [15]. In a sense, this is the precursor idea of what in the sequel would be called a “gauge theoretical framework” [23], after the pioneering work by Cartan in 1925 [16]. But at the time of Einstein, only metric concepts were available to mathematicians and physicists and his solution was the only viable.

It was later clarified that the three principles of relativity, equivalence and covariance, together with causality, just require that the space-time structure has to be determined by either one or both of two fields, a Lorentzian metric $g_{\mu\nu}$ and a linear

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connection $\Gamma_{\mu\nu}^\alpha$, assumed at the beginning to be torsionless for the sake of simplicity. The metric $g_{\mu\nu}$ fixes the causal structure of space-time (the light cones) as well as its metric relations measured by clocks and rods and the lengths of four-vectors; the connection $\Gamma_{\mu\nu}^\alpha$ determines the laws of free fall, the four-dimensional space-time trajectories followed by locally inertial observers.

They have, of course, to satisfy a number of compatibility relations which amount to require that photons follow null geodesics of $\Gamma_{\mu\nu}^\alpha$, so that $\Gamma_{\mu\nu}^\alpha$ and $g_{\mu\nu}$ can be independent, *a priori*, but constrained, *a posteriori*, by some physical restrictions. These, however, do not necessarily impose that $\Gamma_{\mu\nu}^\alpha$ has to be the Levi-Civita connection of $g_{\mu\nu}$ [17].

This justifies, at least on a purely theoretical basis, the fact that one can envisage the so-called “alternative theories of gravitation”, that we prefer to call “Extended Theories of Gravitation” (ETGs) since their starting points are exactly those considered by Einstein and Hilbert in the construction of GR. These are theories in which gravitation is described by either a metric (*purely metric theories*), or by a linear connection (*purely affine theories*), or by both fields (metric-affine theories, also known as *first order formalism theories*). In these theories, the Lagrangian is a scalar density of the curvature invariants constructed out of both $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\alpha$. The choice of Hilbert-Einstein Lagrangian is by no means unique and it turns out that this Lagrangian is in fact the only choice that produces an invariant that is linear in second derivatives of the metric (or first derivatives of the connection). Unfortunately, this Lagrangian is rather singular from the Hamiltonian point of view in the same way of Lagrangians linear in canonical momenta are singular in Classical Mechanics (see *e.g.* [18]).

A number of attempts to generalize GR and unify it with electromagnetism along these lines were followed by Einstein and many others, including Eddington, Weyl and Schrödinger to mention a few. These attempts were eventually abandoned in the 1950s, mainly because of a number of difficulties related to the definitely more complicated structure of a non-linear theory (where by “non-linear” we mean one based on non-linear invariants of the curvature tensor), and also because of the discovery of two new physical interactions, the strong and the weak nuclear forces that required the more general framework of gauge theory [21]. Still, sporadic investigations of al-

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ternative theories continued after 1960. The search for a coherent quantum theory of gravitation, or the belief that gravity has to be considered as a sort of low-energy limit of string or other quantum theories [22] something that we will not discuss here has more recently revived the idea that it is not mandatory to adhere to the simple prescription of Einstein and Hilbert and to assume that classical gravity is governed by a Lagrangian linear in the curvature. Further curvature invariants or non-linear functions of them can also be contemplated, especially in view of the fact that their inclusion is required in both the semiclassical expansion of a quantum Lagrangian and in the low-energy limit of stringy actions. Moreover, it is clear from recent astrophysical observations and from the current cosmological investigations that it is legitimate to doubt the paradigmatic role played by the Einstein equations at Solar System, galactic, extragalactic, and cosmological scales, unless one is willing to admit that the right hand side of Eq. (1.9) contains some types of exotic energy, the *dark matter* (DM) and *dark energy* (DE) components of our universe.

The idea discussed here is, in principle, much simpler. Instead of changing the matter side of the Einstein equations (1.9) and introducing the missing matter-energy content of the observed universe (up to 95% of its total energy content), while adding mysterious and odd-behaving states of the matter fields, we contemplate the fact that it is a priori simpler and more convenient to change the geometric/gravitational sector of these equations by inserting non-linear corrections to the Lagrangian. This procedure could be regarded as a mere matter of taste; however, there is no reason to discard this approach a priori, and this possibility is intriguing and worth exploring. In principle, the action belongs to a vast family of permissible actions and, from the purely phenomenological point of view, this freedom allows it to be chosen on the basis of its best-fit with the available observational data at all scales (solar, galactic, extragalactic, and cosmological). The down side of this approach is that too many models fit well the observations because of the relatively large number of free functions and parameters that they contain, and predictive power may be lost. However, it is hoped that theoretical work will provide guidelines pointing to a preferred action and will discriminate between huge classes of models which fit the data, of which already too many are known. From the theoretical point of view, it makes perfect sense to give

serious consideration to rather well-motivated non-linear theories of gravity based on non-singular Lagrangians. Instead, the Λ CDM (abbreviation for Lambda-Cold Dark Matter) model is accompanied by exotic matter completely different from the known baryons, never detected in our laboratories, and segregated at astrophysical scales.

1.4 The Field Equations of General Relativity

Variational principles are used to formulate the equations of motion of particles and fields in theoretical physics and GR is not an exception. Einstein derived the field equations intuitively but our goal is to show how field equations can be derived from a variational principle. Consider a four-dimensional space-time manifold \mathcal{M} endowed with a Lorentzian metric $g_{\mu\nu}$ and assume that the connection ∇_μ be the symmetric Levi-Civita connection, i.e $\nabla_\lambda g_{\mu\nu} = 0$; hence (\mathcal{M}, g) is a pseudo-Riemannian manifold. In order to obtain second order field equations (Einstein equations) the Lagrangian density is assumed to be a function of the metric and of its derivatives up to second order. We find, in vacuum,

$$S = \int_U \sqrt{-g} \mathcal{L} d^4x , \quad (1.14)$$

where U is a compact region of the manifold \mathcal{M} , \mathcal{L} is the Lagrangian density and $\sqrt{-g} d^4x$ is defined as the invariant volume element. In fact, if we consider the transformation

$$x^\alpha = x^\alpha(\bar{x}^\mu) ,$$

where \bar{x}^μ are the local inertial coordinates (i.e. $\bar{g}_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$), we have

$$dx^\alpha = J d\bar{x}^\mu, \quad J = \det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right) ,$$

where J is the Jacobian determinant of transformation, and

$$\bar{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} g_{\mu\nu} .$$

In the new coordinate system the invariant volume element is

$$\sqrt{-\bar{g}} d^4\bar{x} = \sqrt{-\det\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} g_{\mu\nu}\right)} d^4\bar{x} = \det\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\right) \sqrt{-g} d^4\bar{x} =$$

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$$= \det \left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha} \right) \sqrt{-g} \det \left(\frac{\partial \bar{x}^\lambda}{\partial x^\nu} \right) d^4 x = \sqrt{-g} d^4 x ,$$

from which we derive that

$$\sqrt{-\bar{g}} d^4 \bar{x} = \sqrt{-g} d^4 x . \quad (1.15)$$

The field equations are obtained by requiring the action be stationary under arbitrary variations such that the metric and its first derivatives can be held fixed on the boundary ∂U . Now we need to find the Lagrangian density. As usual, we require that the relevant action should be a scalar. If we want to derive second order equations for the gravitational field it is necessary that the Lagrangian density contains at most the first derivatives of metric tensor. However it is impossible to construct a scalar quantity with only $g_{\alpha\beta}$ and $\Gamma_{\beta\gamma}^\alpha$. Therefore we are induced to choose an expression, for the Lagrangian density \mathcal{L} , that contain high order derivatives, running the risk of obtaining field equations of order higher than the second. The natural choice is the Ricci scalar curvature R in which there are the first and the second derivatives of the the metric tensor. The variational principle is

$$\delta S_{HE} = \frac{1}{16\pi G} \delta \int_U \sqrt{-g} R d^4 x = 0 . \quad (1.16)$$

Recalling the relations

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} ,$$

from which

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} . \quad (1.17)$$

Using relation (1.17), (1.16) becomes

$$\begin{aligned} & \int_U [(\delta \sqrt{-g}) R + \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}] d^4 x = \\ & \int_U \sqrt{-g} \delta g^{\mu\nu} [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}] d^4 x + \int_U \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} d^4 x = 0 . \end{aligned} \quad (1.18)$$

The second integral can be evaluated in the local inertial frame. In this frame the following relations hold

$$R_{\mu\nu}(0) = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha ,$$

$$\delta R_{\mu\nu}(0) = \frac{\partial}{\partial x^\alpha} \delta \Gamma_{\mu\nu}^\alpha - \frac{\partial}{\partial x^\nu} \delta \Gamma_{\mu\alpha}^\alpha ,$$

$$\begin{aligned} g^{\mu\nu}(0) \delta R_{\mu\nu}(0) &= g^{\mu\nu} \frac{\partial}{\partial x^\alpha} \delta \Gamma_{\mu\nu}^\alpha - g^{\mu\nu} \frac{\partial}{\partial x^\nu} \delta \Gamma_{\mu\alpha}^\alpha = \\ &= g^{\mu\nu} \frac{\partial}{\partial x^\rho} \delta \Gamma_{\mu\nu}^\rho - g^{\mu\rho} \frac{\partial}{\partial x^\rho} \delta \Gamma_{\mu\alpha}^\alpha = \\ &= \frac{\partial}{\partial x^\rho} [g^{\mu\nu} \delta \Gamma_{\mu\nu}^\rho - g^{\mu\rho} \delta \Gamma_{\mu\alpha}^\alpha] . \end{aligned}$$

Thus, we can write

$$g^{\mu\nu}(0) \delta R_{\mu\nu}(0) = \frac{\partial W^\rho}{\partial x^\rho}, \quad W^\rho = g^{\mu\nu} \delta \Gamma_{\mu\nu}^\rho - g^{\mu\rho} \delta \Gamma_{\mu\alpha}^\alpha , \quad (1.19)$$

and in general coordinates

$$g^{\mu\nu} \delta R_{\mu\nu} = \frac{\partial}{\partial x^\rho} [g^{\mu\nu} \delta \Gamma_{\mu\nu}^\rho - g^{\mu\rho} \delta \Gamma_{\mu\alpha}^\alpha] = \frac{\partial W^\rho}{\partial x^\rho} . \quad (1.20)$$

So, using Gauss theorem, the second integral in eq. (1.18) can be discarded since its argument is a pure divergence. Then eq. (1.18) becomes

$$\int_U \sqrt{-g} \delta g^{\mu\nu} [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}] d^4x = 0 . \quad (1.21)$$

From this equation, recalling the definition of the Einstein tensor $G_{\mu\nu}$, we obtain the vacuum field equations of GR

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 , \quad (1.22)$$

as Euler-Lagrange equations of the Hilbert-Einstein action.

The Field Equations in presence of matter

Suppose now that exists matter described the action

$$S_M = \int_U \mathcal{L}_M \sqrt{-g} d^4x . \quad (1.23)$$

Taking the variation of this action with respect to the metric $g_{\mu\nu}$ we obtain the *stress-energy tensor* $T^{\mu\nu}$ defined by

$$\delta S_M = \frac{1}{2} \int_U T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} d^4x , \quad (1.24)$$

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with

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} . \quad (1.25)$$

Since $\delta g_{\mu\nu}$ is symmetric, $T_{\mu\nu}^{(m)}$ is also taken to be so. For example, as we have seen, a *perfect fluid* defined to be a continuous distribution of matter with stress-energy tensor $T_{\mu\nu}^{(m)}$ of the form

$$T_{\mu\nu}^{(m)} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu) , \quad (1.26)$$

where u^μ is a unit timelike vector field representing the 4-velocity of the fluid. According to the above interpretation of $T_{\mu\nu}^{(m)}$, the functions ρ and P are, respectively, the mass-energy density and pressure of the fluid as measured in its rest frame. The fluid is called “perfect” because of the absence of heat conduction terms and stress terms corresponding to viscosity. This tensor satisfies the conservation law:

$$\nabla^\mu T_{\mu\nu}^{(m)} = 0 , \quad (1.27)$$

where ∇^μ denotes the covariant derivative operator of the metric $g_{\mu\nu}$.

Now the variational principle is

$$\delta(S_{HE} + S_M) = 0 . \quad (1.28)$$

From this we obtain the field equations in presence of matter

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}^{(m)} , \quad (1.29)$$

where $G_{\mu\nu}$ is the Einstein tensor of $g_{\mu\nu}$ and $\kappa^2 = 8\pi G$ is the coupling constant. The field equations may be written in an equivalent form. In fact, taking the trace of equation (1.29), we find

$$R = -\kappa^2 T^{(m)} ,$$

and thus,

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu}^{(m)} - \frac{1}{2}g_{\mu\nu}T^{(m)} \right) . \quad (1.30)$$

This second representation of Einstein’s field equations is completely equivalent to (1.29). However, it is useful in order to point out the feature of the stress-energy tensor that is the same feature of Einstein tensor. As a final remark, we note that the

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Cauchy problem for GR is well-formulated and well-posed *in vacuo* and in the presence of “reasonable” forms of matter (perfect fluids, minimally coupled scalar fields, etc.), see Appendix A.1 and references therein for more details. The consequence of the well-posedness is that GR is a “stable” theory with a robust causal structure where singularities can be classified [259, 260].

Chapter 2

Extended Theories of Gravity

2.1 Successes and shortcomings of General Relativity

According to the presentation reported in the previous chapter, Einstein's theory of GR provides a comprehensive and coherent description of space, time, gravity, and matter at a macroscopic level. It is formulated in such a way that space and time are not the absolute entities of classical mechanics, but, rather, dynamical quantities determined together with the distribution and motion of matter and energy. As a consequence, Einstein's approach gave rise to a new conception of the universe which, for the first time in the history of physics, can be considered as a dynamical system susceptible of precise mathematical modeling and physical measurement. Cosmology, thus, left the realm of philosophy where it had been relegated until Einstein's work and was legitimately incorporated into that of science. Over the years, the possibility of investigating the universe scientifically has led to the formulation of the Standard Big Bang model of the universe [20] which matched the available cosmological observations until recently. However, in the last thirty years several shortcomings of Einstein's theory were found and scientists began wondering whether GR is the only fundamental theory capable of successfully explaining the gravitational interaction. This new point of view comes mainly from the study of cosmology and quantum field theory. In cosmology, the presence of the Big Bang singularity, together with the flatness, horizon, and monopole problems [24] led to the realization that the standard cosmological model based on GR and on the Standard Model of particle physics is inadequate to describe the universe at extreme regimes. On the other hand, GR is a classical the-

ory which cannot work as a fundamental theory when a full quantum description of space-time and gravity is sought for.

2.2 The Dark Side issue: Dark Energy and Dark Matter

According to the recent observations, the number counts of clusters of galaxies, measurements of type Ia supernovae [25, 26, 27], with the discovery of late-time cosmic acceleration, and the cosmic microwave background (CMB) anisotropies, indicate that of the energy density budget of the universe, 5% comprises ordinary matter (baryons, radiation and neutrinos), while the rest, which does not interact electromagnetically, consists of 27% DM and 68% DE [28]. DM is responsible for the gravitational clumping of galaxies, galaxy clusters and large scale structures and the requirement of its existence had been known for some years. DE is a label for the relativistic energy density with negative pressure required to explain the inferred late-time accelerated expansion of the universe. If GR is the correct theory of the gravitational action then its application to cosmology should incorporate these observations. The implication of this description is that we live in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe that is dominated by cold dark matter (CDM) and DE in the form of a positive cosmological constant. This model of the universe is the best fit so far and it is based on the hypothesis that the universe is homogenous on large scale. It is commonly referred to as the Λ CDM (or concordance) model.

2.3 The geometric view: Extended Theories of Gravity

Phenomena recently appeared in fundamental physics, in astrophysics and cosmology, and several issues related to quantum field theories in curved space-time [29] suggest that gravity may not be described exactly by GR. In fact, if GR is the correct theory for the gravitational action then its application to cosmology should incorporate these observations.

For these reasons, and especially because of the lack of a definitive quantum theory of gravity, various alternative gravitational theories were proposed which attempt

to formulate at least a semiclassical scheme in which GR and its successes could be incorporated.

The question naturally arises: what is the importance of considering alternative theories of gravity to GR, as possible explanations to the observations if the Λ CDM model agrees well with the observations. One of the main motivations for the search for alternative theories of gravity arises from the obscure nature of DE candidates. The alternative possibility is to conjecture that the apparent need for DE could simply be because the application of Einstein's equations at cosmological scales is ill-suited. Some of the modified theories of gravity that provide a late time acceleration for the universe without the need for the presence of any exotic fluids are Scalar-Tensor Theories, Dvali-Gabadadze-Porrati (DGP) braneworld model [30], TeVeS (Tensor-Vector-Scalar) [31] and Horava-Lifschitz gravity [32, 33, 34].

One of the most fruitful approaches resulted in the so-called Extended Theories of Gravity (ETGs) which have, in some sense, become a paradigm in the study of the gravitational interaction. ETGs are based on corrections and enlargements of Einstein's theory. Essentially, the paradigm consists of adding higher order curvature invariants and minimally or non-minimally coupled scalar fields into the dynamics emerging from some effective quantum gravity action [35]. Other reasons to modify GR are provided by the attempt to fully incorporate Mach's principle into the theory. In fact, GR contains only some of Mach's ideas and admits solutions that are explicitly anti-Machian, such as the Gödel universe [36] and exact pp-waves [37]. Moreover, every scheme unifying the fundamental interactions, such as superstring, supergravity, or Grand-Unified Theories (GUTs) produces effective actions in which non-minimal couplings to the geometry or higher order terms in the curvature invariants are necessarily present. Such contributions are due to first or higher loop corrections in the high curvature regime approaching the full (and still speculative) quantum gravity regime [35]. This scheme was adopted in the quantization of matter fields on curved spacetimes and the result was that the interactions between quantum scalar fields and the background geometry, or the gravitational self-interactions, yield corrections to the Hilbert-Einstein Lagrangian [38]. Furthermore, it has been realized that these corrective terms are unavoidable in the effective quantum gravity actions [39]. Clearly, all

these approaches do not constitute a full quantum gravity theory, but are necessary as temporary working schemes toward it.

One such theoretical proposal that has recently attracted a considerable amount of attention is fourth order gravity that can accelerate at late times without the presence of DE [40, 41, 42, 43]. In particular, dynamical systems analysis shows that for FLRW models, there exist classes of fourth order theories which admit a transient decelerated expansion phase that is important for structure growth, followed by one with an accelerated expansion rate [44]. These cosmic evolutions therefore mimic the standard Λ CDM cosmic history. Another feature of these fourth order theories is that they are also able to account for the rotation curves of spiral galaxies without the need for DM [45].

To sum up, higher order invariants of the curvature tensor such as R^2 , $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R\Box R$, $R\Box^K R$ or non-minimally coupled terms between matter (especially scalar) fields and geometry such as ϕR have to be added to the effective gravitational Lagrangian as soon as quantum corrections are introduced. For instance, such terms occur in the low-energy limit of the Lagrangian of string theories or in Kaluza-Klein theories when extra spatial dimensions are compactified. On the other hand, from the conceptual point of view, there is no a priori reason to restrict the gravitational Lagrangian to be a linear function of the Ricci scalar R , minimally coupled with matter. Furthermore, the idea that there are no “exact” laws of physics has been contemplated seriously: in such a case, the effective Lagrangians of physical interactions would be average quantities arising from stochastic behaviour at a more microscopic level. This feature would mean that local gauge invariances and the related conservation laws are well approximated only in the low-energy limit and the fundamental constants of physics can vary [46]. In addition to being motivated by fundamental physics, ETGs have been the subject of great interest in cosmology because they naturally exhibit an inflationary behaviour capable of overcoming the shortcomings of the Standard Big Bang model based on GR. The related inflationary scenarios seem realistic and capable of matching the current observations of the cosmic microwave background (CMB). It has been shown that, by means of conformal transformations, the higher order and non-minimally coupled terms always correspond to Einstein grav-

ity plus one (or more) scalar field(s) minimally coupled to the curvature [47, 48, 49]. More precisely, higher order terms always appear as second order contributions to the field equations when they are replaced by equivalent scalar fields (see chapter 4). For example, the term R^2 in the Lagrangian yields fourth order equations of motion, $R\Box R$ gives sixth order equations, $R\Box^2 R$ yields eighth order equations, and so on. By means of a conformal transformation, any second order derivative term corresponds to a scalar field:¹ for example, fourth order gravity is equivalent to Einstein gravity plus a single scalar field; sixth order gravity to GR plus two scalar fields and so on [48]. For example, it is possible to show that $f(R)$ gravity is equivalent not only to a scalar-tensor theory but also to Einstein theory coupled to an ideal fluid [50]. This property is useful if multiple inflationary events are desired because an early stage could produce large scale structures with very long wavelengths which later grow into the clusters of galaxies observed today, while a later stage could select smaller scale structures observed as galaxies today. The philosophy is that each inflationary era is related to the dynamics of a scalar field. Finally, these extended schemes could naturally solve the graceful exit problem, avoiding the shortcomings of previous inflationary models [51].

From the mathematical point of view, the problem of formally reducing more general theories to the Einstein form has been discussed extensively. Through a Legendre transformation on the metric, higher order theories with Lagrangians satisfying minimal regularity conditions assume the form of GR with (possibly multiple) scalar field(s) sourcing the gravitational field. The formal equivalence between models with variable gravitational coupling and Einstein gravity via conformal transformations has also been known for a long time [54, 55]. This has given rise to a debate on whether the mathematical equivalence between different conformal representations of the theory (called Jordan and Einstein conformal frames) is also a physical equivalence ([56] and references therein).

¹The dynamics of all these scalars are usually determined by second order Klein-Gordon-like equations.

2.3.1 Further gravitational degrees of freedom as auxiliary scalar fields

In ETGs, the Einstein field equations are modified in two ways: the geometry can be coupled non-minimally to some scalar field, and/or derivatives of the metric higher than second order appear. In the former case, we generically deal with scalar tensor theories of gravity. A variety of metric theories of gravity have been devised which postulate the presence of a dynamical scalar gravitational field ϕ in addition to the metric. These theories provide a late time acceleration for the universe without the need for the presence of any exotic fluids. Indeed, it has been generally assumed that a scalar field is responsible for the inflationary scenario which in its various versions explains the horizon problem and the spatial flatness observed in the present stage of the cosmological evolution. But the existence of a scalar field has also been postulated, in some models, as a candidate for cold DM or, elsewhere, to incorporate Mach's principle into the theory of gravitation as the Brans-Dicke theory does [52, 53].

2.3.2 Addressing the Dark Side by Extended Gravity

The idea to relax the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar R , as in the Hilbert-Einstein formulation, is one of the most fruitful and economic compared to the attempts which try to solve cosmological problems by adding new and, mostly, unjustified ingredients in order to give a self-consistent picture of the dynamics. One of the problems is that the DE scale appears to be smaller and smaller with respect to the energy scale of any known interactions. The unnatural smallness of DE density constitutes the cosmological constant problem. In this sense, infrared-modified gravity models could be phenomenologically relevant as a possible alternative to DM and DE whose effects at large scales could be originated by geometry, specifically by the further degrees of freedom emerging in alternative theories of gravity [57]. In a qualitative way, we can see that starting from the Hilbert-Einstein Lagrangian it is necessary to add a term in the action in order to address the dark side issue

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R + \mathcal{S}^m + \mathcal{S}^{DS} \quad \longrightarrow \quad G_{\mu\nu} = \kappa^2 (T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(DS)}).$$

Conversely, extensions of GR provide naturally a cosmological component with negative pressure that in this way is originated from the geometry of the Universe

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R, R_{\mu\nu}R^{\mu\nu}, \dots) + \mathcal{S}^m \quad \longrightarrow \quad G_{\mu\nu} = \kappa^2 (T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(curv)}).$$

Thus, enlarging the geometric sector can be useful in view of the dark side since the further gravitational degrees of freedom have, in principle, a role in addressing DE and DM.

2.4 The quantum gravity motivations

One of the main challenges of modern physics is to construct a theory able to describe the fundamental interactions of nature as different aspects of the same theoretical construct. This purpose has led to the formulation of several unification schemes which attempt to describe gravity by putting it on the same level as the other interactions. All these schemes try to describe the fundamental fields in terms of the conceptual apparatus of quantum mechanics. This is based on the association of the states of a physical system with vectors in a Hilbert space \mathcal{H} where the physical fields are represented by linear operators defined on domains of \mathcal{H} . Up to now, any attempt to incorporate gravity in this scheme has either failed or been unsatisfactory. Conceptually, the main problem is that the gravitational field describes at the same time the gravitational degrees of freedom and the background space-time in which these degrees of freedom live. Due to the difficulties of building a complete theory unifying interactions and particles, during the last thirty years the two fundamental theories of modern physics, GR and quantum mechanics, have been critically reanalyzed. From one side, it is assumed that the matter fields (bosons and fermions) come out from superstructures (*e.g.*, Higgs bosons or superstrings) that, undergoing certain phase transitions, have generated the known particles. On the other hand, it is assumed that the geometry (*e.g.*, the Ricci tensor or the Ricci scalar) interacts directly with quantum matter fields which

back-react on it. Clearly, this interaction modifies the standard gravitational theory, that is, the Lagrangian of gravity plus the effective fields is modified with respect to the Hilbert-Einstein one, and this fact leads to the ETGs. Anyway, the condition that must be satisfied in order for such theories to be physically acceptable is that GR is recovered in the low-energy limit. Despite remarkable conceptual progress has been made following the introduction of generalized gravitational theories, at the same time the mathematical difficulties have increased. The corrections introduced into the Lagrangian augment the (intrinsic) non-linearity of the Einstein equations, making them more difficult to study because differential equations of higher order than second are often obtained and because it is impossible to separate the geometric from the matter degrees of freedom without using symmetries.

The need for a quantum theory of gravity was recognized at the end of the 1950s, when for the first time physicist tried to treat all interactions at a fundamental level and describe them in terms of quantum field theory. Naturally, the first attempt to quantize gravity was through the canonical approach and the covariant approach, successfully applied to electromagnetism.

Concerning the application of these methods to GR, many difficulties arise because Einstein's theory cannot be formulated in terms of a quantum field theory on a fixed Minkowski background. More specifically, in GR the geometry of the background space-time cannot be given a priori because space-time is the dynamical variable itself. In order to introduce the notions of causality, time, and evolution, it is necessary to first solve the equations of motion and build the space-time. As an example, to know if a particular initial condition will give rise to a black hole, it is necessary to fully evolve it by solving the Einstein equations. Then, taking into account the causal structure of the solution obtained, the asymptotic metric at future null infinity has to be studied to evaluate whether it is related, in the causal past, with that initial condition. This problem becomes intractable at the quantum level. According to the uncertainty principle, in non-relativistic quantum mechanics particles do not move along well-defined trajectories and only the probability amplitude $\psi(x, t)$ that a measurement at time t detects a particle around the spatial point x can be calculated. In the same way, in quantum gravity, the evolution of an initial state does not provide a specific space-

time. Therefore, the question arises as to how it is possible to introduce basic concepts such as causality, time, elements of the scattering matrix, or black holes in the absence of a space-time.

The canonical and covariant approaches provide different answers to these questions. The first approach is based on the Hamiltonian formulation of GR and aims at using the canonical quantization procedure. The canonical commutation relations are the same that lead to the uncertainty relations; in fact, the commutation of certain operators on a spatial three-manifold of constant time is imposed, and this three-manifold is fixed to preserve the notion of causality. In the limit of asymptotically flat space-time, the motion generated by the Hamiltonian must be interpreted as temporal evolution (in other words, when the background becomes the Minkowski space-time, the Hamiltonian operator assumes again its role as the generator of translations). The canonical approach preserves the geometric features of GR without the need to introduce perturbative methods [58, 59].

The main difficulties arising from this approach are that the quantum equations involve products of operators defined at the same space-time point and, in addition, they imply the construction of distributions whose physical meaning is unclear. However, the main problem is the absence of a Hilbert space of states and, as consequence, a probabilistic interpretation of the quantities calculated is missing. The covariant quantization approach is closer to the known physics of particles and fields in the sense that it has been possible to extend the perturbative methods of Quantum Electrodynamics (QED) to gravitation. This has allowed the analysis of the mutual interaction between gravitons and of the matter-graviton interactions. The formulation of Feynman rules for gravitons and the demonstration that the theory might be unitary at every order of the expansion was achieved by De Witt. Further progress was achieved with Yang-Mills theories, which describe the strong, weak, and electromagnetic interactions of quarks and leptons by means of symmetries. Such theories are renormalizable because it is possible to give the fermions a mass through the mechanism of spontaneous symmetry breaking. Therefore, it is natural to attempt to consider gravitation as a Yang-Mills theory in the covariant perturbation approach and check whether it is renormalizable. However, gravity does not fit into this scheme; it turns out to be

non-renormalizable when one considers the graviton-graviton interactions (two-loops diagrams) and graviton-matter interactions (one-loop diagrams). The covariant method allows one to construct a theory of gravity which is renormalizable at one-loop in the perturbative series [38]. As a consequence of the non-renormalizability of gravity at different orders, its validity is restricted only to the low-energy domain, *i.e.*, to large scales, while it fails at high energy and small scales. This entails that the full unknown theory of gravity has to be invoked near or at the Planck era and that, sufficiently far from the Planck scale, GR and its first loop corrections describe the gravitational interactions. In this context it makes sense to add higher order terms to the Hilbert-Einstein action.

Furthermore, the assumption that GR is the theory of gravitational interaction gives rise to spin-2 massless bosons, *i.e.* the massless gravitons, when the quantization in the linear approximation of Minkowskian limit is considered. A reasonable question to ask is whether gravitons could be massive in some alternative theory of gravity where GR is only limiting or a particular case [66, 67]. Even if the concept of massive gravitons poses some controversial issues that greatly complicate the formulation of self-consistent theories, such as the presence of ghost, instabilities, discontinuity and strong coupling effects at low energy scales [61, 62, 64, 65], massive graviton solutions cannot be simply ruled out if one wants to face coherently the problem of gravitational interaction in the ultraviolet limit [63, 74, 75, 83]. Furthermore, the problem of massive gravitons emerges and has to be consistently considered also at infrared limit [80, 79] when the large amount of alternative theories of gravity, developed in order to match the problem of cosmic acceleration in view of DE, is taken into account [29, 77, 57, 78, 76].

2.5 Effective field theory from Extended Gravity

The challenging problem of reconciling gravity and quantum field theories could be addressed through the study of Effective Field Theories (EFT), that allow to analyze different energy regimes separately (see *e.g.* [85, 86]). In general, since the effective Lagrangians is non-renormalizable, due to an infinite number of counterterms, one re-

tains only a few of them in a phenomenological approach where only leading terms are necessary. This means that the determination of the effective degrees of freedom is a crucial point for any effective theory and this fact is even more important in connection with gravity. Technically, a way to build up an effective Lagrangian is to identify some expansion parameters and classify terms in the Lagrangian according to such parameters. Without knowing the underlying fundamental theory, the coefficients of the expansion are necessarily unknown, and their values have to be determined, in principle, by experiments.

The study of gravitons has a key role in order to face the problem of reconciling gravity and fundamental interactions. The long standing problem of graviton mass [60, 61, 62, 63, 64, 65, 66] has recently excited a renewed interest both at fundamental and cosmological level. From one side, massive gravitational states could be the signature of some effective theory quantization. From the other side, massive gravitons could be the natural candidates for DM capable of structuring self-gravitating astrophysical systems [68, 69].

There have been several experimental searches for massive gravitons, resulting in upper limits for the mass which differ by several orders of magnitude. For example, a limit on the graviton mass ($\sim 8 \times 10^4$ eV) has been achieved by measuring the decay of two photons [81]. Besides, assuming that clusters of galaxies are bound by more-or-less standard gravity, it is possible to obtain an upper limit of $2 \times 10^{-29} h_0$ eV, where h_0 is the Hubble constant in units of $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [82]. Also gravitational waves sector has a prominent role in this discussion. In fact, gravitational waves coming from GR are described by the transverse-traceless gauge, which is a spin-2 tensor under rotations with massless modes. Beside of these standard results, it is possible to construct consistent models where Lorentz invariance is broken and the masses of scalar, vector and tensor perturbations are different from zero. A direct limit on the mass of graviton can be obtained from gravitational waves by binary stellar systems and from the inspiral rate inferred from the timing of binary pulsars. This bound is about 7.6×10^{-20} eV for the binary pulsar PSR B1913+16 [83]. The same limit can be also obtained by studying binary systems in $f(R)$ -gravity [84]. An estimate of the graviton mass upper limit of about 7×10^{-32} eV, is obtained by considering the effect

of gravitons on the power spectrum of weak lensing, with assumptions about DE and other parameters [72]. The recent discovery of gravitational waves from coalescing Black Holes allowed to set the upper limit of 1.2×10^{-22} eV on the graviton mass [73]. This fact opens new perspectives in the research of massive gravity.

In [1], we take into account an effective theory of gravity that follows naturally from ETGs (see *e.g.* [29]). The action can be expanded in powers of the Ricci curvature scalar R satisfying a massive Klein Gordon field equation. In particular, by linearizing $f(R)$ gravity, the Lagrangian describes a massive scalar field where a mass scale m emerges naturally. The theory does not predict the value of this mass, but it does predict its connection with parameters of the ETG Lagrangian. It is possible to identify correlations between the coefficients of the effective Lagrangian, which may, in turn, induce correlations among observables at different scales. A first result is that the assumption of an effective Lagrangian derived from $f(R)$ gravity allows to escape the problem of scalar ghosts in massive theories, as pointed out in [61]. In the limit where $m \gg \Lambda$ (being Λ the cosmological constant), we achieve a physically acceptable scalar field satisfying a homogeneous Klein Gordon equation and then one achieves an effective field Lagrangian bypassing some of the problems raised in [61] where GR, *i.e.* $f(R) = R$, was considered.

The perturbative approach

We consider a 4-dimensional action in vacuum for a generic function $f(R)$ of the Ricci scalar [29, 77, 57, 78] that we will treat in detail later (see Chapter 3)

$$S = \int d^4x \sqrt{-g} f(R), \quad (2.1)$$

where the Ricci scalar is defined as $R = g^{\mu\nu} R_{\mu\nu}$, and g is the determinant of the metric. At this stage, our only assumption is that $f(R)$ is an analytic function, *i.e.* Taylor expandable, in term of the Ricci scalar, namely

$$f(R) = \sum_n \frac{f^n(R_0)}{n!} (R - R_0)^n = f_0 + f'_0 R + \frac{1}{2} f''_0 R^2 + \dots \quad (2.2)$$

The flat-Minkowski background is recovered as soon as $R = R_0 = 0$ and $f_0 = 0$. The notations $f'(R) = \frac{df(R)}{dR}$ and $f''(R) = \frac{d^2 f(R)}{dR^2}$ indicate the derivative with respect to

the Ricci scalar R . We have defined $f_0 = f(R)|_{R=R_0}$, $f'_0 = f'(R)|_{R=R_0}$ and so on. At the second order of approximation in term of R , the above action (2.1) becomes

$$S = \int d^4x \sqrt{-g} \left[f_0 + f'_0 R + \frac{1}{2} f''_0 R^2 \right]. \quad (2.3)$$

This can be seen as an EFT Lagrangian, naturally emerging in the context of ETG. In a bottom-up approach, from the point of view of unconstrained EFT, there is no rationale, like symmetries or renormalizability, for choosing the gravitational action proportional to R like in GR, except indications that the curvature R is rather small. Furthermore, there are infinite terms allowed by general coordinate invariance, such as $R_{\mu\nu}R^{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor, $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, where $R_{\mu\nu\lambda\sigma}$ is the Riemann tensor, derivatives of R , and so on. Where one has to truncate the expansion is somehow a matter of choice, and the coefficients are completely unknown from a theoretical point of view. Instead, the terms in the action (2.3) follow from the underlying ETG, which can also give indications on the coefficients and the order of the series. Here we are choosing the simplest possibility considering an analytical $f(R)$ theory of gravity.

After the variation of the action (2.3) with respect to the metric, we obtain the following field equations

$$-\frac{f_0}{2}g_{\mu\nu} + f'_0G_{\mu\nu} - f''_0 \left[\nabla_\mu \nabla_\nu R - g_{\mu\nu} \square R + R \left(\frac{1}{4} R g_{\mu\nu} - R_{\mu\nu} \right) \right] = 0, \quad (2.4)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad (2.5)$$

is the Einstein tensor and $\square = \nabla_\sigma \nabla^\sigma$ is the d'Alembert operator with ∇_σ indicating covariant derivatives. It is worth noting that rewriting the ETG Lagrangian in the form

$$\mathcal{L} = \sqrt{-g} \left[\frac{f_0}{f'_0} + R + \frac{1}{2} \frac{f''_0}{f'_0} R^2 \right] f'_0, \quad (2.6)$$

the cosmological constant term can be identified as $\frac{f_0}{f'_0} = -2\Lambda$. We are working in Planck units, hence we assume that the Lagrangian in action (2.6) is multiplied by $\frac{1}{2\kappa^2} = \frac{1}{16\pi G_N}$, where G_N is the Newton constant. From now on, we will work in in “modified” Planck units, that is we will assume a multiplicative factor $1/16\pi\tilde{G}$, with $\tilde{G} = G_N/f'_0$, that reduces to the standard one as soon as $f'_0 = 1$. Immediately, we have

$$\Lambda g_{\mu\nu} + G_{\mu\nu} - \frac{f''_0}{f'_0} \left[\nabla_\mu \nabla_\nu R - g_{\mu\nu} \square R + R \left(\frac{1}{4} R g_{\mu\nu} - R_{\mu\nu} \right) \right] = 0. \quad (2.7)$$

Taking the trace of the above field equations, we have

$$\square R - \frac{f'_0}{3f''_0}(R - 4\Lambda) = 0. \quad (2.8)$$

Clearly, setting $f_0 = 0$, *i.e.* discarding the 0^{th} term, is equivalent to set to zero the cosmological constant. Thus, the trace equation takes the form

$$\square R - \frac{f'_0}{3f''_0}R = 0. \quad (2.9)$$

Eqs. (2.8) and (2.9) are Klein-Gordon-like equations. Indeed, by assuming that the ratio f'_0/f''_0 is negative, we can define an effective mass

$$m^2 \equiv -\frac{f'_0}{3f''_0}, \quad (2.10)$$

so that we have

$$\square R + m^2 R = 0, \quad (2.11a)$$

$$\square R + m^2(R - 4\Lambda) = 0. \quad (2.11b)$$

It follows that the curvature R can be considered formally analogous to a massive scalar field [88]. We can neglect the non-homogeneous equation as soon as the condition

$$R \gg \Lambda, \quad (2.12)$$

holds.

Now we want to study the linearized version of such a theory in order to interpret it in the context of EFT. In order to linearize the field equations (2.4) at first order in $h_{\mu\nu}$, we have to expand around the flat space-time metric $\eta_{\mu\nu}$ [89, 90, 91]. We have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \Rightarrow \quad ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \quad (2.13)$$

with $h_{\mu\nu}$ small ($\mathcal{O}(h^2) \ll 1$). It is important to emphasize that the perturbation $h_{\mu\nu}$ is a symmetric tensor. The Ricci scalar, at the first order in metric perturbation, reads

$$R = \partial^\sigma \partial^\tau h_{\sigma\tau} - \square h, \quad (2.14)$$

where $h \equiv h^\mu_\mu$ is the trace of $h_{\mu\nu}$ and $\square = \partial_\sigma \partial^\sigma$ that is reduced now to the standard d'Alembert operator defined on the underlying Minkowski space-time where gravity is assumed as a perturbation. Considering the harmonic gauge condition¹

$$\partial^\mu h_{\mu\nu} = 0, \quad (2.15)$$

we obtain

$$R = -\square h. \quad (2.16)$$

In this approach, the fluctuation of the metric on the background represents the field mediating the gravitational interaction. Now we want to identify its properties by setting the corresponding field equations.

Considering the homogeneous Klein-Gordon Eq. (2.11a) and substituting the expression for R given by Eq. (2.16), we find

$$\square(\square h + m^2 h) = 0. \quad (2.17)$$

We can choose the trivial solution

$$\square h + m^2 h = 0, \quad (2.18)$$

and find the condition

$$\square h = -m^2 h, \quad (2.19)$$

that is a sort of mass shell condition. We can also consider Eq. (2.11b) discussing the role of cosmological constant. As it is well known, the general solution is the sum of the field satisfying the associated homogeneous Eq. (2.11a) plus a particular solution R' , that we can formally write as

$$R'(x) = 4\Lambda m^2 \int G(x, x') dx'. \quad (2.20)$$

Here $G(x, x')$ is a non-local Green function satisfying the field equation

$$(\square + m^2)G(x, x') = \delta(x, x'). \quad (2.21)$$

¹Such a condition is also called Hilbert, or De Donder or Lorentz gauge. In general, the harmonic gauge is defined in a curved background by the condition $\partial_\nu (g^{\mu\nu} \sqrt{-g}) = 0$. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and expanding to linear order, the harmonic gauge reduces to the standard Lorentz gauge.

Only the scale m^2 appears in Eqs. (2.19) and (2.21), while R' is suppressed by Λ . We can reasonably assume that R' can be neglected with respect to the solutions of Eq. (2.11a), as far as the approximation $\Lambda \ll m^2$ holds.

Let us now rewrite the Lagrangian (2.6) in term of the perturbation. It is

$$\mathcal{L} = \sqrt{-g} \left[\frac{f_0}{f'_0} + R + \frac{f''_0}{2f'_0} R^2 \right] = \sqrt{-\det(\eta_{\mu\nu} + h_{\mu\nu})} \left(R - 2\Lambda - \frac{1}{6m^2} R^2 \right), \quad (2.22)$$

where we have explicitly indicated the determinant of the metric. Substituting $R \rightarrow -\square h$, we find

$$\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} + h_{\mu\nu})} \left[-2\Lambda - \square h - \frac{1}{6m^2} (\square h)^2 \right], \quad (2.23)$$

and using the condition (2.19) as a sort of Lagrange multiplier (see also [92]), it becomes

$$\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} + h_{\mu\nu})} \left[m^2 h - 2\Lambda - \frac{m^2}{6} h^2 \right]. \quad (2.24)$$

Working out the square root up to the second order² in $h_{\mu\nu}$, we find

$$\sqrt{-\det(\eta_{\mu\nu} + h_{\mu\nu})} \simeq 1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu}, \quad (2.25)$$

and the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \left(1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) \left(-2\Lambda + m^2 h - \frac{m^2}{6} h^2 \right) \\ &= -2\Lambda + (m^2 - \Lambda) h + \left(\frac{m^2}{3} - \frac{\Lambda}{4} \right) h^2 + \frac{1}{2} \Lambda h_{\mu\nu} h^{\mu\nu} + \frac{m^2}{24} h^3 - \frac{m^2}{48} h^4 \\ &\quad - \frac{m^2}{4} h h_{\mu\nu} h^{\mu\nu} + \frac{m^2}{24} h^2 h_{\mu\nu} h^{\mu\nu}. \end{aligned} \quad (2.26)$$

By truncating up to the second order in h , we get

$$\mathcal{L} = -2\Lambda + (m^2 - \Lambda) h + \left(\frac{m^2}{3} - \frac{\Lambda}{4} \right) h^2 + \frac{1}{2} \Lambda h_{\mu\nu} h^{\mu\nu}. \quad (2.27)$$

The above Lagrangian describes a spin-0 particle and a spin-2 particle. The term proportional to h does not affect the calculation of perturbative observables, since it is linear in the creation and destruction operators and it vanishes once inserted between vacuum states.

²We expand $\sqrt{-\det(\eta_{\mu\nu} + h_{\mu\nu})}$ at the second order in $h_{\mu\nu}$, in agreement with the order of expansion of $f(R)$ in R .

The effective field Lagrangian and the role of massive modes

Eqs. (2.22) and (2.27) can be considered as effective Lagrangians written in different variables. In [87], the EFT is used to select the low energy modes, that are those of the GR, and contributions from quantum physics are analyzed. As in Eq. (2.22), the gravitational action is chosen proportional to powers of the curvature R , but the arbitrary motivation of this choice is the physical smallness of R . On the other hand, the expansion in R comes out naturally from the ETG where the coefficients are fixed from the effective Lagrangian.

Let us compare the effective Lagrangian from linearized $f(R)$ gravity Eq. (2.27) with a Lagrangian derived on purely phenomenologically ground. The free part of the Lagrangian for a massless spin-2 field can be written as

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{2} \partial^\lambda (h_{\lambda\mu} + h_{\mu\lambda}) \partial^\mu h - \frac{1}{4} \partial^\lambda (h_{\lambda\mu} + h_{\mu\lambda}) \partial_\nu (h^{\mu\nu} + h^{\nu\mu}) + \\ & + \frac{1}{8} \partial_\lambda (h_{\mu\nu} + h_{\nu\mu}) \partial^\lambda (h^{\mu\nu} + h^{\nu\mu}) - \frac{1}{2} \partial_\lambda h \partial^\lambda h. \end{aligned} \quad (2.28)$$

This form is derived on the basis of Lorentz invariance and gauge transformations as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \eta_\mu, \quad (2.29)$$

where ξ_ν and η_μ are eight arbitrary functions. Generic mass terms can be added

$$\mathcal{L}_m = -a_1 h^2 - a_2 h_{\mu\nu} h^{\mu\nu} - a_3 h_{\mu\nu} h^{\nu\mu}, \quad (2.30)$$

with a_1 , a_2 and a_3 being arbitrary coefficients. In our case, $h_{\nu\mu}$ is symmetric, therefore the second and the third term coincide, which is equivalent to set, for instance, $a_3 = 0$. The Lagrangian $\mathcal{L}_0 + \mathcal{L}_m$ describes an effective theory with two particles of 0-spin and 2-spin, as the Lagrangian in Eq. (2.27). It has been demonstrated that, when $a_2 \neq a_3$, the condition of null divergence of h is not generally respected by the scalar field, resulting in negative energy, or indefinite metric, which are not physically acceptable [61]. In order to recover null divergence, the coefficients and the masses of Eq. (2.30) need to respect fixed relations among them.

It is not obvious to build sensible descriptions of the gravitational interaction with this characteristic. A standard way is to use the Hilbert-Einstein action for the massless

gravitational field together with mass terms respecting the symmetries leading to the correct Ward identities. In [61], it is observed that such mass terms do not respect the relations necessary to a physically acceptable Lagrangian $\mathcal{L}_0 + \mathcal{L}_m$, and we are forced to conclude that this description of massive gravity is not satisfactory.

In our case, the effective Lagrangian (2.27) evades the condition $a_2 \neq a_3$ assumed in [61]. In fact, the Lagrangian contains, at leading order, only terms proportional to powers of h , which correspond to $a_2 = a_3 = 0$. Additional contributions are suppressed by Λ , in the limit $\Lambda \ll m^2$, which is the same limit where dynamics of the scalar is described by the physical Klein Gordon equation (2.19). In other words, we can say that starting from an analytical $f(R)$ gravity model, it is quite natural to recover an EFT where massive modes emerge at scalar and tensor levels. These results point out that massive modes have not to be excluded *a priori* in relativistic theories of gravity. Such a feature could give rise to a new physic opening the doors to a self-consistent interpretation of dark side of the universe in the framework of gravity.

Chapter 3

Variational principles in Extended Theories of Gravity

Let us now focus on specific gravitational theories that have received attention lately. It is known that variational principles play a prominent role in theoretical physics and any fundamental physical theory can be formulated in terms of an action from which are derived the equations of motion by means of a variational principle. Specification of the Lagrangian function is determined by mathematical and physical requirements like gauge invariance, renormalizability, simplicity, and so forth. We begin with the exploration of the theoretical aspects related to the formulation of different approaches to variational principle in gravitational theories, then we will concentrate on some specific theories showing their Lagrangian functions, equations of motion and the basic assumptions needed in order to obtain them.

3.1 The Metric and the Palatini approaches

The dynamics of gravitation is given by field equations which can be considered in two different approaches: the metric and Palatini formalism [93]. The former approach relies on the usual variation of the action with respect to the metric tensor $g_{\mu\nu}$ whereas the Palatini formalism deals with metric and (usually torsion-free) connection $\Gamma_{\mu\nu}^{\lambda}$ entering the definition of the Ricci tensor as two independent quantities and the variation is taken with respect to both. In the case of GR, the two approaches are equivalent (they provide the same field equations) as a consequence of the fact that the field equations for the connection $\Gamma_{\mu\nu}^{\lambda}$ give the Levi-Civita connection of the metric $g_{\mu\nu}$.

The situation is different in ETGs depending on functions of curvature invariants or for gravity non-minimally coupled to a scalar field. In these cases, the Palatini and the metric variational principle yield different field equations and different physical predictions. As an example, in $f(R)$ gravity, the metric formalism leads to a system of fourth-order partial differential equations while the Palatini one gives second-order equations. Note also that both formalisms are dynamically equivalent to different classes of Brans-Dicke-like theories, which implies that they cannot be equivalent to each other [94].

From the physical point of view, considering the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^\lambda$ as independent fields amounts to decoupling the metric structure of space-time and its geodesic structure with the connection $\Gamma_{\mu\nu}^\lambda$ being distinct from the Levi-Civita connection of $g_{\mu\nu}$. The causal structure of space-time is defined by $g_{\mu\nu}$, while the space-time trajectories of particles are governed by $\Gamma_{\mu\nu}^\lambda$. In principle, this decoupling enriches the geometric structure of space-time and generalizes the purely metric formalism. By means of the Palatini field equations, this dual structure of space-time is naturally translated into a bimetric structure of the theory: instead of a metric and an independent connection, the Palatini formalism can be seen as containing two independent metrics $g_{\mu\nu}$ and $h_{\mu\nu} = f'(R)g_{\mu\nu}$. In Palatini $f(R)$ gravity the new metric $h_{\mu\nu}$ determining the geodesics is related to the connection $\Gamma_{\mu\nu}^\lambda$ by the fact that the latter turns out to be the Levi-Civita connection of $h_{\mu\nu}$. Let us now consider some particular classes of ETGs.

3.2 Scalar - Tensor Gravity

Let concentrate on theories which include a scalar field as an extra field mediating the gravitational interaction. From now on we adopt Planck units.

3.2.1 Brans-Dicke-like gravity

The Brans-Dicke-like theory of gravity is the prototype of scalar-tensor theories which include a scalar field, non-minimally coupled to the gravity. The action is given

by

$$\mathcal{S}_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] + \mathcal{S}^{(m)} \quad (3.1)$$

where g is the determinant of the metric and

$$\mathcal{S}^{(m)} = \int d^4x \sqrt{-g} \mathcal{L}^{(m)} \quad (3.2)$$

is the action of ordinary matter and ω is the dimensionless Brans-Dicke parameter. The factor ϕ in the denominator of the kinetic term of ϕ in the action is purely conventional and has the only purpose of making ω dimensionless. Matter does not couple directly to ϕ , *i.e.*, the Lagrangian density $\mathcal{L}^{(m)}$ is independent of ϕ (“minimal coupling” of matter). However, ϕ couples directly to the Ricci scalar. The gravitational field is described by both the metric tensor $g_{\mu\nu}$ and the Brans-Dicke scalar ϕ which, together with the matter variables, constitute the degrees of freedom of the theory. As usual for scalar fields, the potential $V(\phi)$ generalizes the cosmological constant and may reduce to a constant, or to a mass term. The original motivation for introducing Brans-Dicke theory was the implementation of Mach’s principle. This is achieved in Brans-Dicke-like theory by making the effective gravitational coupling strength $G_{eff} \approx \phi^{-1}$ depending on the space-time position and being governed by distant matter sources. As already remarked, modern interest in Brans-Dicke and scalar-tensor theories is motivated by the fact that they are obtained as low-energy limits of string theories. By varying the action (3.1) with respect to $g_{\mu\nu}$ and using the well known properties

$$\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta(\sqrt{-g}R) = \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \equiv \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu}$$

yield the field equations

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu}^{(m)} + \frac{\omega}{\phi^2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \quad (3.3)$$

$$+ \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi) - \frac{V}{2\phi} g_{\mu\nu} \quad (3.4)$$

where

$$T_{\mu\nu}^{(m)} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}^{(m)}) \quad (3.5)$$

is the energy-momentum tensor of ordinary matter. By varying the action with respect to ϕ , one obtains

$$\frac{2\omega}{\phi}\square\phi + R - \frac{\omega}{\phi^2}\nabla^\alpha\phi\nabla_\alpha\phi - \frac{dV}{d\phi} = 0 \quad (3.6)$$

Taking now the trace of Eq. (3.3)

$$R = \frac{-8\pi T^{(m)}}{\phi} + \frac{\omega}{\phi^2}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{3\square\phi}{\phi} + \frac{2V}{\phi} \quad (3.7)$$

and using the resulting Eq. (7.1) to eliminate R from Eq. (3.6) leads to

$$\square\phi = \frac{1}{2\omega + 3} \left(8\pi T^{(m)} + \phi \frac{dV}{d\phi} - 2V \right). \quad (3.8)$$

According to this equation, the scalar ϕ is sourced by non-conformal matter (*i.e.*, by matter with trace $T^{(m)} \neq 0$), however the scalar does not couple directly to $\mathcal{L}^{(m)}$: the Brans-Dicke scalar ϕ reacts on ordinary matter only indirectly through the metric tensor $g_{\mu\nu}$, as dictated by Eq. (3.3). The term proportional to $\phi dV/d\phi - 2V$ on the right hand side of Eq. (3.8) vanishes if the potential has the form $V(\phi) = m^2\phi^2/2$ familiar from the Klein-Gordon equation and from particle physics. The action (3.1) and the field equation (3.3) suggest that the field ϕ be identified with the inverse of the effective gravitational coupling

$$G_{eff}(\phi) = \frac{1}{\phi} \quad (3.9)$$

a function of the space-time location. In order to guarantee a positive gravitational coupling, only the range of values $\phi > 0$ corresponding to attractive gravity is considered. The dimensionless Brans-Dicke parameter ω is a free parameter of the theory: a value of ω of order unity would be natural in principle (and it does appear in the low-energy limit of the bosonic string theory). However, values of ω of this order of magnitude are excluded by Solar System experiments, for a massless or light field ϕ (*i.e.*, one that has a range larger than the size of the Solar System). The larger the value of ω , the closer Brans-Dicke gravity is to GR [20]. Brans-Dicke theory with a free or light scalar field is viable in the limit of large ω , but the large value of this parameter required to satisfy the experimental bounds is certainly fine-tuned and makes Brans-Dicke theory unappealing. However, this fine-tuning becomes unnecessary if at the scalar field is given a sufficiently large mass and, therefore, a short range. This means that a self-interaction

potential $V(\phi)$ has to be considered in discussing the limits on ω and this fact is an adjustment of the original Brans-Dicke theory [52].

3.2.2 Scalar -Tensor theories: the general case

In four dimensions, the general form for the action of a scalar-tensor theory involving gravity coupled in a non-standard way with a scalar field is

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) \right], \quad (3.10)$$

where R is the Ricci scalar, $V(\phi)$ and $F(\phi)$ are generic functions describing, respectively, the potential for the field ϕ and the coupling of ϕ with gravity.

The variation with respect to $g_{\mu\nu}$ gives the second-order field equations

$$F(\phi)G_{\mu\nu} = F(\phi) \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right] = -\frac{1}{2}T_{\mu\nu}^\phi - g_{\mu\nu}\square F(\phi) + F(\phi)_{;\mu\nu}, \quad (3.11)$$

which are the generalized Einstein equations; here \square is the d'Alembert operator for the metric g , $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}^\phi$ is the energy-momentum tensor relative to the scalar field ϕ . Here and below, semicolon denotes metric covariant derivatives with respect to g . The energy-momentum has the form

$$T_{\mu\nu}^\phi = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} + g_{\mu\nu}V(\phi), \quad (3.12)$$

The variation with respect to ϕ provides the Klein-Gordon equation, *i.e.* the field equation for the scalar field

$$\square\phi - R\mathcal{F}_\phi(\phi) + V_\phi(\phi) = 0, \quad (3.13)$$

where $\mathcal{F}_\phi = dF(\phi)/d\phi$, $V_\phi = dV(\phi)/d\phi$. This last equation is equivalent to the Bianchi contracted identity [103].

- **Minimal coupling**

The theory of gravity minimally coupled to a scalar field is obtained by imposing $F(\phi) = \text{constant}$ in (3.10) and is described by the action

$$\mathcal{S}_{MC} = \int d^4x \sqrt{-g} \left[-\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) \right]. \quad (3.14)$$

The resulting field equations are

$$G_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} + g_{\mu\nu}V(\phi), \quad (3.15)$$

that is

$$G_{\mu\nu} = T_{\mu\nu}^{\phi}. \quad (3.16)$$

The field equation for the scalar field is

$$\square\phi + V_{\phi}(\phi) = 0. \quad (3.17)$$

3.3 Higher-order gravity

ETGs exhibit two main features: first, the geometry can couple non-minimally to some scalar field; second, derivatives of the metric components of order higher than second may appear. In the first case, as we said previously, we have scalar-tensor theories of gravity, and in the second case we have higher order theories. Combinations of non-minimally coupled and higher order terms can also emerge in effective Lagrangians, producing mixed higher-order-scalar-tensor gravity.

The general class of higher-order-scalar-tensor theories in four dimensions is given by the action

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\left(F(R, \square R, \square^2 R, \dots, \square^k R, \phi) - \frac{\epsilon}{2} g^{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi \right) + \mathcal{L}_m \right], \quad (3.18)$$

where F is an unspecified function of curvature invariants and of a scalar field ϕ (recall that we are adopting Planck units). The term \mathcal{L}_m is the minimally coupled ordinary matter contribution and ϵ is a constant which specifies the theory. Actually its values can be $\epsilon = \pm 1, 0$, fixing the nature and the dynamics of the scalar field which can be a standard scalar field, a phantom field or a field without dynamics. By varying the action with respect to $g_{\mu\nu}$ we obtain the field equations:

$$\begin{aligned}
 G^{\mu\nu} = & \frac{1}{\mathcal{P}} \left[T^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (F - \mathcal{P}R) + (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma}) \nabla_\lambda \nabla_\sigma \mathcal{P} \right] + \\
 & + \frac{1}{2} \left[\sum_{i=1}^k \sum_{j=1}^k (g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\nu} g^{\lambda\sigma}) \nabla_\sigma (\square^{j-i}) \nabla_\lambda \left(\square^{i-j} \frac{\partial F}{\partial \square^i R} \right) \right] \\
 & - g^{\mu\nu} g^{\lambda\sigma} \left[\nabla_\lambda \left(\nabla_\sigma (\square^{j-1} R) \square^{i-j} \frac{\partial F}{\partial \square^i R} \right) \right]
 \end{aligned} \tag{3.19}$$

where $G^{\mu\nu}$ is the Einstein tensor and

$$\mathcal{P} \equiv \sum_{j=0}^n \square^j \left(\frac{\partial F}{\partial \square^i R} \right). \tag{3.20}$$

The differential equations (3.19) are of order $(2k + 4)$. The stress-energy tensor is due to the kinetic part of the scalar field ϕ and to the ordinary matter:

$$T_{\mu\nu} = T_{\mu\nu}^{(m)} + \frac{\epsilon}{2} \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi \right]. \tag{3.21}$$

The possible contribution of a potential $V(\phi)$ is contained in the definition of F . From now on, we indicate by a capital F a Lagrangian density containing also the contribution of a potential $V(\phi)$ and by $F(\phi)$, $f(R)$, or $f(R, \square R)$ a function of such fields without potential. It is worth noticing that the Cauchy problem for these theories could be extremely difficult to formulate and its well-formulation and well-posedness strictly depends on the source fluid considered. An example of this kind of problem is reported in Appendix A.2 and references therein.

By varying with respect to the scalar field ϕ , we obtain the Klein-Gordon equation:

$$\epsilon \square \phi = - \frac{\partial F}{\partial \phi} \tag{3.22}$$

Several approaches can be used to deal with such equations. For example, as we said, by a conformal transformation, it is possible to reduce an extended theory to a multi scalar-tensor theory of gravity. From the action (3.18), by choosing

$$F = F(\phi) - V(\phi) \quad \epsilon = -1. \tag{3.23}$$

we get the action (3.10), *i.e.* the action of scalar-tensor theories with non-minimal coupling.

$f(R)$ gravity

We now examine one of the simplest modifications to GR that is $f(R)$ gravity achieved imposing $F = f(R)$ and $\epsilon = 0$ in the action (3.18). The salient feature of these theories is that the field equations are of fourth order and that GR is recovered as the special case $f(R) = R$. Due to their higher order, these field equations admit a much richer variety of solutions than the Einstein equations. The 4-dimensional action is

$$\mathcal{S} = \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m], \quad (3.24)$$

where $f(R)$ is an arbitrary analytic¹ function of the Ricci curvature scalar R and L_m is a matter Lagrangian¹ that depends on $g_{\mu\nu}$ and matter fields. Varying with respect to $g_{\mu\nu}$ we get the field equation equation

$$G_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} [f(R) - R f'(R)] + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \square f'(R) \right\} + \frac{T_{\mu\nu}^{(m)}}{f'(R)}, \quad (3.25)$$

where $f'(R) = \partial f / \partial R$ and $T_{\mu\nu}^{(m)}$ is the energy-momentum tensor of the matter fields defined by the variational derivative of L_m in terms of $g^{\mu\nu}$. The trace of Eq. (3.25) gives

$$3\square f'(R) + f'(R)R - 2f(R) = T, \quad (3.26)$$

where $T = g^{\mu\nu} T_{\mu\nu}^{(m)}$ and $\square f'(R) = (1/\sqrt{-g}) \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu f'(R))$. Einstein gravity, without the cosmological constant, corresponds to $f(R) = R$ and $f'(R) = 1$, so that the term $\square f'(R)$ in Eq. (3.26) vanishes. In this case we have $R = -T$ and hence the Ricci scalar R is directly determined by the matter (the trace T). In modified gravity the term $\square f'(R)$ does not vanish in Eq. (3.26), which means that there is a propagating scalar degree of freedom, $\phi \equiv f'(R)$. The trace equation (3.26) determines the dynamics of the scalar field ϕ (dubbed ‘‘scalon’’ [88]).

The field equation (3.25) can be written in the following form

$$G_{\mu\nu} = T_{\mu\nu}^{(curv)} + \frac{T_{\mu\nu}^{(m)}}{f'(R)}, \quad (3.27)$$

¹This assumption is not, strictly speaking, necessary and is sometimes relaxed in the literature

¹Note that we do not take into account a direct coupling between the Ricci scalar and matter.

where $T_{\mu\nu}^{(curv)}$ is regarded an effective stress-energy tensor constructed by the extra curvature terms, which we call ‘‘curvature fluid’’ energy-momentum tensor sourcing the effective Einstein equations. This is a geometrical source with vanishing four-divergence. Although this interpretation is questionable in principle because the field equations describe a theory different from GR, and one is forcing upon them the interpretation as effective Einstein equations, this approach is quite useful in practice.

Let discuss $f(R)$ theory in Palatini formalism [93] deriving field equations by treating $g_{\mu\nu}$ and $\Gamma_{\beta\gamma}^\alpha$ as independent variables. For clarity, it is convenient to write action (3.24) in the following form

$$\mathcal{S} = \int d^4x \sqrt{-g} [f(\mathcal{R}) + \mathcal{L}_m], \quad (3.28)$$

where $\mathcal{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor corresponding to the connections $\Gamma_{\beta\gamma}^\alpha$ and $\mathcal{R} \equiv \mathcal{R}(g, \Gamma) = g^{\mu\nu} \mathcal{R}_{\mu\nu}(\Gamma)$. Note that $\mathcal{R}_{\mu\nu}(\Gamma)$ is in general different from the Ricci tensor calculated in terms of metric connections $R_{\mu\nu}(g)$. Varying (3.28) with respect to $g_{\mu\nu}$, we obtain

$$f'(\mathcal{R}) \mathcal{R}_{\mu\nu}(\Gamma) - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = T_{\mu\nu}^{(m)}. \quad (3.29)$$

The trace of Eq. (3.29) is

$$f'(\mathcal{R}) \mathcal{R} - 2f(\mathcal{R}) = T \quad (3.30)$$

where $T = g^{\mu\nu} T_{\mu\nu}^{(m)}$. From (3.30) is clear that the Ricci scalar in Palatini formalism $\mathcal{R}(T)$ is algebraically related to T and it is different from the Ricci scalar of the metric formalism $R(g)$. We have the following condition

$$\mathcal{R} = R + \frac{3}{2} \frac{1}{(f'(\mathcal{R}))^2} (\nabla_\mu f'(\mathcal{R})) (\nabla^\mu f'(\mathcal{R})) + \frac{3}{f'(\mathcal{R})} \square f'(\mathcal{R}). \quad (3.31)$$

where a prime represent a derivative in terms of \mathcal{R} . Let consider the variation with respect to the connection. As stated before, we are considering space-time without torsion so the quantities $\delta\Gamma_{\beta\gamma}^\alpha$ are symmetric with respect to the indices β and γ . Taking into account this symmetry condition the variation of the action (3.28) with respect to the connection leads to the following equations

$$\nabla_\sigma (\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}) = 0, \quad \sigma \neq \nu \neq \mu. \quad (3.32)$$

In conclusion, the Palatini field equations are

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu}(\Gamma) - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = T_{\mu\nu}^{(m)} \quad (3.33)$$

$$\nabla_{\sigma}(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0 \quad (3.34)$$

Note that ∇_{σ} are the covariant derivatives with respect to the connection Γ . It is possible to identify

$$\sqrt{-g}f'(\mathcal{R})g^{\mu\nu} = \sqrt{-h}h^{\mu\nu}. \quad (3.35)$$

Therefore a bi-metric structure naturally arises in this formulation. The functions Γ are identified as the Levi-Civita connection for the metric $h_{\mu\nu}$ and can be expressed as

$$\Gamma_{\mu\nu}^{\lambda} = h^{\lambda\sigma}(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}). \quad (3.36)$$

We can now straightforwardly deduce that when $f(\mathcal{R}) = R$, the theory reduce to GR. This means that until we use Hilbert- Einstein Lagrangian, geodetic structure and metric structure must coincide, or in other words there no difference between Hilbert-Einstein (metric) variational principle and Palatini (metric-affine) variational principle. However, the difference appears for the $f(R)$ models which include non-linear terms in R . While the kinetic term $\square f'(R)$ is present in Eq. (3.26), such a term is absent in Palatini $f(R)$ gravity. This has the important consequence that the oscillatory mode, which appears in the metric formalism, does not exist in the Palatini formalism.

3.3.1 Hybrid gravity

Hybrid metric-Palatini gravity or $f(X)$ gravity is an approach to modified theories of gravity where the action is taken as the standard Hilbert-Einstein (linear in Ricci scalar R) plus a nonlinear term in the Palatini curvature scalar [104, 105].

Similarly as for the metric and Palatini formalism, $f(X)$ gravity can be transformed into scalar-tensor theory [104, 105, 106, 107] as we will show in the following (section 6.5). Using the respective dynamically equivalent scalar-tensor representation, it was shown that the theory can pass the Solar System observational constraints even if the scalar field is very light. This implies the existence of a long-range scalar field, which is able to modify the cosmological and galactic dynamics, but leaves the Solar System

unaffected. The absence of instabilities in perturbations was also verified and explicit models, consistent with local tests, lead to the late-time cosmic speed-up.

Let us consider the action for the hybrid metric-Palatini gravity in the following form [104, 105] :

$$S = \int d^4x \sqrt{-g} [R + f(\mathcal{R})] + S_m, \quad (3.37)$$

where $R = R(g)$ is the metric curvature scalar and $f(\mathcal{R}(\hat{\Gamma}))$ is the function of the Palatini curvature scalar (denoted by \mathcal{R}) which is constructed through an independent connection $\hat{\Gamma}$. The variation of the above action with respect to the metric yields the gravitational field equations

$$G_{\mu\nu} + f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = T_{\mu\nu}, \quad (3.38)$$

where $G_{\mu\nu}$ is the Einstein tensor for the metric $g_{\mu\nu}$ (with Lorentzian signature), while $\mathcal{R}_{\mu\nu}$ is a Ricci tensor constructed by the conformally related metric $h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu}$, where the conformal factor is given by $f'(\mathcal{R}) = df(\mathcal{R})/d\mathcal{R}$. The trace of Eq. (3.38) is the hybrid structural equation, where one can algebraically express the Palatini curvature \mathcal{R} in terms of a quantity X assuming that $f(\mathcal{R})$ has analytic solutions, that is

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = T + R \equiv X. \quad (3.39)$$

The variable X measures the deviation from the GR trace equation $R = -T$, that is GR is fully recovered for $X = 0$ [105].

3.3.2 Gauss-Bonnet Gravity

Recently, the possibility to include the Gauss-Bonnet topological invariant into the Lagrangian have been considered. The action is given by general functions of the Ricci scalar R and the Gauss-Bonnet topological invariant \mathcal{G} , that is $F(R, \mathcal{G})$. These theories are stable and capable of describing the present acceleration of the universe as well as the phantom behavior, the quintessence behavior and the transition from acceleration to deceleration phases. In this sense, they are effective theories working also at infrared scales. In principle, this kind of ETGs can reproduce models able to mimic the Λ -CDM model, as well as other cosmological solutions and suitable perturbation schemes

within different standard scenarios [108, 109, 110, 111, 112]. Also, the Parametrized Post-Newtonian expansion of generalized Gauss-Bonnet models has been worked out [113] as well as spherically symmetric solutions [114].

Let us start by writing the most general action for modified Gauss-Bonnet gravity in 4-dimensions

$$\mathcal{S} = \int d^4x \sqrt{-g} [F(R, \mathcal{G}) + \mathcal{L}_m] . \quad (3.40)$$

This Lagrangian is constructed by considering only the metric tensor and no extra vector or spin degree of freedom is considered. The symbol \mathcal{G} indicates the Gauss-Bonnet invariant

$$\mathcal{G} \equiv R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} , \quad (3.41)$$

that is a combination of the Riemann tensor $R_{\mu\nu\lambda\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. It is important to stress that, in 4-dimension, any linear combination of the Gauss-Bonnet invariant does not contribute to the effective Lagrangian. Furthermore, in 4-dimensions, we have only two non-zero Lovelock scalars [112, 115, 116]. The variation of the action (3.40) with respect to the metric tensor $g_{\mu\nu}$ provides the following gravitational field equations [113, 117]

$$\begin{aligned} G_{\mu\nu} = \frac{1}{F_R} & \left[\nabla_\mu \nabla_\nu F_R - g_{\mu\nu} \square F_R + 2R \nabla_\mu \nabla_\nu F_{\mathcal{G}} - 2g_{\mu\nu} R \square F_{\mathcal{G}} \right. \\ & - 4R_{\mu}^{\lambda} \nabla_\lambda \nabla_\nu F_{\mathcal{G}} - 4R_{\nu}^{\lambda} \nabla_\lambda \nabla_\mu F_{\mathcal{G}} + 4R_{\mu\nu} \square F_{\mathcal{G}} + 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta F_{\mathcal{G}} \\ & \left. + 4R_{\mu\alpha\beta\nu} \nabla^\alpha \nabla^\beta F_{\mathcal{G}} - \frac{1}{2} g_{\mu\nu} (R F_R + \mathcal{G} F_{\mathcal{G}} - F(R, \mathcal{G})) \right] + T_{\mu\nu}^{(m)} , \quad (3.42) \end{aligned}$$

where $F_R = \partial F / \partial R$, $F_{\mathcal{G}} = \partial F / \partial \mathcal{G}$, ∇ is for the covariant derivative and $T_{\mu\nu}^{(m)} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ the energy momentum tensor. Let us note that GR is immediately recovered for $F(R, \mathcal{G}) = R$.

Chapter 4

Conformal Transformations

The frame dependence of the gravity theories is an important object for study. On classical level the theory may manifest different physical properties in different frames that leads to the nontrivial problems related with the correct choice of the field variables. On the other hand, the study of different frames enables one to explore the relation between the different physical theories and thus generate the new exact solutions.

4.1 The meaning of conformal transformations

Conformal transformations are mathematical tools very useful in ETGs as well as in GR [95, 96, 97]. Under a conformal transformation the space-time metric rescaled $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$ and when a scalar field is present in the theory, besides the metric rescaling there is a non-linear redefinition of this field $\phi \rightarrow \tilde{\phi}$. The new dynamical variables $(\tilde{g}_{\mu\nu}, \tilde{\phi})$ are thus obtained. The redefinition of the scalar field is necessary in order to write the kinetic energy density of this field in canonical form. The new set of variables $(\tilde{g}_{\mu\nu}, \tilde{\phi})$ is called the Einstein conformal frame, while $(g_{\mu\nu}, \phi)$ constitute the Jordan frame. When a scalar degree of freedom ϕ is present in the theory, as in scalar-tensor or $f(R)$ -gravity, it generates the transformation to the Einstein frame in the sense that the rescaling is completely determined by a function of ϕ . Infinitely many conformal frames could be introduced in principle, giving rise to as many representations of the theory. Let consider the space-time $(\mathcal{M}, g_{\mu\nu})$, with \mathcal{M} a smooth manifold of dimension $n \geq 2$ and $g_{\mu\nu}$ a Lorentzian or Riemannian metric on \mathcal{M} . The point-dependent

rescaling of the metric tensor

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad (4.1)$$

where the conformal factor $\Omega(x)$ is a nowhere vanishing, regular function, that is called a Weyl or conformal transformation. As a consequence this metric rescaling, the lengths of space-like and time-like intervals and the norms of space-like and time-like vectors are changed, while null vectors and null intervals of the metric $g_{\mu\nu}$ remain null in the rescaled metric $\tilde{g}_{\mu\nu}$. The light cones are left unchanged by the transformation (4.1) and the space-times $(\mathcal{M}, g_{\mu\nu})$ and $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ exhibit the same causal structure; the converse is also true. A vector that is time-like, space-like, or null with respect to the metric $g_{\mu\nu}$ has the same character with respect to $\tilde{g}_{\mu\nu}$, and vice versa.

4.2 The Jordan frame and the Einstein frame

After performing a conformal transformation we use the term ‘conformal frame’ to distinguish the new, rescaled metric from the original. Among the infinite possible conformal frames there are two which are most commonly used and have specific interpretations: the Jordan frame and the Einstein frame. The Jordan frame is the one in which the energy-momentum tensor is covariantly conserved and in which test-particles follow geodesics of the space-time metric. For example, the Brans-Dicke theory [52] is most usually formulated in the Jordan frame. The Einstein frame is the conformal frame in which the field equations of the theory take the form of the Einstein equations (unlike the Jordan frame, the Einstein frame can only be defined for some theories). In the Einstein frame the field equations are second order but the energy-momentum tensor of the matter fields is not always covariantly conserved and test-particles do not necessarily follow geodesics of the space-time metric. Therefore, the Einstein frame is particularly useful for finding vacuum solutions, but less useful for finding solutions with matter fields present.

The transformation properties of different geometrical quantities are useful. They are

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}, \tilde{g} = \Omega^{2n} g, \quad (4.2)$$

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for the inverse metric and the metric determinant,

$$\tilde{\Gamma}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + \Omega^{-1} (\delta_{\beta}^{\alpha} \nabla_{\gamma} \Omega + \delta_{\gamma}^{\alpha} \nabla_{\beta} \Omega - g_{\beta\gamma} \nabla^{\alpha} \Omega), \quad (4.3)$$

for the Christoffel symbols,

$$\begin{aligned} \tilde{R}_{\alpha\beta\gamma}^{\delta} &= R_{\alpha\beta\gamma}^{\delta} + 2\delta_{[\alpha}^{\delta} \nabla_{\beta]} \nabla_{\gamma} (\log \Omega) - 2g^{\delta\sigma} g_{\gamma[\alpha} \nabla_{\beta]} \nabla_{\sigma} (\log \Omega) + \\ &+ 2\nabla_{[\alpha} (\log \Omega) \delta_{\beta]}^{\delta} \nabla_{\gamma} (\log \Omega) - 2\nabla_{[\alpha} (\log \Omega) g_{\beta]\gamma} g^{\delta\sigma} \nabla_{\sigma} (\log \Omega) + \\ &- 2g_{\gamma[\alpha} \delta_{\beta]}^{\delta} g^{\sigma\rho} \nabla_{\sigma} (\log \Omega) \nabla_{\rho} (\log \Omega), \end{aligned} \quad (4.4)$$

for the Riemann tensor,

$$\begin{aligned} \tilde{R}_{\alpha\beta} &= R_{\alpha\beta} - (n-2) \nabla_{\alpha} \nabla_{\beta} (\log \Omega) - g_{\alpha\beta} g^{\rho\sigma} \nabla_{\sigma} \nabla_{\rho} (\log \Omega) + \\ &+ (n-2) \nabla_{\alpha} (\log \Omega) \nabla_{\beta} (\log \Omega) + \\ &- (n-2) g_{\alpha\beta} g^{\rho\sigma} \nabla_{\sigma} (\log \Omega) \nabla_{\rho} (\log \Omega), \end{aligned} \quad (4.5)$$

for the Ricci tensor, and

$$\begin{aligned} \tilde{R} &= \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} = \\ &= \Omega^{-2} \left[R - 2(n-1) \square (\log \Omega) - (n-1)(n-2) \frac{g^{\alpha\beta} \nabla_{\alpha} \Omega \nabla_{\beta} \Omega}{\Omega^2} \right], \end{aligned} \quad (4.6)$$

for the Ricci scalar. In the case of $n = 4$ space-time dimensions, the transformation property of the Ricci scalar can be written as

$$\begin{aligned} \tilde{R} &= \Omega^{-2} \left[R - \frac{6 \square \Omega}{\Omega} \right] = \\ &= \Omega^{-2} \left[R - \frac{12 \square(\sqrt{\Omega})}{\sqrt{\Omega}} + \frac{3g^{\alpha\beta} \nabla_{\alpha} \Omega \nabla_{\beta} \Omega}{\Omega^2} \right]. \end{aligned} \quad (4.7)$$

The Weyl tensor $C_{\alpha\beta\gamma}^{\delta}$ with the last index contravariant, is conformally invariant,

$$\tilde{C}_{\alpha\beta\gamma}^{\delta} = C_{\alpha\beta\gamma}^{\delta}, \quad (4.8)$$

but the same tensor with indices raised or lowered with respect to $C_{\alpha\beta\gamma}^{\delta}$ is not. This property explains the name *conformal tensor* used for $C_{\alpha\beta\gamma}^{\delta}$. If the original metric $g_{\alpha\beta}$ is Ricci-flat (*i.e.*, $R_{\alpha\beta} = 0$), the conformally transformed metric $\tilde{g}_{\alpha\beta}$ is not (see equation (4.5)). In the conformally transformed world the conformal factor Ω plays the role of an effective form of matter and this fact has consequences for the physical interpretation of the theory. A vacuum metric in the Jordan frame is not such in the Einstein

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frame, and the interpretation of what is matter and what is gravity becomes frame-dependent. Nevertheless, if the Weyl tensor vanishes in one frame, it also vanishes in the conformally related frame. Conformally flat metrics are mapped into conformally flat metrics, a property used in cosmology when mapping Friedman- Robertson-Walker (FRW) universes (which are conformally flat) into each other. Since, in general, tensorial quantities are not invariant under conformal transformations, neither are the tensorial equations describing geometry and physics. An equation involving a tensor field ψ is said to be conformally invariant if there exists a number w (the conformal weight of ψ) such that, if ψ is a solution of a tensor equation with the metric $g_{\mu\nu}$ and the associated geometrical quantities, then $\tilde{\psi} = \Omega^w \psi$ is a solution of the corresponding equation with the metric $\tilde{g}_{\mu\nu}$ and the associated geometry. Besides the geometric quantities, it is necessary to consider the behaviour of common forms of matter under conformal transformations. There is no need to say that most forms of matter or fields are not conformally invariant: invariance under conformal transformations is a very special property. Generally, the covariant conservation equation for a (symmetric) stress-energy tensor $T_{\alpha\beta}^{(m)}$ representing ordinary matter,

$$\nabla^\beta T_{\alpha\beta}^{(m)} = 0, \quad (4.9)$$

is not conformally invariant. The conformally transformed $\tilde{T}_{\alpha\beta}^{(m)}$ satisfies the equation

$$\tilde{\nabla}^\beta \tilde{T}_{\alpha\beta}^{(m)} = -\tilde{T}^{(m)} \tilde{\nabla}_\alpha (\log \Omega). \quad (4.10)$$

Obviously, the conservation equation (4.9) is conformally invariant only for a matter component that has vanishing trace $T^{(m)}$ of the energy-momentum tensor. This feature is associated with light-like behaviour; examples are the electromagnetic field and a radiative fluid with equation of state $P^{(m)} = \rho^{(m)}/3$. Unless $T^{(m)} = 0$, (4.10) describes an exchange of energy and momentum between matter and the scalar field Ω , reflecting the fact that matter and the geometric factor Ω are directly coupled in the Einstein frame description. Since the geodesic equation ruling the motion of test particles in GR can be derived from the conservation equation (4.9) (geodesic hypothesis), it follows that time-like geodesics of the original metric $g_{\alpha\beta}$ are not geodesics of the rescaled metric $\tilde{g}_{\alpha\beta}$ and vice versa. Particles in free fall in the world $(\mathcal{M}, g_{\alpha\beta})$ are subject to

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a force proportional to the gradient $\tilde{\nabla}^\alpha \Omega$ in the rescaled world $(\tilde{\mathcal{M}}, \tilde{g}_{\alpha\beta})$ (this is often identified as a fifth force acting on all massive particles and, then, it can be said that no massive test particles exist in the Einstein frame). The stress-energy tensor definition in terms of the matter action $\mathcal{S}^{(m)} = \int d^4x \sqrt{-g} \mathcal{L}^{(m)}$, yields

$$\tilde{T}_{\alpha\beta}^{(m)} = \Omega^{-2} T_{\alpha\beta}^{(m)}, \quad (4.11)$$

$$\tilde{T}^{(m)\alpha}{}_{\beta} = \Omega^{-4} T^{(m)\alpha}{}_{\beta}, \quad (4.12)$$

$$\tilde{T}^{(m)\alpha\beta} = \Omega^{-6} T^{(m)\alpha\beta}, \quad (4.13)$$

$$\tilde{T}^{(m)} = \Omega^{-4} T^{(m)}. \quad (4.14)$$

From the last equation it is clear that the trace vanishes in the Einstein frame if and only if it vanishes in the Jordan frame.

4.3 Conformal Transformations applied to extended gravity

We will now use these transformations to show how the scalar-tensor and some higher-order theories can be transformed from the Jordan frames to the Einstein frame.

4.3.1 Scalar-tensor gravity

As showed in section 3.2, in four dimensions, the action involving gravity coupled in a non-standard way with a scalar field is

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[F(\phi) R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right], \quad (4.15)$$

where R is the Ricci scalar, $V(\phi)$ and $F(\phi)$ are generic functions describing, respectively, the potential for the field ϕ and the coupling of ϕ with gravity.

The conformal transformation on the metric $g_{\mu\nu}$ is

$$\bar{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}, \quad (4.16)$$

where $e^{2\omega}$ is the conformal factor. Under this transformation, the Lagrangian density

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in (4.15) becomes

$$\begin{aligned} \sqrt{-g} \left(FR + \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V \right) &= \sqrt{-\bar{g}} e^{-2\omega} \left(F \bar{R} - 6F \square_{\bar{g}} \omega + \right. \\ &\left. - 6F \omega_{;\alpha} \omega^{;\alpha} + \frac{1}{2} \bar{g}^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - e^{-2\omega} V \right), \end{aligned} \quad (4.17)$$

where \bar{R} and $\square_{\bar{g}}$ are the Ricci scalar and the d'Alembert operator relative to the metric \bar{g} respectively. If we require that the theory in the metric $\bar{g}_{\mu\nu}$ appear as a standard Einstein theory, then the conformal factor has to be related to F [118], namely

$$e^{2\omega} = -2F. \quad (4.18)$$

F must be negative to restore physical coupling. Using this condition, the Lagrangian density (4.17) becomes

$$\sqrt{-g} \left(FR + \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V \right) = \sqrt{-\bar{g}} \left(-\frac{1}{2} \bar{R} + 3 \square_{\bar{g}} \omega + \frac{3F_{\phi}^2 - F}{4F^2} \phi_{;\alpha} \phi^{;\alpha} - \frac{V}{4F^2} \right). \quad (4.19)$$

With the introduction of a new scalar field $\bar{\phi}$ and of the potential \bar{V} , respectively, defined by

$$\bar{\phi}_{;\alpha} = \sqrt{\frac{3F_{\phi}^2 - F}{2F^2}} \phi_{;\alpha}, \quad \bar{V}(\bar{\phi}(\phi)) = \frac{V(\phi)}{4F^2(\phi)}, \quad (4.20)$$

we obtain

$$\sqrt{-g} \left(FR + \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V \right) = \sqrt{-\bar{g}} \left(-\frac{1}{2} \bar{R} + \frac{1}{2} \bar{\phi}_{;\alpha} \bar{\phi}^{;\alpha} - \bar{V} \right), \quad (4.21)$$

which is the usual Hilbert-Einstein Lagrangian density plus the standard Lagrangian density relative to the scalar field $\bar{\phi}$. Accordingly, every non-minimally coupled scalar-tensor theory, in absence of ordinary matter, *i.e.* perfect fluid, is conformally equivalent to an Einstein theory, if the conformal transformation and the potential are suitably defined by (4.18) and (4.20). The converse is also true: for a given $F(\phi)$, such that $3F_{\phi}^2 - F > 0$, it is possible to transform a standard Einstein theory into a non-minimally coupled scalar-tensor theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get the solutions for the scalar-tensor theories, assigned by the coupling $F(\phi)$, via the conformal transformation (4.18) with

the constraints given by Eqs.(4.20). This is precisely what we want to discuss in the cosmological context in cases where the potentials as well as the couplings are relevant from the point of view of the fundamental physics. In the situation under consideration, the Einstein frame is the framework of the Einstein theory with the minimal coupling and the Jordan frame is that of the non-minimally coupled theory [119].

4.3.2 $f(R)$ gravity

In general, fourth-order theories of gravity are given by the action

$$\mathcal{S} = \int d^4x \sqrt{-g} f(R), \quad (4.22)$$

where $f(R)$ is an analytic function of the Ricci curvature scalar R . This is the simplest case of fourth-order gravity but we can construct such kind of theories also using the invariants $R_{\mu\nu} \circ R_{\gamma\mu\nu}^\alpha$. Nevertheless, for cosmological considerations, theories like (4.22) are sufficiently general [120]. Hilbert–Einstein action is recovered for $f(R) = R$. By performing the variation with respect to $g_{\alpha\beta}$, we obtain the following field equations

$$f'(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} = f'(R)^{;\mu\nu} (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}), \quad (4.23)$$

which are fourth-order equations thanks to the term $f'(R)^{;\mu\nu}$. The prime indicates the derivative with respect to R . Putting in evidence the Einstein tensor, the field equations become

$$G_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2}g_{\alpha\beta} [f(R) - Rf'(R)] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\square f'(R) \right\}, \quad (4.24)$$

where the gravitational contributions in the stress-energy tensor can be interpreted, via conformal transformations, as scalar field contributions and then as “matter” terms. Performing the conformal transformation (4.16), we obtain

$$\begin{aligned} \bar{G}_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2}g_{\alpha\beta} [f(R) - Rf'(R)] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\square f'(R) \right\} + \\ + 2 \left(\omega_{;\alpha\beta} + g_{\alpha\beta}\square\omega - \omega_{;\alpha\beta} + \frac{1}{2}g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma} \right). \end{aligned} \quad (4.25)$$

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We can choose the conformal factor

$$\omega = \frac{1}{2} \ln |f'(R)|, \quad (4.26)$$

which has to be substituted into ((4.25)). Rescaling ω in such a way that

$$k\phi = \omega, \quad (4.27)$$

and $k = \sqrt{1/6}$, we obtain the Lagrangian equivalence

$$\sqrt{-g}f(R) = \sqrt{-\bar{g}} \left(-\frac{1}{2}\bar{R} + \frac{1}{2}\bar{\phi}_{;\alpha}\bar{\phi}^{;\alpha} - \bar{V} \right), \quad (4.28)$$

and the Einstein equations in standard form

$$\bar{G}_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\phi_{;\gamma}\phi^{;\gamma} + \bar{g}_{\alpha\beta}V(\phi), \quad (4.29)$$

with the potential

$$V(\phi) = \frac{e^{-4k\phi}}{2} [f(\phi) - \mathcal{F}(e^{2k\phi})e^{2k\phi}] = \frac{1}{2} \frac{f(R) - Rf'(R)}{f'(R)^2}. \quad (4.30)$$

\mathcal{F} is the inverse function of $f'(\phi)$ and $f(\phi) = \int \exp(2k\phi)d\mathcal{F}$. Nevertheless, the problem is completely solved if $f'(\phi)$ can be analytically inverted. To sum up, a fourth-order theory is conformally equivalent to the standard second-order Einstein theory plus a scalar field (see also[121, 122]).

4.3.3 Higher-order theories

Considering a theory higher than fourth order, we have Lagrangian densities of the form [123, 124, 125],

$$\mathcal{L} = \mathcal{L}(R, \square R, \dots \square^k R). \quad (4.31)$$

Every \square operator introduces two further terms of derivation into the field equations.

As an example, a theory like

$$\mathcal{L} = R\square R, \quad (4.32)$$

is a sixth-order theory. The above approach can be pursued considering a conformal factor of the form

$$\omega = \frac{1}{2} \ln \left| \frac{\partial \mathcal{L}}{\partial R} + \square \frac{\partial \mathcal{L}}{\partial \square R} \right|. \quad (4.33)$$

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In general, increasing two orders of derivation in the field equations (*i.e.* every term $\square R$), corresponds to add a scalar field in the conformally transformed frame [123] and then, a sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields. Thus, a $2n$ -order theory can be, in principle, reduced to an Einstein theory + $(n - 1)$ -scalar fields. However, these considerations can be directly generalized to higher-order-scalar-tensor theories in any number of dimensions as shown in [49].

It is worth noting that conformal transformations works at three levels: *i*) on the Lagrangian of the given ETG-theory; *ii*) on the field equations; *iii*) on the solutions. Conformal transformations correlate these levels but, at this point of the discussion, there is no absolute criterion capable of stating what is the “physical” framework since all the frames are equivalent from a mathematical point of view (see also [97] for a detailed discussion).

Chapter 5

Extended Gravity Cosmology

Observations of physical processes occurring in an expanding universe can be used to constrain the underlying gravitational theory in a number of ways. These observations can range from galaxy surveys in the nearby Universe to the results of processes occurring in the very early Universe. We will concentrate in this chapter on a procedure that could give us the possibility to constrain ETGs with the help of processes like primordial nucleosynthesis and microwave background formation. Both of these processes occur early in the Universe's history and the results of both are well observed by astronomers and astrophysicists[5].

5.1 How to select reliable models

As pointed out before, several models of $f(R)$ gravity have been proposed in order to address the dark side problem in cosmology. However, these models should be constrained also at ultraviolet scales in order to achieve some correct fundamental interpretation. In [5], we analyze this possibility comparing quantum vacuum states in given $f(R)$ cosmological backgrounds. Specifically, we compare the Bogolubov transformations associated to different vacuum states for some $f(R)$ models. The procedure consists in fixing the $f(R)$ free parameters by requiring that the Bogolubov coefficients can be correspondingly minimized to be in agreement with both high redshift observations and quantum field theory predictions. In such a way, the particle production is related to the value of the Hubble parameter and then to the given $f(R)$ model.

Reliable classes of $f(R)$ models should be constrained at fundamental level [135]. Specifically, bounds on $f(R)$ models could be derived by taking into account different vacuum states via Bogolubov transformations [136, 137, 138]. Indeed, in quantum field theories, the Bogolubov coefficients drive the different choices of vacuum states. So, requiring that different classes of $f(R)$ functions change vacuum states according to Bogolubov transformations is a basic requirement to guarantee the $f(R)$ viability at fundamental level. This procedure somehow fixes the $f(R)$ free parameters and so it is of some help in reconstructing the $f(R)$ form by means of basic requirements of quantum field theory [77, 139]. For this purpose, one has to confront with the problem of quantizing the space-time in a curved background and then to provide relations between quantization and $f(R)$ gravity at least at semiclassical level. Therefore, the Bogolubov coefficients allow to pass from a vacuum state to another through a semiclassical procedure where the rate of particle production is minimized. If the rate is minimized, one can fix, indeed, the free parameters of a given $f(R)$ model. We assume that the rate is minimized to be consistent with cosmological high redshift observations, from one side, and with quantum field theory predictions, from the other side. Moreover, one can relate the rate of particle production with the Hubble parameter and thus with the redshift z . In this way, it is possible to frame the Bogolubov coefficients in terms of observable cosmological quantities as H_0 , the today observed Hubble constant, or $R_0 \sim \rho_0$, the value of the today curvature or density.

5.1.1 Bogolubov transformations and vacuum states

Let us start considering the derivation of the Bogolubov coefficients as semiclassical quantities in the context of quantum field theory. A strategy to derive the particle production rate in curved space is to fix a background with a constant curvature *i.e.* $R = R_0$. This situation is usually named as the *de-Sitter phase* [38, 140]. From the field Eqs. for $f(R)$ gravity in presence of standard perfect fluid matter

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] f'(R) = T_{\mu\nu}, \quad (5.1)$$

it is easy to derive an effective cosmological constant term $\Lambda_{eff} = \frac{f(R_0)}{2f'(R_0)} = \frac{R_0}{4}$, which, in principle, depending on the value of R_0 , can give rise to an accelerating

expansion phase [141]. The choice of R_0 allows to simplify the calculations thanks to the symmetries of de Sitter space-time. In order to constrain the form of $f(R)$ function, a possible method is to fix the range of free parameters by the transition to different vacuum states. Such a procedure relies on the definition of the Bogolubov coefficients. To define them, let us consider the quantization on a curved background.

Since we are considering extended theories of gravity, a model where a scalar field ϕ is non-minimally coupled to geometry, *i.e.* $\propto R\phi$ can be assumed. The Klein-Gordon equation is

$$[\square - m^2 + \xi R(x)] \phi = 0, \quad (5.2)$$

where m is the effective mass of the field, ξ is the coupling¹.

The general solution can be expressed as a complete set of mode-solutions for the field ϕ [38]

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)], \quad (5.3)$$

where it is possible to adopt a specific set of mode solutions $u_i(x)$, although it is always possible to rewrite $\phi(x)$ for a different set \bar{u}_j as

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)]. \quad (5.4)$$

In other words, one can pass through different decompositions of ϕ , defining a corresponding form of the vacuum solution that is, in general, $\bar{a}_j|0\rangle \neq 0$, in a curved space background. Indeed, expressing the new modes, \bar{u}_j in terms of the old ones u_i , we have

$$\begin{aligned} u_i &= \sum_j (\alpha_{ji} \bar{u}_j - \beta_{ji} \bar{u}_j^*), \\ \bar{u}_j &= \sum_i (\alpha_{ji}^* u_i + \beta_{ji}^* u_i^*), \end{aligned} \quad (5.5)$$

where α_{ji} and β_{ji} are defined as $\alpha_{ij} = (\bar{u}_i, u_j)$, $\beta_{ij} = -(\bar{u}_i, u_j^*)$ and satisfy the relations

$$\begin{aligned} \sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) &= \delta_{ij}, \\ \sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) &= 0. \end{aligned} \quad (5.6)$$

¹It is worth mentioning that any $f(R)$ theory of gravity can be recast as a non-minimally coupled theory through the identification $\phi \rightarrow f'(R)$ and the coupling $f'(R)^{-1}$.

In particular, if we consider the vacuum $|0\rangle$, then $a_i|0\rangle = 0, \forall i$ but, in general, it is $a_i|\bar{0}\rangle \neq 0$. Therefore, from the definition of the particle number given by $N_i \equiv a_i^\dagger a_i$, it results

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2. \quad (5.7)$$

Accordingly, the physical meaning of such coefficients is associated to the rate of particle production. In fact, generic coefficients β_{ji} are associated to the particle number count for given set of modes. Specifically, α_{ji} and β_{ji} are referred to as the *Bogolubov coefficients* which identify the *Bogolubov transformations* and allow to pass from a vacuum state to another one. Since the form of the $f(R)$ function is not known *a priori*, by adopting the above semiclassical procedure and evaluating the different vacuum states for some classes of $f(R)$, we can minimize the rate of particle production. In this way, we can constrain the free parameters of a given $f(R)$ model and in particular, we will see that Bogolubov coefficients strictly depend on the form of $f(R)$.

5.2 Particle production in non-minimally coupled theories of gravity

Non-minimally coupled scalar-tensor theories are the generic prototypes of extended theories of gravity. As we pointed out before, $f(R)$ theories and any extended theory can be recast as GR with some non-minimal couplings and further contribution in the stress-energy momentum tensor (see [142] for the general procedure). The Bogolubov transformations can be discussed in the context of homogeneous and isotropic cosmologies, to define the rate of particle production and then to constrain the functional form of $f(R)$ gravity.

The particle production rate is a mostly universal feature, in the sense that it has not to depend on the particular gravitational background. Indeed, assuming a different gravitational theory² we expect that it is the same and can be consistently used to fix the parameters of the theory itself. This property is extremely relevant since it allows to consider the Bogolubov transformations for different gravitational backgrounds. We limit to the case of $f(R)$ gravity.

²For example, $f(R, G)$, $f(T)$, and so forth.

As we already said, the simplest choice to construct the Bogolubov transformations is assuming a de Sitter phase with a constant curvature R_0 . A spatially flat FRW conformal metric is [143]

$$ds^2 = \frac{1}{H^2 \eta^2} (d\eta^2 - dx_1^2 - dx_2^2 - dx_3^2), \quad (5.8)$$

where we adopted the conformal time $\eta = -\frac{1}{Ha(t)}$, which varies in the interval $-\infty < \eta < 0$. Introducing a scalar field $\phi(\eta, \mathbf{x})$ depending on η and $\mathbf{x} \equiv (x_1, x_2, x_3)$, the corresponding Klein-Gordon equation Eq. (5.2), in terms of space-time modes is

$$(\square - m^2 + \xi R)\phi(\eta, \mathbf{x}) = 0. \quad (5.9)$$

This equation is formally equivalent to Eq. (5.2), although the functional dependence on the variables η and \mathbf{x} is explicit. Without choosing ξ *a priori*, the corresponding class of solutions is

$$\phi(\eta, \mathbf{x}) = \phi_k(\eta) e^{i(\mathbf{k} \cdot \mathbf{x})}, \quad (5.10)$$

where the wave vector is decomposed as $\mathbf{k} \equiv (k_1, k_2, k_3)$. By scaling $\phi(\eta, \mathbf{x}) = \frac{\tilde{\phi}(\eta, \mathbf{x})}{a}$, the Klein-Gordon differential equation for the FRW metric (5.8) becomes

$$\tilde{\phi}_k''(\eta) + \omega^2(\eta, k, \xi) \tilde{\phi}_k(\eta) = 0, \quad (5.11)$$

where the prime stands for the derivative with respect to the conformal time η .

The above equation is analogue to the harmonic oscillator with ω depending on the conformal time η . The ω parameter takes the form

$$\omega(\eta) = \sqrt{k^2 + a^2 [m^2 + 2f(\xi)H^2]}, \quad (5.12)$$

where $f(\xi) \equiv 6\xi - 1$. For our purposes, the function $f(\xi)$ can be conventionally positive-definite assuming $\xi > \frac{1}{6}$. Furthermore, it is convenient to define an effective mass M_{eff} as

$$\frac{M_{eff}^2}{H^2} \equiv \frac{m^2}{H^2} + 2f(\xi), \quad (5.13)$$

where m is the state of mass of the scalar field and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. Since $\xi > \frac{1}{6}$, M_{eff} is always positive. The frequency dependence is

$$\omega(\eta) = \sqrt{k^2 + \frac{M_{eff}^2}{H^2 \eta^2}}. \quad (5.14)$$

which is always positive for $\xi \geq \frac{1}{6}$. The functions $\omega(k)$ and $\omega(\eta)$ are plotted in Fig. (5.1) for some cases of interest.

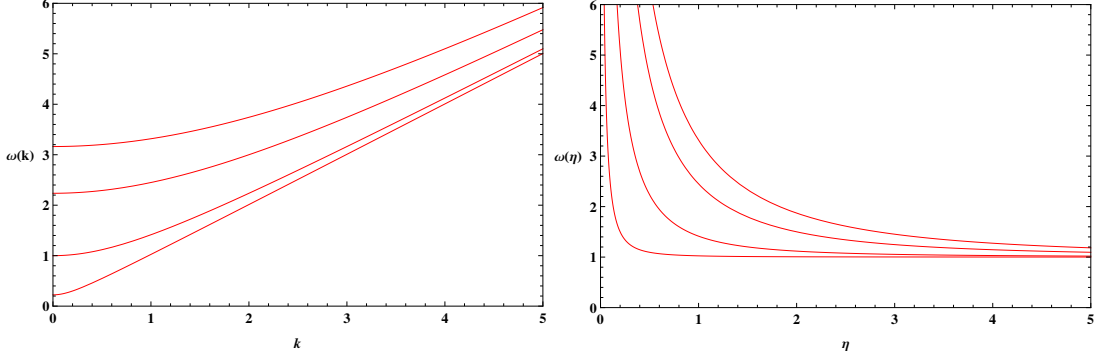


Figure 5.1: Plots of the ω profiles in function of k and η respectively on the left and right. The left plots show the $\omega(k)$ behaviors for four indicative ratios $\frac{M_{eff}^2}{H^2\eta^2} = 0.05, 1, 5, 10$. The figures on the right, on contrary, show the $\omega(\eta)$ behaviors in function of η , with indicative ratios $\frac{M_{eff}^2}{H^2} = 0.05, 1, 5, 10$ and conventionally with $k = 1$.

The general solution $\tilde{\phi}_k(\eta)$ for (5.11) reads

$$\tilde{\phi}_k(\eta) = \sqrt{\eta} \left(A_k H_{k,\nu}^{(1)}(\eta) + B_k H_{k,\nu}^{(2)}(\eta) \right), \quad (5.15)$$

where $H_{k,\nu}^{(1)}$ and $H_{k,\nu}^{(2)}$ are Hankel's functions of first and second type respectively, with the position [144, 145]

$$\nu \equiv \sqrt{\frac{1}{4} - \frac{M_{eff}^2}{H^2}}. \quad (5.16)$$

The corresponding asymptotic behavior is relevant to infer the particle production rate.

In the case $\eta \rightarrow 0^-$, we have

$$\tilde{\phi}_k(\eta) \rightarrow \frac{\sqrt{|\eta|}}{\pi\nu} \left\{ \sin(\pi\nu)\Gamma(1-\nu) \left(\frac{k\eta}{2}\right)^\nu - \Gamma(1+\nu) \left(\frac{k\eta}{2}\right)^{-\nu} \right\}, \quad (5.17)$$

and the square modulus of β_k is [146, 147]

$$|\beta_k(\eta)|^2 = \frac{\omega_k}{2} \left| \phi_k(\eta) - \frac{i}{\omega_k(\eta)} \dot{\phi}_k(\eta) \right|^2. \quad (5.18)$$

We are interested in the case $\frac{M_{eff}}{H} \gg 1$, which corresponds either to the situation where the effective mass dominates over the Hubble rate or H is small at late times of

the universe evolution. We find out

$$|\beta_k(\eta)|^2 \sim \frac{H^3}{32\pi m^3} \left| \Gamma\left(1 - i\frac{m}{H}\right) \right|^2 \exp(\pi m), \quad (5.19)$$

where Γ is the Euler function.

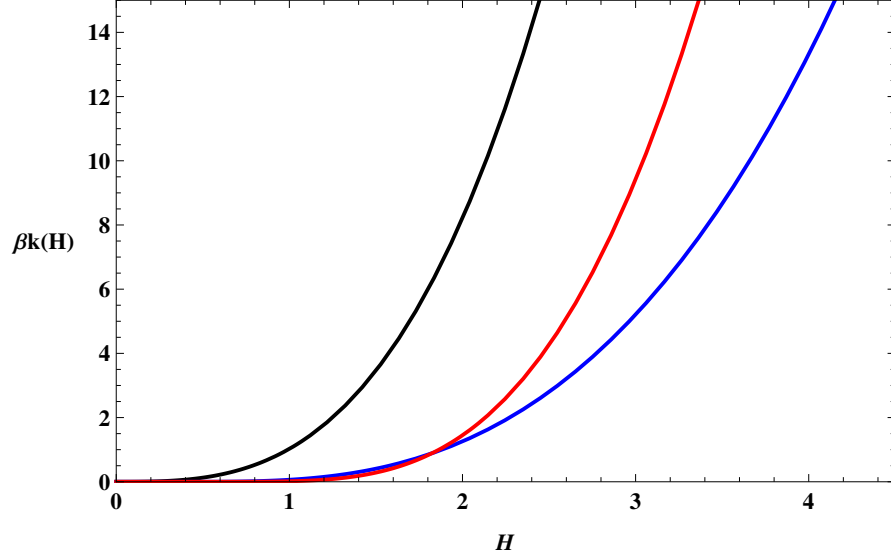


Figure 5.2: Plot of the Bogolubov coefficient $|\beta_k|^2$ varying in the range $H = 0 \dots 5$, with $m = 0.01; 1; 2$ respectively for the black, blue and red lines.

In the case $\frac{M_{eff}}{H} \ll 1$, we obtain that the Bogolubov coefficients are negligibly small [148], that is

$$|\beta_k(\eta)|^2 \ll 1. \quad (5.20)$$

Therefore, by assuming that $\frac{M_{eff}}{H} \ll 1$, we can distinguish two different cases. The first is $H \gg m$, with $m \rightarrow 0$. The second is un-physical, since it provides a diverging Bogolubov coefficient β_k [159]. Hence, by assuming the validity of the above results, we are able to relate the $f(R)$ gravity to Bogolubov coefficients constraining the free parameters of the models. For this purpose, we assume to pass through different vacuum states. Clearly, different $f(R)$ gravity models means different couplings ξ .

Minimizing the rate of particle production in $f(R)$ gravity

Let us consider the physical case $\frac{m}{H} \ll 1$ in the context of $f(R)$ gravity. We investigate such a case in the de Sitter phase with $R = R_0$ and minimize the Bogolobuv coefficients obtaining, correspondingly, the minimum of particle production rate.

Essentially, such a quantity has to be minimized for two reasons. The first concerns cosmological observations at high energy regimes. As an example, taking into account the cosmic microwave background, cosmological measurements could not be compatible with particle production rate, so the condition $\frac{m}{H} \ll 1$ is required in order to guarantee that any theory of gravity works at high z . Furthermore, assuming to pass from different vacuum states, it is important to test if cosmological models describe the vacuum according to observations. In this context, minimizing Bogolobuv coefficients is a powerful tool in order to discriminate among different models (see also [160]).

In particular, our aim is to infer physical bounds on the free parameters of some classes of $f(R)$ models by Bogolobuv transformations. We start by writing H from the cosmological equations derived in the $f(R)$ framework. In a FRW universe, in metric and Palatini formalism respectively, we obtain

$$H^2 = \frac{1}{3} \left[\rho_{curv} + \frac{\rho_m}{f'(R)} \right], \quad (5.21a)$$

$$H^2 = \frac{1}{6f'(R)} \left[\frac{2\rho + Rf'(R) - f(R)}{G(R)} \right]. \quad (5.21b)$$

where ρ_m is the standard matter density. The effective curvature density term is [149]

$$\rho_{curv} = \frac{1}{2} \left[\frac{f(R)}{f'(R)} - R \right] - 3H\dot{R} \left[\frac{f''(R)}{f'(R)} \right], \quad (5.22)$$

and the function $G(R)$ is given by

$$G(R) = \left[1 - \frac{3f''(R)(Rf'(R) - 2f(R))}{2f'(R)(Rf''(R) - f'(R))} \right]^2. \quad (5.23)$$

As we pointed out before, $f(R)$ gravity can be recast in term of a scalar-tensor theory as soon as the identifications $\phi \rightarrow f'(R)$, for the field, and $G_{eff} \rightarrow f'(R)^{-1}$, for the coupling, are adopted.

Moreover, the particle production rate can be achieved at first order by a Taylor expansion of the Bogolubov coefficient β_k ,

$$|\beta_k|^2 = e^{\pi m} \left[\frac{H^3}{32\pi m^3} + \gamma^2 \frac{H}{32\pi m} \right], \quad (5.24)$$

where we adopted the $\Gamma(1 - ix)$ function and its Taylor series in case $x \ll 1$, obtaining $\Gamma \sim 1 + i\gamma x$. The constant γ is the Euler constant and reads $\gamma \sim 0.577$.

As a first step, one can compare such Bogolubov coefficients with the Hubble rate expressed as function of the redshift z . In this way, the form of β_k becomes a function of the redshift as well. This has been reported in the left plot of Fig.5.3, whereas in the right plot we draw the variation of β_k as the redshift increases, *i.e.* its first derivative with respect to the redshift z . The reported three models are: (i) the Λ CDM model [151]; (ii) a cosmographic expansion where the deceleration parameter variation, namely the jerk parameter, is $j(z) \geq 1$ [152]; (iii) the Chaplygin gas where DE and DM are considered under the standard of a single fluid [153, 154, 155]. These models can be considered as relevant paradigms for describing DE [150].

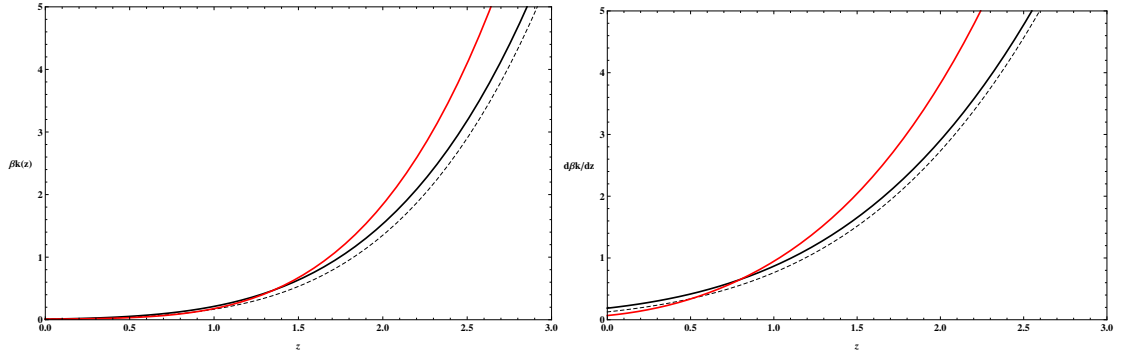


Figure 5.3: Bogolubov coefficients (left figure) evaluated for the Λ CDM model (black line), the cosmographic expansion (dashed line) and the Chaplygin gas (red line), with matter density $\rho = 0.27$, and the Chaplyging gas coefficients: $A = 0.9$, $\beta = 0.8$, (see [153, 154, 155]) with the normalized Hubble rate $H_0 = 0.68$. Derivatives of $\beta_k(z)$ (right figure) have been reported for the same cases, *i.e.* Λ CDM, cosmographic and Chaplygin gas.

In the case of constant curvature $R = R_0$ related to a de-Sitter phase, we obtain

$$|\beta_k|^2 = e^{\pi m} \left[\frac{1}{3^{\frac{3}{2}} 32\pi m^3} \left(\frac{\rho_0}{f'_0} - \Lambda_{eff} \right)^{3/2} + \frac{\gamma^2}{32\sqrt{3}\pi m} \left(\frac{\rho_0}{f'_0} - \Lambda_{eff} \right)^{1/2} \right] \quad (5.25)$$

and

$$|\beta_k|^2 = e^{\pi m} \left[\frac{1}{32\pi m^3} \left(\frac{(2\rho_0 + R_0 - 2\Lambda_{eff})(R_0 - \frac{f'_0}{f''_0})}{R_0 - \frac{f'_0}{f''_0} - \frac{3}{2}R_0 + 6\Lambda_{eff}} \right)^{3/2} \right. \quad (5.26)$$

$$\left. + \frac{\gamma^2}{32\pi m} \left(\frac{(2\rho_0 + R_0 - 2\Lambda_{eff})(R_0 - \frac{f'_0}{f''_0})}{R_0 - \frac{f'_0}{f''_0} - \frac{3}{2}R_0 + 6\Lambda_{eff}} \right)^{1/2} \right], \quad (5.27)$$

respectively for metric and Palatini formalism. From now on $f_0 \equiv f(R = R_0)$, $f'_0 \equiv f'(R = R_0)$, and $f''_0 \equiv f''(R = R_0)$ and ρ_0 is the value of standard matter-energy density for R_0 . Since the form of $f(R)$ is not known *a priori*, we need to consider some cases of particular interest [140, 42] as

$$f_1(R) = R^{1+\delta} + \Lambda, \quad (5.28a)$$

$$f_2(R) = R + \epsilon R^2 \dots, \quad (5.28b)$$

$$f_3(R) = R + R^n + \sigma R^{-m}, \quad (5.28c)$$

$$f_4(R) = R - \frac{\alpha(R)^n}{1 + \beta(R)^n}. \quad (5.28d)$$

Therefore, we need to fix the coefficients $\delta, \epsilon, \sigma, n, m, \alpha$ and β via Eq.(5.24). In order to do this, we require the rate of particle production to be negligible or essentially as small as possible [156]. Thus, the strategy to follow is to assume that the free parameters of Eq.(5.28) minimize the Bogolubov coefficients. Using this procedure, we obtain the results of Tabs. 5.1 and 5.2.

$f_n(R)$	$n.par.$	<i>minimiz.</i>
$f_{1M}(R)$	1	$1 + \delta \leq \frac{4\rho_0}{R_0^{1+\delta}}$
$f_{2M}(R)$	1	$\epsilon \leq \frac{2\rho_0}{R_0^2} - \frac{1}{2R_0}$
$f_{3M}(R)$	3	$nR_0^n \left[1 - \frac{m}{n} \sigma R_0^{-(m+n)} \right] \leq 4\rho_0 - R_0$
$f_{4M}(R)$	3	$(R_0 - nR_0^n \alpha + 2R_0^{1+n} \beta + R_0^{1+2n} \beta^2) (1 + R_0^n \beta^2)^{-1} \leq 4\rho_0 - R_0$

Table 5.1: Table of minimizing conditions for the free parameters of $f(R)$ models from Eq. (5.28) in the metric formalism. Here the subscript M stands for *metric*. The above equalities correspond to the case of vanishing Bogolubov coefficients, whereas the inequalities to more general cases where the Bogolubov coefficients are not zero.

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$f_n(R)$	$n.par.$	$minimiz.$
$f_{1P}(R)$	1	$\forall \delta < 0, \bigvee \delta \geq 1, \delta \neq -1$
$f_{2P}(R)$	1	$\epsilon < 0$
$f_{3P}(R)$	3	$\left(1 + nR_0^{n-1} - m\sigma R_0^{-(m+1)}\right) \left(n(n-1)R_0^{n-2} + m(m+1)\sigma R_0^{-(m+2)}\right)^{-1} \leq R_0$
$f_{4P}(R)$	3	$R_0^{1-n}(1 + R_0^n\beta) \left(R_0 + R_0^{1+2n}\beta^2 + R_0^n(2\beta R_0 - \alpha n)\right) \left(n\alpha(1-n + (1+n)R_0^n\beta)\right)^{-1} \leq R_0$

Table 5.2: Table of minimizing conditions for the free parameters of $f(R)$ models from Eq. (5.28) in the Palatini formalism. Here the subscript P stands for *Palatini*. Inequalities and equalities follow the same considerations of Tab. 5.1. Here, we assumed that $\rho_0 + \frac{R_0}{4} > 0$.

One can calibrate the constraints over the free parameters in Tabs. 5.1 and 5.2 using also late-time and CMBR cosmological constraints [152, 160]. In general, any consistent choice of $f(R)$ gravity leads to

$$f'(R) \leq \frac{4\rho_0}{R_0}, \quad (5.29)$$

in the metric formalism (where the equality requires vanishing Bogolubov coefficients) and

$$R_0 \geq -4\rho_0, \quad (5.30a)$$

$$f'_0 \neq R_0 f''_0, \quad (5.30b)$$

in the Palatini formalism. Again, the equalities lead to vanishing Bogolubov coefficients. Furthermore, Eq. (5.30a) represents a natural constraint on R_0 . If one wants to pass through different vacuum states without a significant particle production rate, these conditions have to be satisfied. In principle, once evaluated the above constraints, it would be also possible to numerically constrain the derivatives of $f(R)$ models. For example, to guarantee that the gravitational constant does not significantly depart from the Solar System limits, one needs that $4\rho_0 \sim R_0$. Hence, observations on ρ_0 open the possibility to constrain R_0 and may be compared to cosmological constraints over R_0 itself [157, 158]. Similarly, in the Palatini case, R_0 is somehow comparable to $-4\rho_0$. Thus, a correct determination of the limits over R_0 could also discriminate between the metric and Palatini approaches.

5.3 Matching Extended Theories of Gravity with PLANCK results

As stated before, the issue of Modified Gravity and DE is far to be fully addressed and different theoretical scenarios need to be carefully compared with data in order to discriminate among them (in particular the case of ETGs). This effort is in its early stages, given the variety of theories and parametrizations that have been suggested, together with a lack of well tested numerical codes that allow to make detailed predictions to constrain the parameters. PLANCK Collaboration [160] investigated the implications of cosmological data for models of DE and Modified Gravity, beyond the standard cosmological scenario. When estimating the density of DE at early times, they significantly improve present constraints and find that it has to be below $\approx 2\%$ (at 95% confidence) of the critical density. They also consider the general parametrizations of the DE or Modified Gravity perturbations that encompass both effective field theories and the phenomenology of gravitational potentials in Modified Gravity models and finally, test a range of specific models, such as k-essence, $f(R)$ theories and coupled DE. Additional probes for the analysis coming from baryonic acoustic oscillations, type-Ia supernovae and local measurements of the Hubble constant, are considered. These are important tools in order to test models and to break degeneracies that are still present in the combination of PLANCK and background data sets.

In any case, constraints given in [160] are consistent with Λ CDM model and with constraints on DE models (including minimally-coupled scalar field models or evolving equation of state models) and Modified Gravity models (including effective field theory, phenomenological parametrizations, $f(R)$ and coupled DE models) that are significantly improved compared to the past analyses.

As a concrete example of universally coupled theories, $f(R)$ models are considered. Results are compatible with Λ CDM. Such theories assume that some screening mechanism is in place, in order to satisfy current bounds on baryonic physics at solar system scales [161]. Such universal coupling could be improved considering the remaining degrees of freedom related to curvature and then the most general theory to be compared with data in this perspective (*i.e.* where all the curvature budget is

considered) is $f(R, \mathcal{G})$ gravity.

Chapter 6

Searching for exact solutions

The problem of selecting viable models cannot be posed only at a phenomenological level but should be considered also at the fundamental level. To this end, symmetries are extremely useful to fix self-consistent models.

Furthermore, any modification to GR invariably results in a set of field equations that is considerably more complicated than Einstein's field equations. In dealing with this complexity there appear in the literature two different ways to proceed. The first is to look for approximate solutions of a specific theory of interest. It is the approach that is most frequently taken up when investigating theories that are motivated by a desire to overcome the perceived shortcomings of the standard theory. The second approach is to look for exact solutions of modifications of GR. The idea behind this approach is to understand as well as possible the effect of modifying the standard theory. Once this behaviour is well understood it can then be used as an approximation to more complicated theories of specific interest, as well for considerations of that particular modification. It is the second approach that will be taken into account in this chapter. The program is to investigate the form of solutions for some alternative theories of cosmological interest; these will be the scalar-tensor theories and fourth-order theories as Hybrid Gravity and Gauss-Bonnet Gravity. Particular attention will be focussed on highly symmetric situations. High symmetry space-times are the most readily solved for and are often the ones of most physical interest. These solutions will then be used to model physical processes that occur in the Universe. Comparing these models with observation allows to constrain the theory, which limits the deviations from GR.

6.1 The Noether Symmetry Approach

Noether symmetries are useful to reduce dynamical systems and find out exact solutions. In cosmology, the so-called **Noether Symmetry Approach** revealed extremely useful to work out physically motivated models related to conservation laws [53]. The Noether symmetry approach has been applied in cosmology by many authors in various contexts, such as in scalar-tensor gravity [103], higher order gravity [110, 200], and in teleparallel gravity [184]. For instance, new exact solutions for cosmological models with a minimally coupled scalar field were first found by requiring the existence of a Noether symmetry for a Lagrangian description on a two-dimensional “configuration space” [176]. Furthermore, the evolution of two dimensional minisuperspace cosmological models, at the classical and quantum levels, was investigated and exact solutions achieved by using the Noether Symmetry Approach considering, as phase space variables, the FRW scale factor and the scalar field [162]. Furthermore, the Noether Symmetry Approach can be applied to quantum cosmology [53, 163], phantom quintessence cosmology [164], to spinor and scalar field models [165]. Finally, the dynamics of homogeneous cosmologies with a scalar field source with an arbitrary self-interaction potential was also explored in [166]. Bianchi universes and related Noether symmetries have been considered in [167].

In the higher gravity context, the application of the Noether theorem allows in principle to select the functional form of $f(R)$, $f(\mathcal{G})$, $f(R, \mathcal{G})$...-gravity models compatible with the symmetry and to find analytical solutions for the field equations, so it can be seen as a physical criterion since the conserved quantities are a sort of Noether charges. Therefore such a criterion might be to look for those solutions which have *cosmological* Noether charge. Although this criterion somehow “breaks” Lorentz-invariance because we need the FRW background to formulate it, however Lorentz-invariance is evidently broken in our universe by the presence of the CBMR radiation which, by itself, fixes a preferred reference frame.

Before proceeding further, we briefly review the basic definitions concerning Noether symmetries [53, 171]. In general, Noether Theorem states that conserved quantities are related to the existence of cyclic variables into dynamics [168, 169, 170].

Searching for exact solutions

Let $\mathcal{L}(q^i, \dot{q}^i)$ be a canonical, non-degenerate point-like Lagrangian where

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0; \quad \det H_{ij} \stackrel{\text{def}}{=} \det \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0, \quad (6.1)$$

with H_{ij} the Hessian matrix related to \mathcal{L} . The dot indicates derivatives with respect to the affine parameter λ which, in our case, corresponds to the cosmic time t . In analytical mechanics, \mathcal{L} is of the form

$$\mathcal{L} = T(q^i, \dot{q}^i) - V(q^i), \quad (6.2)$$

where T and V are the “kinetic” and “potential energy” respectively. T is a positive definite quadratic form in \dot{q}^j . The energy function associated with \mathcal{L} is

$$E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}, \quad (6.3)$$

which is the total energy $T + V$. In any case, $E_{\mathcal{L}}$ is a constant of motion. Since our cosmological problems have a finite number of degrees of freedom, we are going to consider only point-transformations. Any invertible transformation of the “generalized positions” $Q^i = Q^i(q^j)$ induces a transformation of the “generalized velocities” such that

$$\dot{Q}^i(q^j) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j; \quad (6.4)$$

the matrix $\mathcal{J} = \|\partial Q^i / \partial q^j\|$ is the Jacobian of the transformation on the positions, and it is assumed to be nonzero. The Jacobian $\tilde{\mathcal{J}}$ of the induced transformation is easily derived and $\mathcal{J} \neq 0 \rightarrow \tilde{\mathcal{J}} \neq 0$. In general, this condition is not satisfied in the whole space but only in the neighbor of a point. It is a local transformation.

A point transformation $Q^i = Q^i(q^j)$ can depend on one (or more than one) parameter. We can assume that a point transformation depends on a parameter ϵ , *i.e.* $Q^i = Q^i(q^j, \epsilon)$, and that it gives rise to a one-parameter Lie group. For infinitesimal values of ϵ , the transformation is then generated by a vector field: for instance, $\partial/\partial x$ is a translation along the x axis, $x(\partial/\partial y) - y(\partial/\partial x)$ is a rotation around the z axis and so on. The induced transformation (6.4) is then represented by

$$X = \alpha^i(q^j) \frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(q^j) \right) \frac{\partial}{\partial \dot{q}^i}. \quad (6.5)$$

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X is called the “complete lift” of X [170]. A function $F(\mathbf{q}, \dot{\mathbf{q}})$ is invariant under the transformation X if

$$L_X F \stackrel{\text{def}}{=} \alpha^i(q^j) \frac{\partial F}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(q^j) \right) \frac{\partial F}{\partial \dot{q}^i} = 0, \quad (6.6)$$

where $L_X F$ is the Lie derivative of F . Specifically, if $L_X \mathcal{L} = 0$, X is a *symmetry* for the dynamics derived from \mathcal{L} .

Let us consider now a Lagrangian \mathcal{L} and its Euler-Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} = 0. \quad (6.7)$$

Let us consider also the vector field (6.5). Contracting (6.7) with the α^i 's gives

$$\alpha^j \left(\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} \right) = 0. \quad (6.8)$$

Being

$$\alpha^j \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} = \frac{d}{d\lambda} \left(\alpha^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) - \left(\frac{d\alpha^j}{d\lambda} \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^j}, \quad (6.9)$$

from (6.8), we obtain

$$\frac{d}{d\lambda} \left(\alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}. \quad (6.10)$$

The immediate consequence is the *Noether Theorem* which states:

If $L_X \mathcal{L} = 0$, then the function

$$\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}, \quad (6.11)$$

is a constant of motion.

Some comments are necessary at this point. Eq.(6.11) can be expressed independently of coordinates as a contraction of X by a Cartan one-form

$$\theta_{\mathcal{L}} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i. \quad (6.12)$$

For a generic vector field $Y = y^i \partial / \partial x^i$, and one-form $\beta = \beta_i dx^i$, we have, by definition, $i_Y \beta = y^i \beta_i$. Thus Eq.(6.11) can be written as

$$i_X \theta_{\mathcal{L}} = \Sigma_0. \quad (6.13)$$

By a point-transformation, the vector field X becomes

$$\tilde{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left(\frac{d}{d\lambda} (i_X dQ^k) \right) \frac{\partial}{\partial \dot{Q}^k}. \quad (6.14)$$

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We see that \tilde{X}' is still the lift of a vector field defined on the “space of positions.” If X is a symmetry and we choose a point transformation such that

$$i_X dQ^1 = 1 ; \quad i_X dQ^i = 0 \quad i \neq 1 , \quad (6.15)$$

we get

$$\tilde{X} = \frac{\partial}{\partial Q^1} ; \quad \frac{\partial \mathcal{L}}{\partial Q^1} = 0 . \quad (6.16)$$

Thus Q^1 is a cyclic coordinate and the dynamics results *reduced* [168, 169].

Furthermore, the change of coordinates given by (6.15) is not unique and then a clever choice could be very important. In general, the solution of Eq.(6.15) is not defined on the whole space. It is local in the sense explained above. Besides, it is possible that more than one X is found, *e.g.* X_1, X_2 . If they commute, *i.e.* $[X_1, X_2] = 0$, then it is possible to obtain two cyclic coordinates by solving the system

$$i_{X_1} dQ^1 = 1 ; i_{X_2} dQ^2 = 1 ; i_{X_1} dQ^i = 0 ; i \neq 1 ; i_{X_2} dQ^i = 0 ; i \neq 2 . \quad (6.17)$$

The transformed fields will be $\partial/\partial Q^1, \partial/\partial Q^2$. The procedure can not be applied if they do not commute since commutation relations are preserved by diffeomorphisms. If the relation $X_3 = [X_1, X_2]$ holds, also X_3 is a symmetry, being $L_{X_3} \mathcal{L} = L_{X_1} L_{X_2} \mathcal{L} - L_{X_2} L_{X_1} \mathcal{L} = 0$. If X_3 is independent of X_1, X_2 , we can go on until the vector fields close the Lie algebra. The usual approach to this situation is to make a Legendre transformation, going to the Hamiltonian formalism, and then derive the Lie algebra of Poisson brackets.

Let us now assume that \mathcal{L} is of the form (6.2). As X is of the form (6.5), $L_X \mathcal{L}$ will be a homogeneous polynomial of second degree in the velocities plus a inhomogeneous term in the q^i . Since such a polynomial has to be identically zero, each coefficient must be independently zero. If n is the dimension of the configuration space, we get $\{1 + n(n + 1)/2\}$ partial differential equations. The system is overdetermined, therefore, if any solution exists, it will be expressed in terms of integration constants instead of boundary conditions. It is also obvious that an overall constant factor in the Lie vector X is irrelevant. In other words, the Noether Symmetry Approach can be used to select functions which assign the models and such functions (and then the models) can be physically relevant.

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As an example, considering the specific case of $f(R)$ cosmology, the situation is the following. The configuration space is $\mathcal{Q} = \{a, R\}$ while the tangent space for the related tangent bundle is $\mathcal{TQ} = \{a, \dot{a}, R, \dot{R}\}$. The Lagrangian is an application

$$\mathcal{L} : \mathcal{TQ} \longrightarrow \mathfrak{R} \quad (6.18)$$

where \mathfrak{R} is the set of real numbers. The generator of symmetry is

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}}. \quad (6.19)$$

As stated above, a symmetry exists if the equation $L_X \mathcal{L} = 0$ has solutions. Then there will be a constant of motion on shell, *i.e.* for the solutions of the Euler equations. This means that a symmetry exists if at least one of the functions α or β in Eq.(6.19) is different from zero.

• Noether symmetries for systems of second order ordinary differential equations

Let us now briefly review the basic definitions concerning Noether symmetries for systems of second order ordinary differential equations (ODEs) of the form

$$\ddot{q}^\alpha = \omega^\alpha (t, q^\beta, \dot{q}^\beta). \quad (6.20)$$

Let the system of ODEs (6.20) result from a first order Lagrangian $\mathcal{L} = \mathcal{L}(t, q^\beta, \dot{q}^\beta)$. Then the vector field

$$X = \xi (t, q^\beta) \partial_t + \eta^\alpha (t, q^\beta) \partial_\alpha,$$

in the space $\{t, q^i\}$ is a generator of a Noether point symmetry for the ODEs system (6.20), if the additional condition

$$X^{[1]} \mathcal{L} + \mathcal{L} \frac{d\xi}{dt} = \frac{dg}{dt}, \quad (6.21)$$

holds [172], where $g = g(t, q^\beta)$ is the gauge function and $X^{[1]}$ is the first prolongation of X , *i.e.*,

$$X^{[1]} = X + \left(\frac{d\eta^\beta}{dt} - \dot{q}^\beta \frac{d\xi}{dt} \right) \partial_{\dot{q}^\beta}.$$

For every Noether point symmetry there exists a first integral (a Noether integral) of the system of equations (6.20) given by

$$I = \xi E_H - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \eta^\alpha + g, \quad (6.22)$$

where E_H is the Hamiltonian of \mathcal{L} .

6.2 The Hojman Symmetry Approach

Besides Noether approach, other approaches, based on symmetries, can be adopted in order to reduce dynamics and obtain exact solutions. For example, the Hojman conservation theorem [173] can provide a further method to find exact solutions. As we shall see below, the Hojman approach does not require, a priori, the need for Lagrangians and Hamiltonian functions. The symmetry vectors and the corresponding conserved quantities can be obtained by using directly the equations of motion. The interesting fact is that these two approaches may give rise to different conserved quantities that, eventually, can coincide. For example, in the case of minimally coupled scalar-tensor gravity models, the Noether symmetry exists only for exponential potential $V(\phi)$ [174], while the Hojman symmetry exists for a wide range of potentials[176].

In order to formulate the Hojman conservation theorem, let us consider a system of second-order differential equations that, specifically, can represent the equations of motion of a given dynamical system and that should not necessarily result from a Lagrangian [173],

$$\ddot{q}^i = F^i(q^j, \dot{q}^j, t), \quad i, j = 1, \dots, n. \quad (6.23)$$

Dot is the derivative with respect to the time. A symmetry vector X^i for Eqs. (6.23) is defined according to the transformation

$$q'^i = q^i + \epsilon X^i(q^j, \dot{q}^j, t), \quad (6.24)$$

that maps solutions q^i of Eqs. (6.23) into solutions q'^i of the same equations. Such a vector has to satisfy the equations

$$\frac{d^2 X^i}{dt^2} - \frac{\partial F^i}{\partial q^j} X^j - \frac{\partial F^i}{\partial \dot{q}^j} \frac{dX^j}{dt} = 0, \quad (6.25)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial \dot{q}^i}. \quad (6.26)$$

The Hojman conservation theorem [173] states:

If the function F in Eqs. (6.23) satisfies the condition

$$\frac{\partial F^i}{\partial \dot{q}^i} = 0, \quad (6.27)$$

then

$$Q = \frac{\partial X^i}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left(\frac{dX^i}{dt} \right), \quad (6.28)$$

is a conserved quantity, that is

$$\frac{dQ}{dt} = 0. \quad (6.29)$$

Furthermore, if F satisfies

$$\frac{\partial F^i}{\partial \dot{q}^i} = -\frac{d}{dt} \ln \gamma, \quad (6.30)$$

where γ is a function of q^i , then also

$$Q = \frac{1}{\gamma} \frac{\partial(\gamma X^i)}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left(\frac{dX^i}{dt} \right). \quad (6.31)$$

is a conserved quantity.

As previously discussed in [174], such a theorem can be suitably applied to dynamical equations describing cosmological models. Specifically, any cosmological model can be considered a minisuperspace $\mathcal{Q} \equiv \{q_j\}$ whose dynamics is defined on the tangent space $\mathcal{TQ} \equiv \{q_j, \dot{q}_j\}$. If the Hojman theorem is satisfied, conserved quantities related to couplings and potentials can be found. This feature allows the reduction of dynamics and the possibility to obtain exact solutions, as we will discuss below.

6.3 Minimally coupled scalar-tensor gravity

As showed in [174], Hojman theorem can be applied to dynamical systems describing cosmological models. In [174], starting from the point-like Lagrangian (6.48) with equations of motion (6.49) and (6.50), introducing the variable $x = \ln a$ and by combining equations (6.50) and (6.51), it is found the equation of motion

$$\ddot{x} = -f(x)\dot{x}^2, \quad (6.32)$$

where

$$f(x) = \frac{1}{2}\phi'(x)^2. \quad (6.33)$$

Assuming that $a(t)$ and $\phi(t)$ are invertible functions of t , dynamics can be reduced to a one dimensional motion. Eq. (6.51) can be considered as a constraint. From Eq.(6.32), it is clear that $F(x, \dot{x}) = -f(x)\dot{x}^2$, thus

$$\gamma(x) = \gamma_0 e^{\int f(x) dx}, \quad (6.34)$$

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where γ_0 is an integration constant. Another relation is achieved by dividing Eqs.(6.51) and (6.50). We have

$$\frac{V'(\phi)}{V(\phi)} = \frac{f(x)\phi'(x) - \phi''(x) - 3\phi'(x)}{3 - \frac{1}{2}\phi'(x)^2}. \quad (6.35)$$

Eq. (6.25) becomes

$$\begin{aligned} & \left(f(x) \frac{\partial X}{\partial x} + f'(x)X + \frac{\partial^2 X}{\partial x^2} \right) + \dot{x}^2 f(x)^2 \frac{\partial^2 X}{\partial \dot{x}^2} - \\ & \dot{x} \left(2f(x) \frac{\partial^2 X}{\partial x \partial \dot{x}} + f'(x) \frac{\partial X}{\partial \dot{x}} \right) = 0, \end{aligned} \quad (6.36)$$

where we assumed that X does not depend explicitly on time. If $X = X(\dot{x})$, the only solution for (6.36) is [174]

$$X(\dot{x}) = A_0 \dot{x}^n + A_1 \dot{x}, \quad (6.37)$$

and then

$$f(x) = - \left(\frac{1}{nx + f_0} \right). \quad (6.38)$$

Considering the meaning of the above variables, the generic potential with respect to $\varphi = \phi - \phi_c$ is

$$V(\varphi) = \lambda \varphi^{\frac{4}{n}} - \frac{8\lambda}{3n^2} \varphi^{\frac{4}{n}-2}, \quad (6.39)$$

where

$$\lambda = 3V_0 \left(\frac{n^2}{8} \right)^{\frac{2}{n}}. \quad (6.40)$$

The exact solutions $a(t)$ and $\varphi(t)$ for the potential (6.39) are

$$\begin{aligned} a(\bar{t}) &= e^{-\frac{f_0}{n}} e^{-\frac{1}{n} \left[\left(1 - \frac{1}{n}\right) \bar{t} \right]^{\frac{n}{n-1}}}, \\ \varphi(\bar{t}) &= \pm \frac{\sqrt{8}}{n} \left[\left(1 - \frac{1}{n}\right) \bar{t} \right]^{\frac{n}{2(n-1)}}, \end{aligned} \quad (6.41)$$

where the parameter \bar{t} is defined as

$$\bar{t} = y_0 - n|Q_0|^{\frac{1}{n}} t, \quad (6.42)$$

Q_0 is the Hojman conserved quantity and \bar{t} can be seen as a sort conformal time ruled by Q_0 . In the next section we will see how these results can be used in order to find out solutions for non-minimally coupled scalar- tensor cosmologies ruled by the forms of the potential $V(\phi)$ and of the coupling $F(\phi)$ [3].

6.4 Non-minimally coupled scalar-tensor gravity

As seen in section 4.3.1, conformal transformations allow to transform a non-minimally coupled scalar-tensor theory into a minimally one. We want to show this also in the cosmological case and this will be very useful for our purposes.

Consider the spatially flat FRW space-time with a time-dependent scale factor $a(t)$ and

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2). \quad (6.43)$$

The Lagrangian density in (3.10) takes the form

$$\mathcal{L}_{NMC} = 6F(\phi)a\dot{a}^2 + 6F'(\phi)a^2\dot{a}\dot{\phi} + \frac{1}{2}a^3\dot{\phi}^2 - a^3V(\phi), \quad (6.44)$$

where the prime ' denotes the derivative with respect to ϕ and the dot denotes the derivative with respect to the time t . Eq. (6.44) is a point-like Lagrangian on the configuration space (a, ϕ) . The Euler-Lagrange equations relative to (6.44) are

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{2F'}{F} \left(\frac{\dot{a}}{a}\right) \dot{\phi} + \frac{F'}{F} \ddot{\phi} + \left(\frac{F''}{F} - \frac{1}{4F}\right) \dot{\phi}^2 - \frac{V}{2F} = 0, \quad (6.45)$$

$$\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right) \dot{\phi} + 6F' \left(\frac{\dot{a}}{a}\right)^2 + 6F' \left(\frac{\ddot{a}}{a}\right) + V' = 0, \quad (6.46)$$

which correspond respectively to the Einstein-Friedman equation and to the Klein-Gordon equation for the FRW case. The energy function, corresponding to the Einstein (0, 0) equation, is

$$6Fa\dot{a}^2 + 6F'a^2\dot{a}\dot{\phi} + \frac{1}{2}a^3\dot{\phi}^2 + a^3V = 0. \quad (6.47)$$

Besides, the Lagrangian of the action (3.14) is

$$\mathcal{L}_{MC} = -3a\dot{a}^2 + \frac{a^3}{2}\dot{\phi}^2 - a^3V(\phi), \quad (6.48)$$

and the equations of motion are given by

$$\ddot{a} = \frac{a}{3} [V(\phi) - \dot{\phi}^2] \quad (6.49)$$

$$\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right) \dot{\phi} + V'(\phi) = 0, \quad (6.50)$$

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and the energy function is

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3} \left[\frac{\dot{\phi}^2}{2} + V(\phi) \right]. \quad (6.51)$$

The possibility to construct a conformal Lagrangian corresponding to a minimally coupled scalar field can be achieved by introducing the following transformations [99]

$$\begin{aligned} \bar{a} &= \sqrt{-2F(\phi)}a, \\ \frac{d\bar{\phi}}{d\bar{t}} &= \sqrt{\frac{3F'(\phi)^2 - F(\phi)}{2F(\phi)^2}} \frac{d\phi}{dt}, \\ d\bar{t} &= \sqrt{-2F(\phi)}dt. \end{aligned} \quad (6.52)$$

Under transformations (6.52), we obtain

$$\begin{aligned} \frac{1}{\sqrt{-2F}} \mathcal{L} &= \frac{1}{\sqrt{-2F}} (6Fa\dot{a}^2 + 6F'a^2\dot{a}\dot{\phi} + \frac{a^3}{2}\dot{\phi}^2 - a^3V(\phi)) \\ &= -3\bar{a}\dot{\bar{a}}^2 + \frac{1}{2}\bar{a}^3\dot{\bar{\phi}}^2 - \bar{a}^3\bar{V}(\bar{\phi}) = \mathcal{L}_{MC}, \end{aligned} \quad (6.53)$$

where the dot over barred quantities means the derivative with respect to \bar{t} and

$$\bar{V}(\bar{\phi}(\phi)) = \frac{V(\phi)}{4F^2(\phi)}. \quad (6.54)$$

Hence, under transformations (6.52), the non-minimal coupled Lagrangian becomes a conformally related minimal coupled Lagrangian. This means that for any non-minimally coupled scalar field, we may associate a unique minimally coupled scalar field in the conformally related space by deriving the correct relation between the coupling and the potential as (6.54).

Such a property can be used as a solution generator in the sense that by achieving solutions in the Einstein frame through the Hojman Theorem, as in [174], it is possible to derive solutions in the Jordan frame and vice versa.

The quadratic coupling case

Let us start with the general case of a quadratic coupling of the form

$$F(\phi) = \frac{\xi}{4}(k + \phi)^2, \quad (6.55)$$

where k and ξ are arbitrary constants (with $\xi < 0$ in order to recover physical cases).

In this case equations (6.45) and (6.46) in the variable \bar{t} become

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{6\dot{a}}{a} \frac{\dot{\phi}}{(k + \phi)} + \frac{2\ddot{\phi}}{(k + \phi)} + \frac{(4\xi - 1)\dot{\phi}^2}{\xi(k + \phi)^2} - \frac{4V}{\xi^2(k + \phi)^4} = 0, \quad (6.56)$$

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$$\ddot{\phi} + 3\xi \frac{\ddot{a}}{a}(k + \phi) + 3(1 + \xi) \frac{\dot{a}\dot{\phi}}{a} + 3\xi \frac{\dot{a}^2}{a^2}(k + \phi) + \frac{\dot{\phi}^2}{k + \phi} - \frac{2V'}{\xi(k + \phi)^2} = 0. \quad (6.57)$$

Transformations (6.52) gives

$$\begin{aligned} \bar{a} &= \sqrt{-\frac{\xi}{2}} (k + \phi)a, \\ \bar{\phi} &= \sqrt{\frac{6\xi - 2}{\xi}} \ln[k + \phi] + c_1, \\ d\bar{t} &= \sqrt{-\frac{\xi}{2}} (k + \phi)dt. \end{aligned}$$

Solutions (6.41) become

$$\phi(\bar{t}) = -k + \phi_0 e^{\left(\frac{\sqrt{8\xi}}{n\sqrt{6\xi-2}} \left[(1-\frac{1}{n})\bar{t}\right]^{2(n-1)}\right)}, \quad (6.58)$$

$$a(\bar{t}) = \sqrt{-\frac{2}{\xi} \frac{e^{-\frac{f_0}{n}}}{\phi_0}} e^{\left(-\frac{1}{n} \left[(1-\frac{1}{n})\bar{t}\right]^{\frac{n}{n-1}} - \frac{\sqrt{8\xi}}{n\sqrt{6\xi-2}} \left[(1-\frac{1}{n})\bar{t}\right]^{2(n-1)}\right)} \quad (6.59)$$

and the potential is

$$\begin{aligned} V(\phi) &= \left(\frac{n^2(3\xi - 1)}{4\xi}\right)^{\frac{2}{n}} \xi^2 (k + \phi)^4 \left\{ \frac{3}{4n^2} \left[\ln\left(\frac{k + \phi}{\phi_0}\right) \right]^{\frac{4}{n}} \right. \\ &\quad \left. - \frac{\xi}{n^4(3\xi - 1)} \left[\ln\left(\frac{k + \phi}{\phi_0}\right) \right]^{\frac{4}{n}-2} \right\}. \end{aligned} \quad (6.60)$$

To have an idea of what is going on, let us plot the potential (6.60) in Fig. 6.1.

The cosmology of this model is most easily studied in the Einstein frame where the gravity sector is standard taking advantage from the potential transformation (6.54). For example, the inflationary dynamics is determined by the shape of the potential $\bar{V}(\bar{\phi})$. It is worth noticing that $\bar{\phi}$ (and not ϕ) has a canonical kinetic term. Therefore the slow-roll parameters, which control the first and second derivatives of the potential respectively, are

$$\epsilon_{\bar{\phi}} = \frac{1}{2} \left(\frac{\bar{V}_{\bar{\phi}}}{\bar{V}} \right)^2, \quad \eta_{\bar{\phi}} = \frac{\bar{V}_{\bar{\phi}\bar{\phi}}}{\bar{V}}, \quad (6.61)$$

where the subscript $\bar{\phi}$ means $d/d\bar{\phi}$. In the usual way, we can formally define the first and second slow-roll parameters for the field ϕ that are related to the slow roll parameters $\epsilon_{\bar{\phi}}$ and $\eta_{\bar{\phi}}$ via [175]

$$\epsilon_{\bar{\phi}} = \left(\frac{d\phi}{d\bar{\phi}} \right)^2 \epsilon_{\phi}, \quad (6.62)$$

$$\eta_{\bar{\phi}} = \left(\frac{d\phi}{d\bar{\phi}} \right)^2 \eta_{\phi} - \left(\frac{d^2\phi}{d\bar{\phi}^2} \right) \sqrt{\frac{\epsilon_{\phi}}{2}}. \quad (6.63)$$

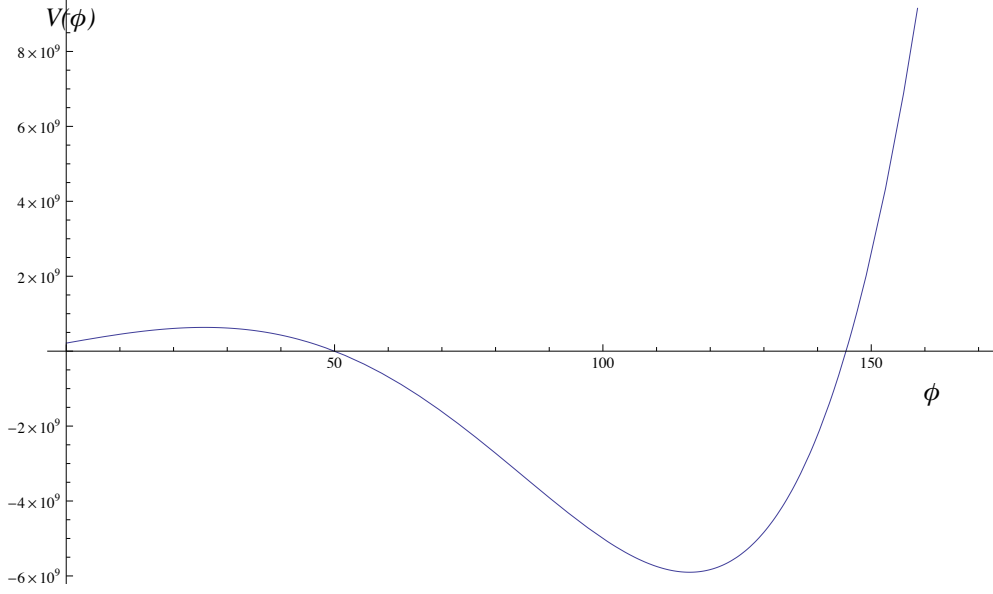


Figure 6.1: Plot of the potential $V(\phi)$ for some specific values of the constants, $k = 100$, $\xi = -10$, $\phi_0 = 150$, $n = 4/3$.

The slow-roll approximation requires that the constraints

$$\epsilon_{\bar{\phi}} \ll 1, \quad (6.64)$$

$$|\eta_{\bar{\phi}}| \ll 1, \quad (6.65)$$

be satisfied. A plot of $\epsilon_{\bar{\phi}}$ is reported in Fig.6.2 and a plot of $|\eta_{\bar{\phi}}|$ in Fig.6.3. The conditions $\epsilon_{\bar{\phi}} \ll 1$ and $|\eta_{\bar{\phi}}| \ll 1$ are satisfied for $0 < \phi < 50$ that is the range of values of the slow-roll phase as it can be expected from Fig.6.1 Therefore, for potential (6.60), with $k = 100$, $\xi = -10$, $\phi_0 = 150$, $n = 4/3$, the conditions $\epsilon_{\bar{\phi}} \ll 1$ and $|\eta_{\bar{\phi}}| \ll 1$ are satisfied for $0 < \phi < 50$ that is the range of values expected for the slow-roll phase as it can be seen from Fig.6.1.

The conformally coupled case

We consider also the case of conformal coupling where

$$F(\phi) = \xi\phi^2, \quad k = 0, \quad (6.66)$$

hence,

$$\dot{\bar{\phi}} = \sqrt{\frac{3F'^2 - F''}{2F^2}} \dot{\phi} = \sqrt{\frac{12\xi - 1}{2\xi}} \begin{pmatrix} \dot{\phi} \\ \phi \end{pmatrix}. \quad (6.67)$$

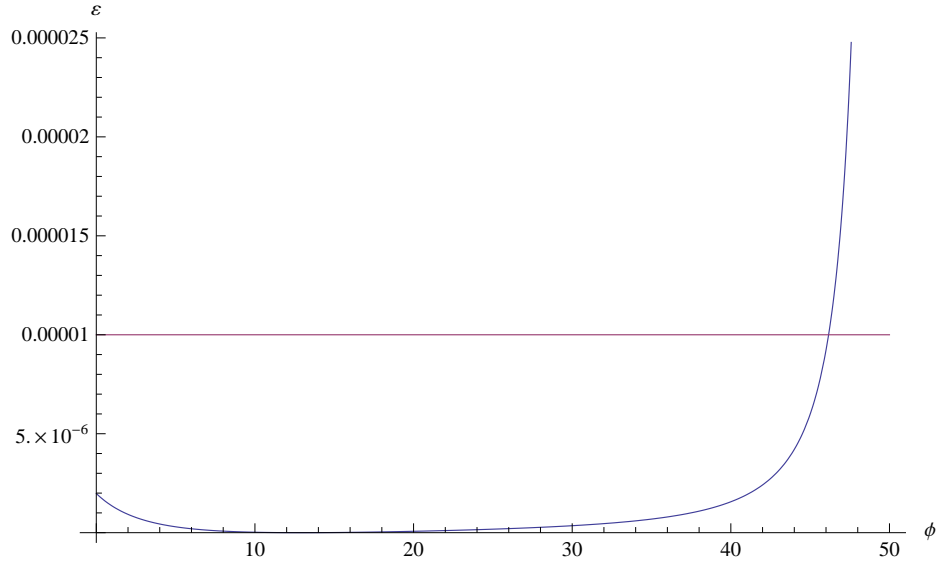


Figure 6.2: Plot of $\epsilon_{\bar{\phi}}$ with $k = 100$, $\xi = -10$, $\phi_0 = 150$, $n = 4/3$. The straight line corresponds to $\epsilon = 10^{-5}$.

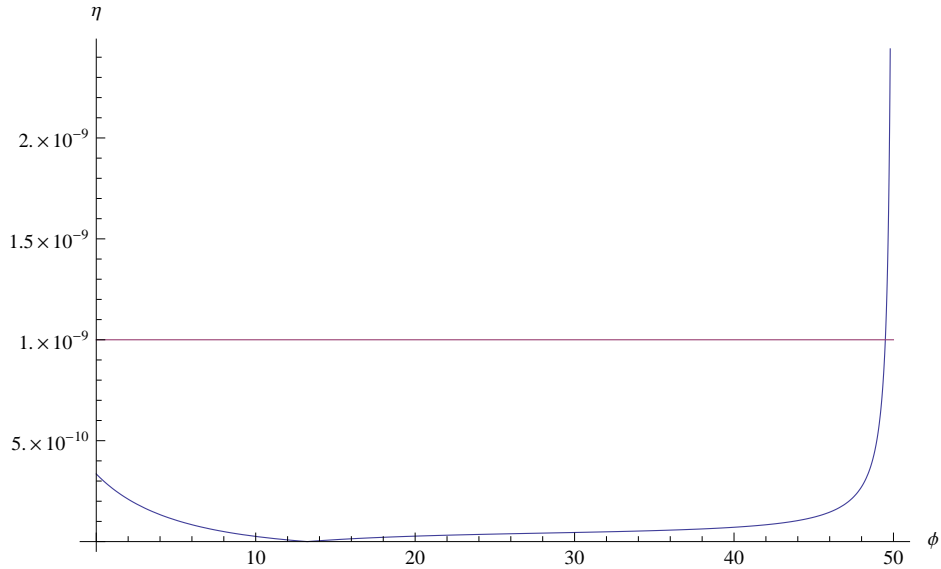


Figure 6.3: Plot of $\epsilon(\phi)$ with $k = 100$, $\xi = -10$, $\phi_0 = 150$, $n = 4/3$. The straight line corresponds to $\eta = 10^{-9}$.

Integrating this relation we have

$$\bar{\phi} = \sqrt{\frac{12\xi - 1}{2\xi}} \ln(\phi) + C_1 = \sqrt{c} \ln(\phi) + C_1. \quad (6.68)$$

Moreover we have

$$\bar{a} = \sqrt{-2F(\phi)}a = \sqrt{-2\xi} \phi a, \quad (6.69)$$

Searching for exact solutions

and

$$d\bar{t} = \sqrt{-2F(\phi)}dt = \sqrt{-2\xi} \phi dt. \quad (6.70)$$

Now using solutions (6.41) for the minimally coupled case, we obtain the following solutions for the non-minimal coupling case

$$\phi(\bar{t}) = \phi_0 e^{\left(\frac{\sqrt{8}}{n\sqrt{c}}[(1-\frac{1}{n})\bar{t}]^{\frac{n}{2(n-1)}}\right)}, \quad (6.71)$$

$$a(\bar{t}) = \frac{e^{-\frac{f_0}{n}}}{\sqrt{-2\xi}\phi_0} e^{\left(-\frac{1}{n}[(1-\frac{1}{n})\bar{t}]^{\frac{n}{n-1}} - \frac{\sqrt{8}}{n\sqrt{c}}[(1-\frac{1}{n})\bar{t}]^{\frac{n}{2(n-1)}}\right)}. \quad (6.72)$$

The potential takes the form

$$V(\phi) = \xi^2 \phi^4 \left(\frac{n^2(12\xi - 1)}{16\xi}\right)^{\frac{2}{n}} \left[\frac{12}{n^2} \left[\ln\left(\frac{\phi}{\phi_0}\right)\right]^{\frac{4}{n}} - \frac{64\xi}{n^4(12\xi - 1)} \left[\ln\left(\frac{\phi}{\phi_0}\right)\right]^{\frac{4}{n}-2}\right]. \quad (6.73)$$

In Fig. 6.4, we represent the potential for some specific values of the integration constants. Clearly the above inflationary analysis works also in this case.

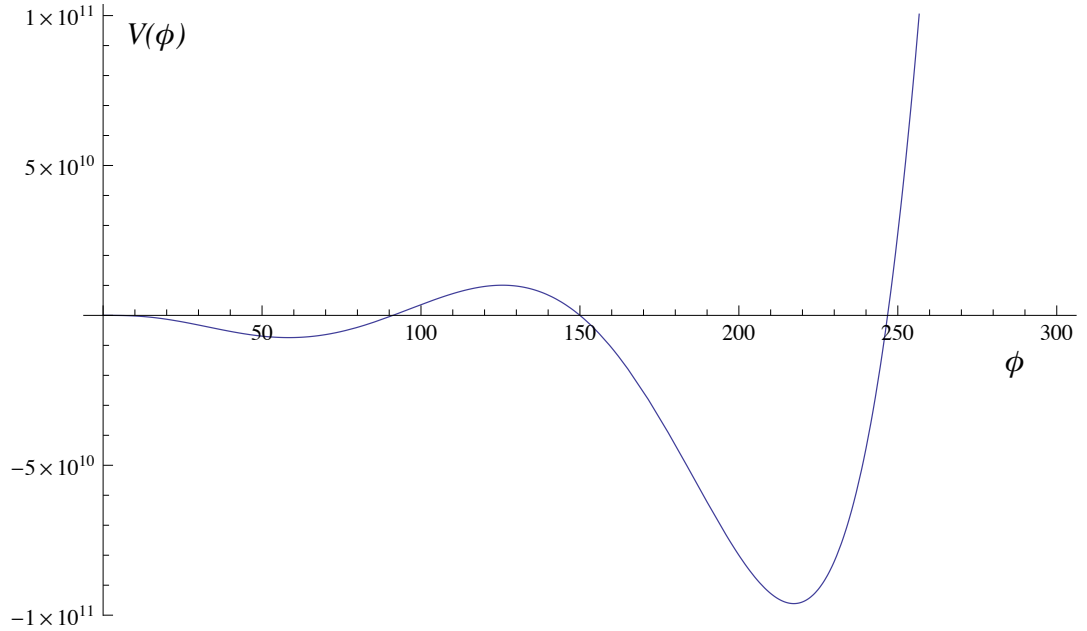


Figure 6.4: The potential $V(\phi)$ with $k = 0$, $\xi = -10$, $\phi_0 = 150$, $n = 4/3$

The string-like case

Finally, let us take into account the low energy limit of string Lagrangian, that is

$$\mathcal{L} = \sqrt{-g}e^{-2\psi} [-R + 4g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - W(\psi)], \quad (6.74)$$

where ψ is the dilaton and $W(\psi)$ is the potential which leads dynamics. Note that, due to the coupling $e^{-2\psi}$, modes associated with the dilaton and with the graviton are non-minimally coupled. Lagrangian (6.74) can be immediately rewritten as a non-minimal coupled Lagrangian (3.10) if we assume the transformation [53]

$$\phi(\psi) = e^{-\psi}, \quad F(\phi) = -\frac{1}{8}e^{-2\psi}, \quad V(\phi) = e^{-2\psi}W(\psi). \quad (6.75)$$

This can be written as

$$\phi = -\ln\psi, \quad F(\phi) = -\frac{1}{8}\phi^2, \quad V(\phi) = \phi^2W(\psi(\phi)). \quad (6.76)$$

We are again in the above case with $\xi = -\frac{1}{8}$ and $\phi_0 = 1$. Starting from the exact solutions (6.73), we have that the class of potentials $W(\psi)$ which satisfy conditions (6.75) are

$$W(\psi) = e^{-2\psi} \left(\frac{5}{4}n^2\right)^{\frac{2}{n}} \left(\frac{3}{16n^2}\psi^{\frac{4}{n}} - \frac{1}{20n^4}\psi^{\frac{4}{n}-2}\right). \quad (6.77)$$

and the exact solutions are

$$\phi(\bar{t}) = e^{\left(\frac{2}{n\sqrt{5}}[(1-\frac{1}{n})\bar{t}]^{\frac{n}{2(n-1)}}\right)}, \quad (6.78)$$

$$a(\bar{t}) = 2e^{-\frac{f_0}{n}} e^{-\left(\frac{1}{n}[(1-\frac{1}{n})\bar{t}]^{\frac{n}{n-1}} + \frac{2}{n\sqrt{5}}[(1-\frac{1}{n})\bar{t}]^{\frac{n}{2(n-1)}}\right)}. \quad (6.79)$$

It is straightforward to see that also in this case, the above inflationary analysis easily applies.

6.5 Hybrid gravity

In order to obtain exact solutions of the field equations and invariant solutions for the hybrid gravity Wheeler-DeWitt Equation in a spatially flat FRW space-time we consider point symmetries [2].

Searching for exact solutions

Indeed, the infinitesimal generator of a point transformation, which leaves invariant the field equations, is a Noether symmetry. This feature provides integrals of motions useful to reduce the related dynamical system and then get exact solutions. In order to determine the Noether symmetries of the classical Lagrangian, we will apply the geometric procedure outlined in [178], where the Noether symmetries of the Lagrangian are connected to the collineations of the second order tensor which is defined by the kinematic part of the Lagrangian. Hence, the Noether symmetry is not only a criterion for the integrability of the system but also a geometric criterion that allows to select the free functions of the theory. This approach has been applied in [179, 180, 181, 182, 183, 184]. Furthermore, in order to solve exactly the Wheeler-DeWitt equation we will apply the theory of Lie invariants that allows to determine the Lie point symmetries of the Wheeler-DeWitt equation. As shown in [185], the Lie point symmetries for the Wheeler-DeWitt equation (or the Klein Gordon equation) are connected to the conformal Lie algebra of the minisuperspace which defines the Laplace operator.

As stated in 3.3.1, for the pure metric and Palatini case [29, 94], the action (3.37) can be transformed into a scalar-tensor theory. We introduce an auxiliary field E such that

$$S = \int d^4x \sqrt{-g} [R + f(E) + f'(E)(\mathcal{R} - E)]. \quad (6.80)$$

The field E is dynamically equivalent to the Palatini scalar \mathcal{R} if $f''(\mathcal{R}) \neq 0$. If the quantities

$$\phi \equiv f'(E), \quad V(\phi) = Ef'(E) - f(E), \quad (6.81)$$

are defined, the action becomes

$$S = \int d^4x \sqrt{-g} [R + \phi\mathcal{R} - V(\phi)]. \quad (6.82)$$

Using relation (3.31) between R and \mathcal{R} with $f(\mathcal{R}) = \phi\mathcal{R}$

$$\mathcal{R} = R + \frac{3}{2\phi^2} \partial_\mu \phi \partial^\mu \phi - \frac{3}{\phi} \square \phi, \quad (6.83)$$

(see [105] for details) one finally obtains the scalar-tensor form of the action

$$S = \int d^4x \sqrt{-g} \left[(1 + \phi)R + \frac{3}{2\phi} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right]. \quad (6.84)$$

Searching for exact solutions

It is important to note that Eq. (6.81) is a Clairaut differential equation [177], that is,

$$E f'(E) - f(E) = V(f'(E)). \quad (6.85)$$

It admits a general linear solution

$$f(E) = cE - V(c) \quad (6.86)$$

for arbitrary $V(\phi)$ and a singular solution followed from the equation

$$\frac{\partial V(f'(E))}{\partial f'} - E = 0. \quad (6.87)$$

Let us consider the FRW spatially flat metric

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2). \quad (6.88)$$

Therefore, from action (6.84) one obtains the pointlike Lagrangian

$$\mathcal{L} = 6a\dot{a}^2(1 + \phi) + 6a^2\dot{a}\dot{\phi} + \frac{3}{2\phi}a^3\dot{\phi}^2 + a^3V(\phi), \quad (6.89)$$

from which one can calculate the following field equations

$$\ddot{a} + \frac{1 - \phi}{2a}\dot{a}^2 - \frac{1}{2}\dot{a}\dot{\phi} - \frac{a}{3\phi}\dot{\phi}^2 - \frac{1}{12}a(3V - 2\phi V_{,\phi}) = 0, \quad (6.90)$$

and

$$\begin{aligned} \ddot{\phi} + \frac{\phi(\phi + 1)}{a^2}\dot{a}^2 + \frac{\phi + 3}{a}\dot{a}\dot{\phi} + \frac{\phi - 2}{4\phi}\dot{\phi}^2 \\ + \frac{\phi}{6}(3V(\phi) - 2(\phi + 1)V_{,\phi}) = 0. \end{aligned} \quad (6.91)$$

Note that Eq. (6.91) is the Klein-Gordon equation for the scalar field ϕ . The energy condition is given by

$$6a\dot{a}^2(1 + \phi) + 6a^2\dot{a}\dot{\phi} + \frac{3}{2\phi}a^3\dot{\phi}^2 - a^3V(\phi) = 0. \quad (6.92)$$

Equations (6.90) and (6.92) can be written in the form of modified Friedmann equations for the scale factor $a(t)$

$$\begin{aligned} 3H^2 &= \rho_{\text{eff}}, \\ (2\dot{H} + 3H^2) &= -p_{\text{eff}}, \end{aligned}$$

Searching for exact solutions

where $H = \dot{a}/a$ is a Hubble parameter and ρ_{eff} and p_{eff} are the total effective energy density and pressure, given by

$$\rho_{\text{eff}} = \frac{2\phi V(\phi) - 12H\phi\dot{\phi} - 3\dot{\phi}^2}{6\phi(1 + \phi)} \quad (6.93)$$

$$p_{\text{eff}} = \frac{2\phi^2 V_{,\phi} - 3\phi V(\phi) - 6\phi^2 H^2 - 6H\phi\dot{\phi} - 4\dot{\phi}^2}{6\phi}, \quad (6.94)$$

respectively.

Following the Noether approach explained in sec. 6.1, it is possible to see that the vector field X for the Lagrangian (6.89) is

$$X = \xi(t, a, \phi) \partial_t + \eta_a(t, a, \phi) \partial_a + \eta_\phi(t, a, \phi) \partial_\phi, \quad (6.95)$$

and the first prolongation is given by

$$X^{[1]} = \xi \partial_t + \eta_a \partial_a + \eta_\phi \partial_\phi + \left(\dot{\eta}_a - \dot{a} \dot{\xi} \right) \partial_{\dot{a}} + \left(\dot{\eta}_\phi - \dot{\phi} \dot{\xi} \right) \partial_{\dot{\phi}}. \quad (6.96)$$

We will apply these considerations to different cases in hybrid gravity.

Searching for Noether Symmetries in Hybrid Gravity

Lagrangian (6.89) is in the standard form $\mathcal{L} = T - V_{\text{eff}}$, where $T = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ is the kinetic energy, with a “kinetic” metric

$$ds_{(2)}^2 = 12a(1 + \phi) da^2 + 12a^2 da d\phi + \frac{3}{\phi} a^3 d\phi^2, \quad (6.97)$$

and an effective potential

$$V_{\text{eff}} = -a^3 V(\phi). \quad (6.98)$$

In order to search for special forms of the potential $V(\phi)$, where the Lagrangian admits Noether point symmetries, we will apply the geometric approach developed in [178].

Since the Lagrangian is time-independent, it admits the Noether symmetry ∂_t with the Hamiltonian as a conservation law, that is

$$E_H = 6a(1 + \phi) \dot{a}^2 + 6a^2 \dot{a} \dot{\phi} + \frac{3}{2\phi} a^3 \dot{\phi}^2 - a^3 V(\phi). \quad (6.99)$$

Due to the constraint coming from Einstein field equation $G_0^0 = 0$ then $E_H = 0$ in vacuum.

Searching for exact solutions

Following the results of [178], in the case of a constant potential $V(\phi) = V_0$ the Lagrangian (6.89) admits an extra Noether symmetry. In this case the Noether symmetry is given by

$$X_1 = \frac{\sqrt{\phi}}{a} \partial_\phi,$$

and the corresponding Noether integral has the form

$$I_1 = 3 \frac{a}{\sqrt{\phi}} \left(2\phi\dot{a} + a\dot{\phi} \right).$$

Under the coordinate transformation

$$a = u^{\frac{2}{3}}, \quad \phi = v^2 u^{-\frac{4}{3}},$$

the Lagrangian becomes

$$\mathcal{L}(u, v, \dot{u}, \dot{v}) = \frac{8}{3} \dot{u}^2 + 6u^{\frac{2}{3}} \dot{v}^2 + V_0 u^2.$$

The field equations are given by

$$\frac{8}{3} \dot{u}^2 + 6u^{\frac{2}{3}} \dot{v}^2 - V_0 u^2 = 0, \quad (6.100)$$

$$\ddot{u} - \frac{3}{4} u^{-\frac{1}{3}} \dot{v}^2 - \frac{3}{8} V_0 u = 0, \quad (6.101)$$

$$\ddot{v} + \frac{2}{3u} \dot{u} \dot{v} = 0, \quad (6.102)$$

respectively. The extra Noether integral in the $\{u, v\}$ variables can be written as $\bar{I}_1 = u^{\frac{2}{3}} \dot{v}$ (where $\bar{I}_1/I_1 = \text{const}$) so one has $\dot{v} = \bar{I}_1 u^{-\frac{2}{3}}$. The general solution of the above system is

$$\int \frac{du}{\sqrt{\frac{3}{8} V_0 u^2 - \frac{9}{4} \bar{I}_1^2 u^{-\frac{2}{3}}}} = \int dt. \quad (6.103)$$

Furthermore, for the Hubble function $H = \dot{a}/a$, we have

$$\frac{H^2}{H_0^2} = (\Omega_\Lambda + \Omega_r a^{-4}), \quad (6.104)$$

where

$$\Omega_\Lambda = \frac{1}{6} \frac{V_0}{H_0^2}, \quad \text{and} \quad \Omega_r = -\frac{\bar{I}_1^2}{H_0^2}, \quad (6.105)$$

Searching for exact solutions

indicate the density parameters for the cosmological constant and radiation, respectively. Observe that in order to have a physical solution, it has to be $\bar{I}_1 \in \mathbb{C}$ and $\text{Re}(\bar{I}_1) = 0$.

The Hubble function (6.104) corresponds to the model with a cosmological constant and a radiation fluid. However, if we introduce dust in our model, $\rho_D = \rho_{m0}a^{-3}$, Eq. (6.100) becomes

$$\frac{8}{3}\dot{u}^2 + 6u^{\frac{2}{3}}\dot{v}^2 - V_0u^2 = \rho_{m0}.$$

Therefore, the analytical solution takes the form

$$\int \frac{du}{\sqrt{\frac{3}{8}V_0u^2 + \frac{3}{8}\rho_{m0} - \frac{9}{4}\bar{I}_1^2u^{-\frac{2}{3}}}} = \int dt,$$

and the Hubble function is

$$\frac{H^2}{H_0^2} = (\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}),$$

where now $\Omega_m = \frac{\rho_{m0}}{6H_0^2}$. Thus, the Hybrid Gravity introduces a further ‘‘radiation’’ term.

Since the linear case is trivial, in the next section, we will perform a conformal transformation for the Lagrangian (6.89) in order to apply the results of [179, 180]. We will consider two separate cases with respect to a lapse function N , one case in which the lapse is a function of the scale factor, $d\tau = N(a)dt$, and another case where it is a function of the scalar field, *i.e.* $d\tau = N(\phi)dt$. For the latter case we will show that Hybrid Gravity is conformally related to a Brans-Dicke-like scalar field theory.

Conformal transformations and Noether Symmetries

The dynamical system described by the Lagrangian (6.89) is conformally invariant, with $E_H = 0$. Therefore, we can apply conformal transformations to Lagrangian (6.89) in order to use the results in [179, 180, 185] and determine new solutions in conformal frames. However, since the minisuperspace described by the metric (6.97) is two-dimensional, it admits an infinite conformal algebra, so that, in order to simplify the problem, we have to provide some ansatz. As a first one, we will consider conformal

Searching for exact solutions

transformation of the form $d\tau = N(a) dt$, so that the space-time metric (6.88) has the form

$$ds^2 = -N^{-2}(a(\tau)) d\tau^2 + a^2(\tau) (dx^2 + dy^2 + dz^2). \quad (6.106)$$

Then we will study the case of conformal transformations of the form $d\tau = N(\phi) dt$.

It is now important to stress that the transformations (6.81) and the relation between the curvature scalars (6.83), that led to the action (6.84), are always in the Jordan frame since the effective gravitational coupling is varying, being $(1 + \phi)$, and the kinetic term is non-canonical. As discussed above, they allow to distinguish between the Palatini degrees of freedom, related to the scalar field ϕ , and the standard metric degrees of freedom, related to GR. This is true also in the cosmological dynamical system given by the pointlike Lagrangian (6.89). A conformal transformation, as the above one, allows to carry the problem into the Einstein frame where a conformal kinetic metric and a new Ricci curvature scalar can be defined. Noether symmetries for conformal Lagrangians can be easily achieved and discussed in this frame as we will see below. Moreover, the physical meaning of the degrees of freedom, related to Hybrid Gravity, is evident in the Einstein frame where dynamics is led by an effective potential which vanishes as soon as GR is restored.

The Lagrangian for the conformal FRW space-time has the form (6.106) is given by

$$\begin{aligned} \mathcal{L}(a, \phi, a', \phi') &= \frac{a^3 V(\phi)}{N(a)} \\ &+ N(a) \left[6a(1 + \phi) a'^2 + 6a^2 a' \phi' + \frac{3}{2\phi} a^3 \phi'^2 \right], \end{aligned} \quad (6.107)$$

where the prime denotes $d/d\tau$ (it should not be confused with the conformal time that requires a special choice of the lapse function $N(a)$). The conformal kinetic metric and the related Ricci scalar are given by

$$d\bar{s}_{(2)}^2 = N(a) \left(12a(1 + \phi) da^2 + 12a^2 da d\phi + \frac{3}{\phi} a^3 d\phi^2 \right), \quad (6.108)$$

and

$$R_{(2)} = -\frac{a^2 N N_{,aa} - a^2 N_{,a}^2 - N^2}{12a^3 N^3},$$

respectively. Since the kinetic metric (6.108) is two-dimensional, the space is an Einstein space. If the Ricci scalar is constant, the Einstein space has a constant curvature.

Searching for exact solutions

In order to reduce the problem to the dynamics of Newtonian physics [180], we consider $R_{(2)} = 0$, so that

$$N(a) = a^{-1}e^{N_0 a}. \quad (6.109)$$

Hence, by applying the geometric approach in [178], one gets that the Lagrangian (6.107), with the solution (6.109), admits extra Noether symmetries. The first one is of the form

$$X_1 = -\frac{1}{2}\partial_a + \frac{\phi + V_1\sqrt{\phi}}{a}\partial_\phi, \quad (6.110)$$

with the corresponding conservation law

$$I_{X_1} = 6 \left(V_1\sqrt{\phi} - 1 \right) \dot{a} + 3\frac{a}{\sqrt{\phi}}V_1\dot{\phi}, \quad (6.111)$$

for the potential

$$V(\phi) = V_0 \left(\sqrt{\phi} + V_1 \right)^4. \quad (6.112)$$

The second symmetry vector and corresponding conservation law are given by

$$X_2 = 2\tau\partial_\tau + a \left(\sqrt{\phi}V_1 + 1 \right) \partial_a - 2V_1\sqrt{\phi}(\phi + 1)\partial_\phi, \quad (6.113)$$

and

$$I_{X_2} = 12a(1 + \phi)\dot{a} + 6a^2 \left(1 - \frac{V_1}{\sqrt{\phi}} \right) \dot{\phi}, \quad (6.114)$$

respectively, with the potential given by

$$V(\phi) = V_0(1 + \phi)^2 \exp\left(\frac{6}{V_1} \arctan \sqrt{\phi}\right). \quad (6.115)$$

We have chosen $N_0 = 0$ for both cases. If $N_0 \neq 0$, one finds that Lagrangian (6.107) admits extra Noether symmetries only in the case of the trivial potential $V(\phi) = 0$.

The Noether Integrals (for both cases), the Hamiltonian E_H and I_X are independent geometrical objects and the relation $\{E_H, I_X\} = 0$ holds. Hence, the dynamical systems are Liouville integrable. Furthermore, the Clairaut Eq. (6.81) for the potential (6.112) is given by

$$Ef'(E) - f(E) = V_0 \left(\sqrt{f'(E)} + V_1 \right)^4. \quad (6.116)$$

Thus, Eq. (6.87) becomes

$$\frac{2V_0}{\sqrt{f'(E)}} \left(\sqrt{f'(E)} + V_1 \right)^3 + E = 0, \quad (6.117)$$

Searching for exact solutions

and hence, by setting $y = \sqrt{f'(E)}$, one obtains the polynomial equations

$$(y + V_1)^3 - \frac{E}{2V_0}y = 0. \quad (6.118)$$

A real solution of Eq. (6.118) is

$$\int df = \int \left(EF(E) + \frac{1}{6V_0F(E)} - V_1 \right)^2 dE, \quad (6.119)$$

where

$$F^3(E) = 6EV_0^2 \left(\sqrt{81V_1^2 - \frac{6E}{V_0}} - 9V_1 \right). \quad (6.120)$$

Let us simplify the equation. For instance, considering $V_1 = 0$, the singular solution of the Clairaut equation yields

$$f(E) = \frac{E^2}{4V_0}. \quad (6.121)$$

It should be noticed that if one substitutes the solution (6.121) into the hybrid master equation (3.39) one finds that the variable $X = 0$, namely, the solution is the GR case.

We can proceed in the same way for the potential given by (6.115)

$$Ef'(E) - f(E) = V_0 [1 + f'(E)]^2 \times \exp\left(\frac{6}{V_1} \arctan \sqrt{f'(E)}\right), \quad (6.122)$$

so that the singular solution follows from the equation

$$[1 + f'(E)] \left(2 + \frac{3}{V_1 \sqrt{f'(E)}} \right) \times \exp\left(\frac{6}{V_1} \arctan \sqrt{f'(E)}\right) + E = 0. \quad (6.123)$$

The physical meaning of potentials (6.112) and (6.115) has to be discussed in detail. The term $\sqrt{\phi}$ seems a limitation on the range of validity of the whole theory. Nevertheless, a careful investigation shows that $\phi > 0$ can be interpreted as a *quintessence field* while $\phi < 0$ is a *phantom field*. In terms of the previous variable $f'(E)$, quintessence-phantom regimes are divided by GR, restored for $\phi = 0$. In other words, the apparent limitation due to the square root into the potential points out nothing else but a change of regime into dynamics.

Hybrid Gravity as a Brans-Dicke-like theory

We now apply the conformal transformation $\bar{g}_{ij} = N(\phi)^{-2} g_{ij}$ in the FRW space-time (6.88). Under this transformation, the action of Hybrid Gravity (6.84) becomes

$$S = \int d^4x \sqrt{-\bar{g}} [(1 + \phi) \bar{R} + \frac{3}{2\phi} \bar{g}^{ij} \phi_{,i} \phi_{,j} - V(\phi)], \quad (6.124)$$

where the conformal Ricci scalar is given by

$$\bar{R} = N^{-2} R - 6N^{-3} g^{ij} N_{;ij}.$$

Substituting it into the action (6.124) one finds

$$\begin{aligned} S = & \int d^4x \sqrt{-g} [(1 + \phi) N^2 R - 6(1 + \phi) N g^{ij} N_{;ij} \\ & + \frac{3}{2\phi} N^2 g^{ij} \phi_{,i} \phi_{,j} - N^4 V(\phi)]. \end{aligned} \quad (6.125)$$

Taking into account the following lapse function

$$N(\phi) = \sqrt{\frac{F(\phi)}{1 + \phi}}, \quad (6.126)$$

then

$$N_{;i} = \frac{1}{2} \sqrt{\frac{1 + \phi}{F(\phi)}} \left(\frac{F_{,\phi}}{1 + \phi} - \frac{F}{(1 + \phi)^2} \right) \phi_{,i}. \quad (6.127)$$

Substituting the results into the various terms of Eq. (6.125), we get the following relations

$$\int d^4x \sqrt{-g} [(1 + \phi) N^2 R] = \int d^4x [F(\phi) R], \quad (6.128)$$

$$\begin{aligned} \int d^4x \sqrt{-g} \frac{3}{2\phi} N^2 g^{ij} \phi_{,i} \phi_{,j} = \\ \int d^4x \sqrt{-g} \frac{3F(\phi)}{2(1 + \phi)\phi} g^{ij} \phi_{,i} \phi_{,j}, \end{aligned} \quad (6.129)$$

and a lengthy, but straightforward calculation leads to

$$\begin{aligned} \int d^4x \sqrt{-g} 6(1 + \phi) N g^{ij} N_{;ij} = \\ - \int d^4x \sqrt{-g} \left[\frac{3}{2} \frac{F(\phi) - (1 + \phi)^2 F_{,\phi}^2}{(1 + \phi) F(\phi)} g^{ij} \phi_{,i} \phi_{,j} \right]. \end{aligned} \quad (6.130)$$

Using the above relations, the action (6.125) takes the form

$$S = \int d^4x \sqrt{-g} \left[F(\phi) R + \frac{3}{2} \left(\frac{F(\phi) - (1+\phi)^2 F_{,\phi}^2}{(1+\phi) F(\phi)} + \frac{3F(\phi)}{2(1+\phi)\phi} \right) g^{ij} \phi_{,i} \phi_{,j} - \frac{F^2(\phi)}{(1+\phi)^2} V(\phi) \right]. \quad (6.131)$$

Moreover, defining a new scalar field Φ (*i.e.* a coordinate transformation)

$$d\Phi = \sqrt{3 \left[\frac{F(\phi) - (1+\phi)^2 F_{,\phi}^2}{(1+\phi) F(\phi)} + \frac{3F(\phi)}{2(1+\phi)\phi} \right]} d\phi,$$

the action becomes

$$S = \int d^4x \sqrt{-g} \left[F(\Phi) R + \frac{1}{2} g^{ij} \Phi_{,i} \Phi_{,j} - \bar{V}(\Phi) \right], \quad (6.132)$$

with

$$\bar{V}(\Phi) = \frac{F^2(\Phi)}{(1+\Phi)^2} V(\Phi). \quad (6.133)$$

The classification of Noether symmetries for the Lagrangian in the action (6.132) has been completely achieved in [180] and previously in [53]. When $F(\Phi) = F_0 \Phi^2$ we have a Brans-Dicke-like scalar field with a potential. Nevertheless, for $F_0 = -1/12$ the Brans-Dicke scalar field gives $\omega_0 = -3/2$ and the Lagrangian of the field equations is singular [103]. In that case the theory is equivalent to the Palatini $f(\mathcal{R})$ [94]. Moreover, when $F(\phi) = F_0$, *i.e.*, $N(\phi) = \sqrt{F_0/(1+\phi)}$, the action (6.132) describes a minimally coupled scalar field and the results in [183] can be applied.

In order to obtain exact solutions, one can apply a conformal transformation of the form $\bar{g}^{\mu\nu} = N^{-2}(a, \phi) g^{\mu\nu}$. It is important to stress that the Wheeler-DeWitt equation, coming from the Hamiltonian of the theory, is conformally invariant and this means that the solutions that we find remain invariant under conformal transformations. The same feature holds for classical solutions. However, it is not always possible to write exact solutions in a close form in any conformal frame.

6.5.1 Exact and invariant solutions

Let us now determine the exact solution of the field equations for the models with potentials (6.112) and (6.115).

Searching for exact solutions

I. The case of potential $V(\phi) = V_0(\sqrt{\phi} + V_1)^4$

Lagrangian, Hamiltonian, and field equations

We consider the following coordinate transformation

$$a = Cv + u, \quad \phi = \left(\frac{v}{Cv + u} - V_1 \right)^2, \quad (6.134)$$

where $C = V_1/(1 + V_1^2)$. In this new coordinates, the Lagrangian (6.107) becomes

$$\mathcal{L}(u, v, u', v') = 6(V_1^2 + 1)u'^2 + \frac{6}{(V_1^2 + 1)}v'^2 + V_0v^4. \quad (6.135)$$

Performing a second coordinate transformation

$$x = \sqrt{12(V_1^2 + 1)}u, \quad (6.136)$$

$$y = \sqrt{\frac{12}{(V_1^2 + 1)}}v, \quad (6.137)$$

the Lagrangian (6.135) is

$$\mathcal{L}(x, y, x', y') = \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \bar{V}_0y^4, \quad (6.138)$$

where $\bar{V}_0 = \frac{V_0}{144}(V_1^2 + 1)^2$. The Hamiltonian is given by

$$\tilde{H} = \frac{1}{2}p_x'^2 + \frac{1}{2}p_y'^2 - \bar{V}_0y^4, \quad (6.139)$$

where p_x, p_y are the momenta. The field equations are the Hamilton equations

$$x' = p_x, \quad y' = p_y \quad (6.140)$$

$$p_x' = 0, \quad p_y' = 4\bar{V}_0y^3, \quad (6.141)$$

and the Hamiltonian constraint is $\tilde{H} = 0$. Moreover, the Hamilton-Jacobi equation for the Hamiltonian (6.139) provides the following action

$$S = c_1x + \int \sqrt{2\bar{V}_0y^4 - c_1^2} + S_0, \quad (6.142)$$

then the field equations reduces to

$$x' = c_1, \quad y' = \varepsilon \sqrt{2\bar{V}_0y^4 - c_1^2}. \quad (6.143)$$

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Exact solutions for these equations are given by

$$x(\tau) = x_1\tau + x_2 \quad (6.144)$$

and

$$\int \frac{dy}{\sqrt{2V_0y^4 - c_1^2}} = \varepsilon(\tau - \tau_0), \quad (6.145)$$

respectively, where $\varepsilon = \pm 1$. In the simplest case where $V_1 = 0$, we have the solution $a(\tau) = a_0\tau$. From the condition $dt = a(\tau)d\tau$, we find $\tau = \sqrt{t}$, that is $a(t) = a_0\sqrt{t}$ which is the radiation solution.

In the case where $c_1 = 0$ and $V_1 \neq 0$ from Eqs. (6.144) and (6.145), we have that

$$y(\tau) = -\varepsilon \frac{1}{\sqrt{2V_0}} \frac{1}{(\tau - \tau_0)}. \quad (6.146)$$

Therefore, the scale factor assumes the following form

$$a(\tau) = a_0(\tau - \tau_0) + a_1 - a_2 \frac{1}{\tau - \tau_0}. \quad (6.147)$$

From this result, we have

$$\tau - \tau_0 = \frac{1}{2a_0} \left(a - a_1 + \varepsilon \sqrt{a^2 - 2aa_1 + a_1^2 + 4a_0a_2} \right), \quad (6.148)$$

and for the Hubble function¹ $H(\tau) = a'/a^2$,

$$H(a) = a_0a^{-2} + 4a_0^2a_2 \left(a^3 - a_1a^2 + \varepsilon a^2 \sqrt{(a - a_1)^2 + 4a_0a_2} \right)^{-2}. \quad (6.149)$$

Thus, in order to have a real solution, the condition

$$(a - a_1)^2 + 4a_0a_2 \geq 0, \quad a \in \mathbb{R}, \quad (6.150)$$

must hold. This means that $a_0a_2 \geq 0$. Hence, if $a_2 = 0$, *i.e.*, $V_1 = 0$, we again obtain the radiation solution.

However, when $a_0 = 0$, from Eq. (6.147), we have that $(\tau - \tau_0) = a_2/(a_1 - a)$ and for the Hubble function

$$H(a) = a_2^{-1} (a_1a^{-1} - 1)^2. \quad (6.151)$$

¹Recall that $H = \frac{1}{a} \frac{da}{dt} = \frac{1}{a^2} \frac{da}{d\tau}$

Searching for exact solutions

Assuming at the present time ($a = 1$), we have that $H^2(a = 1) = H_0$ and from Eq. (6.151), we deduce $a_2^{-1} = H_0 / (|a_1| + 1)^2$. Finally, the Hubble function can be written in the following form

$$\frac{H^2(a)}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_f a^{-1} + \Omega_\Lambda, \quad (6.152)$$

where

$$\Omega_f = \frac{|4a_1|}{(|a_1| + 1)^4}, \quad \Omega_\Lambda = \frac{1}{(|a_1| + 1)^4}, \quad (6.153)$$

$$\Omega_r = \frac{|a_1|^4}{(|a_1| + 1)^4}, \quad \Omega_m = \frac{|4a_1|^3}{(|a_1| + 1)^4}, \quad (6.154)$$

$$\Omega_k = \frac{|6a_1|^2}{(|a_1| + 1)^4}, \quad (6.155)$$

where the meaning of the symbols is straightforward. That means that each power term of $\sqrt{\phi}$ of the power law potential (6.112), $V(\phi) = V_0 (\sqrt{\phi} + V_1)^4$, introduces a power term of the scale factor into the Hubble function. The corresponding fluids are: radiation, dust, curvature-like fluid, a DE fluid with equation of state $p_f = -\frac{2}{3}\rho_f$ and a cosmological constant. We note that the curvature-like fluid follows from the hybrid gravity and not from the geometry of the space-time, since we have considered a spatially flat FRW space-time. Furthermore, for large redshifts z the Hubble function (6.152) behaves like the radiation solution.

Moreover, from the conformal transformation $dt = a(\tau) d\tau$, we have that

$$\tau - \tau_0 = \exp [a_1 a_2^{-1} \tau_0 - a_2^{-1} t - W(w(t))] = (X(t))^{-1} \quad (6.156)$$

where $w(t) = -a_1 a_2^{-1} \exp [a_2^{-1} (a_1 \tau_0 - t)]$ and $W(t)$ is the Lambert W -function [186] by which we can write the exact solution for the scale factor $a(t)$. By substituting Eq. (6.156) in Eq. (6.147), we find the scale factor expressed in terms of the proper time t

$$a^2(t) = [a_2 X(t) - a_1]^2. \quad (6.157)$$

However, from the singularity constraint $a(t \rightarrow 0) = 0$, we find the constraint $\tau_0 = a_1^{-1} a_2 [\ln(a_1^{-1} a_2) - 1]$.

In Figure 6.5 we compare the behavior of the scale factor (6.157) with that of the standard Λ CDM-cosmology and the radiation solution. It can be observed that the

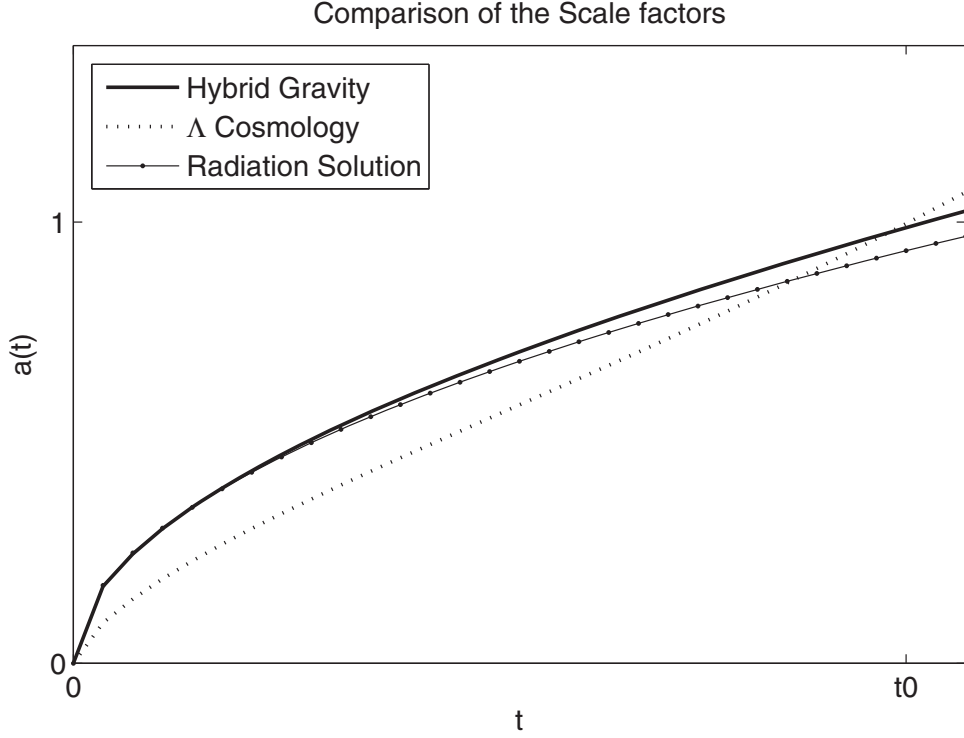


Figure 6.5: Comparison of the scale factor (6.157) with that of Λ CDM-cosmology $a_\Lambda(t)$. and the radiation solution $a_r(t) = a_{0r}\sqrt{t}$ where t_0 is the present time, $a_\Lambda(t_0) = 1$. For the solution (6.157) of the Hybrid Gravity, we set $|a_1| > 1$.

behavior of the scale factor (6.157) of the Hybrid Gravity is similar to the radiation solution in the early universe. Finally, it is important to stress that Hybrid Gravity contribution has a twofold meaning. When the condition $\phi > 0$ holds, it can be read as quintessence, while the condition $\phi < 0$ means phantom field. In both cases, it contributes to the bulk of DE in Eq.(6.152) and disappears as soon as GR is recovered. On the other hand, being GR related to the value $\phi = 0$, it corresponds to a sort of *phantom-quintessence divide* of the theory.

The Wheeler-DeWitt equation

We can define the Wheeler-DeWitt equation (recall that the dimension of the minisuperspace is two and the minisuperspace is flat) from the Hamiltonian (6.139)

$$\Psi_{,xx} + \Psi_{,yy} - 2V_0 y^4 \Psi = 0, \quad (6.158)$$

where Ψ is the Wave Function of the Universe [187]. Following the results in [185],

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one finds that Eq. (6.158) admits Lie point symmetries for the vector fields

$$X_\Psi = c_1 \partial_x + (c_2 \Psi + b(x, y)) \partial_\Psi, \quad (6.159)$$

$$X_b = b(x, y) \partial_\Psi, \quad (6.160)$$

where $b(x, y)$ is a function that satisfies the Wheeler-DeWitt equation (6.158). Hence, we can apply the zero order invariants to reduce Eq. (6.158).

From the Lie point symmetry X_Ψ , the invariant functions are $\{y, Y e^{\mu x}\}$, with $\mu \in \mathbb{C}$ [172], therefore Eq. (6.158) reduces to the following second-order ODE

$$Y_{,yy} + (\mu^2 - 2\bar{V}_0 y^4) Y = 0. \quad (6.161)$$

This equation is the one-dimensional time-dependent oscillator and it admits eight Lie point symmetries [188] which are all Type II hidden symmetries [189, 190]. Thus, we have that

$$Y(y) = y_1 e^{w(y)} + y_2 e^{-w(y)},$$

where

$$w(y) = \frac{\sqrt{2}}{2} \int \sqrt{(2\bar{V}_0 y^4 - \mu^2)} dy. \quad (6.162)$$

Finally, the invariant solution of the Wheeler-DeWitt equation (6.158) is given by

$$\Psi(x, y) = \sum_{\mu} [y_1 e^{\mu x + w(y)} + y_2 e^{\mu x - w(y)}]. \quad (6.163)$$

Following De Witt [187], this solution is the so-called Wave Function of the Universe. It is related to the probability that a universe, in particular our *observed universe*, emerges with some specific initial conditions. In the present case, according to the definition and the sign of the variables, such a solution can assume exponential or oscillatory behaviors, being $\mu \in \mathbb{C}$ and x, y depending on ϕ that can be positive or negative-defined. Thanks to the *Hartle criterion*, in the oscillatory case the wave function results peaked on conserved momenta and observable universes (*i.e.* classical cosmological solutions) come out (see [191] for a detailed discussion). Clearly, a singularity emerges for $\phi = 0$. This means that GR regime is restored and Big Bang singularity cannot be avoided.

II. The case of the potential $V(\phi) = V_0(1 + \phi)^2 \exp\left(\frac{6}{V_1} \arctan \sqrt{\phi}\right)$

Lagrangian, Hamiltonian, and field equations

Like before, we apply the following coordinate transformations for the considered potential

$$a = \frac{1}{\sqrt{12}} \frac{e^u}{\sqrt{\tan^2(v - V_1 u) + 1}}, \quad (6.164)$$

$$\phi = \tan^2(v - V_1 u). \quad (6.165)$$

Lagrangian (6.107) becomes

$$\begin{aligned} \mathcal{L}(r, \theta, r', \theta') &= \frac{1}{2} e^{2u} [(1 + V_1^2) u'^2 - 2V_1 u' v' + v'^2] \\ &\quad + \bar{V}_0 e^{-2u} e^{\frac{6}{V_1} v}, \end{aligned} \quad (6.166)$$

where $\bar{V}_0 = V_0/144$. The Hamiltonian of the system is

$$\tilde{H} = \frac{1}{2} e^{-2u} [p_u^2 + 2V_1 p_u p_v + (1 + V_1^2) p_v^2] - \bar{V}_0 e^{-2u} e^{\frac{6}{V_1} v}. \quad (6.167)$$

The Hamilton equations are

$$u' = e^{-2u} (p_u + V_1 p_v), \quad (6.168)$$

$$v' = e^{-2u} (V_1 p_u + (1 + V_1^2) p_v), \quad (6.169)$$

$$p'_v = \frac{6\bar{V}_0}{V_1} e^{-2u} e^{\frac{6}{V_1} v}, \quad (6.170)$$

$$p'_u = e^{-2u} (p_u^2 + 2V_1 p_u p_v + (1 + V_1^2) p_v^2) - 2\bar{V}_0 e^{-2u} e^{\frac{6}{V_1} v}, \quad (6.171)$$

respectively, and the Hamiltonian constraint provides $\tilde{H} = 0$. Moreover, from the Hamilton-Jacobi equation for Eq. (6.167), we have the action

$$\begin{aligned} S(u, v) &= \frac{c_1}{1 + V_1^2} u - c_1 \frac{V_1}{1 + V_1^2} v \\ &\quad + \frac{V_1}{3(1 + V_1^2)} \left(S_1(v) - c_1 \arctan \frac{S_1(v)}{c_1} \right), \end{aligned} \quad (6.172)$$

where

$$S_1(v) = 2(1 + V_1^2) V_0 e^{\frac{6}{V_1} v} - c_1^2. \quad (6.173)$$

Searching for exact solutions

Finally, the reduced dynamical system has the form

$$e^{2u}u' = c_1 \frac{1 - V_1^2}{1 + V_1^2} + \frac{V_1}{1 + V_1^2} S_1(v), \quad (6.174)$$

$$e^{2u}v' = S_1(v) - c_1 \frac{V_1^3}{1 + V_1^2}, \quad (6.175)$$

which is a system of two nonlinear first order differential equations. Nevertheless, in order to simplify the reduced system of Eqs. (6.174) and (6.175) and in order to write the exact solution of the field equations, we apply a second conformal transformation $ds = e^{2u}d\tau$, and the dynamical system becomes

$$\begin{aligned} \frac{du}{ds} &= C_1 + C_2 + C_3 e^{\frac{6}{V_1}v}, \\ \frac{dv}{ds} &= C_4 e^{\frac{6}{V_1}v} + C_5, \end{aligned}$$

where the constants are $C_{1..5} = C_{1..5}(V_0, V_1, c_1^2)$. The solution of the system can be written as follows

$$u(s) = -\frac{V_1 C_3}{6C_4} \ln \left\{ C_4 \left[\exp \left(\frac{6C_5}{V_1} (s + I_0) - u_1 \right) - 1 \right] \right\} + (C_1 + C_2)s + u_2, \quad (6.176)$$

$$v(s) = -\frac{V_1}{6} \ln \left\{ \frac{1}{C_5} \left[1 - \exp \left(\frac{6C_5}{V_1} (s + I_0) - u_1 \right) \right] \right\} + C_5 (s + u_1). \quad (6.177)$$

Observe that the second conformal transformation $\tau \rightarrow s$ is of the form $ds = \bar{N}(a, \phi) d\tau$.

The Wheeler-DeWitt equation

Let us define the Wheeler-DeWitt equation from the Hamiltonian (6.167)

$$\Psi_{,uu} + 2V_1 \Psi_{,uv} + (1 + V_1^2) \Psi_{,vv} - 4\bar{V}_0 e^{\frac{6}{V_1}v} \Psi = 0. \quad (6.178)$$

Following [185], we find that Eq. (6.178) admits the following vector fields as Lie point symmetries

$$X_1 = \partial_u, \quad X_\Psi = \Psi \partial_\Psi, \quad X_b = b(u, v) \partial_\Psi,$$

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$$\begin{aligned}
X_2 &= e^{-\frac{3v}{V_1}} [\cos(V_C u) \cos(3v) + \sin(V_C u) \sin(3v)] \partial_u \\
&\quad + e^{-\frac{3v}{V_1}} \left[(V_1 \cos(3v) - \sin(3v)) \cos(V_C u) \right. \\
&\quad \left. + (\cos(3v) + V_1 \sin(3v)) \sin(V_C u) \right] \partial_v, \tag{6.179}
\end{aligned}$$

$$\begin{aligned}
X_3 &= e^{-\frac{3v}{V_1}} [\cos(V_C u) \sin(3v) + \sin(V_C u) \cos(3v)] \partial_u \\
&\quad + e^{-\frac{3v}{V_1}} \left[(\cos(3v) + V_1 \sin(3v)) \cos(V_C u) \right. \\
&\quad \left. + (\sin(3v) - V_1 \cos(3v)) \sin(V_C u) \right] \partial_v, \tag{6.180}
\end{aligned}$$

where $V_C = 3(1 + V_1^2)/V_1$. Note that only the symmetry vector $X_1 = \partial_u$ is the generator of the Noether symmetry for the Lagrangian (6.166).

We apply the invariant symmetry vector $X = X_1 + \mu \Psi \partial_\Psi$, where the invariants are $\{v, Y e^{\mu u}\}$. Hence, Eq. (6.178) becomes

$$(1 + V_1^2) Y_{,vv} + 2\mu V_1 Y_{,v} + \left(\mu^2 - 4\bar{V}_0 e^{\frac{6}{V_1}v} \right) Y = 0. \tag{6.181}$$

This equation describes a time-dependent damped oscillator. It is well known that there exists a transformation $(v, Y) \rightarrow (\bar{v}, \bar{Y})$ where it can be written in the form $\bar{Y}_{,\bar{v}\bar{v}} = 0$, since it admits eight Lie point symmetries.

Therefore, the solution of Eq. (6.181) can be expressed in terms of Bessel functions

$$Y(v) = \exp(-3\bar{N}y) \left[c_1 J_{\bar{N}} \left(V_\mu e^{\frac{3}{V_1}v} \right) + c_2 Y_{\bar{N}} \left(V_\mu e^{\frac{3}{V_1}v} \right) \right],$$

where

$$\bar{N} = -\frac{V_1 \mu}{3(1 + V_1^2)}, \quad V_\mu = \frac{2}{3} \frac{V_1 \sqrt{V_0}}{\sqrt{1 + V_1^2}} i.$$

In conclusion, the invariant solution of Eq. (6.178) is

$$\begin{aligned}
\Psi(u, v) &= \sum_{\mu} \exp(\mu u - 3\bar{N}v) \left[c_1 J_{\bar{N}} \left(V_\mu e^{\frac{3}{V_1}v} \right) \right. \\
&\quad \left. + c_2 Y_{\bar{N}} \left(V_\mu e^{\frac{3}{V_1}v} \right) \right]. \tag{6.182}
\end{aligned}$$

We note that it is possible to apply the other Lie symmetries, *e.g.*, X_2, X_3 or any linear combination, in order to determine the invariant solution of the Wheeler-DeWitt equation (6.178). The interpretation of the Wave Function of the Universe is similar to the previous one but the presence of the Bessel functions has to be analyzed

for both asymptotic oscillatory and exponential regimes, depending on the sign of the arguments. Like above, the sign of the scalar field ϕ , *i.e.* of the Hybrid Gravity contribution, plays a main role in order to determine where and when the Hartle criterion is applicable (see also [192]).

6.6 Gauss-Bonnet Gravity

Let focus on the application of Noether approach to $f(R, \mathcal{G})$ gravity [193]. It is possible to select physically interesting forms of $f(R, \mathcal{G})$ asking for the existence of Noether symmetries and the existence of symmetries allows to select constants of motion that reduce dynamics. Furthermore, reduced dynamics results exactly solvable by a straightforward change of variables where a cyclic coordinate is present. As showed in [193], the method allows from one side to solve exactly the dynamics and from the other side, the Noether charge can always be related to some observable quantity. The procedure is based on the fact that both R and \mathcal{G} behave like effective scalar fields as soon as suitable Lagrange multipliers are introduced into dynamics [194]. This allows to define a suitable configuration space $\mathcal{Q} \equiv \{a, R, \mathcal{G}\}$, where a is the FRW scale factor. Therefore, one can search for the invariance of a Lie vector field X by the Lie derivative L_X acting on the point-like Lagrangian $\mathcal{L}(\dot{a}, a, \dot{R}, R, \dot{\mathcal{G}}, \mathcal{G})$.

Field equations of $F(R, \mathcal{G})$ -gravity

Let start by writing, as above, the most general action for modified Gauss-Bonnet gravity (3.40) without the contribution of standard matter Lagrangian \mathcal{L}_m that we will reconsider below,

$$\mathcal{S} = \int d^4x \sqrt{-g} F(R, \mathcal{G}), \quad (6.183)$$

where, as we said before, $F(R, \mathcal{G})$ is a function of the Ricci scalar and Gauss-Bonnet invariant defined in (3.41). The gravitational field equations without matter read [113]

$$\begin{aligned}
 G_{\mu\nu} = \frac{1}{F_R} & \left[\nabla_\mu \nabla_\nu F_R - g_{\mu\nu} \square F_R + 2R \nabla_\mu \nabla_\nu F_G \right. \\
 & - 2g_{\mu\nu} R \square F_G - 4R_\mu^\lambda \nabla_\lambda \nabla_\nu F_G - 4R_\nu^\lambda \nabla_\lambda \nabla_\mu F_G \\
 & + 4R_{\mu\nu} \square F_G + 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta F_G + 4R_{\mu\alpha\beta\nu} \nabla^\alpha \nabla^\beta F_G \\
 & \left. - \frac{1}{2} g_{\mu\nu} (R F_R + \mathcal{G} F_G - F(R, \mathcal{G})) \right]. \tag{6.184}
 \end{aligned}$$

6.6.1 Gauss-Bonnet cosmology

Cosmological equations can be obtained by deducing a point-like canonical Lagrangian $\mathcal{L}(a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}})$ from the action (6.183). The configuration space for the canonical Lagrangian will be $\mathcal{Q} \equiv \{a, R, \mathcal{G}\}$ and the tangent space on which \mathcal{L} is defined will be $\mathcal{TQ} \equiv \{a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}}\}$. The variables $a(t)$, $R(t)$ and $\mathcal{G}(t)$ are the scale factor, the Ricci scalar and the Gauss-Bonnet invariant defined in the FRW metric, respectively. The corresponding Euler-Lagrange equations will be

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{\partial \mathcal{L}}{\partial a}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{\partial \mathcal{L}}{\partial R}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{G}}} = \frac{\partial \mathcal{L}}{\partial \mathcal{G}}, \tag{6.185}$$

with the energy condition

$$E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{R}} \dot{R} + \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{G}}} \dot{\mathcal{G}} - \mathcal{L} = 0. \tag{6.186}$$

Here the dot indicates the derivatives with respect to the cosmic time t .

The method of the Lagrange multipliers can be used to set R and \mathcal{G} as constraints for dynamics. Then, integrating by parts to eliminate higher than one time derivatives, the Lagrangian \mathcal{L} becomes canonical. Using the signature $(+, -, -, -)$, we have

$$\mathcal{S} = \int dt a^3 \left\{ F(R, \mathcal{G}) - \lambda_1 \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] - \lambda_2 \left[\mathcal{G} - 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right) \right] \right\} \tag{6.187}$$

Here a spatially flat FRW space-time has been considered and the definitions of the Ricci scalar and the Gauss-Bonnet invariant on this metric has been adopted, that is

$$R = -6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \quad \mathcal{G} = 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right). \tag{6.188}$$

The Lagrange multipliers $\lambda_{1,2}$ are obtained by varying the action with respect to R and \mathcal{G} , that is

$$\lambda_1 = \frac{\partial F(R, \mathcal{G})}{\partial R}, \quad \lambda_2 = \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}}, \tag{6.189}$$

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then the above action becomes

$$\begin{aligned} \mathcal{S} = \int dt \left\{ a^3 F(R, \mathcal{G}) - a^3 \frac{\partial F(R, \mathcal{G})}{\partial R} \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \right. \\ \left. - a^3 \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \left[\mathcal{G} - 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right) \right] \right\}. \end{aligned} \quad (6.190)$$

After an integration by parts, the point-like Lagrangian assumes the form

$$\begin{aligned} \mathcal{L} = 6a\dot{a}^2 \frac{\partial F(R, \mathcal{G})}{\partial R} + 6a^2 \dot{a} \frac{d}{dt} \left(\frac{\partial F(R, \mathcal{G})}{\partial R} \right) - 8\dot{a}^3 \frac{d}{dt} \left(\frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right) \\ + a^3 \left[F(R, \mathcal{G}) - R \frac{\partial F(R, \mathcal{G})}{\partial R} - \mathcal{G} \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right], \end{aligned} \quad (6.191)$$

that is a canonical function of the coupled fields a , R and \mathcal{G} depending on the cosmic time t . It is important to underline that the Lagrange multipliers have been properly chosen by considering the definition of the Ricci curvature scalar R and the Gauss-Bonnet invariant \mathcal{G} . This allows to consider as canonical the constrained dynamics.

The Euler-Lagrange equations from Eqs. (6.185 - 6.186) are [193]

$$\begin{aligned} \left[\left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} \right] \frac{\partial F(R, \mathcal{G})}{\partial R} + \frac{d^2}{dt^2} \left(\frac{\partial F(R, \mathcal{G})}{\partial R} \right) + 2 \frac{\dot{a}}{a} \frac{d}{dt} \left(\frac{\partial F(R, \mathcal{G})}{\partial R} \right) \\ - 8 \frac{\ddot{a}\dot{a}}{a^2} \frac{d}{dt} \left(\frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right) - 4 \left(\frac{\dot{a}}{a} \right)^2 \frac{d^2}{dt^2} \left(\frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right) - \frac{1}{2} \left[F(R, \mathcal{G}) \right. \\ \left. - R \frac{\partial F(R, \mathcal{G})}{\partial R} - \mathcal{G} \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right] = 0, \end{aligned} \quad (6.192)$$

$$\left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + \left[\mathcal{G} - 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right) \right] \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0, \quad (6.193)$$

$$\left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + \left[\mathcal{G} - 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right) \right] \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} = 0. \quad (6.194)$$

It is worth noticing that the form of Eqs. (6.193) and (6.194) show a symmetry in the variables R and \mathcal{G} . Finally the energy condition (6.186), corresponding to the

00-Einstein equation, is

$$\begin{aligned} & \left(\frac{\dot{a}}{a}\right)^2 \frac{\partial F(R, \mathcal{G})}{\partial R} + \left(\frac{\dot{a}}{a}\right) \frac{d}{dt} \left(\frac{\partial F(R, \mathcal{G})}{\partial R} \right) - 4 \left(\frac{\dot{a}}{a}\right)^3 \frac{d}{dt} \left(\frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right) \\ & - \frac{1}{6} \left[F(R, \mathcal{G}) - R \frac{\partial F(R, \mathcal{G})}{\partial R} - \mathcal{G} \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right] = 0. \end{aligned} \quad (6.195)$$

For consistency, considering \mathcal{G} , R and a as variables, then R and \mathcal{G} coincides with the definitions of the Ricci scalar and Gauss-Bonnet invariant in the FRW metric, respectively, *i.e.* these are Euler's constraints of the dynamics.

6.6.2 Noether symmetries in Gauss-Bonnet cosmology

Following [193] let see how solutions of the system (6.192-6.195) can be achieved by asking for the existence of Noether symmetries.

As discussed in sec. 6.1 Noether symmetry for the Lagrangian (6.191) exists if the condition $L_X \mathcal{L} = 0 \rightarrow X \mathcal{L} = 0$ holds. Here L_X is the Lie derivative with respect to the Noether vector X . This condition is nothing else but the contraction of the Noether vector X , defined on the tangent space $\mathcal{TQ} = \{q_i, \dot{q}_i\} = \{a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}}\}$ of the Lagrangian $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i) = \mathcal{L}(a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}})$. In our case, the generator of symmetry is

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial \mathcal{G}} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}} + \dot{\gamma} \frac{\partial}{\partial \dot{\mathcal{G}}}. \quad (6.196)$$

The functions α, β, γ depend on the variables a, R, \mathcal{G} and then

$$\begin{aligned} \dot{\alpha} &= \frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial R} \dot{R} + \frac{\partial \alpha}{\partial \mathcal{G}} \dot{\mathcal{G}}, & \dot{\beta} &= \frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial R} \dot{R} + \frac{\partial \beta}{\partial \mathcal{G}} \dot{\mathcal{G}}, \\ \dot{\gamma} &= \frac{\partial \gamma}{\partial a} \dot{a} + \frac{\partial \gamma}{\partial R} \dot{R} + \frac{\partial \gamma}{\partial \mathcal{G}} \dot{\mathcal{G}}. \end{aligned} \quad (6.197)$$

As mentioned above, a Noether symmetry exists if at least one of them is different from zero. Their analytic forms corresponds to a set of partial differential equations obtained by equating to zero the terms in $\dot{a}^2, \dot{R}^2, \dot{\mathcal{G}}^2, \dot{a}\dot{R}, \dot{a}\dot{\mathcal{G}}, \dot{R}\dot{\mathcal{G}}$ and so on. A system of thirteen partial differential equations is found [193] considering the fact that canonical Lagrangian is defined by the Lagrange multipliers

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$$\alpha \frac{\partial F(R, \mathcal{G})}{\partial R} + \beta a \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + \gamma a \frac{\partial F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + 2a \left(\frac{\partial \alpha}{\partial a} \right) \frac{\partial F(R, \mathcal{G})}{\partial R} + a^2 \left(\frac{\partial \beta}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + a^2 \left(\frac{\partial \gamma}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$2\alpha \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + \beta a \frac{\partial^3 F(R, \mathcal{G})}{\partial R^3} + \gamma a \frac{\partial^3 F(R, \mathcal{G})}{\partial R^2 \partial \mathcal{G}} + a \left(\frac{\partial \alpha}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + 2 \left(\frac{\partial \alpha}{\partial R} \right) \frac{\partial F(R, \mathcal{G})}{\partial R} + a \left(\frac{\partial \beta}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + a \left(\frac{\partial \gamma}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$\beta \frac{\partial^3 F(R, \mathcal{G})}{\partial \mathcal{G}^2 \partial R} + \gamma \frac{\partial^3 F(R, \mathcal{G})}{\partial \mathcal{G}^3} + 3 \left(\frac{\partial \alpha}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} + \left(\frac{\partial \beta}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + \left(\frac{\partial \gamma}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} = 0$$

$$\left(\frac{\partial \alpha}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} = 0$$

$$\left(\frac{\partial \alpha}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} + \left(\frac{\partial \alpha}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$\left(\frac{\partial \alpha}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + \left(\frac{\partial \alpha}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} = 0$$

$$2\alpha \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + \beta a \frac{\partial^3 F(R, \mathcal{G})}{\partial \mathcal{G} \partial R^2} + \gamma a \frac{\partial^3 F(R, \mathcal{G})}{\partial \mathcal{G}^2 \partial R} + a \left(\frac{\partial \alpha}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + 2 \left(\frac{\partial \alpha}{\partial \mathcal{G}} \right) \frac{\partial F(R, \mathcal{G})}{\partial R} + a \left(\frac{\partial \beta}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + a \left(\frac{\partial \gamma}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$\left(\frac{\partial \beta}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + \left(\frac{\partial \gamma}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} = 0$$

$$\beta \frac{\partial^3 F(R, \mathcal{G})}{\partial \mathcal{G} \partial R^2} + \gamma \frac{\partial^3 F(R, \mathcal{G})}{\partial \mathcal{G}^2 \partial R} + \left(\frac{\partial \beta}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} + \left(\frac{\partial \gamma}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} + 3 \left(\frac{\partial \alpha}{\partial a} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$\left(\frac{\partial \alpha}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} = 0$$

$$\left(\frac{\partial \alpha}{\partial R} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$\left(\frac{\partial \alpha}{\partial \mathcal{G}} \right) \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0$$

$$3\alpha \left[F(R, \mathcal{G}) - R \frac{\partial F(R, \mathcal{G})}{\partial R} - \mathcal{G} \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} \right] - \beta a \left[R \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + \mathcal{G} \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G} \partial R} \right] - \gamma a \left[\mathcal{G} \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} + R \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} \right] = 0$$

(6.198)

Searching for exact solutions

The system (6.198) is overdetermined and, if solvable, enables one to assign α, β, γ and $F(R, \mathcal{G})$. The analytic form of $F(R, \mathcal{G})$ can be fixed by imposing, in the last equation of system (6.198), the conditions

$$\begin{cases} F(R, \mathcal{G}) - R \frac{\partial F(R, \mathcal{G})}{\partial R} - \mathcal{G} \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} = 0 \\ R \frac{\partial^2 F(R, \mathcal{G})}{\partial R^2} + \mathcal{G} \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G} \partial R} = 0 \\ \mathcal{G} \frac{\partial^2 F(R, \mathcal{G})}{\partial \mathcal{G}^2} + R \frac{\partial^2 F(R, \mathcal{G})}{\partial R \partial \mathcal{G}} = 0 \end{cases} \quad (6.199)$$

where the second and third equations are symmetric. It is clear that this is nothing else but an arbitrary choice since more general conditions are possible. In [193] the following functional forms are chosen

$$F(R, \mathcal{G}) = f(R) + f(\mathcal{G}), \quad F(R, \mathcal{G}) = f(R)f(\mathcal{G}), \quad (6.200)$$

and the functional forms of $F(R, \mathcal{G})$ compatible with the system (6.199) are

$$F(R, \mathcal{G}) = F_0 R + F_1 \mathcal{G}, \quad F(R, \mathcal{G}) = F_0 R^n \mathcal{G}^{1-n}. \quad (6.201)$$

This allows to work out cosmological models compatible with the Noether symmetries [193].

6.6.3 Examples of exact cosmological solutions

Let us consider some examples showed in [193], where the existence of the symmetry allows a suitable reduction of the dynamical system and a full control of the problem based on first principles.

The case $F(R, \mathcal{G}) = F_0 R + F_1 \mathcal{G}$

Considering the functional form $F(R, \mathcal{G}) = F_0 R + F_1 \mathcal{G}$ The first equation of system (6.198) gives

$$\alpha + 2a \frac{d\alpha}{da} = 0. \quad (6.202)$$

Searching for exact solutions

Except the last one, the others equations of the system (6.198) are identically zero. The Noether symmetry is given by

$$\alpha = \frac{\alpha_0}{\sqrt{a}}, \quad \beta = 0, \quad \gamma = 0. \quad (6.203)$$

This is nothing else but GR as it can be expected from the fact that any linear combination of the Gauss-Bonnet invariant does not contribute to the effective Lagrangian in 4 dimensions. Therefore, we have

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} = 0, \quad (6.204)$$

$$R + 6 \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} \right] = 0, \quad (6.205)$$

$$\mathcal{G} - 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right) = 0, \quad (6.206)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = 0, \quad (6.207)$$

and hence the Minkowski space-time is recovered in vacuum while standard Friedman solutions are recovered when standard perfect fluid matter is considered.

The case $F(R, \mathcal{G}) = F_0 R^n \mathcal{G}^{1-n}$,

We consider now a more interesting case where the Noether symmetry rules the relation between R and \mathcal{G} . Taking into account the simplest non-trivial case $n = 2$, the functional form of $F(R, \mathcal{G})$ is $F(R, \mathcal{G}) = F_0 \frac{R^2}{\mathcal{G}}$. Accordingly, the point-like Lagrangian is

$$\mathcal{L} = \frac{4F_0\dot{a}}{\mathcal{G}} \left[3a\dot{a}R + 3a\dot{R} - 3a^2\dot{\mathcal{G}}\frac{R}{\mathcal{G}} + 4\dot{a}^2\dot{R}\frac{R}{\mathcal{G}} - 4\dot{a}^2\dot{\mathcal{G}}\left(\frac{R}{\mathcal{G}}\right)^2 \right]. \quad (6.208)$$

To solve the system, the variable $\frac{R}{\mathcal{G}} = \zeta$ has been chosen in [193] and then the Lagrangian (7.34) becomes

$$\mathcal{L} = 12a\dot{a}^2F_0\zeta + 12a^2\dot{a}F_0\dot{\zeta} + 16\dot{a}^3F_0\zeta\dot{\zeta}. \quad (6.209)$$

Searching for exact solutions

The Euler-Lagrange equations are

$$6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] + 24 \left(\frac{\ddot{a}\dot{a}^2}{a^3} \right) \zeta = 0, \quad (6.210)$$

$$\left[2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] \zeta + 2 \frac{\dot{a}}{a} \dot{\zeta} + \ddot{\zeta} + 8 \left(\frac{\dot{a}\ddot{a}}{a^2} \right) \zeta \dot{\zeta} + 4 \left(\frac{\dot{a}}{a} \right)^2 \dot{\zeta}^2 \quad (6.211)$$

$$+ 4 \left(\frac{\dot{a}}{a} \right)^2 \zeta \ddot{\zeta} = 0. \quad (6.212)$$

and the energy condition is

$$\left(\frac{\dot{a}}{a} \right)^2 + \left(\frac{\dot{a}}{a} \right) \left(\frac{\dot{\zeta}}{\zeta} \right) + 4 \left(\frac{\dot{a}}{a} \right)^3 \dot{\zeta} = 0. \quad (6.213)$$

Clearly, eq. (6.210) is immediately verified as soon as definitions (6.188) are replaced and hence is a consistency condition. A class of power law solutions is found,

$$a(t) = a_0 t^s, \quad \zeta = \zeta_0 t^2, \quad \text{with } s = 3, \quad (6.214)$$

with $\zeta_0 = -5/72$. Another solution has the form

$$a(t) = a_0 \exp(\Lambda t), \quad \ln \left(\frac{\zeta}{\zeta_0} \right) + 4\Lambda^2 (\zeta - \zeta_0) = -\Lambda t, \quad (6.215)$$

where Λ is a constant [193].

Chapter 7

The cosmic history by Extended gravity

Inflationary cosmology has been developed to remedy serious problems and shortcomings in the Cosmological Standard Model at early stages of its evolution [88, 207, 208, 209, 210].

The general aim is to address problems like the initial singularity, the cosmological horizon, the cosmic microwave background isotropy (and the related anisotropies generated, in principle, with initial quantum fluctuations), the large scale structure formation and evolution, the absence of magnetic monopoles and so on [159, 211, 212, 213, 214]. Then, inflationary theory allowed us to understand why our universe is so large and flat, why it is homogeneous and isotropic, why its different parts started their expansion simultaneously. According to this theory, the universe at the very early stages of its evolution rapidly expanded (inflated) in a slowly changing vacuum-like state, which is usually associated with a scalar field with a large energy density. However, the inflationary mechanism can be achieved in several different ways considering not only a primordial scalar fields but also geometric corrections into the effective gravitational action.

The main ingredient of all these scenarios is the claim that an inflationary phase occurs at some stage in the early universe and that one or more sources, different from standard ordinary matter, give rise to accelerated cosmic expansion. Such an expansion can be a single or a multiple event often related to the formation of structure at large and at very large scale. Generally, inflationary scenarios originated from some

fundamental theory like quantum gravity, strings, M-theory or GUT models. Reversing the argument, inflationary models and observables related to inflation can be used to probe fundamental theories (see, for example the latest results of the PLANCK and BICEP2 collaborations [204, 205, 215]).

In particular, quantum fluctuations of a given scalar field *i.e.* the *inflaton*, gives a mechanism for the origin of large scale structure. In other words, inflation gives rise to density perturbations that exhibit a scale invariant spectrum. Such a feature, in principle, is directly observed by measuring the temperature anisotropies in cosmic microwave background [205, 217, 218, 219, 220, 221]. Apart the general features, the possibilities to realize inflation are several. For example, in the *old inflation*, inflaton is trapped in a false vacuum phase through a first order transition, while, in the *new inflation*, expansion ends up with a second order phase transition after a slow rolling phase [207, 208, 209]. According to the problems to address, there are several different inflationary models, for example the power law inflation, the hybrid inflation, the oscillating inflation, the trace- anomaly driven inflation, the *k*-inflation, the ghost-inflation, the tachyon inflation and so on [222, 223, 224, 225, 226, 227, 228]. Furthermore, some of these models have no potential minimum and the inflationary mechanism appears different compared to the standard one. See for example the quintessential inflation [229] or the tachyon inflation [230, 231, 232, 233, 234, 235].

A natural way to achieve inflation is considering higher-order curvature corrections in the Hilbert-Einstein Lagrangian [29, 77, 236, 237, 57, 238, 239, 94, 240, 241, 242]. The first and well-known example of this approach is the Starobinsky model [88] where inflation is essentially driven by R^2 contributions, being R the Ricci curvature scalar. After this preliminary model, other higher-order curvature terms have been taken into account [243, 245, 244, 42, 246, 247, 248, 49]. The philosophy is that, in the early higher-curvature regime, such further curvature invariants come out as renormalization terms in quantum field theories in curved spacetime [38]. Furthermore, under conformal transformations, the theory becomes minimally coupled in the Einstein frame. In this frame, the conformal scalar field assumes the role of *inflaton* and leads the primordial acceleration [49]. However, more than one scalar field can be achieved by conformal transformations disentangling the degrees of freedom present in the Jordan

frame.

In order to explain both the early and the late-time acceleration in a geometrical way [253], without invoking huge amount of dark energy or, sometime, ill-defined scalar fields, several combinations of curvature invariants, like $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$... can be considered into the action [249, 250, 251, 252]. Among these attempt, a key role can be played by the Gauss-Bonnet topological invariant \mathcal{G} that naturally arises in the process of quantum field theory regularization and renormalization in curved space-time [38].

In particular, it contributes to the trace anomaly where higher-order curvature terms are present [254]. In some sense, considering a theory where both R and \mathcal{G} are non-linearly present exhausts the budget of curvature degrees of freedom needed to extend GR since the Ricci scalar and both the Ricci and the Riemann tensors are present in the definition of \mathcal{G} . From the inflation point of view, introducing \mathcal{G} beside R gives the opportunity to achieve a double inflationary scenario where the two acceleration phases are led by \mathcal{G} and R respectively. As we will see below, this happens as soon as both R and \mathcal{G} appear in non-linear combinations since linear R means just GR (and then no inflation) and linear \mathcal{G} identically vanishes in four dimensional gravitational action, being an invariant. On the other hand, the combination of both terms seems to improve the inflationary mechanism since one achieves a R -dominated phase and a \mathcal{G} -dominated phase. The second leads the Universe at very early stages of its evolution because \mathcal{G} is quadratic in curvature invariants and then it is dominant in stronger curvature regimes. Specifically, using a non linear function of \mathcal{G} , inserted into the $f(R)$ approach, that is a $F(R, \mathcal{G})$ function, *extends* the Starobinsky model since the whole curvature “interactions”, present in the early Universe, are taken into account. In view of the recent results by the PLANCK [215] and BICEP2 [204] collaborations, the potential advantages of this class of models, compared to the original Starobinsky one, could be that curvature degrees of freedom (in particular the scalaron R) result better constrained (see [205] for a detailed discussion). A first study in this sense is in the paper by Ivanov and Toporensky [216], where cosmological dynamics of fourth order gravity is studied in presence of Gauss-Bonnet term.

In [4] we discuss the possibility to obtain inflation considering a generic $F(R, \mathcal{G})$

theory where, in principle, both R and \mathcal{G} are non-linear in the action. All the recent studies on models of this type [255, 256, 257, 193, 113, 203, 108, 109] put in evidence the fact that the Gauss- Bonnet topological invariant can solve some shortcomings of the original $f(R)$ gravity and contributes, in non trivial way, to the accelerated expansion.

7.1 Cosmological inflation in Gauss-Bonnet Gravity

Considering again the above Gauss-Bonnet Action (6.183) and field equations (6.184), the trace equation has the form

$$3[\square F_R + V_R] + R[\square F_G + W_G] = 0, \quad (7.1)$$

where \square is the d'Alembert operator in curved space-time and

$$F_R \equiv \frac{\partial F(R, \mathcal{G})}{\partial R}, \quad F_G \equiv \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}}, \quad (7.2)$$

are the partial derivatives with respect to R and \mathcal{G} . It is possible to define two different potentials that depend on the scalar curvature and the Gauss-Bonnet invariant that enter the trace equation with their partial derivatives

$$V_R = \frac{\partial V}{\partial R} = \frac{1}{3}[RF_R - 2F(R, \mathcal{G})], \quad (7.3)$$

$$W_G = \frac{\partial W}{\partial \mathcal{G}} = 2\frac{\mathcal{G}}{R}F_G. \quad (7.4)$$

It is important to underline that, from Eqs.(6.184)-(7.1), GR is recovered as soon as $F(R, \mathcal{G}) = R$. Moreover, when \mathcal{G} is not considered, we are exactly in the $f(R)$ gravity context. Clearly, as in the case of the Starobinsky R scalaron, \mathcal{G} plays the role of a further scalar field whose dynamics is given by the Klein-Gordon-like Eq. (7.1). This means that we can expect a natural double inflation where both geometric fields play a role. As for the R scalaron, we can expect a *mass* for the \mathcal{G} scalaron which determine the “strength” of the \mathcal{G} -dominated inflation.

$F(R, \mathcal{G})$ double inflation

Let focus on the general features of $F(R, \mathcal{G})$ cosmology and inflation. Consider a flat FRW metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (7.5)$$

where $a(t)$ is the scale factor of the Universe. Inserting this metric into the action (1.14) and, as explained in the previous chapter, applying the method of Lagrange multipliers we obtain the point-like Lagrangian [193]. Hence, assuming suitable Lagrange multipliers for R and \mathcal{G} the point-like Lagrangian is

$$\mathcal{L} = 6a\dot{a}^2 F_R + 6a^2 \dot{a} \dot{F}_R - 8\dot{a}^3 \dot{F}_G + a^3 [F(R, \mathcal{G}) - R F_R - \mathcal{G} F_G], \quad (7.6)$$

which is a canonical function defined on the tangent space $\mathcal{TQ} \equiv \{a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}}\}$. Specifically, the Lagrangian (7.6) has a canonical form thanks to the Lagrange multipliers

$$R = 6 \left(2H^2 + \dot{H} \right), \quad (7.7)$$

$$\mathcal{G} = 24H^2 \left(H^2 + \dot{H} \right), \quad (7.8)$$

that are the field equations for the related dynamical system [193] (consider that now the Lagrange multipliers have been defined for the metric with signature $(-, +, +, +)$). Here $H = \frac{\dot{a}}{a}$ is the Hubble parameter and the overdot denotes the derivative with respect to the cosmic time t . The cosmological equations in term of H , are

$$\dot{H} = \frac{1}{2F_R + 8H\dot{F}_G} \left[H\dot{F}_R - \ddot{F}_R + 4H^3\dot{F}_G - 4H^2\ddot{F}_G \right], \quad (7.9)$$

$$H^2 = \frac{1}{6F_R + 24H\dot{F}_G} \left[F_R R - F(R, \mathcal{G}) - 6H\dot{F}_R + \mathcal{G}F_G \right], \quad (7.10)$$

where Eq. (7.10) is the energy condition, that is the $(0, 0)$ Einstein equation. The full dynamical system of $F(R, \mathcal{G})$ cosmology is given by Eqs. (7.7), (7.8), (7.9), (7.10).

In order to obtain inflation, the following conditions have to be satisfied

$$\left| \frac{\dot{H}}{H^2} \right| \ll 1, \quad \left| \frac{\ddot{H}}{H\dot{H}} \right| \ll 1. \quad (7.11)$$

It means, that the magnitude of the *slow-roll parameters*

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = -\frac{\ddot{H}}{2H\dot{H}}, \quad (7.12)$$

has to be small during inflation. Moreover, $\epsilon > 0$ is necessary to have $H < 0$. The acceleration is expressed as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \quad (7.13)$$

and then the accelerated expansion ends only when the slow-roll parameter ϵ is of the unit order.

In order to discuss a possible inflationary scenario, we choose the following Lagrangian [4]

$$F(R, \mathcal{G}) = R + \alpha R^2 + \beta \mathcal{G}^2, \quad (7.14)$$

where α and β are constants with dimension of a length squared and a length to the fourth power respectively. The linear term in R is included to produce the correct weak-field limit. Clearly, we have considered a R^2 model with a correction which adds new degrees of freedom due to the presence of the Gauss- Bonnet term. In the above Lagrangian, the term \mathcal{G}^2 is the first significant term in \mathcal{G} since the linear one gives no contribution¹. It is well known that a theory like $f(R) = R + \alpha R^2$ is capable of producing an inflationary scenario [88] not excluded from the last PLANCK release [160]. In [4] we concentrate on the question if such an inflationary scenario can be improved considering the whole curvature budget that can be encompassed by adding a non linear function of the Gauss-Bonnet invariant. As pointed out before, in such a case we can have a R -driven inflation led by the R^2 term and a \mathcal{G} -driven inflation led by \mathcal{G}^2 term. Nevertheless, this is nothing else but a toy model that should be improved by realistic forms of the $F(R, \mathcal{G})$ function.

To develop the above considerations, we consider the point-like Lagrangian (7.6). From analytical mechanics it is well known that any Lagrangian can be decomposed

¹Recall that in four dimensions, we have

$$\int d^4x \sqrt{-g} \mathcal{G} = 0. \quad (7.15)$$

This means that only a function of the Gauss- Bonnet invariant makes this integral non-trivial. On the other hand, in five or higher dimensions Eq.(7.15) is different from zero.

as

$$L = K(q_i, \dot{q}_j) - U(q_i), \quad (7.16)$$

where K and U are the kinetic energy and potential energy respectively with $q_i \equiv \{a, R, \mathcal{G}\}$ and $\dot{q}_j \equiv \{\dot{a}, \dot{R}, \dot{\mathcal{G}}\}$. For the Lagrangian density L of the point-like Lagrangian (7.6), related by $\mathcal{L} = a^3 L$, we have

$$K(a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}}) = 6 \left(\frac{\dot{a}}{a}\right)^2 F_R + 6 \left(\frac{\dot{a}}{a}\right) \dot{F}_R - 8 \left(\frac{\dot{a}}{a}\right)^3 \dot{F}_{\mathcal{G}}, \quad (7.17)$$

$$U(R, \mathcal{G}) = -[F(R, \mathcal{G}) - R F_R - \mathcal{G} F_{\mathcal{G}}], \quad (7.18)$$

and by assuming the specific model (7.14), we obtain

$$L = \overbrace{6 \left(\frac{\dot{a}}{a}\right)^2 (2\alpha R + 1) + 12\alpha \left(\frac{\dot{a}}{a}\right) \dot{R} - 16\beta \left(\frac{\dot{a}}{a}\right)^3 \dot{\mathcal{G}}}_{\text{kinetic energy}} - \underbrace{[\beta \mathcal{G}^2 + \alpha R^2]}_{\text{potential energy}}, \quad (7.19)$$

A qualitative shape of the potential $U(R, \mathcal{G})$ is reported in Fig. 7.1 and a possible slow-roll trajectory is shown.

Let us now stress the effective behaviour of the Lagrangian (7.14)

$$F(R) \simeq R + \alpha R^2 + \beta R^4. \quad (7.20)$$

In other words, as we already pointed out in the previous chapter, the correction to the R^2 model due to the presence of topological \mathcal{G}^2 term can be seen as a sort of $\sim R^4$ correction. Nevertheless, it is important to emphasize that \mathcal{G}^2 and R^4 have roughly the same dynamical role only at background level for the homogeneous and isotropic FRW metric. As soon as one takes into account anisotropies and inhomogeneities, \mathcal{G}^2 and R^4 assume different roles since extra diagonal components of the Ricci and Riemann tensors cannot be discarded. In other words, considering the definition of the Gauss-Bonnet invariant, Eq. (3.41), $\mathcal{G} \sim R^2$ only in the FRW context. In more

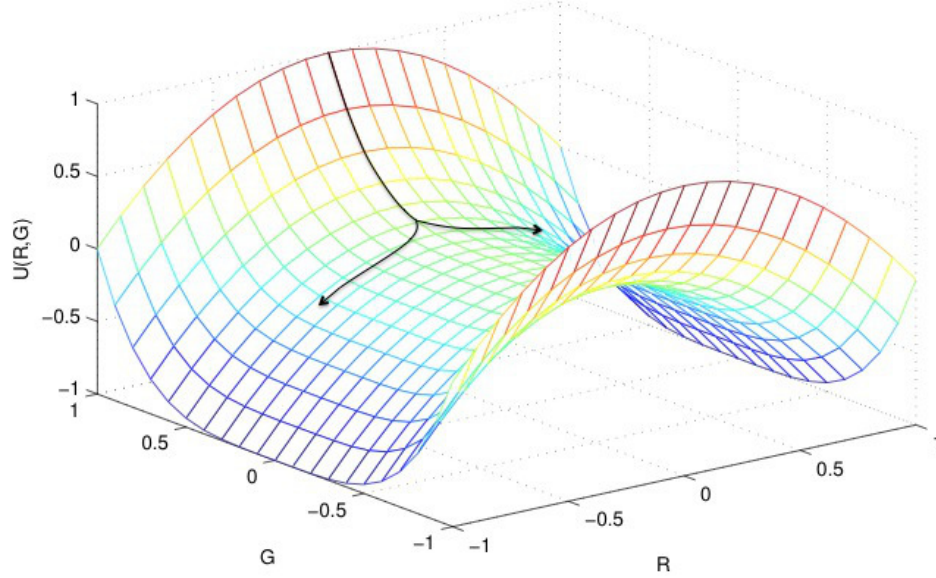


Figure 7.1: Plot of $U(R, \mathcal{G}) = \alpha R^2 + \beta \mathcal{G}^2$. We note that the two fields can both cooperate to the slow rolling phase. We assumed α and β of the order unit with negative α and positive β . The choice of negative α is due to the stability conditions for the R^2 model discussed in [195].

general situations this approximation no longer holds. This means that \mathcal{G}^2 and R^4 can be observationally distinguished only evaluating anisotropies and inhomogeneities resulting from perturbations where, as we already said, extra diagonal components of the Ricci and Riemann tensors are not negligible.

Let us describe now the qualitative evolution of the model. In Fig. 7.2, the trends of $U(R, \mathcal{G})$ sections are reported according to the dominance of the terms in the potential. The behaviour is different depending on the “strength” of R^2 or \mathcal{G}^2 terms. In fact, they give rise to a potential with two minima that can be separated by a barrier (see Fig. 7.2 in the bottom). This represent a double inflationary scenario where \mathcal{G} -scalar dominates at early epochs, at moderate early epochs dominate R -scalar and finally the model converges towards standard GR. Due to the fact that \mathcal{G} runs as $\mathcal{G} \simeq R^2$, it is dominant at very high curvature improving, in some sense, the Starobinsky inflation. In the present simple toy-model, we considered \mathcal{G}^2 and this means, as pointed out

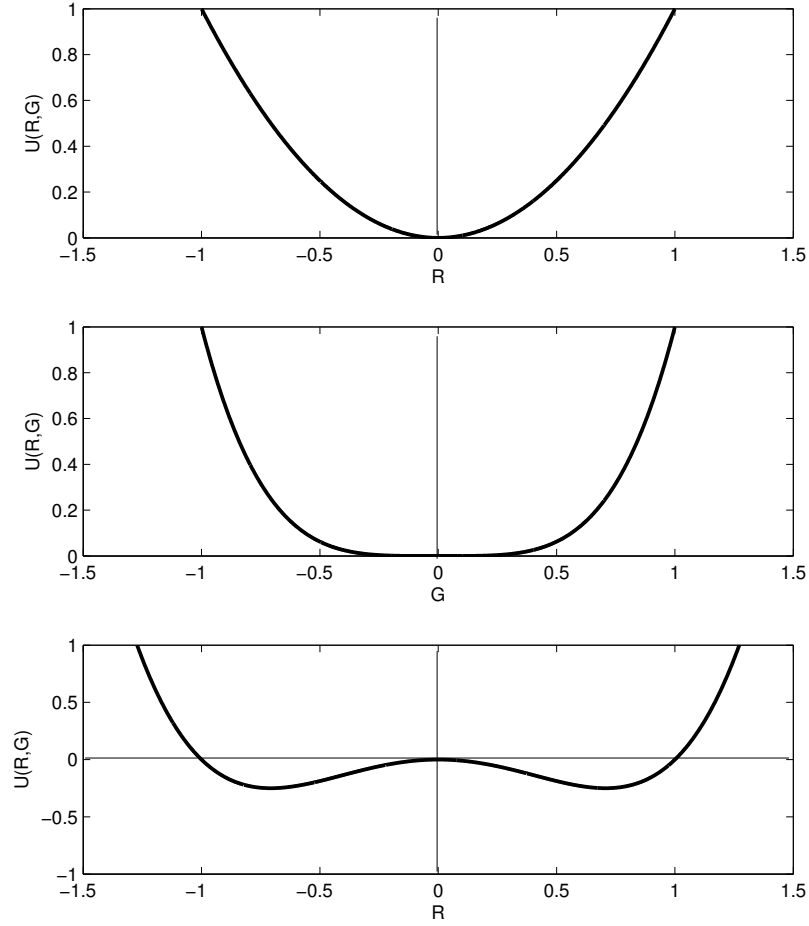


Figure 7.2: Plots of sections of the potential $U(R, \mathcal{G}) = \alpha R^2 + \beta \mathcal{G}^2$. In the top panel, is reported the section of the potential when the R^2 term is dominant. In the central panel, the case where the term \mathcal{G}^2 is dominant. In bottom panel, there is the behavior of $U(R, \mathcal{G}) = \alpha R^2 + \beta \mathcal{G}^2 \sim \alpha R^2 + \beta R^4$ with respect to the Ricci scalar. It is evident a symmetry breaking and a phase transition. The values of α and β are the same as in Fig.1.

above, that $\mathcal{G}^2 \sim R^4$. From the energy condition, given by Eq. (7.10), we have

$$\begin{aligned}
 &12\alpha H\ddot{H} + H^2 + 36\alpha H^2\dot{H} + 288\beta H^4\dot{H}^2 \\
 &+ 192\beta H^5\ddot{H} + 576\beta H^6\dot{H} - 96\beta H^8 - 6\alpha\dot{H}^2 = 0,
 \end{aligned} \tag{7.21}$$

and from (7.9), we obtain

$$\begin{aligned}
 & 576\beta H^2 \dot{H}^3 + 768\beta H^3 \dot{H} \ddot{H} + \beta H^4 (1728\dot{H}^2 + 96\ddot{H}) \\
 & + 288\beta H^5 \ddot{H} - 384\beta H^6 \dot{H}^2 \\
 & + 18\alpha H \ddot{H} + 24\alpha \dot{H}^2 + 6\alpha \ddot{H} + \dot{H} = 0.
 \end{aligned} \tag{7.22}$$

Considering the slow-roll conditions $\dot{H} \ll H^2$ and $\ddot{H} \ll H\dot{H}$, this implies that $\frac{\ddot{H}}{H} \ll \dot{H}$. From Eq.(7.21), one has

$$\begin{aligned}
 & H^2 + 6\alpha (2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2) \\
 & + 96\beta H^4 (3\dot{H}^2 + 2H\ddot{H} + 6H^2\dot{H} - H^4) = 0.
 \end{aligned} \tag{7.23}$$

In order to study the evolution of the model, we have to distinguish among the various regimes. Let us suppose that

$$6\alpha \gg 96\beta H^4 \tag{7.24}$$

Then Eq. (7.23) takes the form

$$H^2 + 6\alpha (2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2) \cong 0 \tag{7.25}$$

and we obtain that

$$m_R^2 = \frac{1}{6\alpha} \tag{7.26}$$

and the solution for the scale factor is

$$a(t) \sim \exp \left[\frac{t}{\sqrt{6\alpha}} \right]. \tag{7.27}$$

This is nothing else but the well known R^2 inflation where the sign and the value of α determine the number of e-foldings [196].

On the other hand, we can consider the regime

$$96\beta H^4 \gg 6\alpha, \tag{7.28}$$

where

$$H^2 + 96\beta H^4 (3\dot{H}^2 + 2H\ddot{H} + 6H^2\dot{H} - H^4) \cong 0. \tag{7.29}$$

Inflation is recovered for

$$H^6 \sim \frac{1}{96\beta}, \quad (7.30)$$

and then it is

$$a(t) \sim \exp \left[\frac{t}{\sqrt[6]{96\beta}} \right]. \quad (7.31)$$

From the above considerations, we can introduce a further mass term

$$m_{\mathcal{G}}^2 = \frac{1}{2\sqrt[3]{12\beta}}, \quad (7.32)$$

due to the Gauss-Bonnet correction that leads another earlier inflationary behaviour. In conclusion, it seems that considering the whole curvature budget in the effective action (*i.e.* the further combinations of curvature invariants more than the linear R) means to introduce two effective masses that lead the dynamics.

Let stress that the parameters α and β have to be consistent with the Solar System constraints according to the chameleon mechanism. In the low energy regime, GR has to be recovered and then the quadratic and quartic terms in R must be negligible. Essentially, starting from very early epochs, one has first to recover the Starobinsky model and then the Einstein regime. Furthermore, this means that the two-scalaron regimes, leading the two early inflationary phases have to become negligible for $R \rightarrow 0$ in order to recover the standard Newtonian potential. Therefore, the analysis in [161, 197] leads to assuming the values of the parameters α and β of the order unit to achieve the consistency with the chameleon mechanism and the Solar System experiments.

$F(R, \mathcal{G})$ power-law inflation

In the framework of $F(R, \mathcal{G})$ gravity also power-law inflation can be easily achieved. As pointed out in the previous chapter, using the Noether Symmetry Approach [53] in the generic action (1.14) and choosing appropriate Lagrangian multipliers that make the point-like Lagrangian canonical, models where conserved quantities emerge can be selected (see also [198]-[202] for analogue cases). This means to impose $L_X \mathcal{L} = 0$ as we have already seen. A possible choice is to consider the class of Lagrangians

$$F(R, \mathcal{G}) = F_0 R^n G^{1-n}, \quad (7.33)$$

The cosmic history by Extended gravity

related to the presence of the Noether symmetries [193]. For $n = 2$, it is $F(R, \mathcal{G}) = F_0 R^2 G^{-1}$. and with this choice the point-like Lagrangian (7.6) becomes

$$\begin{aligned} \mathcal{L} = & \frac{4F_0 \dot{a}}{\mathcal{G}} \left[3a \dot{a} R + 3a \dot{R} - 3a^2 \dot{\mathcal{G}} \left(\frac{R}{\mathcal{G}} \right) \right. \\ & \left. + 4\dot{a}^2 \dot{R} \left(\frac{R}{\mathcal{G}} \right) - 4\dot{a}^2 \dot{\mathcal{G}} \left(\frac{R}{\mathcal{G}} \right)^2 \right]. \end{aligned} \quad (7.34)$$

The same choice can be done into the cosmological Eqs. (7.9) and (7.10) that are nothing else but the Euler- Lagrange equations of the Lagrangian (7.34) together with the Lagrange multipliers (7.7) and (7.8). Power law solutions for these class of Lagrangians are easily found [193, 203]. For example, we have

$$a(t) = t^s, \quad \text{with } n = 2 \quad \text{and} \quad s = 3. \quad (7.35)$$

A further interesting solution is

$$a(t) = t^s, \quad \text{with } n = \frac{3}{4} \quad \text{and} \quad s = \frac{1}{2}. \quad (7.36)$$

The general conditions between the exponents n and s are

$$n = \frac{1+s}{2} \quad \text{and} \quad n = \frac{1}{1+2s(s-1)} - 2s. \quad (7.37)$$

When one of these conditions is satisfied, the constraints on R and \mathcal{G} are satisfied. It can be easily verified that solutions (7.35) and (7.36) are in one of these cases.

Let us now discuss inflation. For this purpose, we have to consider Eqs. (7.9) and (7.10) that now ore of the form

$$\dot{H} = -\frac{s(n-1)[n(6s-4) - 3s(s+1) + 4]}{[s(s-5) + 2n(2s-1) + 2]t^2}, \quad (7.38)$$

$$H^2 = -\frac{2s^2(s-1)(n-1)}{[s(s-5) + 2n(2s-1) + 2]t^2}. \quad (7.39)$$

The slow roll conditions are

$$\epsilon = \frac{2s(1-n) + 2(s-1) - s(s-1)}{2s(s-1)} \ll 1, \quad (7.40)$$

$$\eta = \frac{1}{\sqrt{2}} \sqrt{\frac{s^2(s-1)(n-1)}{s(s-5) + n(4n-2) + 2}} \ll 1. \quad (7.41)$$

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Considering the relation $n = \frac{(1+s)}{2}$, we obtain the the slow roll conditions ϵ and η are satisfied for $s > 2.171$. In conclusion, we can easily see that, for relatively large s , slow-roll conditions are satisfied. In Figs. 7.3 and 7.4, qualitative pictures of the parameter space regions where inflation is allowed are reported.

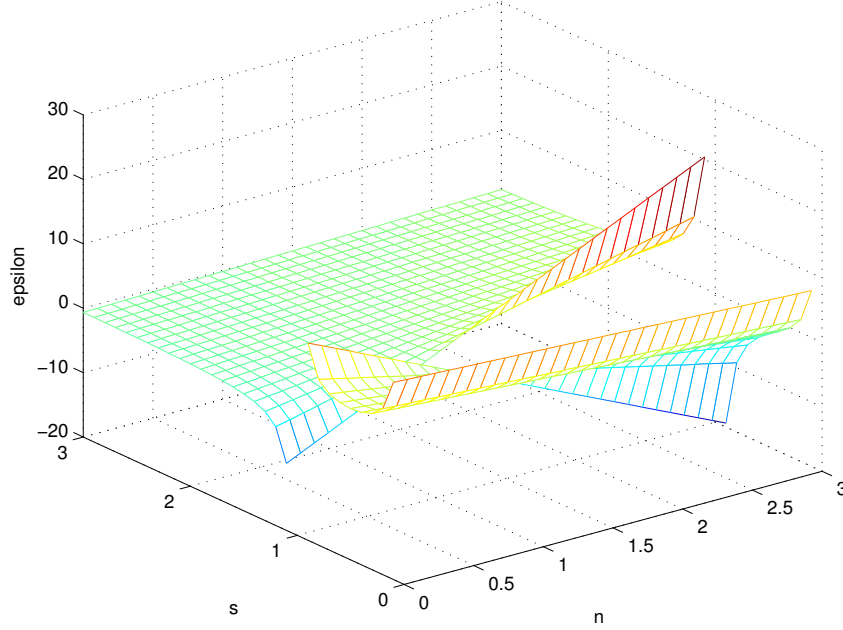


Figure 7.3: Plot of $\epsilon(n, s)$. The allowed region for inflation is the green one, in that region the value of ϵ is less than 1.

In addition, we can evaluate the anisotropies and the power spectrum coming from inflation using the slow-roll parameters. The spectral index n_s and the tensor-to-scalar ratio r are respectively

$$n_s = 1 - 6\epsilon + 2\eta, \quad r = 16\epsilon, \quad (7.42)$$

while the amplitude of the primordial power spectrum is

$$\Delta_{\mathcal{R}}^2 = \frac{\kappa^2 H^2}{8\pi^2 \epsilon}. \quad (7.43)$$

We obtain that the values $n_s \sim 1.01$ and $r \sim 0.10$ are in good agreement with the observational values of spectral index estimated by PLANCK data, *i.e.* $n_s = 0.9603 \pm 0.0073$ (68% CL) and $r < 0.11$ (95% CL) [160, 205]. These results are consistent also with the values measured by the BICEP2 collaboration [204].

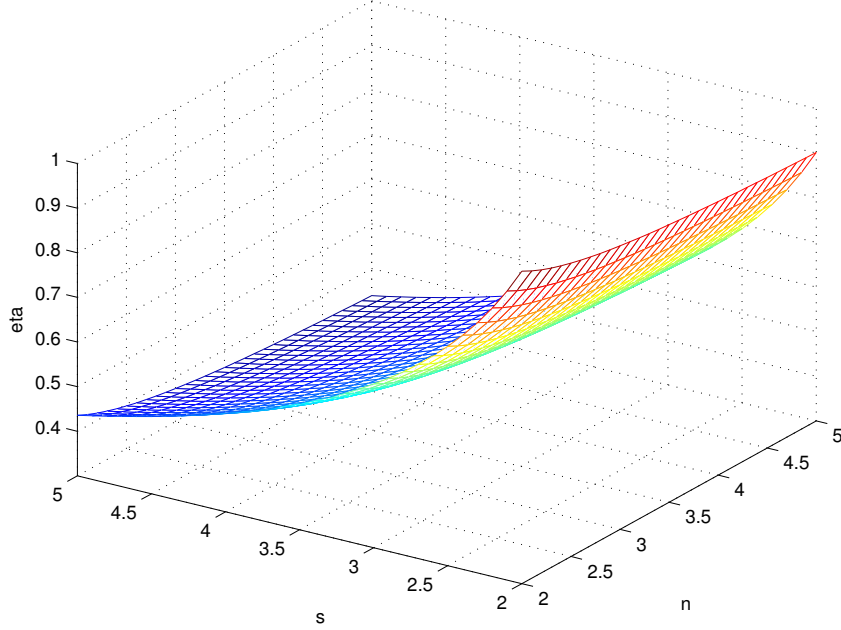


Figure 7.4: Plot of $\eta(n, s)$ parameters. The blue and green part in the figure are the allowed solutions.

Finally, it is possible to estimate the grow factor for the class of models $F(R, \mathcal{G}) = F_0 R^n G^{1-n}$. The equation which describes the evolution of the matter fluctuations in the linear regime is

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m = 0, \quad (7.44)$$

where ρ_m is the matter density and G_{eff} is the effective Newton coupling which, in our case, is

$$G_{\text{eff}} = \frac{G_N}{F_R(R, \mathcal{G})}, \quad (7.45)$$

where G_N is the Newton gravitational constant. However, we are considering perfect fluid matter that enters minimally coupled in action (1.14). We use Eq. (7.10) with matter density contribution as follow

$$4\pi G\rho_{(m)} = \frac{3H^2}{2} - 4\pi G\rho_{(\mathcal{G}B)}, \quad (7.46)$$

with

$$\rho_{(\mathcal{G}B)} = \frac{RF_R - F(R, \mathcal{G}) - 6H\dot{F}_R + \mathcal{G}F_{\mathcal{G}} - 24H^3\dot{F}_{\mathcal{G}}}{16\pi G_N}. \quad (7.47)$$

Inserting Eqs. (7.45) and (7.46) into Eq.(7.44), we obtain the equation

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{RF_R - F(R, \mathcal{G}) - 6H\dot{F}_R + \mathcal{G}F_{\mathcal{G}} - 24H^3\dot{F}_{\mathcal{G}}}{4F_R}\delta_m = 0. \quad (7.48)$$

Now, considering relations (7.37), we have $a(t) = a_0 t^s = a_0 t^{2n-1}$ and consequently $H = \frac{2n-1}{t}$, therefore, Eq.(7.48) becomes

$$\ddot{\delta}_m + \frac{2n-1}{t}\dot{\delta}_m + \frac{3(6n^2 - 6n - 1)}{2t^2}\delta_m = 0. \quad (7.49)$$

Eq.(7.49) is an Euler equation whose general solution is

$$\delta_m(t) = t^{\frac{1}{2}(-\sqrt{3-8n^2}-4n+3)} \left(c_2 t^{\sqrt{3-8n^2}} + c_1 \right). \quad (7.50)$$

Since $a(z) = (1+z)^{-1}$ we have that

$$H = H_0 a^{-\frac{1}{2n-1}} = H_0 \left(\frac{1}{1+z} \right)^{\frac{1}{2n-1}}, \quad (7.51)$$

where H_0 is the Hubble constant that can be chosen as a prior in agreement with data.

The deceleration parameter q is

$$q = -1 - \frac{d \ln H}{d \ln a} = -1 + \frac{1}{2n-1}. \quad (7.52)$$

In Fig. 7.5, the comparison between a $F(R, \mathcal{G})$ model with the Λ CDM analogue is reported.

By a rapid inspection of the figure, it is evident that there is no change in the evolution of the curve since, for any $F(R, \mathcal{G}) = F_0 R^n \mathcal{G}^{1-n}$ model the deceleration parameter preserves sign, and therefore the universe always accelerates or always decelerates depending on the value of n . Clearly, for $n = 1$, the solution is an Einstein-de Sitter model as it has to be. On the other hand, the accelerated expansion of the universe ($q < 0$) is recovered for $n > 1$, but, in this case, the universe accelerates forever without the possibility of structure formation. In conclusion, it is important to underline that more realistic models are necessary in order to fit the observations.

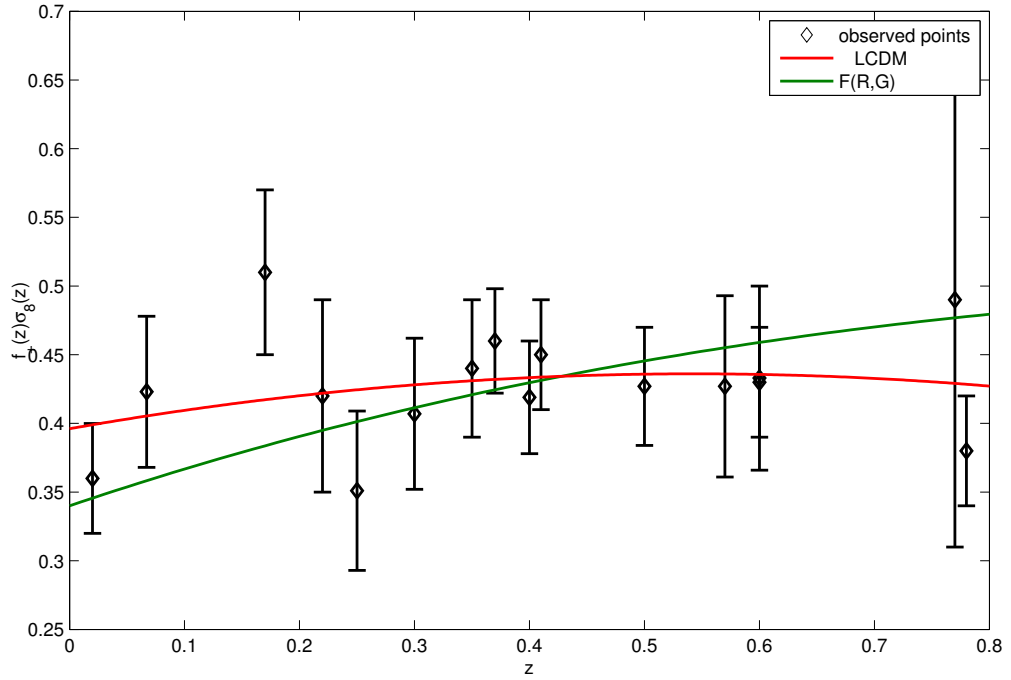


Figure 7.5: The plot shows the comparison of the growth rate $f_+(z)\sigma_8(z)$ for $F(R, \mathcal{G}) = F_0 R^n \mathcal{G}^{1-n}$ (green line) compared to that of Λ CDM (red line). The solid points are the observed one [206]. For $F(R, \mathcal{G})$ we consider the value $n = 2$. The parameter F_0 is assumed as a “prior” normalized at the Λ CDM value of the gravitational constant. This means that, in our units, it can be assumed of order unit. See also [184] for details.

Conclusions and Perspectives

The goal of this thesis has been to develop, in the framework of extended theories of gravity, methods to find out exact solutions for the field equations and to identify possible approaches to constrain these theories. The approach has then been used to identify new behaviours and to place observational constraints on deviations from the standard theory. To this aim, a number of methods have been adopted in both cosmological and weak-field frameworks. The focus of the analysis of cosmological solutions has been on FRW universes. These universes are appealing for a number of reasons. As well as having the same symmetries as our own observable Universe (on the largest scales), their high degree of symmetry is particularly useful for finding exact solutions to the field equations of the theory. In this way we have been able to see explicitly what effect such modifications to GR would have on the evolution of the Universe, as well as on the physical processes occurring within it.

First of all, we investigated the issue of the consistency of a field theory for massive gravitons that can be settled by extending the Einstein gravity through generic functions of curvature invariants.

Starting from the minimal extension of the Hilbert-Einstein action, that means that further degrees of freedom of gravitational field have to be taken into account, the possibility of massive gravitons naturally emerges. In particular, massive scalar modes results from the linearization of $f(R)$ gravity. The main result is that it is possible to obtain massive terms which indeed emerge naturally if one breaks spontaneously the diffeomorphism invariance of GR, and, in this case, for a certain range of parameters, it is possible to evade ghosts and discontinuities. Furthermore, it is possible to identify a natural mass scale m directly related to the expansion parameters of the theory. This fact could avoid to fix *by hand* the graviton mass since it comes directly from the struc-

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ture of the theory. Upper limits (or mass ranges) could directly come by experimental constraints. Finally, in the limit $m \gg \Lambda$ the theory results naturally regularized and the massive scalar satisfies a physically acceptable Klein Gordon equation.

Then, starting from the fact that ETGs are good candidates to solve several shortcomings of modern astrophysics and cosmology we emphasize that a “final” alternative theory solving all the issues has not been found yet and that the search for a procedure able to constrain the huge amount of extended theories proposed is still open. To this end, we investigated the role played by the particle production rate in the context of the simplest modifications to GR, $f(R)$ gravity. We considered both the metric and Palatini approaches reproducing particle production in both cases.

Specifically, we derived the Bogolubov coefficients, which allow to pass from a vacuum state to another. These coefficients can be related to the Hubble parameter which strictly depends on the functional form of a given $f(R)$ model. In this sense, the particle production rate depends on the specific form of $f(R)$ gravity. Hence, this is a method to constrain the free parameters of a given model, invoking a semi-classical scheme. Indeed, since the function $f(R)$ is not defined a priori, it is necessary to determine some theoretical conditions on $f(R)$ parameters at some fundamental level. Thus, we assumed to minimize the Bogolubov coefficients, *i.e.* the particle production rate, allowing us to pass through different vacuum states, once postulated the background. In particular, we considered a de Sitter phase $R = R_0$ and derived the Bogolubov transformations for some class of $f(R)$ models taking advantage from the fact that such models can be easily recast as scalar-tensor models. The Bogolubov coefficients have been evaluated for a homogeneous and isotropic universe, postulating that the particle production rate is negligibly small. This fact provides conditions on the form of $f(R)$. As a result, constraints can be derived on free parameters of different classes of $f(R)$ functions. Such constraints can be combined with Solar System constraints under suitable conditions. In particular, we demonstrated that cosmological measurements of R_0 would discriminate between metric or Palatini approaches.

Besides, without leaving the context of ETGs, the problem to find out exact solutions has been faced. We investigated some approaches to find out exact solutions in the framework of ETGs. Specifically, we considered the Noether symmetry approach

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and the Hojman symmetry approach. The first relies on the well-known Noether theorem while the second is based on the Hojman conservation theorem for a system of second-order differential equations, where a Lagrangian formulation is not strictly requested.

The Hojman Symmetry Approach can be a useful tool to find out exact solutions in dynamical systems as soon as a suitable Hojman vector is identified. Considering conformal transformations, the method works well in both Einstein and Jordan frame with the only shrewdness that conformal transformations have to be non-singular. We have shown that physical solutions achieved in the Einstein frame by Hojman symmetry can be easily transformed into the Jordan frame once a relation between the potential and the coupling is found out. As examples, we derived potentials with a clear inflationary meaning whose parameters can, in principle, be confronted with cosmological observations.

Afterwards, we considered the application of point symmetries in the Hybrid Gravity in order to select the $f(\mathcal{R})$ function and to find analytical solutions of the field equations and of the Wheeler-DeWitt equation for Quantum Cosmology. We showed that, in order to find nonlinear, integrable $f(\mathcal{R})$ models, we have to apply conformal transformations in the Lagrangian. Conformal transformations of the forms $d\tau = N(a)dt$ and $d\tau = N(\phi)dt$ allow us to achieve the results. In the second case, the Lagrangian of the field equations is reduced to a Brans-Dicke-like theory with a general coupling function; then the results from for scalar-tensor models can be applied. For the first conformal transformation we find two cases of the $f(\mathcal{R})$ function where the field equations admit Noether symmetries. For each case, we transform the field equation by means of normal coordinates to simplify the dynamical system and write exact solutions. Furthermore, we have written the Wheeler-DeWitt equation for the two-dimensional minisuperspace. The Lie point symmetries for the Wheeler-DeWitt equations can be determined and applied in order to find invariant solutions of the Wheeler-DeWitt equations. However, it is possible to apply another more general conformal transformation of the form $d\tau = N(a, \phi)dt$. If we do not consider matter, the field equations are always conformally invariant. Furthermore, the Wheeler-DeWitt equation is also conformally invariant; hence the solutions that we obtained

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hold for any frame. It is interesting to stress that, in the case of the power-law potential $V(\phi) = V_0(\sqrt{\phi} + V_1)^4$ the Hubble function $H^2(z)$ is a fourth-order polynomial with non vanishing coefficients. More specifically, every power-law term of $\sqrt{\phi}$ in the potential produces a corresponding fluid in the model. Furthermore, the nature of such a fluid can be quintessence or phantom field depending on the sign of ϕ . For $\phi = 0$, GR is restored. However, we recall that, in the case where $V_1 = 0$, the solution of the scale factor is the radiation solution as it has to be for conformally invariant solutions. Finally, we have to stress that the only power-law Hybrid Gravity which admits Noether symmetries is $f(\mathcal{R}) \propto \mathcal{R}^2$. This result is different for $f(R)$ -metric gravity and $f(\mathcal{R})$ -Palatini gravity where the power-law functions which admit Noether symmetries are $f(R) = R^n$ and $f(\mathcal{R}) = \mathcal{R}^n$.

Finally, as an example of the cosmological evolution ruled by ETGs we consider $f(R, \mathcal{G})$ cosmology. We have considered the possibility to obtain cosmological inflation starting from a generic function $F(R, \mathcal{G})$ of the Ricci curvature scalar R and the GaussBonnet topological invariant \mathcal{G} . Such a kind of theory, due to the algebraic relation among the curvature invariants in \mathcal{G} can exhaust the whole curvature budget of effective gravitational theories where derivatives of curvature invariants are not present. The main feature that emerges by this approach is the fact that two effective masses have to be considered, one related to R and the other related to \mathcal{G} . These masses define two different scales that drive dynamics at early and very early epochs, giving rise to a natural double inflationary scenario. We sketched the essential characteristics of this picture considering exponential and power-law inflation.

More work is required to fully understand how these solutions should be interpreted and what their physical effects could be. There are several directions in which future research on these subjects can proceed. First, future work may focus on the generalization of the method involving Bogolubov transformations for space-times with variable curvature; also in those cases, one may check how to minimize the rate of particle production in order to pass from a vacuum to another one. Furthermore, it would be possible to evaluate Bogolubov coefficients in other modified gravity theories to seek for constraints on free parameters.

Then, we may focus on the generalization of Noether and Hojman approaches to

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other alternative theories of gravity and on the detailed matching with data of the features achieved in the framework of $F(R, \mathcal{G})$ cosmology. Extracting reliable physical effects from these models will be the arguments of future studies.

Appendix A

The Cauchy Problem in Relativistic Theories of Gravity

A.1 The Cauchy Problem in General Relativity

In this appendix we show that the initial value formulation of GR [265] is well-formulated (and also well-posed as shown in [260]). Let us consider a system of Gaussian normal coordinates [260]. In these coordinates, the time components of metric tensor are $g_{00} = -1$ and $g_{0i} = 0$ with the signature $(-, +, +, +)$. These are particularly useful to split the spatial hypersurface Σ_3 from the orthogonal time-geodesics in a given space-time M . Given a second rank symmetric tensor $W_{\mu\nu}$, defined on the globally hyperbolic space-time manifold $(M, g_{\mu\nu})$, it is possible to define the symmetric tensor $W_{\mu\nu}^*$ as

$$W_{\mu\nu}^* = W_{\mu\nu} - \frac{1}{2}W g_{\mu\nu} \quad (\text{A.1})$$

where $W = W^{\mu\nu}g_{\mu\nu}$ is the trace of $W_{\mu\nu}$. Furthermore, if Σ_4 is a space-time domain in M where $g_{00} \neq 0$ and Σ_3 is the three-surface given by the equation $x^0 = 0$, then the following statements are equivalent

- (a) $W_{\mu\nu} = 0$ in Σ_4
- (b) $W_{ij}^* = 0$ and $W_{4\alpha} = 0$ in Σ_4
- (c) $W_{ij}^* = 0$ and $\nabla_\nu W_\mu^\nu = 0$ in Σ_4 with $W_{0\mu} = 0$ in Σ_3

The Cauchy Problem in Relativistic Theories of Gravity

where ∇_ν denotes the covariant derivative with respect to the Levi-Civita connection induced by $g_{\mu\nu}$. Let us take into account now the Einstein equations in the form

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(m)} \quad (\text{A.2})$$

with the contracted Bianchi identities

$$\nabla^\nu T_{\mu\nu}^{(m)} = 0 \quad (\text{A.3})$$

where $G_{\mu\nu}$ is the Einstein tensor. We can define the tensor

$$W_{\mu\nu} \equiv G_{\mu\nu} - \kappa^2 T_{\mu\nu}^{(m)} \quad (\text{A.4})$$

The tensor $W_{\mu\nu}^*$ is

$$W_{\mu\nu}^* = R_{\mu\nu} - \kappa^2 T_{\mu\nu}^* \quad (\text{A.5})$$

and the Einstein equations are

$$W_{\mu\nu} = 0 \quad (\text{A.6})$$

These are 10 equations for the 20 unknown functions $g_{\mu\nu}$ and $T_{\mu\nu}^{(m)}$. Let us assign now the 10 functions $g_{0\mu}$ and $T_{ij}^{(m)}$. The remaining 10 functions g_{ij} and $T_{0\mu}^{(m)}$ are determined by Eq. (A.6). These functions can be expressed in the equivalent form

$$R_{ij} - \kappa^2 T_{ij}^* = 0, \quad \nabla^\nu W_{\mu\nu} = \nabla^\nu T_{\mu\nu}^{(m)} = 0 \quad (\text{A.7})$$

with the condition

$$G_{0\mu} - \kappa^2 T_{0\mu}^{(m)} = 0 \quad (\text{A.8})$$

on the hypersurface $x^0 = 0$. Equation (A.7) can be rewritten as

$$g_{ij,00} = 2\bar{R}_{ij} - \frac{A}{2}g_{ij,0} + g^{lm}g_{il,0}g_{jm,0} + 2\kappa^2 T_{ij}^* \quad (\text{A.9})$$

$$T_{0\nu}^{(m)} = -T_{\nu,0}^{(m)} = T_{\nu,i}^{i(m)} + \Gamma_{\mu i}^\mu T_\nu^{i(m)} - \Gamma_{\mu\nu}^i T_i^{\mu(m)} \quad (\text{A.10})$$

where \bar{R}_{ij} is the intrinsic Ricci tensor of the hypersurface $x^0 = 0$ and $\Gamma_{\mu\nu}^\rho$ is the Levi-Civita connection of the metric $g_{\mu\nu}$, and

$$A \equiv g^{ij}g_{ij,0} \quad (\text{A.11})$$

In the same way, the constraint equation (A.8) becomes

$$A_{,i} - D^j g_{ij,0} + 2\kappa^2 T_{0i}^{(m)} = 0 \quad (\text{A.12})$$

$$\bar{R} - \frac{1}{4}A^2 + \frac{1}{4}B + 2\kappa^2 T_{00}^{(m)} = 0 \quad (\text{A.13})$$

where \bar{R} is the intrinsic Ricci scalar of the hypersurface $x^0 = 0$, D_i denotes the covariant derivative operator on this hypersurface associated with the Levi-Civita connection of the intrinsic metric $g_{ij}|_{x^0=0}$ and

$$B = g^{ij} g^{lm} g_{il,0} g_{jm,0} \quad (\text{A.14})$$

Let us assign now the set of Cauchy data on the hypersurface $x^0 = 0$

$$g_{ij} \quad g_{ij,0} \quad T_{\mu 0}^{(m)} \quad (\text{A.15})$$

Such data have to satisfy the constraint equations (A.12) and (A.13). Equation (A.9) and (A.10) explicitly gives the values of the quantities

$$g_{ij,00} \quad T_{0\mu,0}^{(m)} \quad (\text{A.16})$$

as functions of the Cauchy data. By differentiating Eqs. (A.9) and (A.10), it is straightforward to obtain the time derivatives of higher order as functions of the initial data. This procedure allows one to locally reconstruct the solution of the field equations as a power series of the time variable x^0 .

In other words this means that the initial three-surface Σ_3 is then a Cauchy hypersurface for the globally hyperbolic space-time $(M, g_{\mu\nu})$ and the initial value problem is well-formulated in GR.

A.2 The Cauchy Problem in $f(R)$ gravity

The generalities of the metric formulation of $f(R)$ gravity are showed in sec. 3.3. Here we show how $f(R)$ -gravity can be re-interpreted as an O'Hanlon scalar-tensor theory, by introducing a suitable scalar field φ which non-minimally couples with the gravity sector [241, 277]. After we discuss the Cauchy problem showing that it is

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both well formulated and well posed [276]. Let us take into account an O'Hanlon Lagrangian of the form [277]

$$\mathcal{S} = \int \sqrt{g} [\varphi R - V(\varphi) + \mathcal{L}_m] ds \quad (\text{A.17})$$

where $V(\varphi)$ is the self-interaction potential. Field equations are derived by varying (A.17) with respect to both g_{ij} and φ which now represents a new dynamical variable.

One obtains

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{1}{\varphi} \left[\Sigma_{ij} + \nabla_i \nabla_j \varphi - g_{ij} g^{pq} \nabla_p \nabla_q \varphi - \frac{1}{2}V(\varphi)g_{ij} \right] \quad (\text{A.18a})$$

$$R - \frac{dV(\varphi)}{d\varphi} = 0 \quad (\text{A.18b})$$

We notice that, replacing (A.18b) into the trace of (A.18a), we obtain the scalar equation

$$g^{pq} \nabla_p \nabla_q \varphi = \frac{1}{3} \left[\varphi \frac{dV(\varphi)}{d\varphi} - 2V(\varphi) + \Sigma \right] \quad (\text{A.19})$$

System (A.18) is then equivalent to eqs. (A.18a) together with (A.19). Given the function $f(R)$ in (3.24), we shall suppose that its first derivative $f'(R)$ is invertible. In such a circumstance, it is easily seen that metric $f(R)$ theories of gravity can be mapped onto O'Hanlon theories and vice-versa. Indeed, defined the scalar field

$$\varphi = f'(R) \quad (\text{A.20a})$$

and the potential

$$V(\varphi) = f' [R(\varphi)]R(\varphi) - f(R(\varphi)) \quad (\text{A.20b})$$

It is straightforward to verify that, under the above hypothesis $f''(R) \neq 0$, eq. (A.20a) expresses the inverse relation of (A.18b), namely

$$R - \frac{dV(\varphi)}{d\varphi} = 0 \quad \iff \quad \varphi = f'(R) \quad (\text{A.21})$$

being the potential $V(\varphi)$ defined by (A.20b). A direct comparison of eqs. (3.25) with eqs. (A.18a) shows then that solutions of (3.25) together with (A.20a) are also solutions of (A.18) and viceversa. As a final remark, we recall that in O'Hanlon theory the standard conservation laws $\nabla^i \Sigma_{ij} = 0$ hold. An explicit proof of the vanishing of the covariant divergence of the energy-momentum tensor in modified theories of gravity can be found in [278].

A.2.1 The Cauchy problem for O'Hanlon gravity

Taking into account the above dynamical equivalence, the Cauchy problem for $f(R)$ gravity can be defined as the Cauchy problem for the corresponding O'Hanlon theory. In this perspective, we discuss the Cauchy problem for the O'Hanlon gravity. Let us show the well-posedness of the Cauchy problem for system (A.18a) and (A.19) in vacuo. As we shall see, the same conclusions hold in presence of matter sources satisfying the standard conservation laws $\nabla^i \Sigma_{ij} = 0$. To this aim, we use generalized harmonic coordinates, given by the conditions

$$F_\varphi^i = F^i - H^i = 0 \quad \text{with} \quad F^i = g^{pq} \Gamma_{pq}^i, \quad H^i = \frac{1}{\varphi} \nabla^i \varphi \quad (\text{A.22})$$

The generalized harmonic gauge (A.22) is a particular case of the one introduced in [279] to prove the well-posedness of the Cauchy problem for more general scalar-tensor theories of gravity. As we shall see, the gauge (A.22) allows us to develop a second order analysis very similar to the one used in GR [272]. We rewrite eqs. (A.18a) in the form

$$R_{ij} = \frac{1}{\varphi} \left[T_{ij} - \frac{1}{2} T g_{ij} \right] \quad (\text{A.23})$$

where

$$T_{ij} = \nabla_i \nabla_j \varphi - g_{ij} g^{pq} \nabla_p \nabla_q \varphi - \frac{1}{2} V(\varphi) g_{ij} \quad (\text{A.24})$$

plays the role of effective energy-momentum tensor. The Ricci tensor can be expressed as [272]

$$R_{ij} = R_{ij}^\varphi + \frac{1}{2} [g_{ip} \partial_j (F_\varphi^p + H^p) + g_{jp} \partial_i (F_\varphi^p + H^p)] \quad (\text{A.25})$$

with

$$R_{ij}^\varphi = -\frac{1}{2} g^{pq} \partial_{pq}^2 g_{ij} + A_{ij}(g, \partial g) \quad (\text{A.26})$$

where only first order derivatives appear in the functions A_{ij} . Assuming $F_\varphi^i = 0$ and taking the expression of H^i into account, we obtain the following representation

$$R_{ij} = -\frac{1}{2} g^{pq} \partial_{pq}^2 g_{ij} + \frac{1}{\varphi} \partial_{ij}^2 \varphi + B_{ij}(g, \varphi, \partial g, \partial \varphi) \quad (\text{A.27})$$

where the functions B_{ij} depend on the metric g , the scalar field φ and their first order derivatives. Analogously, using eq. (A.19) to replace all terms depending on the

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divergence $g^{pq}\nabla_p\nabla_q\varphi$, the right hand side of (A.23) can be expressed as

$$\frac{1}{\varphi} \left[T_{ij} - \frac{1}{2} T g_{ij} \right] = \frac{1}{\varphi} \partial_{ij}^2 \varphi + C_{ij}(g, \varphi, \partial g, \partial \varphi) \quad (\text{A.28})$$

Again, in the functions C_{ij} , only first order derivatives are involved. A direct comparison of eqs. (A.27) with eqs. (A.28) shows that, in the considered gauge, eqs. (A.22) assume the form

$$g^{pq}\partial_{pq}^2 g_{ij} = D_{ij}(g, \varphi, \partial g, \partial \varphi) \quad (\text{A.29})$$

Eqs. (A.29), together with eq. (A.27), form a quasi-diagonal, quasi-linear second-order system of partial differential equations, for which well known theorems by Leray [272, 280, 260] hold. Given initial data on a space-like surface, the associated Cauchy problem is then well-posed in suitable Sobolev spaces [272]. Of course, the initial data have to satisfy the gauge conditions $F_\varphi^i = 0$ as well as the Hamiltonian and momentum constraints

$$G^{0i} = \frac{1}{\varphi} T^{0i} \quad i = 0, \dots, 3 \quad (\text{A.30})$$

on the initial space-like surface. In connection with this, we notice that, from eq. (A.19), we can derive the expression of the second partial derivative $\partial_0^2 \varphi$ and replace it in the right hand side of (A.30), so obtaining constraints involving no higher than first order partial derivatives with respect to the time variable x^0 . To conclude, we have to prove that the gauge conditions $F_\varphi^i = 0$ are preserved in a neighborhood of the initial space-like surface. To this end, we first verify that the divergence of the Einstein-like equations (A.18a) vanishes, namely

$$\nabla^i (\varphi G_{ij} - T_{ij}) = 0 \quad (\text{A.31})$$

A straightforward calculation yields

$$\begin{aligned} \nabla^i (\varphi G_{ij} - T_{ij}) &= (\nabla^i \varphi) R_{ij} - \frac{1}{2} \varphi_j \left(R - \frac{dV}{d\varphi} \right) + \varphi \nabla^i G_{ij} \\ &\quad - (\nabla^i \nabla_i \nabla_j - \nabla_j \nabla^i \nabla_i) \varphi \end{aligned} \quad (\text{A.32})$$

By definition, the Einstein and Ricci tensors satisfy the identities $\nabla^i G_{ij} = 0$ and $(\nabla^i \varphi) R_{ij} = (\nabla^i \nabla_i \nabla_j - \nabla_j \nabla^i \nabla_i) \varphi$. On the other hand, $R - \frac{dV}{d\varphi} = 0$ is assured by field equations (A.18b). Therefore, identities (A.31) follow. If now g_{ij} and φ are

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the solutions of reduced Einstein-like equations (A.29) and field equation (A.19), one has

$$\varphi G^{ij} - T^{ij} = -\frac{\varphi}{2} (g^{ip} \partial_p F_\varphi^j + g^{jp} \partial_p F_\varphi^i - g^{ij} \partial_p F_\varphi^p) \quad (\text{A.33})$$

Identity (A.31) shows then that the functions F_φ^i satisfy necessarily the linear homogeneous system of wave equations

$$g^{pq} \partial_{pq}^2 F_\varphi^i + E_p^{iq} \partial_q F_\varphi^p = 0 \quad (\text{A.34})$$

where E_p^{iq} are known functions on the space-time. Since the constraints (A.30) amount to the condition $\partial_0 F_\varphi^i$ [272] on the initial space-like surface, a well known uniqueness theorem for differential systems like (A.34) [272] assures that $F_\varphi^i = 0$ in the region where solutions of (A.29) and (A.19) exist.

As mentioned above, the well-posedness of the Cauchy problem can be proved also in presence of coupling with standard matter sources, such as electromagnetic or Yang-Mills fields, (charged) perfect fluid, (charged) dust, Klein-Gordon scalar fields. When this is the case, eqs. (A.19) and (A.29) have to be coupled with the matter field equations. Applying the same arguments developed for GR [272, 270, 271, 274], it is easily seen that, in the generalized harmonic gauge (A.22), the matter field equations together with eqs. (A.19) and (A.29) form a Leray hyperbolic and causal differential system admitting a well-posed Cauchy problem. In addition to the well-known results by Bruhat's, the key point is that the field equations of matter field imply the standard conservation laws $\nabla^i \Sigma_{ij} = 0$. This fact allows to verify the validity of eqs. (A.31) in presence of matter too. In conclusion, the Cauchy problem is well-formulated and well-posed for $f(R)$ gravity.

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