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QUANTILE REGRESSION METHODS IN ECONOMICS AND FINANCE

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Table of contents

List of figures	vii
List of tables	xiii
1 Introduction	1
1.1 Overview	1
1.2 Main contributions	2
1.3 Conference and Seminar Presentations	4
2 Asset Allocation Strategies Based on Penalized Quantile Regression	5
2.1 State-of-the art and further developments	5
2.2 Asset allocation based on quantile regression	9
2.2.1 Portfolio performance as function of quantiles levels	9
2.2.2 Simulation exercise	15
2.2.3 The inclusion of the ℓ_1 -norm penalty	19
2.3 Empirical set-up	21
2.3.1 Data description	21
2.3.2 Rolling analysis with in- and out-of-sample assessment	23
2.3.3 Performance indicators	24
2.4 Empirical results	25
2.4.1 The role of the intercept	35
2.5 Concluding remarks	39
3 The Determinants of Equity Risk and their Forecasting Implications: a Quantile Regression Perspective	41
3.1 Literature review and new suggestions	41
3.2 Realized volatilities and conditioning variables	45
3.3 Modelling the realized range conditional quantiles	51

3.4	Density forecast and predictive accuracy	53
3.5	Results	57
3.5.1	Full sample analyses	57
3.5.2	Rolling analysis	62
3.5.3	Evaluation of the predictive power	66
3.5.4	Single asset results	69
3.6	Conclusions	75
4	The Dynamic Impact of Uncertainty in Causing and Forecasting the Distribution of Oil Returns and Risk	79
4.1	Oil movements and uncertainty: an introductory discussion	79
4.2	Causality and forecasting methods	82
4.2.1	Causality in quantiles	82
4.2.2	Quantiles and density forecasting	84
4.2.3	Evaluation of the predictive accuracy	86
4.3	Dataset and rolling analysis	91
4.3.1	Data description	91
4.3.2	Dynamic analysis and rolling window procedure	93
4.4	Empirical findings	96
4.5	Concluding comments	116
5	Further research	117
5.1	The Conditional Quantile-Dependent Autoregressive Value-at-Risk	117
5.2	The Role of Causality in Quantiles and Expectiles in Networks Combinations	118
5.3	The US Real GNP is Trend-Stationary After All	119
	References	121
	Appendix A Volatility estimation in the presence of noise and jumps	131
	Appendix B Quantile regression: estimation and testing	135
	Appendix C Additional tables and figures	141

- 2.4 In-Sample results, obtained without penalizations, from the returns series of the Standard & Poor's 500 index constituents. Rolling analysis with window size of 500 observations. A, B, C denote the strategies built from quantile regression models applied at probabilities levels of 0.1, 0.5 and 0.9, respectively, whereas D refers to the ordinary least squares regression model. From left to right the subfigures report the boxplots of the following statistics: mean (a), standard deviation (b), mean absolute deviation (c), Value-at-Risk at the level of 10% (d), α -risk at $\alpha = 0.1$ (e), $\hat{\Psi}_1(r_p, \psi)$ (f), $\hat{\Psi}_2(r_p, \psi)$ (g), with $\psi = 0.9$, Sharpe Ratio (h). 28
- 2.5 In-Sample results, obtained by imposing the ℓ_1 -norm, from the returns series of the Standard & Poor's 500 index constituents. Rolling analysis with window size of 500 observations. A, B, C denote the strategies built from quantile regression models applied at probabilities levels of 0.1, 0.5 and 0.9, respectively, whereas D refers to the ordinary least squares regression model. From left to right the subfigures report the boxplots of the following statistics: mean (a), standard deviation (b), mean absolute deviation (c), Value-at-Risk at the level of 10% (d), α -risk at $\alpha = 0.1$ (e), $\hat{\Psi}_1(r_p, \psi)$ (f), $\hat{\Psi}_2(r_p, \psi)$ (g), with $\psi = 0.9$, Sharpe Ratio (h). 29
- 2.6 Out-of-Sample results generated by the strategies built from the ordinary least squares (with *LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and from the quantile regression (with *PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 100 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: standard deviation (a), mean absolute deviation (b), Value-at-Risk (c) and α -risk (d) at the level of 10%. 30
- 2.7 Out-of-Sample results generated by the strategies built from the ordinary least squares (with *LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and from the quantile regression (with *PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 500 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: standard deviation (a), mean absolute deviation (b), Value-at-Risk (c) and α -risk (d) at the level of 10%. 31

2.8	Out-of-Sample results generated by the strategies built from the ordinary least squares (with (<i>LASSO</i>) and without (<i>OLS</i>) ℓ_1 -norm penalty) and from the quantile regression (with (<i>PQR</i> (ϑ)) and without (<i>QR</i> (ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 100 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: $\widehat{\Psi}_1(r_p, \psi)$ (a) and $\widehat{\Psi}_2(r_p, \psi)$ (b) at $\psi = 0.9$, mean (c) and Sharpe Ratio (d).	32
2.9	Out-of-Sample results generated by the strategies built from the ordinary least squares (with (<i>LASSO</i>) and without (<i>OLS</i>) ℓ_1 -norm penalty) and from the quantile regression (with (<i>PQR</i> (ϑ)) and without (<i>QR</i> (ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 500 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: $\widehat{\Psi}_1(r_p, \psi)$ (a) and $\widehat{\Psi}_2(r_p, \psi)$ (b) at $\psi = 0.9$, mean (c) and Sharpe Ratio (d).	33
2.10	The evolution of the portfolio value generated by the strategies built from the ordinary least squares (with (<i>LASSO</i>) and without (<i>OLS</i>) ℓ_1 -norm penalty) and from the quantile regression (with (<i>PQR</i> (ϑ)) and without (<i>QR</i> (ϑ)) ℓ_1 -norm penalty, for $\vartheta = \{0.1, 0.5, 0.9\}$) models. The subfigures report different results according to the used dataset and to the fact that the regression models are applied with or without ℓ_1 -norm penalty. The rolling analysis is implemented with window size of 1000 observations.	34
2.11	Turnover of the strategies built from the ordinary least squares (with (<i>LASSO</i>) and without (<i>OLS</i>) ℓ_1 -norm penalty) and from the quantile regression (with (<i>PQR</i> (ϑ)) and without (<i>QR</i> (ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 100 (Subplot (a)) and 500 (Subplot (b)) indices constituents. The rolling analysis is carried out with window size of 1000 observations.	34
3.1	Realized range-based bias corrected bipower variation over time.	47
3.2	Level and boxplot for the first principal component of the range-based bias corrected bipower variations.	48
3.3	Coefficients plots. The red lines represent the coefficients values over τ levels, while the blue areas are the associated 95% confidence intervals.	61
3.4	Rolling analysis for fpc_{t-1}	62
3.5	Rolling analysis for $\overline{fpc}_{5_{t-1}}$	63
3.6	The persistence of volatility over time.	64
3.7	Rolling analysis for vix_{t-1}	65

3.8	Rolling analysis for $sp500_{t-1}$	65
3.9	Rolling analysis for $jump_{t-1}$	66
4.1	The causality of the Economic Policy Uncertainty (in logarithm) on the oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.	97
4.2	The causality of the Market Equity Uncertainty (in logarithm) on the oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.	97
4.3	The causality of Economic Policy Uncertainty (in logarithm) on the squared oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.	98
4.4	The causality of the Market Equity Uncertainty (in logarithm) on the squared oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.	98
4.5	The figure reports the linear correlation coefficients $\rho_{D_{y_t}}(\tau) = \rho(D_{y_t, x_{1,t}}(\tau), D_{y_t, x_{2,t}}(\tau))$ and $\rho_{D_{y_t^2}}(\tau) = \rho(D_{y_t^2, x_{1,t}}(\tau), D_{y_t^2, x_{2,t}}(\tau))$. $D_{y_t, x_{1,t}}(\tau)$ is a dummy variable taking value 1 if the test statistic (4.3), applied for the pair $(x_{1,t}, y_t)$ at the τ level, and computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead, is greater than 1.96, 0 otherwise. $D_{y_t, x_{2,t}}(\tau)$, $D_{y_t^2, x_{1,t}}(\tau)$ and $D_{y_t^2, x_{2,t}}(\tau)$ are computed in the same way.	99
4.6	The dynamic correlations, $\rho(\cdot)$, of the variables $y_t, y_t^2, x_{1,t}, x_{2,t}$. The linear correlations coefficients are computed through a rolling window procedure with window size of 500 observations and step of one period ahead.	99
4.7	Conditional distributions and densities of y_t and y_t^2 . Subfigures (a) and (c) display, respectively, the conditional distributions of y_t and y_t^2 , estimated from the first subsample determined through the rolling window procedure. “Original”, “Adjusted” and “Kernel” stand for the distributions arising directly from Models (4.12)-(4.14), the ones obtained by adjusting the original estimates through the quantile bootstrap method proposed by Chernozhukov et al. (2010), and the ones built by means of the Epanechnikov kernel, respectively. Subfigures (b) and (d) show the conditional densities of y_t and y_t^2 , respectively, estimated from the first and the last windows determined by the rolling window procedure, by applying the Epanechnikov kernel method.	104

- 4.8 The p-values of the Berkowitz (2001) test over the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions. Model 1 includes all the predictors in (4.12) and (4.14), respectively. Model 2 includes just y_{t-1} and y_{t-2} , in “OIL RETURNS”, y_{t-1}^2 and y_{t-2}^2 , in “SQUARED OIL RETURNS”. Model 3 includes y_{t-1} , y_{t-2} , $x_{1,t-1}$ and $x_{1,t-2}$, in “OIL RETURNS”, y_{t-1}^2 , y_{t-2}^2 , $x_{1,t-1}$ and $x_{1,t-2}$ in “SQUARED OIL RETURNS”. Finally, Model 4 includes y_{t-1} , y_{t-2} , $x_{2,t-1}$ and $x_{2,t-2}$, in “OIL RETURNS”, y_{t-1}^2 , y_{t-2}^2 , $x_{2,t-1}$ and $x_{2,t-2}$ in “SQUARED OIL RETURNS”. 107
- 4.9 The Diebold and Mariano (2002) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied at three different τ values: 0.05 (black lines), 0.50 (red lines) and 0.95 (blue lines). 108
- 4.10 The Amisano and Giacomini (2007) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied by placing greater emphasis on the center (red lines), on the right tail (blue lines) and on the left tail (green lines) of the conditional distributions. It is also applied by placing equal weight to the different regions of the distributions (black lines). . . . 109
- 4.11 The Diks et al. (2011) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied by placing greater emphasis on the center (red lines), on the right tail (blue lines) and on the left tail (green lines) of the conditional distributions. 110
- 4.12 The Gneiting and Ranjan (2011) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied by placing greater emphasis on the center (red lines), on the right tail (blue lines) and on the left tail (green lines) of the conditional distributions. 111

4.13	Evidence of the crucial impact of EPU and EMU in forecasting the y_t conditional distribution and quantiles over time, detected by several tests and linked to the y_t (a), $x_{1,t}$ (b), and $x_{2,t}$ (c) series. A, B, C, D, E, F, G, H, I stand for the tests proposed by Diks et al. (2011), with focus on the right and the center parts of the distribution, Gneiting and Ranjan (2011), with focus on the right tail of the distribution, Amisano and Giacomini (2007), with focus on the right, the center and the left parts of the distribution, Diebold and Mariano (2002), with focus on the 95-th and the 5-th percentile, Berkowitz (2001), respectively.	114
4.14	Evidence of the crucial impact of EPU and EMU in forecasting the y_t^2 conditional distribution and quantiles over time, detected by several tests and linked to the y_t (a), $x_{1,t}$ (b), and $x_{2,t}$ (c) series. A, B, C, D, E, F, G stand for the tests proposed by Diks et al. (2011), with focus on the center and the left parts of the distribution, Gneiting and Ranjan (2011), with focus on the right tail of the distribution, Amisano and Giacomini (2007), with focus on the center and the left parts of the distribution, Diebold and Mariano (2002), with focus on the 95-th percentile, Berkowitz (2001), respectively.	115
C.0.1	Autocorrelation and partial autocorrelation functions for the fpc series. . . .	143
C.0.2	Scatter plots.	144
C.0.3	The trend over time of the jump component, measured by $JUMP$, for the companies BAC , CTG , IBM and JPM	144
C.0.4	Distribution of z_t	145
C.0.5	The impacts of $rrv_{i,t-1}$ on the conditional volatility quantiles over τ for each asset.	145
C.0.6	The impacts of $\overline{rrv}_{5i,t-1}$ on the conditional volatility quantiles over τ for each asset.	146
C.0.7	The impacts of vix_{t-1} on the conditional volatility quantiles over τ for each asset.	146
C.0.8	The impacts of $sp500_{t-1}$ on the conditional volatility quantiles over τ for each asset.	147
C.0.9	The impacts of $jump_{i,t-1}$ on the conditional volatility quantiles over τ for each asset.	147

List of tables

2.1	The impact of the ℓ_1 -norm penalty on active and short positions.	26
2.2	Out-of-Sample Analysis partitioned on two sub-periods.	36
2.3	Analysis of the intercepts of the quantile regression models.	38
2.4	Analysis of the out-of-sample residuals distributions.	39
3.1	Descriptive analysis on the estimated daily volatilities series.	46
3.2	Linear correlation coefficients.	50
3.3	Quantile regression results.	58
3.4	Restricted model against unrestricted model.	59
3.5	Quantile regressions for each asset ($\tau = 0.1$).	70
3.6	Quantile regressions for each asset ($\tau = 0.5$).	71
3.7	Quantile regressions for each asset ($\tau = 0.9$).	72
3.8	Berkowitz test for the single asset analysis.	75
3.9	Amisano-Giacomini test for the single asset analysis.	76
3.10	Diebold-Mariano test for the single asset analysis.	77
4.1	Descriptive statistics.	92
4.2	Structural breaks in the conditional distribution and quantiles of y_t	94
4.3	Structural breaks in the conditional distribution and quantiles of y_t^2	95
4.4	Conditional average correlations.	101
4.5	Quantile regression output.	102
4.6	Persistence of significance over the rolled windows.	103
4.7	The asymmetric impact of uncertainty on the oil movements.	106
C.0.1	Jump testing results.	141
C.0.2	Quantile regression results.	142

Chapter 1

Introduction

1.1 Overview

In the recent years, quantile regression methods have attracted relevant interest in the statistical and econometric literature. This phenomenon is due to the advantages arising from the quantile regression approach, mainly the robustness of the results and the possibility to analyse several quantiles of a given random variable. Such as features are particularly appealing in the context of economic and financial data, where extreme events assume critical importance. The present thesis is based on quantile regression, with focus on the economic and financial environment, and is structured in three main parts, developed in Chapters 2, 3 and 4, respectively, summarized in the following.

In particular, in Chapter 2, we propose new approaches in developing asset allocation strategies on the basis of quantile regression and regularization techniques. It is well known that quantile regression model minimizes the portfolio extreme risk, whenever the attention is placed on the estimation of the response variable left quantiles. We show that, by considering the entire conditional distribution of the dependent variable, it is possible to optimize different risk and performance indicators. In particular, we introduce a risk-adjusted profitability measure, useful in evaluating financial portfolios under a pessimistic perspective, since the reward contribution is net of the most favorable outcomes. Moreover, as we consider large portfolios, we also cope with the dimensionality issue by introducing an ℓ_1 -norm penalty on the assets weights.

In Chapter 3 we focus on the the determinants of equity risk and their forecasting implications. Several market and macro-level variables influence the evolution of equity risk in addition to the well-known volatility persistence. However, the impact of those covariates might change depending on the risk level, being different between low and high

volatility states. By combining equity risk estimates, obtained from the Realized Range Volatility, corrected for microstructure noise and jumps, and quantile regression methods, we evaluate, in a forecasting perspective, the impact of the equity risk determinants in different volatility states and, without distributional assumptions on the realized range innovations, we recover both the points and the conditional distribution forecasts. In addition, we analyse how the relationships among the involved variables evolve over time, through a rolling window procedure. The results show evidence of the selected variables' relevant impacts and, particularly during periods of market stress, highlight heterogeneous effects across quantiles.

Finally, in Chapter 4 we analyse the dynamic impact of uncertainty in causing and forecasting the distribution of oil returns and risk. The aim is to analyse the relevance of recently developed news-based measures of economic policy uncertainty and equity market uncertainty in causing and predicting the conditional quantiles and distribution of the crude oil variations, defined both as returns and squared returns. For this purpose, on the one hand, we study the causality relations in quantiles through a non-parametric testing method; on the other hand, we forecast the conditional distribution on the basis of the quantile regression approach and the predictive accuracy is evaluated by means of several suitable tests. Given the presence of structural breaks over time, we implement a rolling window procedure to capture the dynamic relations among the variables.

In addition to the three chapters above introduced, three additional contributions need to be mentioned. Those works are not yet completed or lie beyond the quantile regression framework, and, for this reason, are summarized in a separate, concise, section. In particular, Chapter 5 describes the main ideas and the preliminar findings.

1.2 Main contributions

The main, original, contributions of the thesis are summarized in the following.

1. First of all, we contribute to the literature by introducing new approaches for building asset allocation strategies, on the basis of penalized quantile regression models whose coefficients coincide with the assets weights of financial portfolios.

On the one hand, we show that the quantile regression not only leads to the minimization of the portfolio extreme risk, by focusing on the left tail of the response variable distribution, as well known in the literature, but also to the optimization of other risk and performance indicators. For that purpose, we exploit the entire conditional distribution of the dependent variable, finding that:

- a) under reasonable assumptions, at the median level, the quantile regression solution of the linear model corresponds to the minimization of the mean absolute deviation of the portfolio returns;
- b) at high quantiles levels, the quantile regression solution provides an outstanding performance in terms of profitability and risk-adjusted returns. As a by-product, we introduce a performance measure, completely new in the literature, quantifying the magnitude of all the negative outcomes balanced by a part of positive results, net of the most favorable ones.

On the other hand, in order to meet the financial industry's requirements, we deal with portfolios characterized by large cross-sectional dimension, where the accumulation of estimation errors becomes a problem that investors and portfolio managers must address. Assuming we are interested in maintaining a pessimistic asset allocation strategy, possibly coherent with the investor preferences, we control for the impact of estimation errors by imposing a penalty on the l_1 -norm of the assets weights. That method provides benefits in terms of sparsity of the portfolio, indirectly associated with diversification/concentration and turnover. To the best of our knowledge, the method described above has never been investigated in the literature.

2. Secondly, we propose an innovative approach in analysing the determinants of the equity market risk. For this purpose, we adopt the Realized Range-Based Bias Corrected Bipower Variation as consistent estimator of the integrated variance, even in the presence of microstructure noise and jumps. To the best of our knowledge, quantile regression methods for the analysis of realized range volatility measures is novel in the literature. Besides, we do not restrict the attention on a few quantiles; differently, we forecast a fine grid providing detailed information of the covariates impact. Furthermore, that fine grid represents the basis for recovering the density forecast of the volatility. In particular, after adjusting the original estimates and obtaining non-crossing quantiles, we build the entire density through a nonparametric kernel-based method.
3. Finally, to the best of our knowledge, the present thesis is the first work in the literature which analyses the importance of two recently developed news-based measures of economic policy and equity market uncertainty in causing and forecasting, both in- and out-of-sample, the conditional distribution of both the oil returns and their volatility. By implementing a rolling window scheme, due to the presence of structural breaks over time, and by exploiting the synergies between the causality and the forecasting

exercises, we are able to recover important information about the dynamic impact of the two uncertainty indices, never detected in the literature.

1.3 Conference and Seminar Presentations

From Chapters 2, 3 and 4, three different papers have been defined. They have been submitted to international journals and have been presented at different conferences and seminars.

In particular, “Asset Allocation Strategies Based on Penalized Quantile Regression” has been presented at the “SOFINE-CEQURA Spring Junior Research Workshop” (Nesselwang, Germany), at the seminar organized by the Department of Economics, Management and Statistics of the University of Palermo (Italy) and at the “9th International Conference on Computational and Financial Econometrics” (London, UK).

“The Determinants of Equity Risk and their Forecasting Implications: a Quantile Regression Perspective” has been presented at the “2nd CIdE Workshop for PhD Students in Econometrics and Empirical Economics” (Perugia, Italy), at the “8th International Conference on Computational and Financial Econometrics” (Pisa, Italy) and at the seminar organized by the European Business School (Wiesbaden, Germany).

Finally, “The Dynamic Impact of Uncertainty in Causing and Forecasting the Distribution of Oil Returns and Risk” has been presented at the “9th International Conference on Computational and Financial Econometrics” (London, UK).

Chapter 2

Asset Allocation Strategies Based on Penalized Quantile Regression

2.1 State-of-the art and further developments

Starting with the seminal contribution by Markowitz (1952) on the mean-variance portfolio theory, portfolio estimation and asset selection got increasing attention from both a practitioner's and a research view point. In the finance industry, asset allocation and security selection have a central role in designing portfolio strategies for both private and institutional investors. Differently, the academia focused on developments of the Markowitz approach over different research lines: linking it to market equilibrium as done by Sharpe (1964), Lintner (1965b), Lintner (1965a), and Mossin (1966); modifying the objective function both when it is set as an utility function or when it takes the form of a performance measure (Alexander and Baptista, 2002; Farinelli et al., 2008); developing tools for the estimation and the forecasting of the Markowitz model inputs, with great emphasis on return and risk.

Among the various methodological advancements, we focus on those associated with variations of the objective function or, more generally, based on alternative representations of the asset allocation problem. Some of the various asset allocation approaches proposed in the last decades share a common feature: they have a companion representation in the form of regression models where the coefficients correspond or are linked to the assets weights in a portfolio. Two examples are given by the estimation of efficient portfolio weights by means of linear regression of a constant on asset excess returns (Britten-Jones, 1999) and the estimation of the Global Minimum Variance Portfolio weights through the solution of a specific regression model, see e.g. Fan et al. (2012).

In the previously cited cases, the portfolio variance plays a fundamental role. However, even if we agree with the relevance of variance (or volatility) for risk measurement and management, the financial literature now includes a large number of other indicators that might be more appropriate. As an example, for an investor whose preferences or attitudes to risk are summarized by an utility function where extreme risk is present, volatility might be replaced by tail expectation. Similarly, we can easily identify reward measures and performance indicators differing from the simple average (or cumulated) return and from the Sharpe ratio. On the one side, we aim at keeping those elements in mind but, on the other side, we want to remain confined within the allocation approaches where weights can be associated with a linear model. In such a framework, it is possible to show that the adoption of non standard methods for the estimation of the linear model parameters (and portfolio weights) leads to solutions equivalent to the optimization of performance measures or risk measures differing from the Sharpe ratio and the volatility. Thus, moving away from the least square regression approach for estimating portfolio weights is equivalent to optimizing a non-standard objective function.

The leading example is given by Bassett et al. (2004), which proposed a pessimistic asset allocation strategy relying on the quantile regression method introduced by Koenker and Bassett (1978). In particular, Bassett et al. (2004) start from the linear model whose solution provides the global minimum variance portfolio weights. Then, they show that estimating by quantile regression methods a low quantile (the α -quantile) of the response variable allows minimizing a measure of the portfolio extreme risk that they call α -risk. Therefore, a change in the estimation approach allows moving from Global Minimum Variance portfolio to the minimum α -risk portfolio.

Variants of the α -risk are known under a variety of names, such as “Expected Shortfall” (Acerbi and Tasche, 2002), “Conditional Value-at-Risk” (Rockafellar and Uryasev, 2000), and “Tail Conditional Expectation” (Artzner et al., 1999). Consequently, the pessimistic asset allocation strategy of Bassett et al. (2004) corresponds to an *extreme* risk minimization approach.

The work by Bassett et al. (2004) also represents the starting point of our contributions. Building on quantile regression methods, we aim at introducing innovative asset allocation strategies coherent with the maximization of a risk-reward trade-off. Moreover, we combine quantile regression with regularization methods, such as LASSO (Tibshirani, 1996), in order to cope with the problematic issues arising from the large portfolios dimensionality. Our contributions provide an answer to specific research questions, with a potential application within the financial industry.

The first research question originates from a limitation of the pessimistic asset allocation approach of Bassett et al. (2004), which is a risk-minimization-driven strategy. Is it possible to maintain the focus on the α -risk, as in Bassett et al. (2004), and at the same time maximize a performance measure, thus taking into account also rewards? Our first contribution consists in showing that quantile regression models can be used not only to build financial portfolios with minimum extreme risk, as well known in the financial econometrics literature, but also to optimize other risk and performance measures by exploiting the information contained in the entire support of the response variable conditional distribution. We focus on linear model representation whose coefficients are associated with the Global Minimum Variance portfolio weights, as in Bassett et al. (2004), where quantile regression estimation of the linear model coefficients leads to the minimum α -risk portfolio. We generalize the results in two ways. First, showing that, under reasonable assumptions, at the median level, the quantile regression solution of the linear model corresponds to the minimization of the mean absolute deviation of portfolio returns. Secondly, at high quantiles levels the quantile regression solution provides portfolio weights with an outstanding performance in terms of profitability and risk-adjusted returns. Such a solution corresponds to the maximization of a specific reward measure, which is given as the conditional expected return net of the most favorable outcomes; hence, it is a pessimistic allocation, as in Bassett et al. (2004), but with focus on the right tail rather than the left one. As a by-product, we introduce a performance measure, completely new in the literature; it is a risk-adjusted ratio which quantifies the magnitude of all the negative returns balanced by a part of positive results, net of the most favorable ones.

The second research question stems from empirical evidence and practitioners' needs. Financial portfolios are frequently characterized by a large cross-sectional dimension, i.e. they (potentially) include a large number of assets. Assuming we are interested in maintaining a pessimistic asset allocation strategy, possibly coherent with the investor preferences, we face a clear trade-off: on the one side the large cross-sectional dimension allows taking advantage of the diversification benefits which are anyway relevant even within a pessimistic allocation approach; on the other side, the number of parameters to estimate by a quantile regression approach quickly increases as the portfolio dimension grows. As a result, the accumulation of estimation errors becomes a problem that must be addressed. The question is as follows: can we control estimation error by maintaining the focus on the pessimistic asset allocation approach? Providing a possible solution, we impose a penalty on the ℓ_1 -norm of the quantile regression coefficients along the line of the Least Absolute Shrinkage and Selection Operator (LASSO), introduced by Tibshirani (1996) in a standard linear regression framework. Recent studies show that applications of the LASSO to the mean-variance portfolio framework provide benefits in terms of sparsity of the portfolio (indirectly

associated with diversification/concentration and turnover) and good out-of-sample properties, see e.g. Brodie et al. (2009), DeMiguel et al. (2009), Fan et al. (2012), Yen and Yen (2014) and Fastrich et al. (2014). In the statistical literature, the ℓ_1 penalty became a widely used tool not only in linear regression, but also for quantile regression models, see, e.g. Koenker (2005), Belloni and Chernozhukov (2011), Li and Zhu (2008), while applications in asset allocation are still scarce. Härdle et al. (2014) used penalized quantile regression as a security selection tool, in an index tracking framework, whereas the portfolios weights are estimated by optimizing as objective function the Cornish-Fisher Value-at-Risk. Differently, in the approach we introduce, the penalized quantile regression model automatically selects and estimates the relevant assets weights in a single step. To the best of our knowledge, such an approach has never been investigated in the financial econometrics literature. We therefore suggest to estimate the pessimistic asset allocation strategy of Bassett et al. (2004) and the pessimistic allocation here introduced, by penalizing the ℓ_1 -norm of the assets weights.

We evaluate the two methodological contributions previously outlined with an extensive empirical analysis, in which we compare the performance of the asset allocation strategies built from the quantile regression models, at different quantiles levels, and the ordinary least squares approach. Differently from Bassett et al. (2004), we use both simulated as well as real-world data. Moreover, we analyze both the in-sample and the out-of-sample performances by implementing a rolling window procedure. Finally, we focus on portfolios with a reasonably large cross-sectional dimension, being thus close to the real needs of the financial industry. The in-sample results, on both real-world and simulated data, show that each strategy performs consistently to the expectations, optimizing the respective objective functions, α -risk, mean absolute deviation, or the upper-tail-based reward measure. Indeed, quantile regression applied at low probability levels outperforms the other strategies in terms of extreme risk. Least squares and median regression models turn out to be the best strategies in terms of volatility, as the former minimizes the portfolio variance whereas the latter minimizes the mean absolute deviation of portfolios returns. Finally, quantile regression at high probability levels provides the best results in terms of profitability and risk-adjusted return. Out-of-sample results show that the quantile regression models maintain their in-sample properties but only at high probability levels. Despite such a result might be interesting from a practitioner's point of view, it is quite surprising from a methodological perspective. We studied this phenomenon providing an explanation associated with the role of the model intercept. Finally, we highlight the critical importance of regularizing the quantile regression problem to improve the out-of-sample performance of portfolios characterized by large cross-sectional dimension.

The work is structured as follows. In Section 2.2 we show how the penalized quantile regression model can be used to build asset allocation strategies optimizing different objective functions at different quantiles levels. In Section 2.3 we provide the details about the datasets, the performance indicators, the rolling window procedure and the empirical set-up. In Section 2.4 we discuss the empirical results, and Section 2.5 concludes.

2.2 Asset allocation based on quantile regression

2.2.1 Portfolio performance as function of quantiles levels

Several asset allocation strategies estimate portfolio weights by optimizing a criterion function. The latter could either take the form of an utility function, of a risk measure, of a performance measure or a combination of different measures. A subset of those asset allocation approaches have a companion representation in the form of a regression model where the estimated coefficients correspond to the portfolio weights. The leading example is the Global Minimum Variance Portfolio (*GMVP*), whose composition is the solution of the ordinary least squares regression model.

In the case of a financial portfolio consisting of n stocks, let $\mathbf{R} = [R_1, \dots, R_n]$ be the row vector of the assets returns¹, with covariance matrix $\mathbb{V}[\mathbf{R}] = \mathbf{\Sigma}$, whereas the row vector of weights is denoted by $\mathbf{w} = [w_1, \dots, w_n]$; given the row vector $\mathbf{1}$, with length n and elements equal to 1, $\mathbf{1}\mathbf{w}' = 1$ in order to satisfy the budget constraint. The portfolio return is then $R_p = \mathbf{R}\mathbf{w}'$, but we can also use a companion representation to include the budget constraint. First, we set $R_i^* = R_n - R_i$, for $i = 1, \dots, n-1$, and then use these deviations for the computation of the portfolio return which becomes $R_p = R_n - w_1R_1^* - \dots - w_{n-1}R_{n-1}^*$, where the n -th asset weight ensures the weights sum at 1. By starting from such representation, it is possible to show that the minimization of the portfolio variance can be expressed as

$$\begin{aligned} \min_{\mathbf{w} \in \mathbb{R}^n} \mathbf{w}\mathbf{\Sigma}\mathbf{w}' &= \min_{(\mathbf{w}_{-n}, \xi) \in \mathbb{R}^n} \mathbb{E} [\mathbf{R}\mathbf{w}' - \xi]^2 \\ &= \min_{(\mathbf{w}_{-n}, \xi) \in \mathbb{R}^n} \mathbb{E} [R_n - w_1R_1^* - \dots - w_{n-1}R_{n-1}^* - \xi]^2, \end{aligned} \quad (2.1)$$

where ξ is the intercept of the linear regression model, \mathbf{w}_{-n} denotes the weights vector excluding w_n and $w_n = 1 - \sum_{i=1}^{n-1} w_i$, in order to satisfy the budget constraint.

In Equation (2.1), the portfolio variance, $\mathbf{w}\mathbf{\Sigma}\mathbf{w}'$, is rewritten as $\mathbb{E}[R_n - w_1R_1^* - \dots - w_{n-1}R_{n-1}^* - \xi]^2$. The latter, corresponds to the variance of the errors for the linear regression of asset n returns, R_n , with respect to R_i^* . Therefore, it is possible to minimize $\mathbf{w}\mathbf{\Sigma}\mathbf{w}'$ by

¹To simplify the notation we suppress the dependence of returns on time.

minimizing the sum of squared errors of a linear regression model, with response variable R_n and covariates R_1^*, \dots, R_{n-1}^* . Thus, estimating the coefficients w_1, \dots, w_{n-1} , along with the intercept ξ , is equivalent to finding the *GMVP* weights (Fan et al., 2012). In Model (2.1), the response variable equals the n -th asset returns, R_n . However, the result does not depend on the choice of the numeraire asset.

Despite the portfolio variance is a relevant risk measure, the financial literature presents a large number of other indicators to be considered for profitability and risk analyses. If we move away from the standard linear regression framework above mentioned, it is possible to estimate portfolios compositions by optimizing alternative performance measures. For instance, Bassett et al. (2004) proposed a pessimistic asset allocation strategy that relies on quantile regression in order to minimize a risk measure: the so-called α -risk.

By starting from a monotone increasing utility function $u(\cdot)$, that transforms monetary outcomes into utility terms, Bassett et al. (2004) define the expected utility associated with R_p as

$$\mathbb{E}[u(R_p)] = \int_{-\infty}^{\infty} u(r_p) dF_{R_p}(r_p) = \int_0^1 u\left(F_{R_p}^{-1}(\vartheta)\right) d\vartheta, \quad (2.2)$$

where $F_{R_p}(r_p)$ denotes the distribution function of R_p evaluated at r_p , with density $f_{R_p}(r_p)$, whereas ϑ is the quantile index such that $\vartheta \in \mathcal{U}$, with $\mathcal{U} \subset (0, 1)$.

Such a framework is directly linked with the Choquet expected utility (Schmeidler, 1989), defined as

$$\mathbb{E}_v[u(R_p)] = \int_0^1 u\left(F_{R_p}^{-1}(\vartheta)\right) dv(\vartheta), \quad (2.3)$$

where $v(\cdot)$ is a distortion function which modifies the original probability assessment. In particular, $v(\cdot)$ allows inflating the likelihood associated with the least favorable (i.e. $v(\cdot)$ is concave) or the most favorable outcomes (i.e. $v(\cdot)$ is convex). Bassett et al. (2004) propose to use the function $v_\alpha(\vartheta) = \min\{\vartheta/\alpha, 1\}$, where α is a very low probability level, e.g. $\alpha = \{0.01, 0.05, 0.1\}$, associated with the (negative) returns in the left tail of $f_{R_p}(r_p)$. Then, (2.3) can be rewritten as

$$\mathbb{E}_{v_\alpha}[u(R_p)] = \alpha^{-1} \int_0^\alpha u\left(F_{R_p}^{-1}(\vartheta)\right) d\vartheta, \quad (2.4)$$

where $\mathbb{E}_{v_\alpha}[u(R_p)]$ implies pessimism since it inflates the likelihood associated with the α least favorable outcomes, whereas the remaining $1 - \alpha$ proportion is entirely deflated.

Equation (2.4) is directly linked to a measure of extreme risk, $\rho_{v_\alpha}(R_p)$, defined by Bassett et al. (2004) as α -risk:

$$\rho_{v_\alpha}(R_p) = - \int_0^1 F_{R_p}^{-1}(\vartheta) dv(\vartheta) = -\alpha^{-1} \int_0^\alpha F_{R_p}^{-1}(\vartheta) d\vartheta. \quad (2.5)$$

Such a quantity is a coherent measure of risk according to the definition of Artzner et al. (1999). Many variants of $\rho_{v_\alpha}(R_p)$ have been discussed in the financial literature, under a variety of names: Expected Shortfall (Acerbi and Tasche, 2002), Conditional Value-at-Risk (Rockafellar and Uryasev, 2000), and Tail Conditional Expectation (Artzner et al., 1999).²

Notably, (2.5) might be taken as the target risk measure for portfolio allocation, see e.g. Basak and Shapiro (2001), Krokmal et al. (2002), Ciliberti et al. (2007), Mansini et al. (2007). In such a case, $\rho_{v_\alpha}(R_p)$ can be minimized by resorting to the quantile regression method, as suggested by Bassett et al. (2004), in a framework similar to the estimation of the *GMVP* weights in (2.1), where R_n is the response variable, whereas R_1^*, \dots, R_{n-1}^* are the covariates. Within a quantile regression framework, the conditional ϑ -th quantile of R_n is estimated by minimizing the expected value of the asymmetric loss function:

$$\rho_\vartheta(\varepsilon) = \varepsilon [\vartheta - I(\varepsilon < 0)], \quad (2.6)$$

where $\varepsilon = R_n - \xi(\vartheta) - w_1(\vartheta)R_1^* - \dots - w_{n-1}(\vartheta)R_{n-1}^*$, $\xi(\vartheta)$ is the model intercept, and $I(\cdot)$ denotes the indicator function taking value 1 if the condition in (\cdot) is true, 0 otherwise.

The estimated ϑ -th conditional quantile of R_n is equal to $\widehat{\xi}(\vartheta) + \widehat{w}_1(\vartheta)R_1^* + \dots + \widehat{w}_{n-1}(\vartheta)R_{n-1}^*$, where $[\widehat{\xi}(\vartheta), \widehat{w}_1(\vartheta), \dots, \widehat{w}_{n-1}(\vartheta)]$ is the coefficients vector minimizing (2.6), at a specific quantile level ϑ (Koenker and Bassett, 1978). In the case in which $\vartheta = \alpha$, Bassett et al. (2004) showed that

$$\min_{(\xi(\alpha), \mathbf{w}_{-n}(\alpha)) \in \mathbb{R}^n} \mathbb{E}[\rho_\alpha(\varepsilon)] = \alpha (\mu_p + \rho_{v_\alpha}(R_p)), \quad (2.7)$$

where $\mathbf{w}_{-n}(\alpha) = [w_1(\alpha), \dots, w_{n-1}(\alpha)]$, $\mu_p = \mathbb{E}[R_p]$ and $\rho_{v_\alpha}(R_p)$ as in (2.5).

Let $r_{n,t}$ and $r_{i,t}^*$ be, respectively, the observed values of R_n and R_i^* , for $i = 1, \dots, n-1$, at time t , then, from (2.7), the quantile regression model

$$\min_{(\xi(\alpha), \mathbf{w}_{-n}(\alpha)) \in \mathbb{R}^n} \sum_{t=1}^T \rho_\alpha(r_{n,t} - w_1(\alpha)r_{1,t}^* - \dots - w_{n-1}(\alpha)r_{n-1,t}^* - \xi(\alpha)) \quad (2.8)$$

$$s.t. \mu_p = c$$

allows minimizing the empirical α -risk of a financial portfolio, with the constraints that the expected portfolio return is equal to a target c and that the sum of the assets weights is equal to 1. Similarly to Model (2.1), $[\widehat{w}_1(\alpha), \dots, \widehat{w}_{n-1}(\alpha)]$, the estimated coefficients vector of the covariates R_1^*, \dots, R_{n-1}^* in the quantile regression model, is then the weights vector of

²Although $\rho_{v_\alpha}(R_p)$ could be denominated in different ways, throughout the paper we refer to the (2.5) as α -risk.

R_1, \dots, R_{n-1} for the portfolio with minimum α -risk. Note that the weight of the n -th asset is then derived from the budget constraint: $w_n(\alpha) = 1 - \sum_{i=1}^{n-1} w_i(\alpha)$. In this formulation, the assets weights do not change if we change the numeraire. As the constraint $\mu_p = c$ in (2.8) requires the estimation of expected returns, which is known to be a challenging task due to large estimation errors, see e.g. Brodie (1993) and Chopra and Ziemba (1993), we hereby choose to focus on

$$\min_{(\xi(\alpha), \mathbf{w}_{-n}(\alpha)) \in \mathbb{R}^n} \sum_{t=1}^T \rho_\alpha (r_{n,t} - w_1(\alpha)r_{1,t}^* - \dots - w_{n-1}(\alpha)r_{n-1,t}^* - \xi(\alpha)), \quad (2.9)$$

that is the minimization of the portfolio α -risk, subject only to the budget constraint.

As the portfolio performance does not depend just on extreme risk, but rather on the occurrence of returns over their entire density support, we generalize the approach of Bassett et al. (2004) and allow the construction of portfolios calibrated on different performance measures. First of all, we introduce an approach which makes use of two new performance measures, with potential application also in the performance evaluation area. The main idea stems from observing that (2.5) could be associated with profitability, and no longer only with extreme risk, if we replace the low probability levels α by high probability levels. According to this intuition, we introduce two different indicators, as described next. If we denote the high probability values by ψ , e.g. $\psi = \{0.9, 0.95, 0.99\}$, the α -risk in (2.5) translates into

$$\Psi_1(R_p, \psi) = -\psi^{-1} \int_0^\psi F_{R_p}^{-1}(\vartheta) d\vartheta. \quad (2.10a)$$

Given that $-\Psi_1(R_p, \psi) = \mathbb{E}[R_p | R_p \leq F_{R_p}^{-1}(\psi)]$, the quantile regression model, applied at ψ , allows to minimize $\Psi_1(R_p, \psi)$ and, consequently, to maximize the conditional portfolio expected return. Therefore, by minimizing $\Psi_1(R_p, \psi)$ an agent is taking a pessimistic asset allocation strategy, in the sense that such a choice leads to the maximization of the portfolio expected return net of the most favorable outcomes, since the interval $(\psi, 1]$ is not included in the objective function. Moreover, as $\lim_{\psi \rightarrow 1} -\Psi_1(R_p, \psi) = \int_0^\psi F_{R_p}^{-1}(\vartheta) d\vartheta$, it is possible to obtain benefits in terms of unconditional portfolio expected return, given that we maximize a quantity which approximates $\mathbb{E}[R_p]$.

The second performance indicator we introduce is obtained by decomposing the integral in Equation (2.10a). In particular, let $\bar{\vartheta}$ be the value of ϑ such that $F_{R_p}^{-1}(\bar{\vartheta}) = 0$, at which the integral $\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta$ reaches its lowest value; for instance, $\bar{\vartheta} = 0.5$ when the distribution

is symmetric at 0. Given $\bar{\vartheta} < \psi < 1$, (2.10a) could be rewritten as

$$\begin{aligned}\Psi_1(R_p, \psi) &= -\psi^{-1} \int_0^\psi F_{R_p}^{-1}(\vartheta) d\vartheta \\ &= -\psi^{-1} \left[\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta + \int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta \right],\end{aligned}\quad (2.10b)$$

where $\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta$ is computed from negative realizations and quantifies their magnitude. Differently, $\int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta$ quantifies the magnitude of a part of the positive outcomes, excluding the most favorable ones, given that the region beyond ψ is not considered. The quantile regression model, applied at the ψ -th level, minimizes $\Psi_1(R_p, \psi)$ and, thus, also $-\zeta = -\left(\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta + \int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta\right)$. When $f_{R_p}(r_p)$ is characterized by a null or a negative skewness, ζ is negative, whereas ζ could be positive in the case of positive skewness. In the first case, ζ could be seen as a net loss; differently, in the latter case, ζ is a net profit. Therefore, quantile regression model leads to the minimization of a loss ($\zeta < 0$) or to the maximization of a profit ($\zeta > 0$), since, in (2.10b), ζ is multiplied by the constant $-\psi^{-1} < 0$. In other words, the quantile regression model minimizes $|\zeta|$, if $\zeta < 0$, or maximizes $|\zeta|$, if $\zeta > 0$, with benefits in terms of the ratio:

$$\Psi_2(R_p, \psi) = \frac{\int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta}{\left| \int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta \right|}.\quad (2.11)$$

Hence, the ratio $\Psi_2(R_p, \psi)$ is a risk-adjusted measure because it quantifies the magnitude of all the negative outcomes balanced by a part of positive results, net of the most favorable ones. Although high $\Psi_2(R_p, \psi)$ values correspond to low $\Psi_1(R_p, \psi)$ levels, when different strategies are compared, there are no guarantees that the strategy which minimizes $\Psi_1(R_p, \psi)$ is the one which maximizes $\Psi_2(R_p, \psi)$. In other words, the ranking of different strategies built on the sum between $\int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta$ and $\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta$ may not coincide with the ranking built on the basis of their ratio. For example, suppose that for a certain strategy A , $\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta = -34.04$ and $\int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta = 8.13$; differently, strategy B returns $\int_0^{\bar{\vartheta}} F_{R_p}^{-1}(\vartheta) d\vartheta = -33.74$ and $\int_{\bar{\vartheta}}^\psi F_{R_p}^{-1}(\vartheta) d\vartheta = 7.95$. B is better in terms of $\Psi_1(R_p, \psi)$, but A outperforms B in terms of $\Psi_2(R_p, \psi)$.

Therefore, beside their use within a quantile regression-based portfolio allocation strategy, the two indicators in (2.10a) and (2.11) are also novel contributions for the performance measurement literature. We stress that while (2.10a) resemble tail-based risk measures, and is not a proper absolute performance measure, see Caporin et al. (2014a), indicator (2.11) is novel. It is interesting to notice that $\Psi_2(R_p, \psi)$ is related both to the *Omega* measure

proposed by Keating and Shadwick (2002) and to the modified Rachev Ratio (Ortobelli et al., 2005). Nevertheless, there are some important differences between these quantities. The *Omega* measure (Keating and Shadwick, 2002) is defined as

$$\Omega(R_p) = \frac{\int_0^\infty [1 - F_{R_p}(r_p)dr_p]}{\int_{-\infty}^0 F_{R_p}(r_p)dr_p}. \quad (2.12)$$

$\Psi_2(R_p, \psi)$ differs from Omega because the latter, compares the entire regions associated with negative and positive outcomes, respectively. Differently, (2.11) is more restrictive because its numerator takes into account just a part of positive outcomes, as long as $\psi < 1$.

The modified Rachev Ratio (Ortobelli et al., 2005), $MR(R_p, \alpha, \psi)$, equals

$$MR(R_p, \alpha, \psi) = \frac{-\alpha^{-1} \int_0^\alpha F_{R_p}^{-1}(\vartheta)d\vartheta}{(1 - \psi)^{-1} \int_\psi^1 F_{R_p}^{-1}(\vartheta)d\vartheta}. \quad (2.13)$$

In that case, the difference arises from the fact that (2.13) compares the extreme outcomes associated to the distribution tails, as typically $\alpha = \{0.01, 0.05, 0.1\}$ and $\psi = \{0.9, 0.95, 0.99\}$, thus fully neglecting the impact coming from the central part of the portfolio returns distribution.

In empirical application, we compute the sample counterparts of $\Psi_1(R_p, \psi)$ and $\Psi_2(R_p, \psi)$ as follows:

$$\widehat{\Psi}_1(r_p, \psi) = -\frac{\sum_{t=1}^T r_{p,t} I(r_{p,t} \leq \widehat{Q}_\psi(r_p))}{\sum_{t=1}^T I(r_{p,t} \leq \widehat{Q}_\psi(r_p))}, \quad (2.14)$$

$$\widehat{\Psi}_2(r_p, \psi) = \frac{\sum_{t=1}^T r_{p,t} I(0 \leq r_{p,t} \leq \widehat{Q}_\psi(r_p))}{|\sum_{t=1}^T r_{p,t} I(r_{p,t} < 0)|}, \quad (2.15)$$

where $r_{p,t}$ denotes the portfolio return observed at t , $\widehat{Q}_\psi(r_p)$ denotes the estimated ψ -th quantile of the portfolio returns, $I(\cdot)$ is the indicator function taking value 1 if the condition in (\cdot) is true, 0 otherwise.

Beside highlighting the impact of the quantile regression model in terms of $\Psi_1(R_p, \psi)$ and $\Psi_2(R_p, \psi)$, we go further considering also the central ϑ values. Now we focus on portfolio volatility, quantified by the mean absolute deviation introduced by Konno and Yamazaki (1991):

$$MAD(R_p) = \mathbb{E} [|R_p - \mathbb{E}[R_p]|], \quad (2.16)$$

estimated in empirical applications as

$$\widehat{MAD}(r_p) = \frac{1}{T} \sum_{t=1}^T |r_{p,t} - \bar{r}_p|, \quad (2.17)$$

where \bar{r}_p is the sample mean of portfolio returns in the interval $[1, T]$.

Under the hypotheses that the portfolio mean, $\mathbb{E}[R_p]$, and the median regression intercept $\xi(\vartheta = 0.5)$ are both equal to zero, we show that the median regression allows to minimize the quantity in (2.16). Indeed, the quantile regression model at $\vartheta = 0.5$ minimizes $\mathbb{E}[|R_n - w_1(0.5)R_1^* - \dots - w_{n-1}(0.5)R_{n-1}^* - \xi(0.5)|] = \mathbb{E}[|R_p - \xi(0.5)|]$. Thus, under the assumptions $\mathbb{E}[R_p] = 0$ and $\xi(0.5) = 0$, the median regression minimizes the mean absolute deviation of portfolio returns.

To summarize, the quantile regression model allows reaching different purposes. First of all, we should choose a low probability level, α , when we want to minimize the extreme risk, quantified by the α -risk. When the attention is focused on volatility minimization, quantified by MAD , we should use median regression. Finally, with a high probability level, ψ , we minimize $\Psi_1(R_p, \psi)$, with positive effects in terms of $\Psi_2(R_p, \psi)$.

2.2.2 Simulation exercise

Bassett et al. (2004) applied the model in (2.8) by using simulated returns of 4 assets, showing its better performance in terms of extreme risk with respect to the classic Markowitz (1952) portfolio. Nevertheless, in the real world, investors trade financial portfolios consisting of many more assets, primarily to achieve a satisfactory diversification level, to better deal with the risk-return trade-off (Markowitz, 1952). In order to further motivate the relevance of quantile regression approaches for portfolio allocation, and to show the impact of the methodological improvements previously described, we consider a simulation exercise on portfolios containing 94 assets.³ The returns are simulated from distributions with different features, in terms of kurtosis and skewness, to verify how these differences impact on the performance of the considered strategies. In particular, we test 4 different distributions: the Multivariate Normal, the Multivariate t -Student with 5 degrees of freedom, the Multivariate Skew-Normal with negative skewness and the Multivariate Skew-Normal with positive skewness. In the case of the Multivariate Skew-Normal, we used two different values of the

³The portfolios dimensionality comes from the fact that we simulated returns from a distribution whose covariance matrix and mean vector are estimated from real data. We refer here to the constituents of the Standard & Poor's 100 index at November 21, 2014, and whose time series are continuously available from November 4, 2004 to November 21, 2014. See Section 2.3.1 for further details on the dataset.

skewness parameter, to obtain returns series with average skewness equal to 0.02 and -0.02, respectively.

We simulated, from each distribution, 1000 samples of 94 assets returns for 500 periods, comparing four strategies: the standard as in (2.1), denoted as *OLS*, and the ones arising from the quantile regression models applied at three probability levels, that is $\vartheta = \{0.1, 0.5, 0.9\}$, denoted, respectively, as *QR*(0.1), *QR*(0.5) and *QR*(0.9), respectively. The portfolios weights determined by the four strategies are estimated from each of the 1000 simulated samples and the portfolios returns are computed by means of the in-sample approach.⁴ Thus, for each strategy and for each sample, we obtain 500 portfolio returns from which we compute the following statistics: variance, mean absolute deviation, α -risk (with $\alpha = 0.1$), $\Psi_1(R_p, \psi)$ and $\Psi_2(R_p, \psi)$, at $\psi = 0.9$. We display in Figures 2.1-2.2 the results obtained from the Multivariate Normal and Skew-Normal (right-skewed) distributions.⁵

As expected, *OLS* and *QR*(0.5) provide the best results in terms of portfolio volatility, since the former minimizes the portfolio variance (Subfigure (a)), whereas the latter minimizes the portfolio *MAD* (Subfigure (b)). *QR*(0.1) minimizes the α -risk at $\alpha = 0.1$ (Subfigure (c)), differently, *QR*(0.9) is the best strategy in terms of profitability. Indeed, as it is possible to see from Subfigure (d), it outperforms the other three strategies in terms of $\hat{\Psi}_1(r_p, 0.9)$, which quantifies the average of portfolio returns which are less or equal to their 90-th percentile. It is interesting to observe that, when the assets returns distributions are positive skewed (Figure 2.2(d)), $\hat{\Psi}_1(r_p, 0.9)$ takes negative values, mainly in the case of *QR*(0.9); it means that, on average, the positive returns prevail over the negative ones, even if the most favorable outcomes in the right tail of the distributions are discarded.

Finally, *QR*(0.9) provides the best results in terms of $\hat{\Psi}_2(r_p, 0.9)$ in three out of four cases; the exception occurs in Figure 2.2(e), where the returns are characterized by significant positive skewness, which is typically an unrealistic assumption for financial returns series, typically affected by negative skewness, as discussed in Cont (2001). The results also show that the ranking based on $\Psi_1(R_p, \psi)$, as expected, might not coincide to the one based on $\Psi_2(R_p, \psi)$. Indeed, we checked that *QR*(0.9) always provides, also in the case of the Multivariate Skew-Normal distribution (right-skewed), the highest values of the difference between the numerator and the denominator of the ratio defined in Equation (2.11).

⁴See Section 2.3.2 for further details about the in-sample analysis.

⁵The boxplots obtained in the case of the other distributions are available on request. Results are qualitatively similar to those obtained from the Multivariate Normal distribution: the presence of fatter tails, as in the Multivariate *t*-Student, or of negative asymmetry, as in the Multivariate Skew-Normal with negative skewness, don't lead to different results.

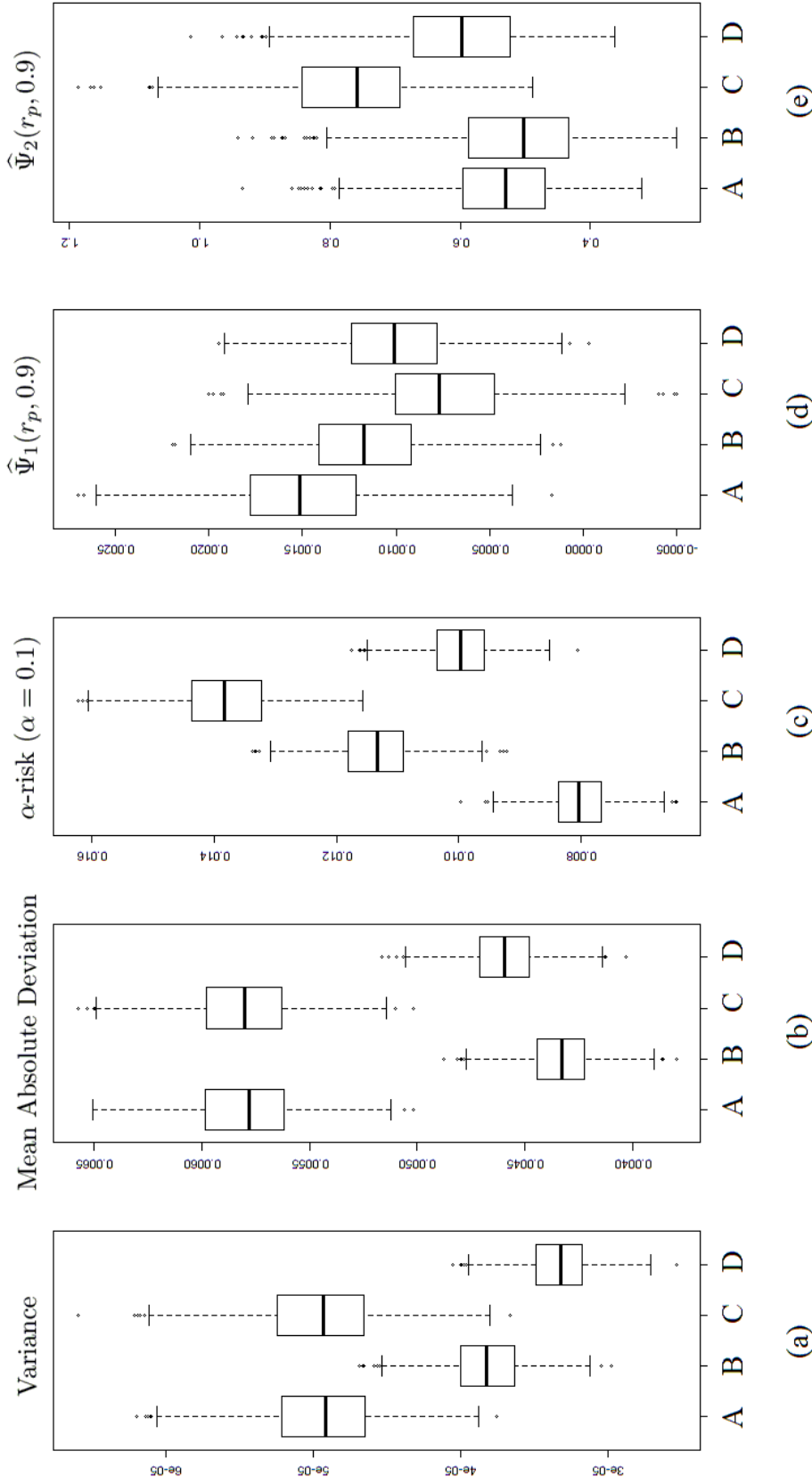


Fig. 2.1 In-Sample results with returns simulated from Multivariate Normal Distribution, whose covariance matrix and mean vector are estimated from real data, i.e. the constituents of the Standard & Poor's 100 index at November 21, 2014, and whose time series are continuously available from November 4, 2004 to November 21, 2014. The analysis is carried out on 1000 simulated samples of 94 assets returns for 500 periods. From left to right the subfigures report the boxplots of the following statistics: variance (a), mean absolute deviation (b), α -risk at $\alpha = 0.1$ (c), $\widehat{\Psi}_1(r_p, \psi)$ at $\psi = 0.9$ (d) and $\widehat{\Psi}_2(r_p, \psi)$ at $\psi = 0.9$ (e). A, B, C denote the strategies built from quantile regression models applied at probabilities levels of 0.1, 0.5 and 0.9, respectively, whereas D refers to the ordinary least squares regression model.

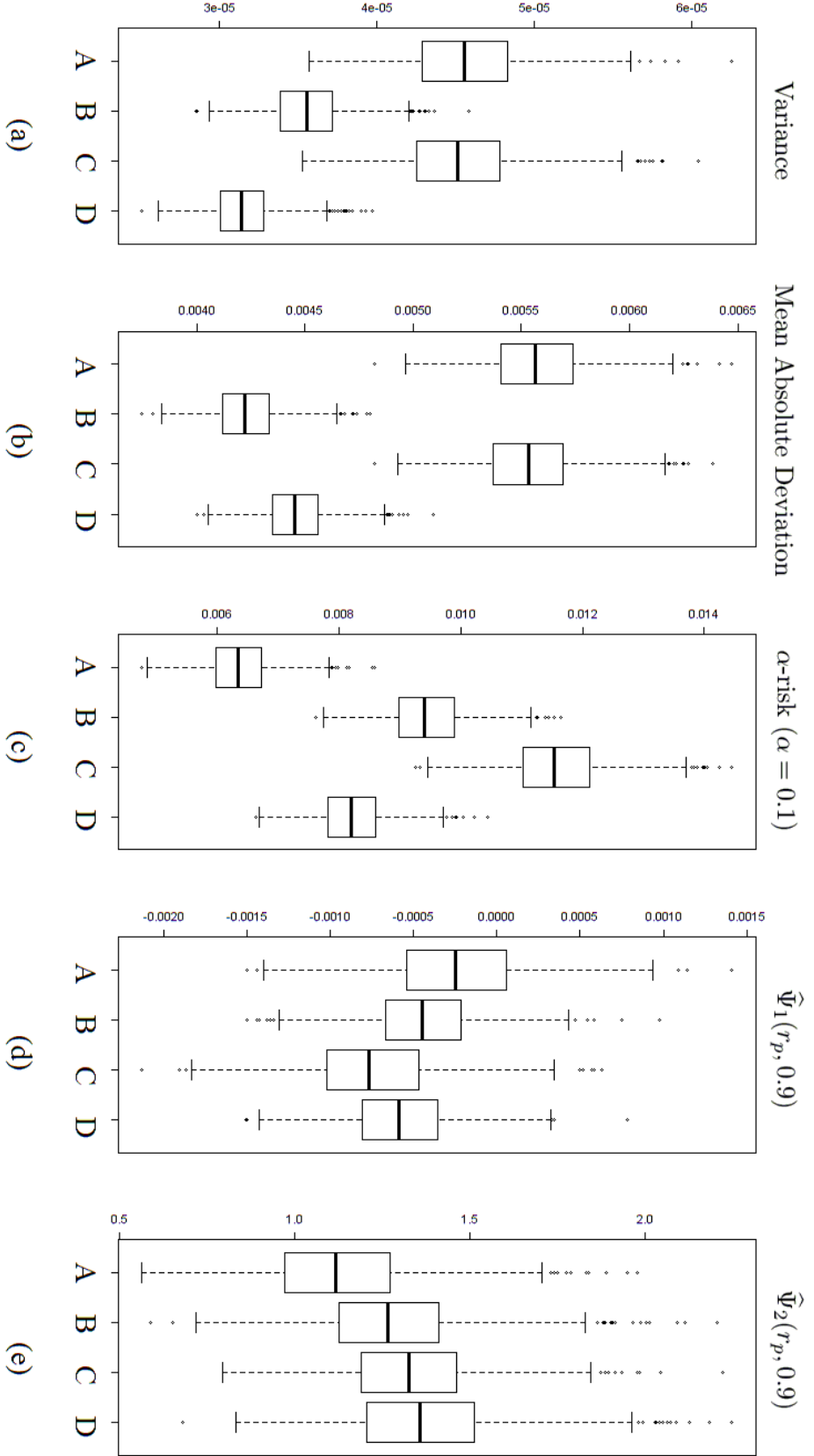


Fig. 2.2 In-Sample results with returns simulated from Multivariate Skew-Normal Distribution (average skewness of 0.02), whose covariance matrix is estimated from real data, i.e. the constituents of the Standard & Poor's 100 index at November 21, 2014, and whose time series are continuously available from November 4, 2004 to November 21, 2014. The analysis is carried out on 1000 simulated samples of 94 assets returns for 500 periods. From left to right the subfigures report the boxplots of the following statistics: variance (a), mean absolute deviation (b), α -risk at $\alpha = 0.1$ (c), $\hat{\Psi}_1(r_p, \psi)$ at $\psi = 0.9$ (d) and $\hat{\Psi}_2(r_p, \psi)$ at $\psi = 0.9$ (e). A, B, C denote the strategies built from quantile regression models applied at probabilities levels of 0.1, 0.5 and 0.9, respectively, whereas D refers to the ordinary least squares regression model.

Nevertheless, in all the cases apart the one in which the distributions of returns are assumed to be right-skewed (this is hardly the case of financial time series), $QR(0.9)$ turns out to be the best strategy in terms of $\Psi_2(R_p, \psi)$ too.

2.2.3 The inclusion of the ℓ_1 -norm penalty

Large portfolios allow taking advantage of the diversification benefits. Nevertheless, Statman (1987) and recently Fan et al. (2012) show that the inclusion of additional assets in the portfolio involves relevant benefits but only up to a certain number of assets. On the other hand, the number of parameters to estimate increases as the portfolio dimensionality grows. As a result, the consequent accumulation of estimation errors becomes a problem that must be carefully addressed. For, instance, Kourtis et al. (2012) defined the estimation error as the price to pay for diversification. Furthermore, when large portfolios are built through regression models, as showed in Section 2.2.1, the assets returns are typically highly correlated; then, the estimated portfolios weights are poorly determined and exhibit high variance.

We propose here a further extension to the Bassett et al. (2004) approach in order to deal with financial portfolios characterized by large cross-sectional dimension, i.e. by a large number of assets. Our solution builds on penalizations techniques, see e.g. Hastie et al. (2009), widely applied in the recent financial literature, see e.g. DeMiguel et al. (2009), Fan et al. (2012), Fastrich et al. (2014), Yen and Yen (2014), Ando and Bai (2014). Among all the possible methods, we make use of the ℓ_1 -norm penalty, useful in the context of variable selection, by which we penalize the absolute size of the regression coefficients. In the last ten years, it became a widely used tool not only in linear regression, but also in quantile regression models, see, e.g. Koenker (2005), Belloni and Chernozhukov (2011), Li and Zhu (2008).

Härdle et al. (2014) used the ℓ_1 -norm penalty in a quantile regression model where the response variable is a core asset, represented by the Standard & Poor's 500 index, whereas the covariates are hedging satellites, i.e. a set of hedge funds. After setting the quantiles levels according to a precise scheme, the aim is to buy the hedge funds whose coefficients, estimated from the penalized quantile regression model, are different from zero. Therefore, in the work by Härdle et al. (2014) the penalized quantile regression is used as a security selection tool, in an index tracking framework. In a second step, by placing the focus on the downside risk, Härdle et al. (2014) determine the optimal weights of the funds previously selected by optimizing the objective function given by the Cornish-Fisher Value-at-Risk (CF-VaR). Differently, we use a penalized quantile regression model which allows to solve in just one step both the security selection and the asset allocation problems. The response and

the covariates are determined from the assets included in the portfolio, without considering external variables (such as market indices) with the aim to optimize different performance measures according to different ϑ levels. In particular, given $1 \leq k \leq n$, we propose the asset allocation strategy based on the following model:

$$\arg \min_{(\mathbf{w}_{-k}(\vartheta), \xi(\vartheta)) \in \mathbb{R}^n} \sum_{t=1}^T \rho_{\vartheta} \left(r_{k,t} - \sum_{j \neq k} w_j(\vartheta) r_{j,t}^* - \xi(\vartheta) \right) + \lambda \sum_{j \neq k} |w_j(\vartheta)|, \quad (2.18)$$

where the parameters $(\xi(\vartheta), \mathbf{w}_{-k}(\vartheta))$ depend on the probability level ϑ , $\mathbf{w}_{-k}(\vartheta)$ is the weights vector which does not include w_k , that is the weight of the k -th asset selected in the (2.18) as numeraire, whereas λ is the penalty parameter; the larger λ , the larger is the portfolio sparsity. Thus, by penalizing the sum of the absolute coefficients values, i.e. ℓ_1 -norm, some of the weights might converge to zero. Moreover, in a financial perspective, this leads to select portfolios with a fewer active positions, with relevant impact on turnover and transactions costs.

Unlike Model (2.8), the solutions of (2.18) depend on the choice of the numeraire R_k . Indeed the R_k weight is not penalized, given that it is equal to $w_k(\vartheta) = 1 - \sum_{j \neq k} w_j(\vartheta)$, to satisfy the budget constraint. Therefore, we need to define a criterion to select the numeraire asset among all the available securities. We propose a naive but intuitive solution: we suggest selecting as numeraire the asset characterized by the lowest in-sample $\widehat{\Psi}_1(R_p, \psi)$ value. This choice is due to the fact that the numeraire plays a prominent role with respect to the other selected assets, therefore it must record the best expected performance in terms of a specific indicator.

Another important issue refers to the choice of the optimal λ value; we remind that the higher λ , the more sparse is the portfolio. For this purpose, we follow the approach proposed by Belloni and Chernozhukov (2011). They considered the problem of dealing with a large number of explanatory variables, with respect to the sample size T , where only at most $s \leq n$ regressors have a non-null impact on each conditional quantile of the response variable. In this context, where the ordinary quantile regression estimates are not consistent, they showed that, by penalizing the ℓ_1 -norm of the regressors coefficients, the estimates are uniformly consistent over the compact set $\mathcal{U} \subset (0, 1)$. In order to determine the optimal λ value, they proposed a data-driven method with optimal asymptotic properties. This method takes into account the correlation among the variables involved in the model and leads to different optimal λ values according to the ϑ level. The penalization parameter is built from the random variable

$$\Lambda = T \sup_{\vartheta \in \mathcal{U}} \max_{j \neq k} \left| \frac{1}{T} \sum_{t=1}^T \left[\frac{r_{j,t}^* (\vartheta - \mathbf{1}_{\{e_t \leq \vartheta\}})}{\hat{\sigma}_j \sqrt{\vartheta(1-\vartheta)}} \right] \right|, \quad (2.19)$$

where e_1, \dots, e_T are i.i.d. uniform $(0, 1)$ random variables independently distributed from the covariates, \mathbf{r}^* , and $\hat{\sigma}_j^2 = T^{-1} \sum_{t=1}^T (r_{j,t}^*)^2$. As recommended in Belloni and Chernozhukov (2011), we simulate the Λ values, by running 100000 iterations. Hence, the optimal penalty parameter is computed as

$$\lambda^* = \frac{\tau \sqrt{\vartheta(1-\vartheta)}}{T}, \quad (2.20)$$

where $\tau = 2\hat{Q}_{0.9}(\Lambda|\mathbf{r}^*)$ and $\hat{Q}_{0.9}(\Lambda|\mathbf{r}^*)$ is the 90-th percentile of Λ conditional on the explanatory variables values. The method given by Equation (2.20) is used in the empirical analysis, discussed in Section 2.4, in order to determine the optimal value of the penalization parameter λ .

2.3 Empirical set-up

2.3.1 Data description

While Bassett et al. (2004) used only simulated data in their work, we go further and provide an empirical evaluation based on real data, taking into account two different datasets. They consist of the daily returns of the firms included, from November 4, 2004 to November 21, 2014, in the baskets of the Standard & Poor's 100 and the Standard & Poor's 500 indices, respectively.⁶ In the first dataset (*S&P100*) we deal with 94 assets, whereas in the second one (*S&P500*) we have 452 stocks. Figure 2.3 allows to analyze the main descriptive statistics for the largest *S&P500* dataset.

From Figure 2.3(a) we can see that the average returns are close to zero and that they tend to be positive, being symmetrically distributed around the median 0.055%; their maximum and minimum values are equal to 0.242% and -0.044% , respectively. The distribution of the standard deviations is centered at the median value of 2.137% and ranges from 1.013% to 5.329% (Figure 2.3(b)), with the presence of a few particularly volatile companies associated to the extreme right-tailed values.

The kurtosis index distribution is right-skewed, with extremely large values in the right tail (Figure 2.3(c)): the median is equal to 12.379, whereas the minimum and the maximum are equal, respectively, to 5.948 and 72.867, pointing out that the returns distributions are affected by heavy tails, as expected (see Cont (2001) for stylized facts of financial returns). Figure 2.3(d) shows that the skewness index is symmetrically distributed around the median value of 0.209. It ranges from -3.088 to 2.640, with the presence of some extreme values in both the tails; hence, the returns have leptokurtic and asymmetric distributions.

⁶The data are recovered from Thomson Reuters Datastream.

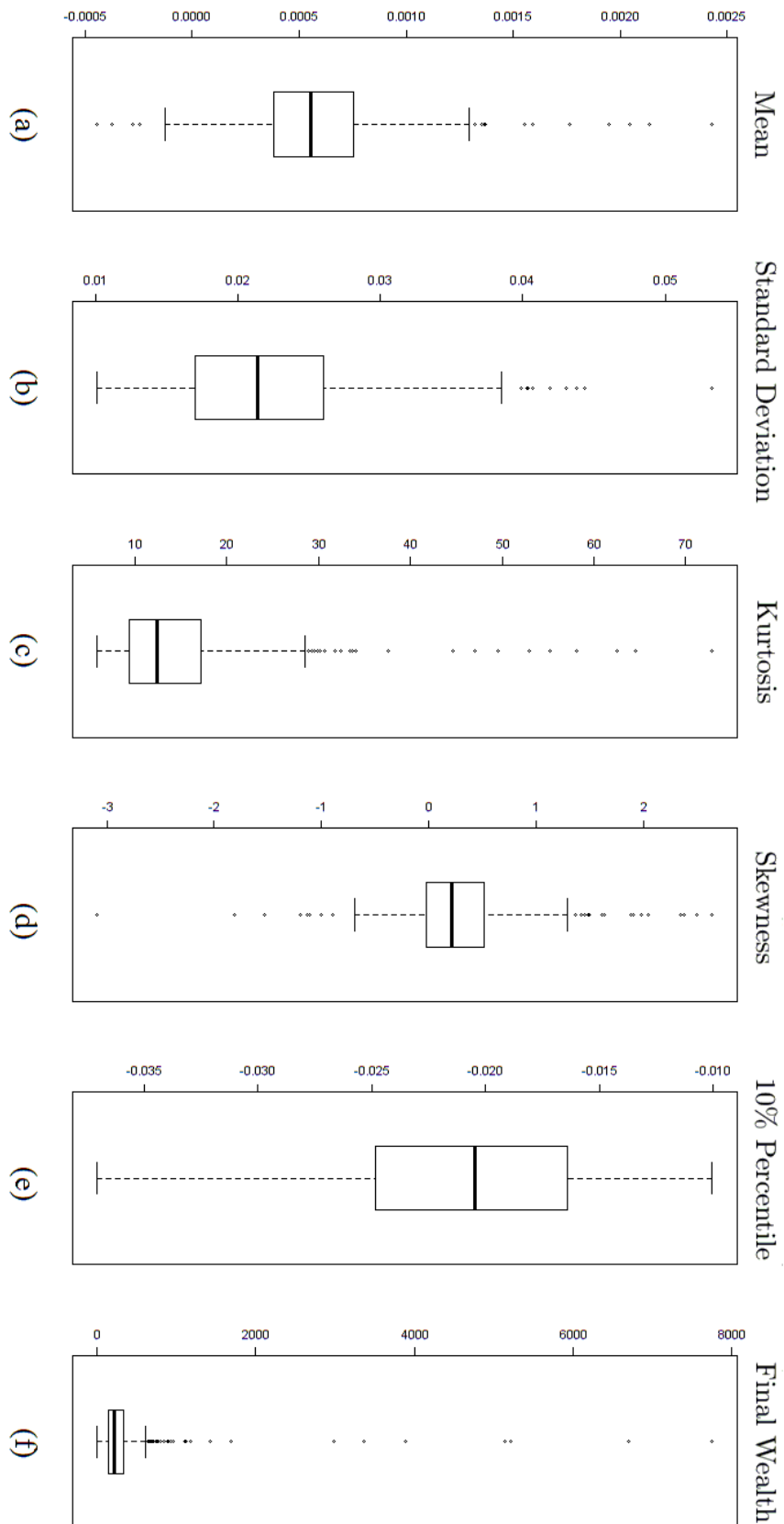


Fig. 2.3 Statistics computed from each of the assets returns generated by the Standard & Poor's 500 constituents. From left to right the boxplots refer to: (a) the average returns, (b) the standard deviations of the assets returns, (c) the Kurtosis index, (d) the skewness index, (e) the 10-th percentile of returns, (f) the final wealth generated by each company at the end of the analysis when the initial wealth is equal to 100 \$.

Figure 2.3(e) displays the 10-th percentile of the returns series, a measure of extreme risk that we used as estimate of the Value-at-Risk, as discussed in Section 2.3.3. From Figure 2.3(e) we can see that the distribution of the 10-th percentile is affected by a slight negative asymmetry, centered at the median of -2.044% , with minimum and maximum equal to -3.703% and -1.004% , respectively. The last measure, reported in Figure 2.3(f), displays the distribution of the wealth we obtain in November 21, 2014 from the single assets, after investing, in November 4, 2004, 100 \$ in each of them. The final wealth distribution, right-skewed, is affected by relevant extreme values in its right tail. It ranges from 2.423 to 7757.073, with median value of 221.650 \$; therefore, on average, we record an increase in stocks values.

2.3.2 Rolling analysis with in- and out-of-sample assessment

We estimate the portfolios weights by using the least squares and the quantile regression models introduced in Section 2.2. At the moment in which portfolios are built, it is just possible to predict their future performance, whereas the actual results are not ensured, being evaluable only ex post. Assuming to behave like an investor, we further extend the work of Bassett et al. (2004) by implementing a rolling window procedure, which allows to analyze the out-of-sample performance. Furthermore, we can assess the stability of the estimates over time.

The rolling window procedure is described as follows. Iteratively, the original assets returns time series, with dimension $T \times n$, are divided in subsamples with window size ws . The first subsample includes the daily returns from the first to the ws -th day. The second subsample is obtained by removing the oldest observations and including those of the $(ws + 1)$ -th day. The procedure goes on till the $(T - 1)$ -th day is reached. In the empirical analysis we make use of two different window sizes, that is $ws = \{500, 1000\}$, to check how the portfolio performance depends on both the portfolio dimensionality and the sample size.

For each window, we estimate the portfolio weights, denoted by $\hat{\mathbf{w}}_t$, for $t = ws, \dots, T - 1$, by means of a given asset allocation strategy. Let $\mathbf{r}_{t-ws+1,t}$ be the $ws \times n$ matrix whose rows are the assets returns vectors recorded in the period between $t - ws + 1$ and t . Then, portfolio returns are computed both in-sample and out-of-sample. In the first case, for each rolled subsample, we multiply each row of $\mathbf{r}_{t-ws+1,t}$ by $\hat{\mathbf{w}}_t$, obtaining ws in-sample portfolio returns, from which we can compute the performance indicators described in Section 2.3.3. Overall, from all the $T - ws$ submatrices $\mathbf{r}_{t-ws+1,t}$, we obtain $T - ws$ values of each performance indicator, whose distribution is analyzed in Section 2.4.

Unlike the in-sample analysis, where we assess estimated performance indicators, the aim of the out-of-sample analysis is to check whether the expectations find confirmation in the

actual outcomes. Therefore, the out-of-sample performance plays a critical role, given that it corresponds to the actual impacts on the wealth obtained by an investor that daily revises his portfolio. In particular, for $t = ws, \dots, T - 1$, $\widehat{\mathbf{w}}_t$ is multiplied by \mathbf{r}_{t+1} , i.e. the assets returns vector observed at $t + 1$, to obtain the out-of-sample portfolio returns. In this way, for each asset allocation strategy, we obtain one series of out-of-sample portfolio returns, that is a vector with length $T - ws$, from which we compute the performance indicators described in Section 2.3.3.

2.3.3 Performance indicators

In the empirical analysis we compare several asset allocation strategies, built from both the ordinary least squares and the quantile regression models. We assess and compare their performance using several indicators, to provide information about profitability, risk and impact of trading fees on the portfolios. Each indicator is computed from both the in-sample and out-of-sample returns.

Some of the performance indicators, namely α -risk, MAD , $\Psi_1(R_p, \psi)$ and $\Psi_2(R_p, \psi)$ are described in Section 2.2.1. In addition, we take into account other measures, typically used in financial studies. The first one is the Sharpe ratio, a risk-adjusted return measure, defined as

$$SR = \frac{\bar{r}_p}{\hat{\sigma}_p}, \quad (2.21)$$

where \bar{r}_p and $\hat{\sigma}_p$ denote, respectively, the sample mean and standard deviation of the portfolio returns.⁷ As stated in Section 2.3.2, in the in-sample case, \bar{r}_p and $\hat{\sigma}_p$ are computed for each of the rolled subsample $\mathbf{r}_{t-ws+1,t}$, for $t = ws, \dots, T - 1$. As result, we obtain $T - ws$ Sharpe ratios. Differently, in the out-of-sample case, we have one portfolio return for each window, obtaining overall a single vector of portfolio returns, with length equal to $T - ws$, from which we compute the Sharpe ratio.

In addition to the α -risk, we also consider the Value-at-Risk, defined as the threshold value such that the probability that the portfolio loss exceeds that value in a given time horizon is equal to α . The importance of Value-at-Risk is due not only to the fact that it is widely used by financial institutions to allocate their capital, but it is also used by financial authorities to define capital requirements in their supervisory activity.⁸ In the present work, the Value-at-Risk is estimated as the α -th quantile, with $\alpha = 0.1$, of the portfolio returns, both in-sample and out-of-sample.

⁷To be precise, the numerator of the (2.21) should be equal to the risk-free excess return. For simplicity, we assume that the risk-free return is equal to zero.

⁸See e.g. "International convergence of capital measurement and capital standards" by Basel Committee on Banking Supervision, available at <http://www.bis.org/publ/bcbs118.pdf>.

Finally, we assess the impact of the trading fees on the portfolio rebalancing activity through the turnover, computed as

$$Turn = \frac{1}{T - ws} \sum_{t=ws+1}^T \sum_{j=1}^n |\hat{w}_{j,t} - \hat{w}_{j,t-1}|, \quad (2.22)$$

where $\hat{w}_{j,t}$ is the weight of the j -th asset determined by an asset allocation strategy at day t . The higher the turnover, the larger is the impact of costs arising from the rebalancing activity.

2.4 Empirical results

The first aspect we analyze refers to the impact of the ℓ_1 -norm penalty on portfolio weights. For the quantile regression model, we estimate the optimal shrinkage parameter λ according to the method proposed by Belloni and Chernozhukov (2011). For each quantile level, we compute the (2.20) by using the full sample data, for both *S&P100* and *S&P500*. Hence, after implementing the rolling window procedure, we compute the number of active and short positions for each rolled sample, whose average values are denoted by \bar{n}_a and \bar{n}_s , respectively. The asset allocation strategy built from Model (2.18) is denoted as $PQR(\vartheta)$, for $\vartheta \in (0, 1)$. We also apply the ℓ_1 -norm on Model (2.1) and the resulting asset allocation strategy is denoted as *LASSO*. For *LASSO*, λ^* is calibrated in order to obtain comparable results, in terms of \bar{n}_a , to those generated by the quantile regression model at $\vartheta = 0.5$. For simplicity, we show in Table 2.1 the λ^* values and the average number of active and short positions over the rolled windows just for $PQR(0.1)$, $PQR(0.5)$, $PQR(0.9)$ and *LASSO*.⁹ We note that λ^* changes according to the ϑ levels, reaching relatively higher values at the center of the ϑ domain. This leads to the attenuation of the quantile regression approach tendency for an increase of active positions around the median, see Table 2.1.

Moreover, we analyzed the evolution over time of the portfolio weights estimated by both the ordinary least squares and the quantile regression approaches. We checked that the weights become more stable with the ℓ_1 -norm penalty and that the effect is more clear with $ws = 1000$. This result is due to the fact that the ℓ_1 -norm penalty shrinks to zero the weights of the highly correlated assets and the larger window size reduces the impact of the estimation errors.

⁹In the present work, the position on a certain stock is considered to be long if the absolute value of its weight is larger than 0.0005. Similarly, a position is considered short if the weight is less than -0.0005. The weights whose values are between -0.0005 and 0.0005 are set equal to zero.

Table 2.1 The impact of the ℓ_1 -norm penalty on active and short positions.

Strategy	λ^* (S&P100)	λ^* (S&P500)	\bar{n}_a (S&P100)		\bar{n}_a (S&P500)		\bar{n}_s (S&P100)		\bar{n}_s (S&P500)	
			$w_s=500$	$w_s=1000$	$w_s=500$	$w_s=1000$	$w_s=500$	$w_s=1000$	$w_s=500$	$w_s=1000$
$PQR(0.1)$	0.3644	0.6964	24	43	32	74	6	19	8	26
$PQR(0.5)$	0.6073	0.1608	29	44	42	77	8	17	10	28
$PQR(0.9)$	0.3644	0.6964	27	41	36	71	7	15	7	23
LASSO	0.0004	0.0004	28	45	42	77	9	20	12	32

The table reports the optimal shrinkage parameters (λ^*), the average numbers of active (\bar{n}_a) and short (\bar{n}_s) positions over the rolled samples, for the ordinary least squares (LASSO) and the quantile regression ($PQR(\vartheta)$) models, regularized through the ℓ_1 -norm penalty, by using different datasets and window sizes.

Now we analyze and compare the in-sample performance of the various allocation strategies. From the in-sample portfolios returns, computed as described in Section 2.3.2, we obtain the following performance measures: mean, standard deviation, Sharpe Ratio, Value-at-Risk at $\alpha = 0.1$ (see Section 2.3.3), α -risk at $\alpha = 0.1$, $\Psi_1(R_p, 0.9)$, $\Psi_2(R_p, 0.9)$ and MAD , defined by (2.5), (2.10a), (2.11) and (2.16), respectively. As stated in Section 2.3.2, overall, from all the $T - ws$ subsamples arising from the rolling procedure, we get $T - ws$ values of each performance indicator. We make use of boxplots in order to see how the in-sample statistics are distributed over the rolled subsamples. We show in Figures 2.4-2.5 the output obtained from the ordinary least squares and the quantile regression (applied at $\vartheta = \{0.1, 0.5, 0.9\}$) models, with and without the imposition of the ℓ_1 -norm penalty, by using as dataset the S&P 500 constituents (*S&P500*) and applying the rolling technique with window size of 500 observations.¹⁰ The ordinary least squares model provides the lowest standard deviation, with (Figure 2.5(b)) and without (Figure 2.4(b)) penalty, as expected, since its objective function is given by the portfolio variance. As pointed out in Section 2.2.1, under the assumptions $\mathbb{E}[R_p] = 0$ and $\xi(0.5) = 0$, the median regression minimizes the portfolio mean absolute deviation. Such a result is clear with $QR(0.5)$ (Figure 2.4(c)); when we impose the ℓ_1 -norm penalty, $PQR(0.5)$ still provides the lowest MAD with respect to the other quantile regression models, but it is outperformed by *LASSO* (Figure 2.5(c)). Indeed, if we impose the ℓ_1 -norm penalty on the median regression, we want to obtain a portfolio characterized by two features: low mean absolute deviation and sparsity, i.e. with a limited number of active positions. Then, the further aim of obtaining sparse portfolios could require sacrifices in terms of higher MAD with respect to the case in which the median regression is not penalized, i.e. when the attention is placed just on the minimization of the quantity defined in Equation (2.16). The same phenomenon occurs in the case of α -risk: the quantile regression model applied at $\vartheta = 0.1$ provides the best results when it is not penalized (Figure 2.4(e)), but its best performance fades when we impose the ℓ_1 -norm penalty (Figure 2.5(e)). In terms of Value-at-Risk there is no strategy which systematically outperforms the other ones; indeed, the ranking based on VaR is not well defined and it varies depending on both the dataset and the window size.

With regard to the Sharpe ratio (Figures 2.4(h)-2.5(h)), the lowest variance values allow *OLS* and *LASSO* to be ranked on top positions even if they do not generate, on average, remarkable portfolio returns, as showed by Figures 2.4(a)-2.5(a). Moreover, when we focus on the average return, there is no strategy systematically dominating the others. Finally, the quantile regression at $\vartheta = 0.9$ always outperforms the other strategies in terms of both $\hat{\Psi}_1(r_p, 0.9)$ and $\hat{\Psi}_2(r_p, 0.9)$ (Subfigures (f) and (g)).

¹⁰We obtained very similar results, available on request, in the other cases.

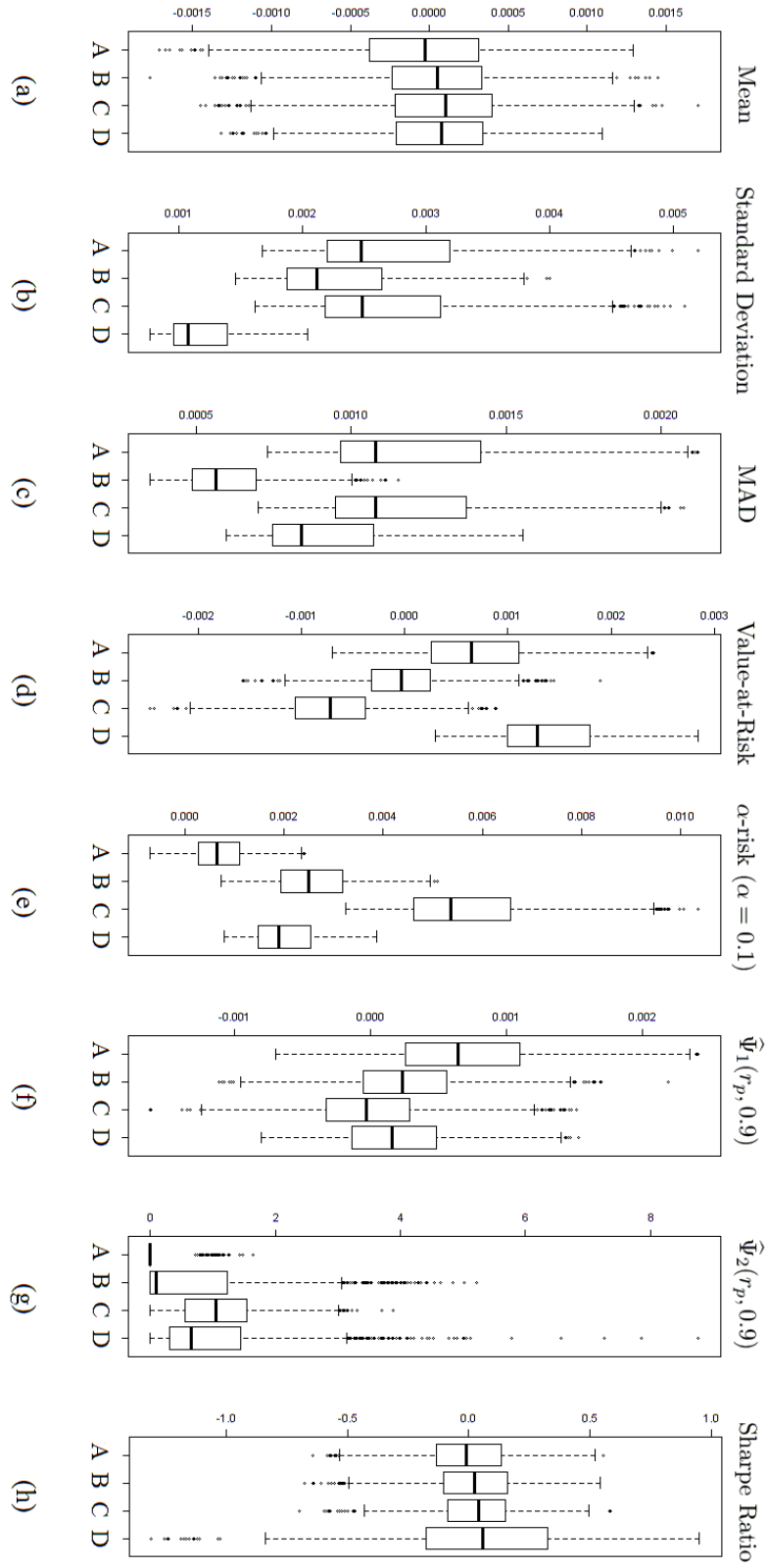


Fig. 2.4 In-Sample results, obtained without penalizations, from the returns series of the Standard & Poor's 500 index constituents. Rolling analysis with window size of 500 observations. A, B, C denote the strategies built from quantile regression models applied at probabilities levels of 0.1, 0.5 and 0.9, respectively, whereas D refers to the ordinary least squares regression model. From left to right the subfigures report the boxplots of the following statistics: mean (a), standard deviation (b), mean absolute deviation (c), Value-at-Risk at the level of 10% (d), α -risk at $\alpha = 0.1$ (e), $\hat{\Psi}_1(r_p, \psi)$ (f), $\hat{\Psi}_2(r_p, \psi)$ (g), with $\psi = 0.9$, Sharpe Ratio (h).

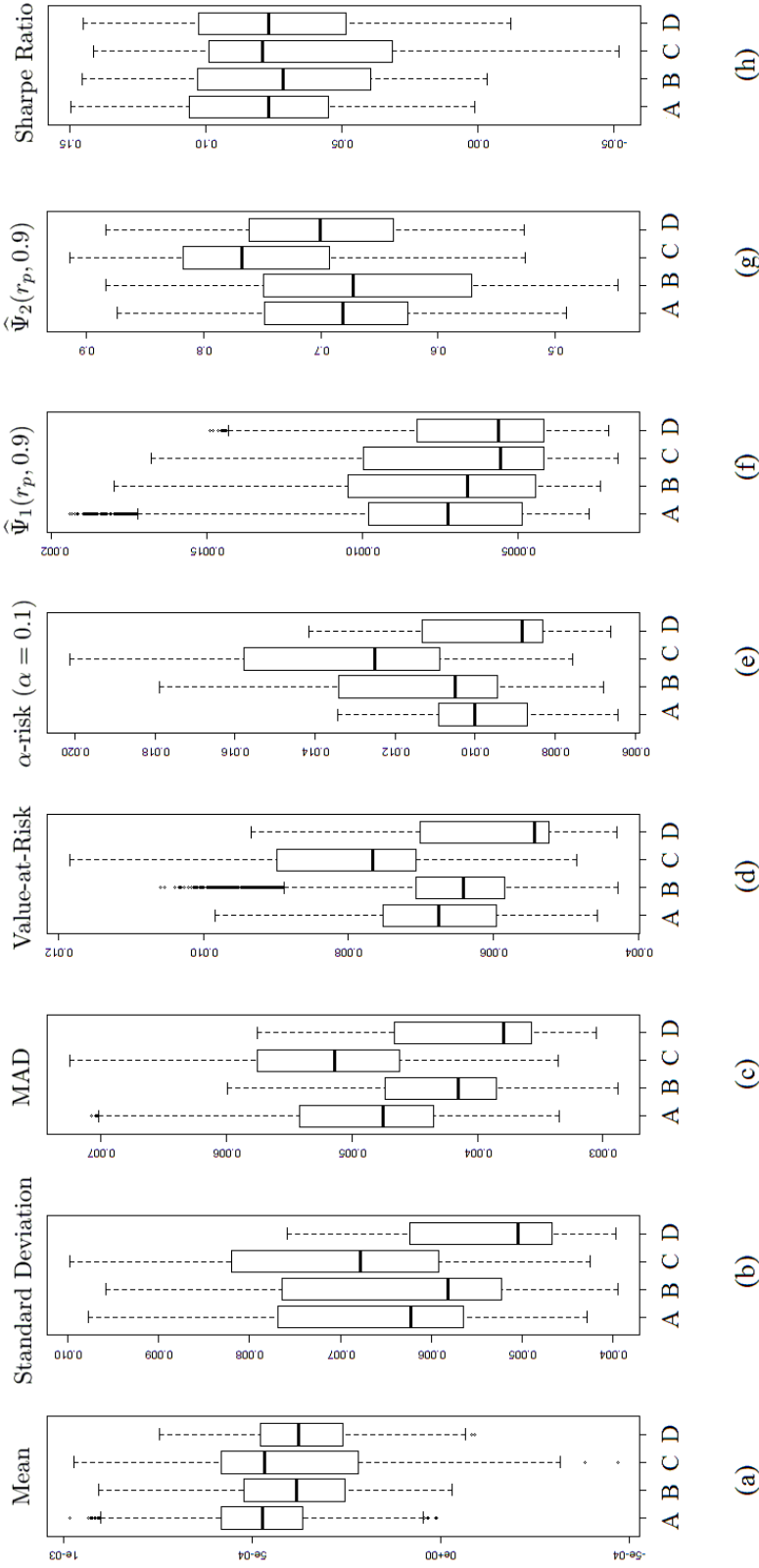


Fig. 2.5 In-Sample results, obtained by imposing the ℓ_1 -norm, from the returns series of the Standard & Poor's 500 index constituents. Rolling analysis with window size of 500 observations. A, B, C denote the strategies built from quantile regression models applied at probabilities levels of 0.1, 0.5 and 0.9, respectively, whereas D refers to the ordinary least squares regression model. From left to right the subfigures report the boxplots of the following statistics: mean (a), standard deviation (b), mean absolute deviation (c), Value-at-Risk at the level of 10% (d), α -risk at $\alpha = 0.1$ (e), $\widehat{\Psi}_1(r_p, \psi)$ (f), $\widehat{\Psi}_2(r_p, \psi)$ (g), with $\psi = 0.9$, Sharpe Ratio (h).

After analyzing the in-sample results, it is important to check whether they are confirmed out-of-sample. The critical role of the out-of-sample analysis is due to the fact that it refers to the actual performance that an investor draws from a financial portfolio. Now we compare the ordinary least squares and the quantile regression (applied at $\vartheta = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$) models, with and without ℓ_1 -norm penalty, showing in Figures 2.6-2.9 just the results obtained by running the rolling method with $ws = 1000$; similar results apply for $ws = 500$. Consistently to the in-sample expectations, the ordinary least squares regression model records the lowest out-of-sample standard deviation. The ℓ_1 -norm penalty implies important improvements for all the strategies. Although the quantile regression model works better in terms of *MAD* at central ϑ values, as expected, the ordinary least squares model records the lowest mean absolute deviations; it is important to notice that the ℓ_1 -norm penalty reduces the *MAD* of all the strategies (Figures 2.6(b)-2.7(b)).

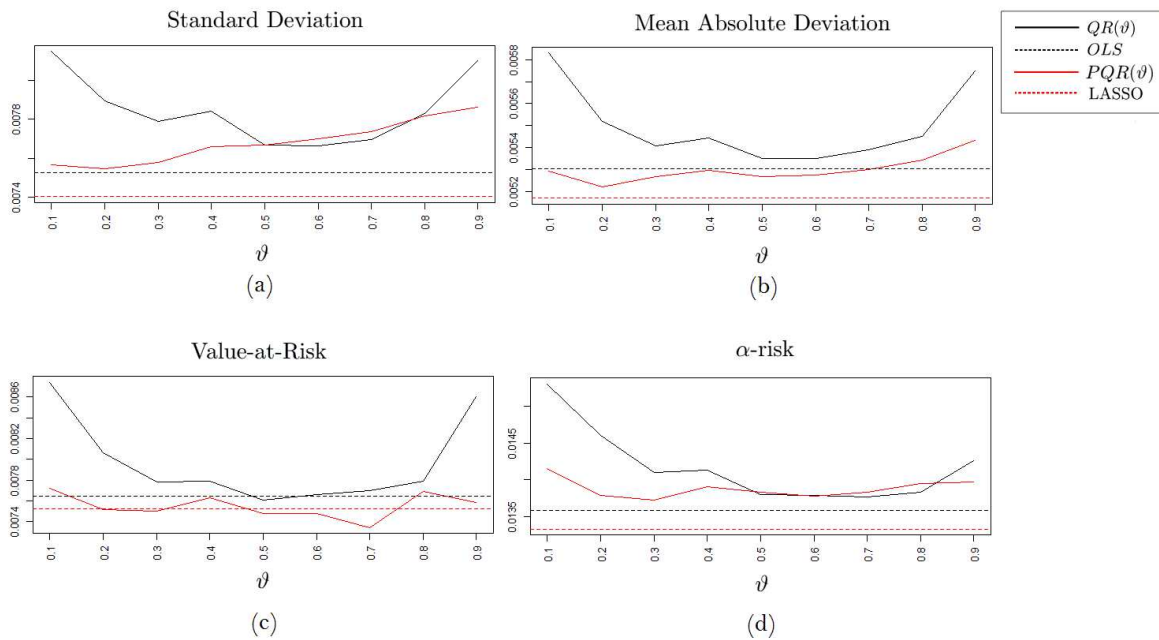


Fig. 2.6 Out-of-Sample results generated by the strategies built from the ordinary least squares (with (*LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and from the quantile regression (with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 100 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: standard deviation (a), mean absolute deviation (b), Value-at-Risk (c) and α -risk (d) at the level of 10%.

When we analyze the Value-at-Risk, the ℓ_1 -norm penalty brings benefits. In most cases, the ordinary least squares model outperforms the quantile regression models, mainly when they are not penalized. The ℓ_1 -norm penalty allows to reduce this gap and (see e.g. Figures 2.6(c)-2.7(c)) *LASSO* is outperformed by *PQR*(ϑ) at medium-high ϑ values. The behaviour

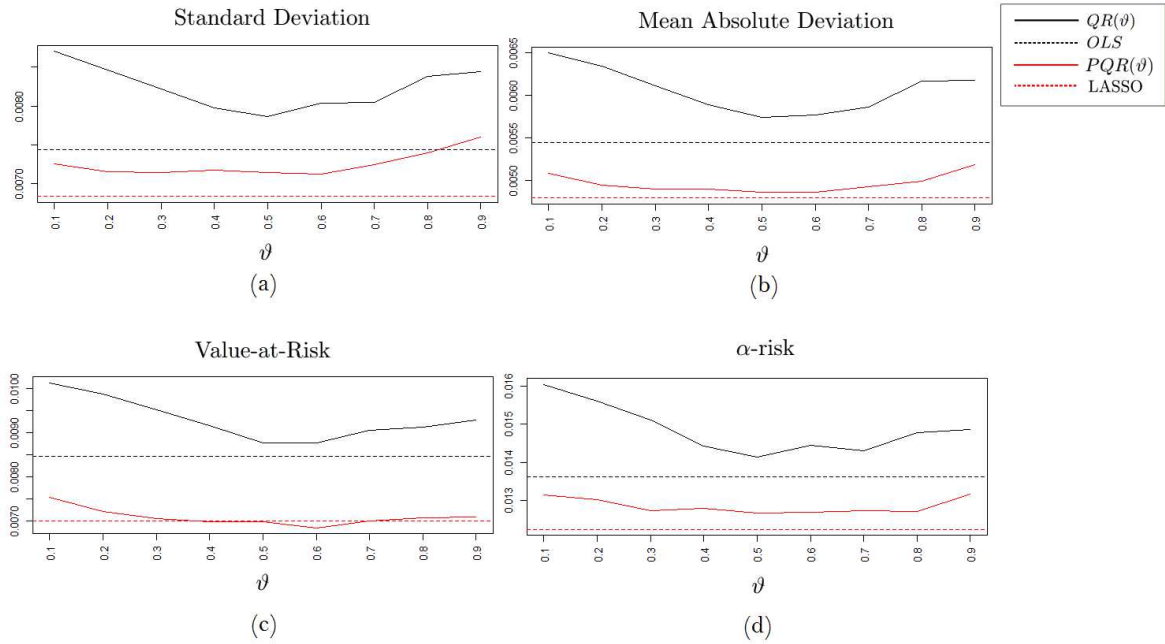


Fig. 2.7 Out-of-Sample results generated by the strategies built from the ordinary least squares (with ($LASSO$) and without (OLS) ℓ_1 -norm penalty) and from the quantile regression (with ($PQR(\vartheta)$) and without ($QR(\vartheta)$) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 500 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: standard deviation (a), mean absolute deviation (b), Value-at-Risk (c) and α -risk (d) at the level of 10%.

of the out-of-sample α -risk changes with respect to the in-sample case, as we can see, for instance, from Figures 2.6(d)-2.7(d). Indeed, the quantile regression model, with and without penalizations, generates disappointing results at low ϑ levels. The ℓ_1 -norm penalty turns out to be very effective, given that it reduces the exposure of all the strategies to the α -risk. $LASSO$ is ranked as the best strategy and the quantile regression model works better at central ϑ levels.

With regard to profitability, the quantile regression model applied at high ϑ levels provides outstanding out-of-sample results, consistently to the in-sample expectations. In case the portfolios weights are estimated by using a sample size not sufficiently large with respect to the portfolio dimensionality (i.e. $ws = 500$ and $n = 452$), $\hat{\Psi}_1(r_p, 0.9)$ and $\hat{\Psi}_2(r_p, 0.9)$ don't exhibit clear trends over ϑ when quantile regression model is not regularized, as we can see from the comparisons between Figures 2.8(a)-(b) and Figures 2.9(a)-(b). Differently, $\hat{\Psi}_1(r_p, 0.9)$ and $\hat{\Psi}_2(r_p, 0.9)$ become, respectively, negative and positive functions of ϑ when we impose the ℓ_1 -norm penalty, also in the case of the $S\&P500$ dataset. Similar conclusions are drawn from the average return and the Sharpe ratio. In particular, in the case of the $S\&P100$ dataset, the quantile regression model applied at high ϑ levels generates

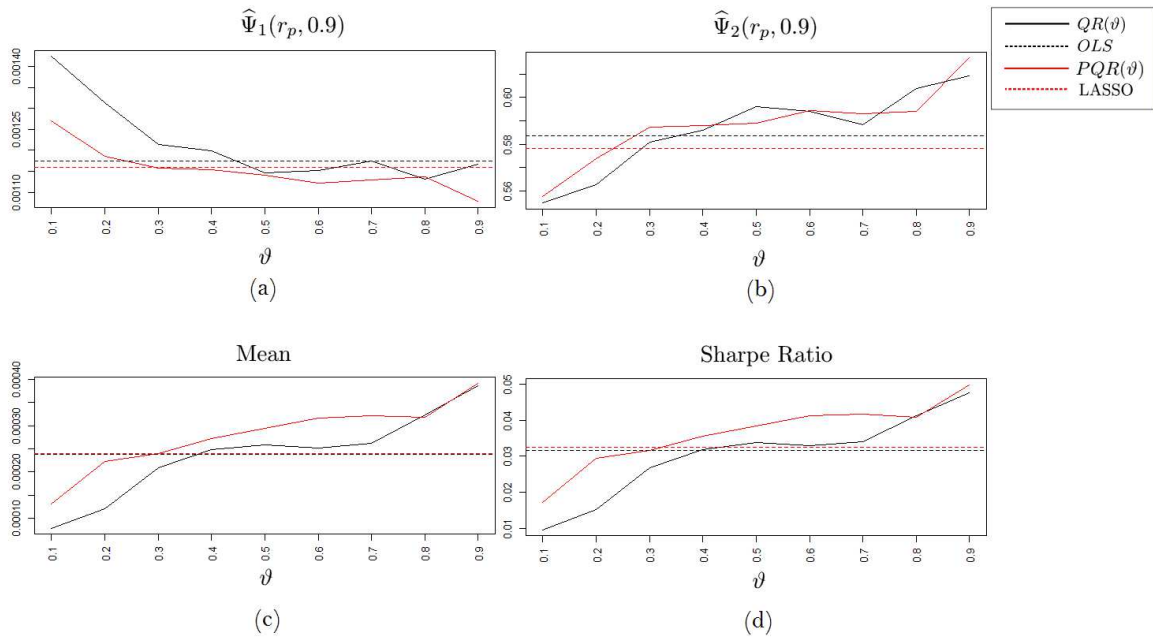


Fig. 2.8 Out-of-Sample results generated by the strategies built from the ordinary least squares (with (*LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and from the quantile regression (with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 100 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: $\widehat{\Psi}_1(r_p, \psi)$ (a) and $\widehat{\Psi}_2(r_p, \psi)$ (b) at $\psi = 0.9$, mean (c) and Sharpe Ratio (d).

the best performance, with and without penalizations, and *PQR*(0.9) outperforms all the other strategies (see e.g. Figures 2.8(c)-(d)). In the case of the *S&P500* dataset, the quantile regression model at $\vartheta = 0.9$ turns out to be the best strategy only if it is applied with the ℓ_1 -norm penalty. In order to analyze the trend over time of the portfolio value generated by each strategy, we assume that the initial wealth is equal to 100 \$ and we update it from $w_S + 1$ to T , according to the out-of-sample portfolio returns. In the case of *S&P100*, the quantile regression model at $\vartheta = 0.9$ not only provides the highest final wealth, but it also dominates the other strategies over time; differently, in the case of the *S&P500* dataset, it systematically outperforms the other strategies when we impose the ℓ_1 -norm penalty (see Figure 2.10).

As stated above, without regularizations, large portfolios are affected by an increasing variability in portfolios weights, with negative effects on trading fees. Consequently, turnover becomes a problem, particularly for the quantile regression model (Figure 2.11). The ℓ_1 -norm penalty allows to obtain sparse portfolios, with stable weights over time, thus it turns out to be very effective in terms of turnover control. In fact, the inclusion of the ℓ_1 -norm penalty causes a sharp drop of turnover in all the analyzed cases. The quantile regression model is

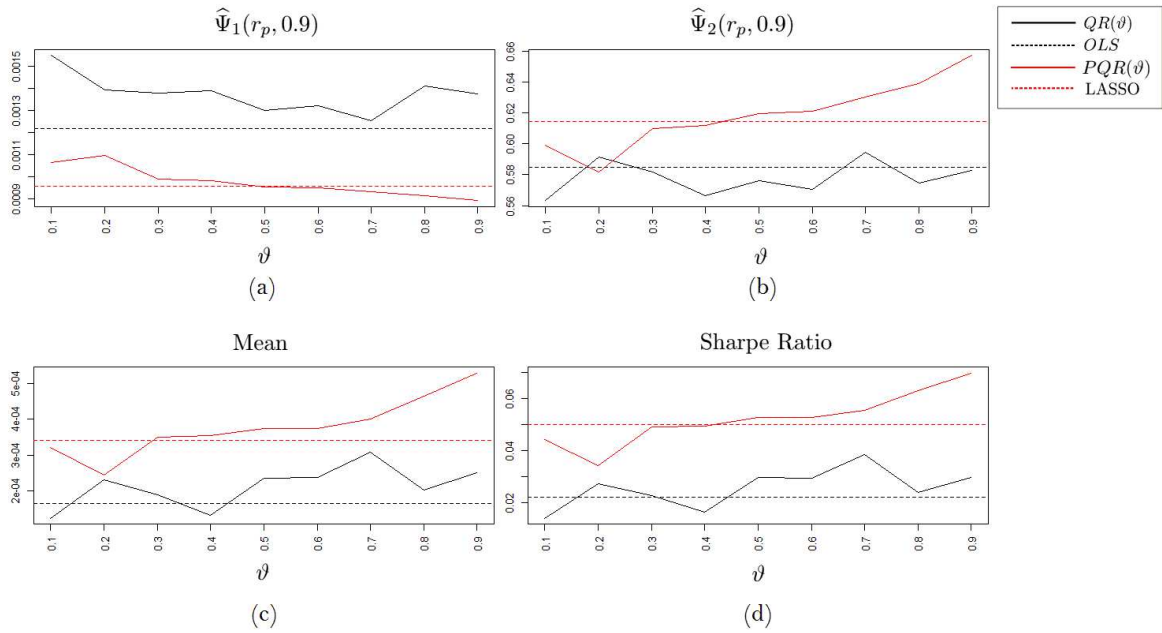


Fig. 2.9 Out-of-Sample results generated by the strategies built from the ordinary least squares (with ($LASSO$) and without (OLS) ℓ_1 -norm penalty) and from the quantile regression (with ($PQR(\vartheta)$) and without ($QR(\vartheta)$) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 500 index constituents and the rolling analysis is carried out with a window size of 1000 observations. In the subfigures we consider the following measures: $\hat{\Psi}_1(r_p, \psi)$ (a) and $\hat{\Psi}_2(r_p, \psi)$ (b) at $\psi = 0.9$, mean (c) and Sharpe Ratio (d).

the one which most benefits from regularization, since the marked gap between OLS and $QR(\vartheta)$ shrinks to become almost irrelevant.

To summarize the out-of-sample results, the ℓ_1 -norm penalty regularizes the portfolio weights, with noticeable positive effects on turnover. Moreover, it leads to clear improvements of both the portfolio risk and profitability. In general, the ordinary least squares model turns out to be the best strategy in terms of risk, given that it implies the lowest levels of volatility and extreme risk; quantile regression model works better at central ϑ values. The quantile regression, applied at low ϑ values, generates unsatisfactory results in terms of extreme risk with respect to the in-sample expectations; differently, when we analyze the portfolio profitability and the risk-adjusted return, it provides outstanding performances at high ϑ levels.

The empirical analysis is applied on data recorded from November 4, 2004 to November 21, 2014. Those years are characterized by special events, namely the subprime crisis, originated in the United States and marked by Lehman Brothers default in September 2008, and the sovereign debt crisis, which hit the Eurozone some years later. Those events had a deep impact on financial markets and, therefore, it is important to check whether they affect

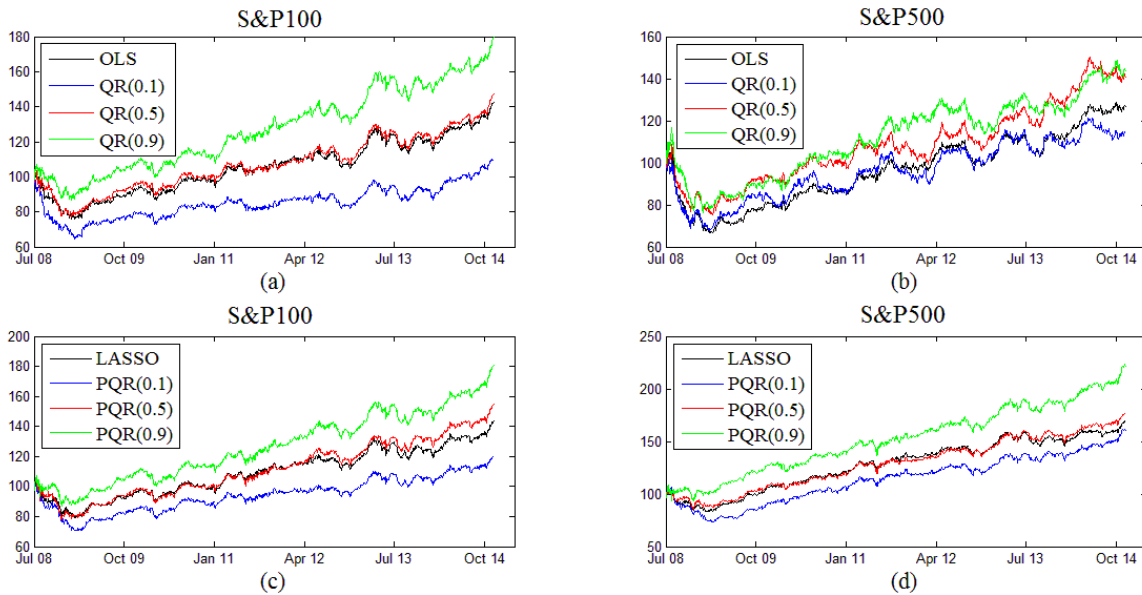


Fig. 2.10 The evolution of the portfolio value generated by the strategies built from the ordinary least squares (with (*LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and from the quantile regression (with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty), for $\vartheta = \{0.1, 0.5, 0.9\}$ models. The subfigures report different results according to the used dataset and to the fact that the regression models are applied with or without ℓ_1 -norm penalty. The rolling analysis is implemented with window size of 1000 observations.

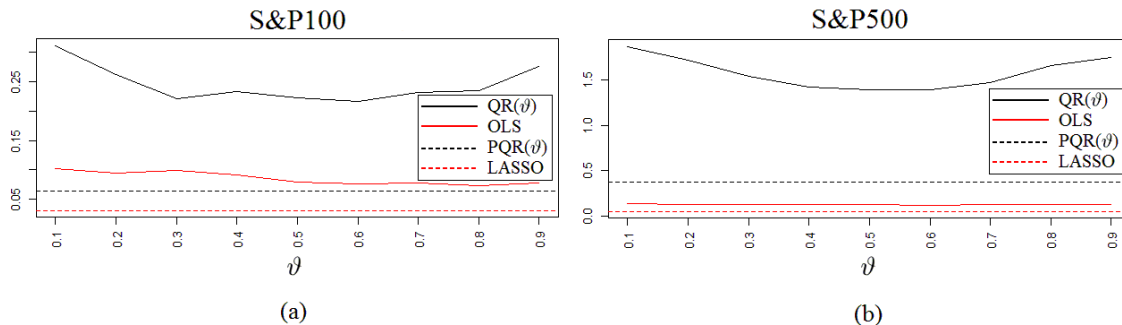


Fig. 2.11 Turnover of the strategies built from the ordinary least squares (with (*LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and from the quantile regression (with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty) models. The strategies are applied on the returns series of the Standard & Poor's 100 (Subplot (a)) and 500 (Subplot (b)) indices constituents. The rolling analysis is carried out with window size of 1000 observations.

the performance of the considered asset allocation strategies. Moreover, in this way, we can analyse whether and how the performance of the strategies depends on the states of the market: the state characterized by financial turmoils and the state of relative calm. To this

purpose, we divided the series of the out-of-sample portfolios returns into two sub-periods. When the window size is equal to 1000, the first sub-period goes from July 31, 2008 to October 31, 2011, whereas it covers the days between October 31, 2006 and October 31, 2010 at $ws = 500$. We can associate this sub-period to the state of financial turmoil, given the proximity to the above mentioned crises. The second sub-period includes the remaining days till November 21, 2014.

As expected, the strategies record a better out-of-sample performance in the second sub-period with respect to the first one, as we can see, for instance, from Table 2.2, where we report the results obtained from the *S&P500* dataset, applying the rolling procedure with window size of 1000 observations.¹¹

Similarly to the analysis on the entire sample, in both the two sub-periods the ordinary least squares regression almost always records the best performance in terms of risk, evaluated by means of standard deviation, mean absolute deviation, Value-at-Risk and α -risk. Differently, the quantile regression model applied at $\vartheta = 0.9$ tends to be the best strategy in terms of profitability and risk-adjusted return, as we checked from the average portfolio return, $\widehat{\Psi}_1(r_p, \psi)$, $\widehat{\Psi}_2(r_p, \psi)$, the Sharpe ratio and the final wealth created after investing an initial amount of 100 \$.

2.4.1 The role of the intercept

We observed that, in the out-of-sample performance, some strategies, built from quantile regression models at different quantiles levels, kept their in-sample properties, whereas other ones failed in achieving that. It is important to study the reason underlying this phenomenon and, in the following, we provide one explanation associated with the model intercept.

Given the numeraire R_k , $1 \leq k \leq n$, and the covariates R_j^* , $j \neq k$, for simplicity, we denote the residual term associated with the quantile regression model, applied at the level ϑ , by $\varepsilon(\vartheta)$. In the rolling window procedure, the estimated parameters change over time, having their own variability. In order to take into account their dependence on time, we denote by $(\widehat{\xi}_t(\vartheta), \widehat{\mathbf{w}}_{-k,t}(\vartheta))$ the coefficients estimated in t , from the data recorded in the interval $[t - ws + 1; t]$, for $t = ws, \dots, T - 1$. The out-of-sample ϑ -th quantile of R_k , computed in $t + 1$, depends on both the estimates obtained in t and the realizations of \mathbf{R} in $t + 1$, being equal to $\widehat{\xi}_t(\vartheta) + \sum_{j \neq k} \widehat{w}_{j,t}(\vartheta) r_{j,t+1}^*$. Therefore the corresponding out-of-sample residual is computed as

$$\varepsilon_{t+1}(\vartheta) = r_{k,t+1} - \left[\widehat{\xi}_t(\vartheta) + \sum_{j \neq k} \widehat{w}_{j,t}(\vartheta) r_{j,t+1}^* \right]. \quad (2.23)$$

¹¹The results obtained in the other cases are available on request.

Table 2.2 Out-of-Sample Analysis partitioned on two sub-periods.

Strategy	Standard Deviation	MAD	Value at Risk	α -risk	$\hat{\Psi}_1(r_p, 0.9)$	$\hat{\Psi}_2(r_p, 0.9)$	Mean	Sharpe Ratio	Final Wealth
FIRST SUB-PERIOD									
<i>QR</i> (0.1)	1.0041	0.7476	1.1421	1.8869	0.1916	0.5388	0.0004	0.0004	97.5315
<i>PQR</i> (0.1)	0.8603	0.7476	0.8922	1.6124	0.1417	0.5566	0.0210	0.0244	117.1751
<i>QR</i> (0.5)	0.9228	0.6693	1.0251	1.6864	0.1730	0.5295	0.0081	0.0088	106.2985
<i>PQR</i> (0.5)	0.8668	0.6693	0.8325	1.5876	0.1228	0.5976	0.0350	0.0404	130.8294
<i>QR</i> (0.9)	0.9772	0.6910	1.0271	1.7274	0.1717	0.5393	0.0212	0.0217	116.9810
<i>PQR</i> (0.9)	0.9167	0.6910	0.9226	1.6384	0.1127	0.6366	0.0557	0.0608	152.9185
<i>OLS</i>	0.8771	0.6296	0.9673	1.6514	0.1649	0.5303	-0.0019	-0.0022	98.0775
<i>LASSO</i>	0.8140	0.6296	0.8102	1.4955	0.1159	0.6057	0.0360	0.0442	133.1528
SECOND SUB-PERIOD									
<i>QR</i> (0.1)	0.7135	0.5539	0.8498	1.2700	0.1169	0.6033	0.0238	0.0334	116.3018
<i>PQR</i> (0.1)	0.5604	0.5539	0.6281	0.9730	0.0702	0.6683	0.0431	0.0768	135.6459
<i>QR</i> (0.5)	0.6215	0.4792	0.7397	1.0842	0.0835	0.6604	0.0389	0.0625	131.2027
<i>PQR</i> (0.5)	0.5197	0.4792	0.5840	0.9021	0.0639	0.6744	0.0402	0.0773	133.4267
<i>QR</i> (0.9)	0.6875	0.5442	0.8839	1.2280	0.0999	0.6524	0.0292	0.0425	121.5124
<i>PQR</i> (0.9)	0.5641	0.5442	0.6055	0.9423	0.0634	0.6990	0.0503	0.0892	143.6969
<i>OLS</i>	0.5819	0.4584	0.7022	1.0162	0.0766	0.6760	0.0350	0.0602	127.9555
<i>LASSO</i>	0.5227	0.4584	0.5862	0.9094	0.0711	0.6490	0.0324	0.0620	125.4998

The table reports the values of different performance indicators generated by the strategies built on the ordinary least squares regression (with *LASSO*) and without (*OLS*) ℓ_1 -norm penalty) and on the quantile regression (with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty, for $\vartheta = \{0.1, 0.5, 0.9\}$) models. The statistics are computed from the Out-of-Sample returns recorded in two different sub-periods: July 2008-October 2011 and October 2011-November 2014. The rolling analysis is applied on the Standard & Poor's 500 Constituents dataset, with window size of 1000 observations. Standard deviation, *MAD*, Value-at-Risk at $\alpha = 0.1$, α -risk (with $\alpha = 0.1$), $\hat{\Psi}_1(r_p, 0.9)$ and mean are expressed in percentage values. The final wealth is the value we obtain at the end of the analysis from each strategy by investing an initial amount equal to 100 \$.

Given that the portfolio return, under the budget constraint, can be written as $R_p(\vartheta) = R_k - \sum_{j \neq k} w_j(\vartheta) R_j^*$, from the (2.23) we obtain

$$r_{p,t+1}(\vartheta) = \varepsilon_{t+1}(\vartheta) + \widehat{\xi}_t(\vartheta). \quad (2.24)$$

From the (2.24) we can see that the out-of-sample portfolio return depends on two components: the intercept and the residual. When all the regressors are equal to zero, the estimated intercept corresponds to the estimated quantile of the response variable and, in general, we should expect that $\widehat{\xi}_t(\vartheta)$ is a positive function of ϑ . This phenomenon is particularly accentuated in case the so-called location-shift hypothesis holds, i.e. when the slopes of the quantile regression models are constant across ϑ , so that the estimated quantiles change according to the intercept levels. Consequently, at high/low ϑ values, we should expect that the intercept term is a positive/negative component of the portfolio return in (2.24). Differently, at high/low ϑ values, the magnitude of the positive residuals is lower/greater than the magnitude of the negative ones; hence, we should expect that $\varepsilon_t(\vartheta)$ is a negative/positive component of the portfolio return in the (2.24).

Given the opposite behaviour of $\varepsilon_{t+1}(\vartheta)$ and $\widehat{\xi}_t(\vartheta)$ over ϑ , it is useful to study their distributions in order to understand the different out-of-sample performances of the strategies built from the quantile regression models. For simplicity, we compare the results obtained from three quantiles levels, i.e. $\vartheta = \{0.1, 0.5, 0.9\}$.

We start by analyzing the intercepts distributions, reporting in Table 2.3 the mean and the standard deviation of $\widehat{\xi}_t(\vartheta)$, for $t = ws, \dots, T - 1$.

We checked that, as expected, the support of the $\widehat{\xi}_t(\vartheta)$ distribution moves on the right as ϑ increases. As a result, from the second and the fourth columns of Table 2.3 it is possible to see that, in average, $\widehat{\xi}_t(\vartheta)$ is a positive component of the out-of-sample portfolio return generated from the quantile regression model at $\vartheta = 0.9$; we have the opposite result at $\vartheta = 0.1$, whereas at the median level the intercept takes, in average, values close to zero. We can see from the third and the fifth columns of Table 2.3 that, at the median level, the intercept distribution is characterized by the lowest dispersion, consistently to the fact that the median regression implies, among all the quantile regression models, the lowest out-of-sample portfolio volatility. In all the cases, the largest window size of 1000 observations reduces the intercepts dispersions, mainly at $\vartheta = \{0.1, 0.9\}$.

After analyzing the impact of the intercept, we now study the behaviour of the out-of-sample residuals. In contrast to the intercept case, the residuals supports move on the right as ϑ decreases, as expected. Consequently, as it is possible to see from the second and the fourth columns of Table 2.4, the residuals are, in average, negative/positive components of

Table 2.3 Analysis of the intercepts of the quantile regression models.

Strategy	S&P100		S&P500	
	Mean	St. Dev.	Mean	St. Dev.
$\vartheta = 0.1$				
<i>QR</i> (0.1); $ws=500$	-0.6403	0.1943	-0.0699	0.0569
<i>PQR</i> (0.1); $ws=500$	-0.7212	0.2045	-0.6910	0.1360
<i>QR</i> (0.1); $ws=1000$	-0.7482	0.0943	-0.3515	0.0371
<i>PQR</i> (0.1); $ws=1000$	-0.7464	0.0913	-0.6571	0.0754
$\vartheta = 0.5$				
<i>QR</i> (0.5); $ws=500$	0.0456	0.0196	0.0026	0.0474
<i>PQR</i> (0.5); $ws=500$	0.0601	0.0194	0.0636	0.0220
<i>QR</i> (0.5); $ws=1000$	0.0474	0.0171	0.0166	0.0156
<i>PQR</i> (0.5); $ws=1000$	0.0583	0.0120	0.0512	0.0096
$\vartheta = 0.9$				
<i>QR</i> (0.9); $ws=500$	0.6680	0.1583	0.0733	0.0520
<i>PQR</i> (0.9); $ws=500$	0.7551	0.1204	0.7832	0.1251
<i>QR</i> (0.9); $ws=1000$	0.7963	0.0769	0.3780	0.0409
<i>PQR</i> (0.9); $ws=1000$	0.7908	0.0553	0.7142	0.0626

The table reports the average values (%) and the standard deviations (%) of the intercepts estimated for the quantile regression models with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty, for $\vartheta = \{0.1, 0.5, 0.9\}$. The rolling window procedure is applied at $ws = \{500, 1000\}$. Datasets: *S&P100* and *S&P500*.

the portfolios returns in (2.24) at high/low ϑ levels. By comparing Tables 2.3 and 2.4, it is important to notice that the residuals distributions have larger volatilities with respect to the intercepts ones.

To summarize, if we build an asset allocation strategy from a quantile regression model with high/low ϑ levels, we can obtain benefits/losses in terms of positive/negative intercept values. Differently, with low/high ϑ levels, we derive benefits/losses from the residuals. The opposite effects are, in average, balanced. Nevertheless, the intercepts distributions have a lower dispersion with respect to the residuals distributions. Then, at high ϑ values, we obtain benefits from a component (the intercept) characterized by greater stability, but, on the other hand, we are penalized by a second component (the residuals) which are more volatile. Differently, when we use quantile regressions models at low quantiles levels, the benefits of positive residuals are more volatile than the losses of negative intercepts. The more stable benefits characterizing the strategies built from the quantile regression models at high ϑ levels support their better out-of-sample performance.

Table 2.4 Analysis of the out-of-sample residuals distributions.

Model	S&P100		S&P500	
	Mean (%)	St. Dev. (%)	Mean (%)	St. Dev. (%)
$\vartheta = 0.1$				
<i>QR</i> (0.1); $ws=500$	0.6487	0.9493	0.1455	1.8449
<i>PQR</i> (0.1); $ws=500$	0.7407	0.8417	0.7225	0.8339
<i>QR</i> (0.1); $ws=1000$	0.7561	0.8217	0.3637	0.8721
<i>PQR</i> (0.1); $ws=1000$	0.7594	0.7637	0.6892	0.7317
$\vartheta = 0.5$				
<i>QR</i> (0.5); $ws=500$	-0.0249	0.8152	0.0480	1.7475
<i>PQR</i> (0.5); $ws=500$	-0.0405	0.8169	-0.0280	0.7822
<i>QR</i> (0.5); $ws=1000$	-0.0215	0.7670	0.0069	0.7867
<i>PQR</i> (0.5); $ws=1000$	-0.0290	0.7668	-0.0137	0.7143
$\vartheta = 0.9$				
<i>QR</i> (0.9); $ws=500$	-0.6348	0.9144	-0.0281	1.8113
<i>PQR</i> (0.9); $ws=500$	-0.7250	0.8540	-0.7398	0.8932
<i>QR</i> (0.9); $ws=1000$	-0.7578	0.8146	-0.3529	0.8446
<i>PQR</i> (0.9); $ws=1000$	-0.7517	0.7877	-0.6612	0.7631

The table reports the average values (%) and the standard deviations (%) of the out-of-sample residuals arising from the quantile regression models with (*PQR*(ϑ)) and without (*QR*(ϑ)) ℓ_1 -norm penalty, for $\vartheta = \{0.1, 0.5, 0.9\}$. The rolling window procedure is applied at $ws = \{500, 1000\}$. Datasets: *S&P100* and *S&P500*.

2.5 Concluding remarks

We have shown how quantile regression based asset allocation corresponds to the minimization of lower tail risk, mean absolute deviation or maximization of a reward measure, depending on the quantile we are looking at. Within such an analyses we introduced a novel performance measure, that is clearly related to specific portfolio return distribution quantiles. In order to cope with the potentially large cross-sectional dimension of portfolio and at the same time to control for estimation error, we combine quantile regression and regularization, based on the ℓ_1 -norm penalty, to estimate portfolio weights. Our empirical evidences, based both on simulations and real data examples, highlights the features and the benefits of our methodological contributions. The new tools provided (asset allocation strategies, performance measures and penalization approaches) will be of potential interest in several areas including performance evaluation and the design of asset allocation frameworks.

Further research high on our agenda is the inclusion of different penalty functions, such as the non-convex ones, that have also a direct interpretation as measures of portfolio diversification (e.g. ℓ_q -norm with $0 \leq q \leq 1$). They not only typically identify investment strategies with better out of sample portfolio performance, but also promote more sparsity

than the ℓ_1 -norm penalty. Moreover, we aim at developing a method to choose simultaneously the optimal quantile level as well as the optimal intensity of the penalty.

Chapter 3

The Determinants of Equity Risk and their Forecasting Implications: a Quantile Regression Perspective

3.1 Literature review and new suggestions

Recent events, such as the subprime crisis, which originated in the United States and was marked by Lehman Brothers' default in September 2008, and the sovereign debt crisis, which hit the Eurozone in 2009, have highlighted the fundamental importance of risk measurement, monitoring, and forecasting. The volatility of asset returns, a commonly used measure of risk, is a key variable in several areas of finance and investment, such as risk management, asset allocation, pricing, and trading strategies. Therefore, estimating and forecasting the volatility point values and distribution play a critical role. The use of predicted volatility levels is central, for instance, in the pricing of equity derivatives, in the development of equity derivative trading strategies, and in risk measurement when risk is associated with volatility, while the volatility distribution is of interest for trading/pricing volatility derivatives, for designing volatility hedges for generic portfolios, and for accounting for the uncertainty on volatility point forecasts.

Many financial applications use constant volatility models (Black and Scholes, 1973), although empirical evidence suggests that variance changes over time. Several approaches have been developed with the purpose of achieving more accurate estimates, such as the class of ARCH (Engle, 1982) and GARCH (Bollerslev, 1986) models and the stochastic volatility models (Harvey et al., 1994; Jacquier et al., 1994; Melino and Turnbull, 1990; Taylor, 1994). Nevertheless, financial data are affected by several features, e.g. the so-called stylized facts

(Cont, 2001), and standard GARCH and stochastic volatility models do not capture all of them (Corsi, 2009).

We might also estimate volatility with non-parametric methods, such as by means of realized measures, which have been shown to perform better than traditional GARCH and stochastic volatility models when forecasting conditional second-order moments (Andersen et al., 2003). This approach has attracted considerable interest because of the availability of high-frequency financial data. In fact, as opposed to models that treat volatility as a latent (non-observable) element, realized measures use additional information from the intraday returns. We believe these methods provide more flexibility than standard GARCH-type models do, so we focus on realized measures here.

Among the various realized volatility measures, we choose a range-based measure for reasons of efficiency. We thus follow Christensen et al. (2009) and adopt the realized range-based bias corrected bipower variation. Christensen et al. (2009) demonstrated that their measure is a consistent estimator of the integrated variance in the presence of microstructure noise¹ and price jumps.

Studies that focus on volatility forecasting have identified key macroeconomic and financial variables as important drivers of volatility, highlighting their power in improving forecast performances. For instance, Christiansen et al. (2012) predicted the asset return volatility by means of macroeconomic and financial variables in a Bayesian Model Averaging framework. They considered several asset classes, such as equities, foreign exchange, bonds, and commodities, over long time spans and found that economic variables provide information about future volatility from both an in-sample and an out-of-sample perspective. Paye (2012) tested the power of financial and economic variables to forecast the volatility at monthly and quarterly horizons and rarely found a statistical difference between the performance of macroeconomic fundamentals and univariate benchmarks. Fernandes et al. (2009) used parametric and semi-parametric Heterogeneous Auto Regressive (HAR) processes to model and forecast the VIX index and found significant results using financial and macroeconomic variables as additional regressors. Caporin and Velo (2011) used an HAR model with asymmetric effects with respect to volatility and return, and GARCH and GJR-GARCH specifications for the variance equation. Caporin et al. (2011) studied the relationship between the first principal component of the volatility jumps, estimated using thirty-six stocks and a set of macroeconomic and financial variables, such as VIX, S&P 500 volume, CDS, and Federal Fund rates, and found that CDS captures a large part of the moves of the expected jumps. Opschoor et al. (2014) used the Bloomberg Financial Conditions Index, which comprises

¹The microstructure noise that arises from peculiar phenomena like non-continuous trading, infrequent trades, and bid-ask bounce (Hasbrouck, 2006; O'Hara, 1998; Roll, 1984) affects the properties of the realized variance estimators.

money, bond, and equity markets, observing that worse financial conditions are associated with both higher volatility and higher correlations. Given the findings of these studies, we, too, use macroeconomic and financial variables in our model.

Our purpose is to generalize the previous contributions further. We model and forecast the conditional quantiles of the realized range-based bias corrected bipower variation by means of the quantile regression method introduced by Koenker and Bassett (1978). Focusing on realized volatility measures, we believe that the quantile regression approach can provide useful new evidence by allowing the entire conditional distribution of the realized volatility measure to be estimated, instead of restricting the attention to the conditional mean. This approach could have a relevant advantage when the impact of covariates is changing, depending on market conditions (say, on low or high volatility states), or when the purpose is to recover density forecasts without making a distributional assumption.

Other authors have applied quantile regression in a financial framework. For instance, Engle and Manganelli (2004) proposed the CAViaR model to estimate the conditional Value-at-Risk, an important measure of risk that financial institutions and their regulators employ.² White et al. (2008) generalized the CAViaR to the Multi-Quantile CAViaR (MQ-CAViaR) model, studying the conditional skewness and kurtosis of S&P 500 daily returns. White et al. (2010) extended the MQ-CAViaR model in the multivariate context to measure the systemic risk, a critical issue highlighted by the recent financial crises, taking into account the relationships among 230 financial institutions from around the world. Li and Miu (2010) proposed, on the basis of the binary quantile regression approach, a hybrid bankruptcy prediction model with dynamic loadings for both the accounting-ratio-based and market-based information. Castro and Ferrari (2014) used the $\Delta CoVaR$ model as a tool for identifying/ranking systemically important institutions. Finally, Caporin et al. (2014b) adopted quantile regressions in detecting financial contagion across bond spreads in Europe.

Our work is closely related to the contribution of Zikes and Barunik (2013). However, our analysis differs in several important points. First of all, Zikes and Barunik (2013) estimated volatility by means of a realized measure that takes into account only the effects of jumps in the price process. Differently, we use the realized range-based bias corrected bipower variation, which considers the impact of microstructure noise as well as that of jumps. To the best of our knowledge, quantile regression methods for the analysis of realized range volatility measures have never been used in the econometric and empirical financial literature, which provides a strong motivation for our study. Secondly, Zikes and Barunik (2013) used a heterogeneous autoregressive quantile model, whereas we also made use of conditioning

²In the present work we do not consider the CAViaR model in order to control the computational complexity of the analysis.

exogenous variables. Third, Zikes and Barunik (2013) performed quantile forecasts, focusing on a few quantiles; we go farther by estimating a fine grid of quantiles in order to recover information about the entire realized volatility conditional distribution. Moreover, on the basis of the obtained volatility quantiles, we build the entire conditional volatility density.

Our work takes then an empiric point of view and focuses on the high-frequency data of sixteen stocks issued by large-cap companies that operate in differing economic sectors. All companies are quoted on the U.S. market. In a first step, we analyze the first principal component of the estimated volatility as a summary of the sixteen series (a kind of market factor).

We provide a first empirical contribution to the literature by showing that some macro-finance-related variables have a significant impact on volatility quantiles; that their impact changes across quantiles, becoming irrelevant in some cases; and that the significance and strength of the relationship changes over time. This last finding is particularly evident when we focus on turbulent market phases, as in these periods we note an increase in the impact of some variables, such as the VIX. The last finding is particularly evident for high-volatility quantiles. The heterogeneity we observe across quantiles can also be interpreted as evidence against the location-shift hypothesis.³ These empirical evidences suggest that the uncertainty on volatility point forecast, as measured by the influence of covariates on different volatility quantiles, is changing depending on market states. Consequently, volatility forecast precision is undangered, as well as the appropriateness of volatility density forecast. Both those elements have a central role in risk management and volatility hedging and trading, thus providing support for our analyses.

Our second empirical contribution comes from the single asset analyses. We develop a specific model for each asset in order to determine how the features of the sixteen companies affect the relationships among the variables involved. We find some heterogeneity in the assets' reactions to the macro-finance variables, which holds across both time and volatility quantiles. However, we find an overall confirmation of the findings associated with the first principal component. Therefore, the relevance of quantile-based covariate impact on volatility is of relevant interest both at aggregated as well as the single asset level.

A third empirical contribution of our analyses stems from a forecasting exercise. We compare the quantile-based density forecasts to those of a benchmark model adapted to the realized range volatility mean and variance. The reference model combines a HAR structure on the realized volatility mean, plus a GJR-GARCH (Glosten et al., 1993) for the mean innovation variances. The HAR structure, inspired by the work of Corsi (2009), captures

³The location-shift hypothesis assumes homogeneous impacts of the covariates across the conditional quantiles of the response variable.

the persistence of realized measures and is consistent with the presence of heterogeneous agents in the market, while the GJR-GARCH is coherent with Corsi et al. (2008)'s volatility of volatility hypothesis. We compare the benchmark model density forecasts and the quantile regression-based forecasts by means of the tests proposed by Berkowitz (2001) and Amisano and Giacomini (2007). Moreover, by using a quantile-based loss function, we also consider the Diebold and Mariano (2002) test. We stress that the Berkowitz (2001) test allows for an absolute evaluation of the density forecasts provided by one model, while the Amisano and Giacomini (2007) approach compares two competing models. The results confirm that our approach performs better, thus providing support for the use of quantile regression methods in all areas where volatility quantiles might have a role. Among the possible applications, we mention volatility trading and volatility hedging (Euan, 2013; Zhang et al., 2010), as well as the evaluation of uncertainty around volatility point forecasts. The latter could be of crucial importance if we plan to combine volatility point and density forecasts derived from realized volatility measures with returns-based volatility models, as in the HEAVY model class (Shephard and Sheppard, 2010).

The remainder of the paper is structured as follows. Section 3.2 includes the description of the data and of the conditioning variables. Section 3.3 presents the model we propose to forecast the range bipower variation conditional quantiles. Section 3.4 is devoted to the density forecast and predictive accuracy and provides the details about the tests we use. The results are analyzed in Section 3.5, and Section 3.6 provides a set of concluding remarks. Appendix A deals with the properties of some of the realized measures we use to estimate volatility in the context of high-frequency data, whereas Appendix B provides a background on quantile regression estimation, diagnostic and testing.

3.2 Realized volatilities and conditioning variables

The database we use includes stock prices recorded with a frequency of one minute, from 9:30 a.m. to 4:00 p.m. of every trading day between January 2, 2003, and June 28, 2013, inclusive. The equities analyzed are those of large companies that operate in various economic sectors of the U.S. market: AT&T Inc. (*ATT*), Bank of America (*BAC*), Boeing (*BOI*), Caterpillar, Inc. (*CAT*), Citigroup, Inc. (*CTG*), FedEx Corporation (*FDX*), Honeywell International, Inc. (*HON*), Hewlett-Packard Company (*HPQ*), International Business Machines Corp. (*IBM*), JPMorgan Chase & Co. (*JPM*), Mondelez International, Inc. (*MDZ*), Pepsico, Inc. (*PEP*), The Procter & Gamble Company (*PRG*), Time Warner, Inc. (*TWX*), Texas Instruments, Inc. (*TXN*), and Wells Fargo & Company (*WFC*). The dataset is drawn from TickData. The prices

are adjusted for extraordinary operations and filtered for errors, anomalies, and outliers that arise from traders' activities.⁴

From the high-frequency data above described, after computing the assets returns, we estimate the daily volatilities of the sixteen assets, through the realized range-based bias corrected bipower variation ($RRV_{BVBC}^{n,m}$) introduced by Christensen et al. (2009). $RRV_{BVBC}^{n,m}$ is a robust estimator of the integrated variance in the presence of both price jumps and microstructure noise.⁵ The main descriptive statistics computed from the 16 volatilities series are given in Table 3.1.

Table 3.1 Descriptive analysis on the estimated daily volatilities series.

Stock	Mean	St. Deviation	1st Quartile	3rd Quartile	IQR	Skewness	Kurtosis
ATT	0.0219	0.0489	0.0074	0.0194	0.0120	16.1606	448.9054
BAC	0.0733	0.2368	0.0068	0.0483	0.0415	12.6079	281.8217
BOI	0.0248	0.0392	0.0095	0.0255	0.0160	8.5553	126.9734
CAT	0.0331	0.0625	0.0116	0.0301	0.0185	10.1863	183.9472
CTG	0.0953	0.3202	0.0092	0.0709	0.0617	12.5506	243.8441
FDX	0.0246	0.0360	0.0093	0.0259	0.0167	6.9830	90.6966
HON	0.0267	0.0508	0.0102	0.0261	0.0159	15.2523	408.7239
HPQ	0.0293	0.0483	0.0122	0.0302	0.0180	11.4175	200.7032
IBM	0.0167	0.0382	0.0059	0.0145	0.0086	12.2683	236.4302
JPM	0.0486	0.1239	0.0091	0.0356	0.0265	9.1167	131.0281
MDZ	0.0143	0.0222	0.0060	0.0143	0.0084	10.6387	218.0799
PEP	0.0132	0.0362	0.0048	0.0124	0.0076	26.6152	960.1705
PRG	0.0144	0.1460	0.0046	0.0112	0.0066	48.4411	2426.2210
TWX	0.0296	0.0516	0.0111	0.0281	0.0170	9.4520	164.3001
TXN	0.0356	0.0393	0.0161	0.0407	0.0246	7.0559	104.7102
WFC	0.0547	0.1502	0.0065	0.0368	0.0303	6.9857	70.8576
FPC	0.0002	0.4283	-0.1283	-0.0413	0.0869	9.2116	126.8468

The table reports for each stock (the ticker is given in the first column) some descriptive statistics computed for the estimated daily volatilities. From left to right we compute: the mean (%), the standard deviation (%), the first quartile (%), the third quartile (%), the interquartile range (%), the skewness and the kurtosis indices. *FPC* stands for the first principal component of the single stocks volatilities.

Table 3.1 shows that the average realized daily volatilities take values from 0.0132% (*PEP*) to 0.0953% (*CTG*), with the financial companies, namely *BAC*, *CTG*, *JPM* and *WFC*, being the most volatile ones, as expected given the sample period. Furthermore, in the case of the four financial companies, the volatilities distributions have larger dispersion, as we can see from both the standard deviation and the interquartile range values. The first quartile ranges between 0.0046% (*PRG*) and 0.0161% (*TXN*), whereas the minimum and the

⁴For additional details see the company website: <http://www.tickdata.com>.

⁵See A for further details on the estimator and its properties.

maximum values of the third quartile are 0.0112% (*PRG*) and 0.0709% (*CTG*), respectively. All the 16 companies have evident right-skewed and leptokurtic volatilities distributions, suggesting the presence of volatility bursts. We also analyzed the trend of the volatilities series over time. We observed that the 16 stocks are affected by the 2008-2009 financial crisis and, as expected, the financial companies are more sensitive to this event; for instance, this phenomenon is clear in Figure 3.1, where we show the $RRV_{BVBC}^{n,m}$ trend of two companies: *JPM* (financial company) and *HPQ* (non-financial company).

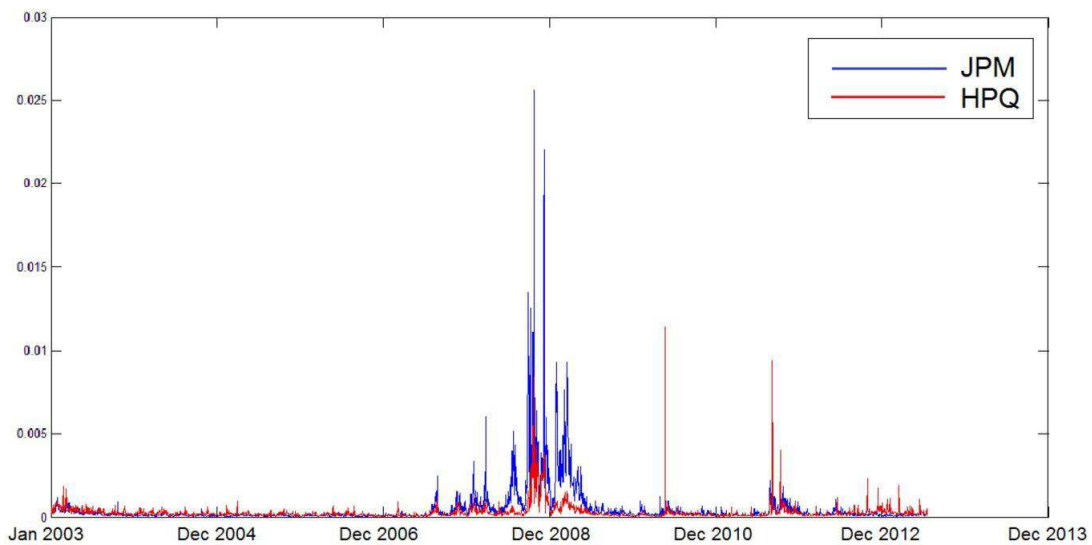


Fig. 3.1 Realized range-based bias corrected bipower variation over time.

A principal component analysis is carried out on the range-based bias corrected bipower variations of the sixteen assets. In particular, the first principal component (*FPC*) explains 77% of the overall variance. The evolution of the assets' volatilities have a strong common behavior that we might interpret as *market* or *systematic* behavior. Therefore, the analysis of the first principal component could produce useful results. Consequently, in the following we model the conditional quantiles of both the first principal component and the single assets realized volatilities.

The last row of Table 3.1 provides the descriptive statistics computed on the first principal component, that captures the most important features of the 16 volatilities distributions, such as right skewness and leptokurtosis. Figure 3.2 shows that *FPC* is strongly affected by the 2008-2009 financial crisis, since it records high values in that period. Moreover, from the boxplot we can see that these peaks are extreme values in the right tail of the *FPC* distribution, suggesting the wisdom of using quantile regression (Koenker and Bassett, 1978) rather than

regression on the mean, because the latter is particularly sensitive to extreme values. The *FPC* autocorrelations keep high values for many lags, indicating volatility persistence, a common finding on realized range/variance sequences.⁶

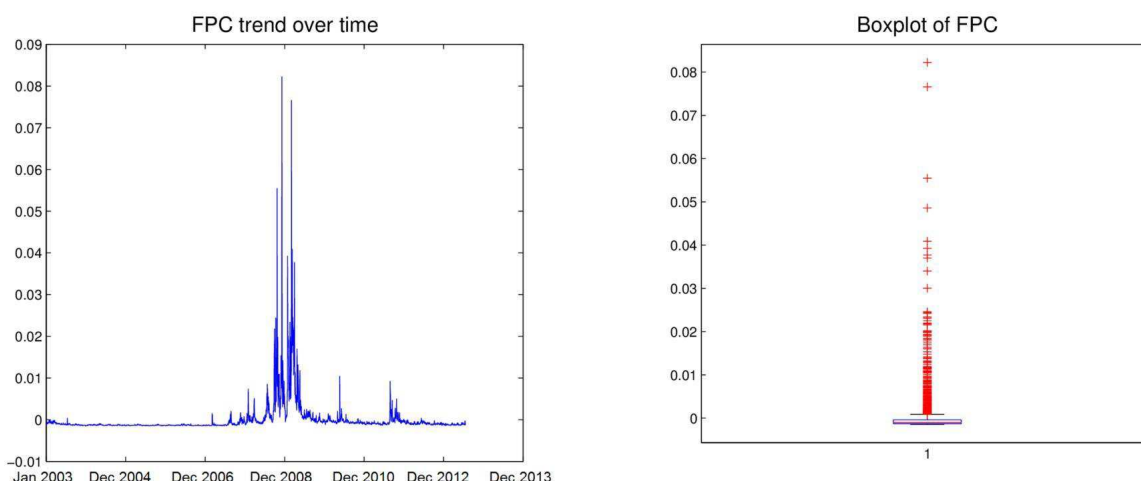


Fig. 3.2 Level and boxplot for the first principal component of the range-based bias corrected bipower variations.

In addition to the data described so far, our analyses take into account some key macroeconomic and financial variables that convey important information about the overall market trend and risk and that will be considered exogenous variables that affect the conditional quantiles. Other studies (Caporin et al., 2011; Caporin and Velo, 2011) have used several indicators to analyze the realized variance and range series, among which we select just two (since the others, such as the logarithmic returns of the U.S. dollar-Euro exchange rate and of oil, were not significant in the present analysis): the daily return of the S&P 500 index (*sp500*), which reflects the trend of the U.S. stock market, and the logarithm of the VIX index (*vix*), a measure of the implied volatility of S&P 500 index options. We expected that negative returns of the S&P 500 would have a positive impact on the market volatility, the well-known leverage effect, while high levels of *vix* reflect pessimism among the economic agents, so a positive relationship between *vix* and the volatility level is expected. The observations associated to *sp500* and *vix* are recorded at a daily frequency and are recovered from Datastream.

Many macroeconomic and financial time series are not stationary and are often characterized by unit-root non-stationarity (Nelson and Plosser, 1982). Building on this evidence, we

⁶C provides some graphic analyses. Additional descriptive statistics are available on request.

must determine whether the data-generating process of vix is affected by unit root.⁷ To this purpose, we consider two standard tests: the augmented Dickey-Fuller (ADF) test (Dickey and Fuller, 1981) and the Phillips-Perron (PP) test (Phillips and Perron, 1988). The ADF test rejects the null hypothesis at the 10% significance level, and the test statistic equals -3.18 (with a p-value of 0.09). On the other hand, the PP test rejects the null hypothesis at the 5% level, and the test statistic is -24.48 (with a p-value of 0.03). Therefore, we have a moderate amount of evidence against the presence of a unit root for the vix series. The rejection of the null is much clearer with the PP test.

Beside vix and $sp500$, we use other quantities to describe the evolution of realized range conditional quantiles. First, we follow Corsi (2009) in introducing among the explanatory variables those commonly adopted in (*HAR*) models of Realized Volatility. Focusing on the first principal component of realized ranges, whose observed value at t is denoted by $fpct$, a first explanatory variable is its lagged value: $fpct_{t-1}$. This variable is usually accompanied by other quantities that are built from local averages of past elements:

$$\overline{fpct}_{m^*t} = \frac{1}{m^*} \sum_{i=0}^{m^*-1} fpct_{t-i}. \quad (3.1)$$

Corsi (2009)'s model used $m^* = 5$ and $m^* = 21$, representing the weekly and monthly horizons. These components allow the heterogeneous nature of the information arrivals (Corsi, 2009) in the market to be considered. In fact, many operators have differing time horizons. For instance, intraday speculators have a short horizon, while insurance companies trade much less frequently. Therefore, agents whose time horizons differ, perceive, react to, and cause different types of volatility components. In our study we use only $m^* = 5$ with the first lag. In fact, the longer horizon component, with $m^* = 21$, was not significant. We might interpret \overline{fpct}_{5t-1} as reflecting the medium-term investors who typically rebalance their positions at a weekly frequency. We found that $fpct_{t-1}$ and \overline{fpct}_{5t-1} are positively correlated with $fpct_t$, suggesting a positive impact at least in the mean.⁸ We will use similar variables (lagged and weekly elements) for each of the company specific realized range sequences.

The last explanatory variable we consider, denoted by *JUMP*, takes into account the impact of jumps in the price process. At the single asset level, the jump intensity could be detected through the test statistic introduced by Christensen and Podolskij (2006), here denoted as $Z_{TP,t}$, by which we test the null hypothesis of no jumps at day t .⁹ For each asset, we test for the presence of jumps by computing the ratio between the number of days in

⁷We do not report the tests on $sp500$, as they provided no useful results: the index level is non-stationary, while the index return is stationary.

⁸Figure C.0.2 shows the positive association among variables.

⁹The details about $Z_{TP,t}$ are given in A.

which the null hypothesis of no jumps is rejected by the total number of days included in our dataset, with significance level set to 0.05. We observed that the ratio ranges from 44.89% (*CTG*) to 91.97% (*MDZ*);¹⁰ therefore, the inferential procedure of Christensen and Podolskij (2006) detects the presence of jumps, with a clear heterogeneity across the 16 assets, and supports the inclusion of a component accounting for jumps in our model. In particular, we compute *JUMP* as the difference between the realized range-based variance ($RRV^{n,m}$) and the realized range-based bias corrected bipower variation ($RRV_{BVBC}^{n,m}$), which are, respectively, jump non-robust and jump robust estimators of the integrated variance.¹¹ For the i -th stock, the observed value of *JUMP* at t is denoted by $jump_{i,t}$; we checked that the $jump_{i,t}$ series have evident peaks in periods of financial turmoils.¹² In the case in which we analyze the first principal component of the realized ranges, i.e. *FPC*, for reasons of consistency, we compute $jump_t$ as the first principal component of $jump_{i,t}$, for $i = 1, \dots, 16$.

To summarize, if we consider the first principal component of realized ranges, the variables of interest are: fpc_t (i.e. the dependent variable), fpc_{t-1} , \overline{fpc}_{5t-1} , vix_{t-1} , $sp500_{t-1}$ and $jump_{t-1}$ (the explanatory variables). Table 3.2 reports the linear correlation coefficients across those quantities. fpc_{t-1} has the highest correlation with fpc_t , and the signs of the correlation coefficients are consistent with expectations, since fpc_{t-1} , \overline{fpc}_{5t-1} , vix_{t-1} , and $jump_{t-1}$ are positively correlated with fpc_t , while $sp500_{t-1}$ has a negative correlation coefficient. We obtain similar results for each realized volatility sequence where the lagged values, the weekly lags and the jump variable are company specific.¹³

Table 3.2 Linear correlation coefficients.

	fpc_t	fpc_{t-1}	\overline{fpc}_{5t-1}	vix_{t-1}	$sp500_{t-1}$	$jump_{t-1}$
fpc_t	1.00	0.75	0.73	0.54	-0.14	0.50
fpc_{t-1}	0.75	1.00	0.83	0.54	-0.05	0.55
\overline{fpc}_{5t-1}	0.73	0.83	1.00	0.62	0.01	0.59
vix_{t-1}	0.54	0.54	0.62	1.00	-0.11	0.36
$sp500_{t-1}$	-0.14	-0.05	0.01	-0.11	1.00	-0.03
$jump_{t-1}$	0.50	0.55	0.59	0.36	-0.03	1.00

The table reports the full-sample linear correlation coefficients across the variables entering in Model (3.5).

¹⁰See all the results in Table C.0.1.

¹¹The details on $RRV^{n,m}$ and $RRV_{BVBC}^{n,m}$ are given in A.

¹²In Figure C.0.3 we show the trend over time of $jump_{i,t}$ for selected series: *BAC*, *CTG*, *IBM* and *JPM*. Similar results are obtained for the other assets.

¹³Results are not reported for space constraints but are available upon request.

3.3 Modelling the realized range conditional quantiles

We now introduce the model we propose to study the conditional quantiles of volatility. As we specified in the introduction, we focus on realized range quantiles, estimated following the quantile regression approach introduced by Koenker and Bassett (1978). Quantile regression overcomes some limitations of linear regression methods, including the sensitivity to outliers and the need for assuming a linearity, and allows focusing on the quantiles of the conditional distribution of a random variable without thus restricting the attention on the conditional mean. Let Y be a real-valued random variable with distribution function $F_Y(y) = P(Y \leq y)$. For any $0 < \tau < 1$, the τ -th quantile of Y is equal to $F_Y^{-1}(\tau) = \inf\{y : F_Y(y) \geq \tau\}$.¹⁴

Let $I(\cdot)$ be the indicator function taking value 1 if the condition in (\cdot) is true, 0 otherwise, the approach introduced by Koenker and Bassett (1978) makes use of the asymmetric loss function

$$\rho_\tau(\varepsilon) = \varepsilon[\tau - I(\varepsilon < 0)], \quad (3.2)$$

showing that the minimizer \tilde{y}_τ of the expected loss function $\mathbb{E}[\rho_\tau(Y - \tilde{y}_\tau)]$ satisfies $F_Y(\tilde{y}_\tau) - \tau = 0$. In particular, \tilde{y}_τ is the conditional quantile function $Q_Y(\tau|X_1, X_2, \dots, X_\delta)$ in the linear quantile regression:

$$\tilde{y}_\tau = Q_Y(\tau|X_1, X_2, \dots, X_\delta) = \beta_0(\tau) + \beta_1(\tau)X_1 + \dots + \beta_\delta(\tau)X_\delta, \quad (3.3)$$

where $\mathbf{X} = (X_1, X_2, \dots, X_\delta)$ is the vector of δ explanatory variables.¹⁵

Given the time index $t = 1, \dots, T$, let y_t and $x_{j,t}$ be, respectively, the realizations of Y and X_j , for $j = 1, \dots, \delta$, at t . Then, the parameter vector $\boldsymbol{\beta}(\tau) = (\beta_0(\tau), \dots, \beta_\delta(\tau))$ is estimated as a solution of the quantile regression problem:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^{\delta+1}} \sum_{t=1}^T \rho_\tau(y_t - \beta_0 - \beta_1 x_{1,t} - \dots - \beta_\delta x_{\delta,t}). \quad (3.4)$$

B includes further details on quantile regression, focusing on model estimation, on the evaluation of parameter standard errors by bootstrap methods, on diagnostic checking and hypothesis testing.

¹⁴In our work, Y coincides with FPC when we study the first principal component of the stocks volatilities, at the single assets levels it is equal to the volatility of each stock.

¹⁵When we analyze the FPC quantiles, the covariates are fpc_{t-1} , $\overline{fpc}_{5_{t-1}}$, vix_{t-1} , $sp500_{t-1}$ and $jump_{t-1}$; differently, at the single asset level, the explanatory variables are $rrv_{i,t-1}$, $\overline{rrv}_{5_{i,t-1}}$, vix_{t-1} , $sp500_{t-1}$, $jump_{i,t-1}$.

The specification adopted for the conditional quantiles of the first principal component is the following:

$$\begin{aligned}
 Q_{fpc_t}(\tau|\mathbf{x}_{t-1}) &= \mathbf{x}'_{t-1}\boldsymbol{\beta}(\tau) = \beta_0(\tau) + \beta_1(\tau)fpc_{t-1} + \beta_2(\tau)\overline{fpc}_{5_{t-1}} + \beta_3(\tau)vix_{t-1} \\
 &+ \beta_4(\tau)sp500_{t-1} + \beta_5(\tau)jump_{t-1},
 \end{aligned} \tag{3.5}$$

where $Q_{fpc_t}(\tau|\mathbf{x}_{t-1})$ denotes the τ -th quantile of fpc_t , conditional to the information included in \mathbf{x}_{t-1} . Although this approach is not novel, as conditional quantiles have already been used in a risk-management framework (e.g., Engle and Manganelli (2004), White et al. (2008) and White et al. (2010)), we stress that, to the best of our knowledge, realized measures based on ranges have never been used. Therefore, even a simple estimation of Model (3.5) would provide useful results in terms of revealing the impact of covariates and the stability of the various coefficients across quantiles. Quantile regressions could also be used to forecast the conditional quantiles of the realized range volatility sequence, as we discuss below.

Certain events, such as the subprime crisis that was born in the U.S. and that was marked by Lehman Brothers' default in September 2008 and the Eurozone sovereign debt crisis in 2010-2011, had considerable effects on the mechanisms that govern the international financial system. These extreme events could have affected the relationship between control variables and conditional quantiles, so it is necessary to determine whether the relationships that characterize Model (3.5) change over time before, during, or after these period of turmoil. For this purpose, we performed a rolling analysis with a step of one day and a window size of 500 observations. Thus, it is possible to determine how the coefficients' values evolve over time and over τ . Further, Engle and Manganelli (2004), within a risk management perspective, adopt quantile regression methods for the estimation of the Value-at-Risk. In their model, they introduce lagged quantiles among the conditioning variables. We do not follow their approach to control the computational complexity given the presence of a rolling method in the evaluation of conditional quantiles.

We also built Model (3.5) to predict the conditional quantiles of the first principal component computed on the realized range-based bias corrected bipower variations of the sixteen assets. The forecasts, produced for a single step ahead, provide relevant details for the covariates' prediction abilities.

As the underlying companies have differing features and operate in differing economic sectors, it is also useful to build a model for each asset. These asset-specific models have the same structure of that one given in Equation (3.5), but the dependent variable is the conditional $RRV_{BVC}^{n,m}$ quantile of the one asset, and fpc_{t-1} , $\overline{fpc}_{5_{t-1}}$, and $jump_{t-1}$ are replaced with the analogous quantities computed for each asset. Therefore, the model built for the i -th asset,

for $i = 1, \dots, 16$, is

$$\begin{aligned} Q_{rrv_{i,t}}(\tau|\mathbf{x}_{i,t-1}) &= \beta_{0,i}(\tau) + \beta_{1,i}(\tau)rrv_{i,t-1} + \beta_{2,i}(\tau)\overline{rrv}_{5i,t-1} + \beta_{3,i}(\tau)vix_{t-1} \\ &+ \beta_{4,i}(\tau)sp500_{t-1} + \beta_{5,i}(\tau)jump_{i,t-1}, \end{aligned} \quad (3.6)$$

where $rrv_{i,t}$ is the observed $RRV_{BVBC}^{n,m}$ related to the i -th company at day t , $\overline{rrv}_{5i,t}$ is the mean of the $rrv_{i,t}$ values recorded in the last 5 days, and $jump_{i,t-1}$ is the difference between $RRV^{n,m}$ and $RRV_{BVBC}^{n,m}$, computed for the i -th stock at day $t - 1$.

The models we propose allow the conditional quantiles of a realized volatility measure to be estimated. We don't know the true, unobserved volatility of the assets returns (they are estimated) so measurement errors might play a role. However, we restrict our attention to the forecast of the realized measure at a given sampling frequency, not a forecast of the future returns volatility. As a consequence, as Zikes and Barunik (2013) discussed, the impact of measurement errors has limited importance.

3.4 Density forecast and predictive accuracy

Differently from the previous section, we prefer providing more details on the forecast evaluation tools we consider given the central role of volatility forecasting in our work. In Section 3.3 we described the models we propose to estimate the dynamic governing the conditional quantiles of the response variable Y , i.e. the first principal component of the assets volatilities in Model (3.5) or the volatilities of the single stocks in Model (3.6). Given the estimated model, we are able to forecast the conditional τ -th quantile of Y , that is $Q_{y_t}(\tau|\mathbf{x}_{t-1})$.

The standard quantile regression approach allows estimating individual quantiles, but it does not guarantee their coherence, i.e. their increasing monotonicity in $\tau \in (0, 1)$. For instance, it might occur that the predicted 95th percentile of the response variable is lower than the 90th percentile. If quantiles cross, corrections must be applied in order to obtain a valid conditional distribution of volatility. For instance, in order to cope with the crossing problem, Koenker (1984) applied parallel quantile planes, whereas Bondell et al. (2010) estimated the quantile regression coefficients with a constrained optimization method.

Here we follow a different approach, proposed by Zhao (2011). Given a collection of ϑ predicted conditional quantiles $(Q_{y_t}(\tau_1|\mathbf{x}_{t-1}), \dots, Q_{y_t}(\tau_\vartheta|\mathbf{x}_{t-1}))$, for $0 < \tau_j < \tau_{j+1} < 1$, $j = 1, \dots, \vartheta - 1$, we first rearrange them into ascending order, by making use of the quantile bootstrap method proposed by Chernozhukov et al. (2010). Then, starting from the rearranged quantiles, denoted by $(Q_{y_t}^*(\tau_1|\mathbf{x}_{t-1}), \dots, Q_{y_t}^*(\tau_\vartheta|\mathbf{x}_{t-1}))$, we estimate the entire conditional

distribution with a nonparametric kernel method. The predicted density equals

$$\hat{f}_{Y_t}(y^*|\mathbf{x}_{t-1}) = \frac{1}{\vartheta h_\vartheta} \sum_{i=1}^{\vartheta} K\left(\frac{y^* - Q_{y_t}^*(\tau_i|\mathbf{x}_{t-1})}{h_\vartheta}\right), \quad (3.7)$$

where y^* are evenly interpolated points that generates the support of the estimated distribution, h_ϑ is the bandwidth, $K(\cdot)$ is the kernel function, and $\hat{f}_{Y_t}(y^*|\mathbf{x}_{t-1})$ is the one-period ahead forecasted density of Y computed at y^* , given the information set available in $t - 1$. Following Gaglianone and Lima (2012), we use as $K(\cdot)$ the Epanechnikov kernel.

With the solution above described, we are able to recover the entire volatility density, that could be of interest if one is dealing with volatility trading or volatility hedging applications. In this case, the analysis takes a density-forecasting perspective, where the assessment of a proposed approach's predictive power (as compared to a benchmark model) and the evaluation of the potential benefits associated with introducing covariates are particularly important. To these purposes, we apply three testing approaches: the tests proposed by Berkowitz (2001) and Amisano and Giacomini (2007), and a loss-functions-based forecast evaluation that builds on the Diebold and Mariano (2002) testing approach.

If, for simplicity, we denote the *ex ante* forecasted conditional density $\hat{f}_{Y_t}(\cdot|\mathbf{x}_{t-1})$, that we estimate in $t - 1$, by $\hat{f}_{t-1}(\cdot)$, the first step of the Berkowitz (2001) test consists in computing, for all of the available days, the variable

$$v_t = \int_{-\infty}^{y_t} \hat{f}_{t-1}(u)du = \hat{F}_{t-1}(y_t), \quad (3.8)$$

where $\hat{F}_{t-1}(\cdot)$ is the distribution function corresponding to the density $\hat{f}_{t-1}(\cdot)$.

Under correct model specification, Rosenblatt (1952) showed that v_t is i.i.d. and uniformly distributed on $(0, 1)$, a result that holds regardless of the underlying distribution of y_t , even when $\hat{F}_{t-1}(\cdot)$ changes over time. Berkowitz (2001) first pointed out that, if $v_t \sim \mathcal{U}(0, 1)$, then

$$z_t = \Phi^{-1}(v_t) \sim \mathcal{N}(0, 1), \quad (3.9)$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal distribution function.

Given that, under correct model specification, z_t should be independent and identically distributed as standard normal, an alternative hypothesis is that the mean and the variance differ from 0 and 1, respectively, with a first-order autoregressive structure. In particular, Berkowitz (2001) considered the model

$$z_t - \mu_b = \rho_b(z_{t-1} - \mu_b) + e_t \quad (3.10)$$

to test the null hypothesis $H_0 : \mu_b = 0, \rho_b = 0, \text{var}(e_t) = \sigma_b^2 = 1$. The test builds on (4.17) is based on the likelihood-ratio statistic

$$LR_b = -2 [L_b(0, 1, 0; z_t) - L_b(\hat{\mu}_b, \hat{\sigma}_b, \hat{\rho}_b; z_t)], \quad (3.11)$$

where $L_b(\hat{\mu}_b, \hat{\sigma}_b, \hat{\rho}_b; z_t)$ is the likelihood function associated with Equation (4.17) and computed from the maximum-likelihood estimates of the unknown parameters μ_b , σ_b and ρ_b . Under the null hypothesis H_0 , the test statistic is distributed as $\chi^2(3)$.

The Berkowitz (2001) test can be applied to models that provide a density forecast for the realized range volatility. The alternative models' specifications for the conditional quantiles (such as with/without the covariates) or density forecast approaches may differ. Obviously, models that do not provide a rejection of the null hypothesis will be correctly specified, so, at least in principle, many alternative specifications could be appropriate for the data at hand.

The approach Berkowitz proposed allows for an absolute assessment of a given model. In fact, it focuses on the goodness of a specific sequence of density forecasts, relative to the unknown data-generating process. However, the Berkowitz test has a limitation in that it has power only with respect to misspecification of the first two moments. As Berkowitz (2001) noted, if the first two conditional moments are specified correctly, then the likelihood function is maximized at the conditional moments' true values. Nevertheless, in practice, models could be misspecified even at higher-order moments. In that case, a viable solution is to compare density forecasts, that is, to perform a relative comparison given a specific measure of accuracy. To cope with this issue, in addition to Berkowitz (2001)'s approach, we consider Amisano and Giacomini (2007)'s test and a similar loss function-based approach that uses the Diebold and Mariano (2002) test statistics.

Amisano and Giacomini (2007) developed a formal out-of-sample test for ranking competing density forecasts that is valid under general conditions. The test is based on a widely adopted metric, the *log-score*. In particular, the *log-score* arising from our approach is equal to $\log(\hat{f}_{t-1}(y_t))$, whereas $\log(\hat{g}_{t-1}(y_t))$ is the *log-score* obtained by a competing model. For the two sequences of density forecasts, we define the quantity

$$WLR_t = w(y_t^{st}) [\log \hat{f}_{t-1}(y_t) - \log \hat{g}_{t-1}(y_t)], \quad (3.12)$$

where y_t^{st} is the realization of Y at day t , standardized using the estimates of the unconditional mean and standard deviation computed from the same sample on which the density forecasts for t are estimated, and $w(y_t^{st})$ is the weight the forecaster arbitrarily chooses to emphasize particular regions of the distribution's support. After computing the quantities WLR_t for all of the samples considered in the forecast evaluation, we compute the mean $\overline{WLR} =$

$(T - ws)^{-1} \sum_{t=ws+1}^T WLR_t$, where ws is the window size adopted for the computation of density forecasts.¹⁶

In order to test for the null hypothesis of equal performance, that is, $H_0 : E [\overline{WLR}] = 0$, against the alternative of a different predictive ability $H_1 : E [\overline{WLR}] \neq 0$, Amisano and Giacomini (2007) proposed the use of a weighted likelihood ratio test:

$$AG = \frac{\overline{WLR}}{\hat{\sigma}_{AG} / \sqrt{T - ws}}, \quad (3.13)$$

where $\hat{\sigma}_{AG}^2$ is a heteroskedasticity- and autocorrelation-consistent (HAC) Newey and West (1987) estimator of the asymptotic variance $\sigma_{AG}^2 = Var [\sqrt{T - ws} \overline{WLR}]$. Amisano and Giacomini (2007) showed that, under the null hypothesis, $AG \xrightarrow{d} \mathcal{N}(0, 1)$.

We applied the Amisano and Giacomini (2007) test by using four designs for the weights in Equation (4.23), which allows us to verify how the results change according to the particular regions of the distribution's support on which we are focusing. We set $w_{CE}(y_t^{st}) = \phi(y_t^{st})$ to give an higher weight to the center of the distribution, $w_{RT}(y_t^{st}) = \Phi(y_t^{st})$ when we focus more on the right tail, $w_{LT}(y_t^{st}) = 1 - \Phi(y_t^{st})$ for the left tail, and $w_{NW}(y_t^{st}) = 1$ when giving equal importance to the entire support ($\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density function and the standard normal distribution function, respectively).

Finally, we carried out a comparison at the single quantile level, focusing on the quantiles that have critical importance in our framework. To this purpose, we built on the approach Diebold and Mariano (2002) proposed and considered the following loss function:

$$L_{\tau,t}^{(i)}(y_t, Q_{y_t}^{(i)}(\tau, \mathbf{x}_{t-1})) = \left[\tau - I(y_t - Q_{y_t}^{(i)}(\tau, \mathbf{x}_{t-1}) < 0) \right] \times (y_t - Q_{y_t}^{(i)}(\tau, \mathbf{x}_{t-1})), \quad (3.14)$$

where $Q_{y_t}^{(i)}(\tau, \mathbf{x}_{t-1})$ is the τ -th forecasted quantile of Y , obtained from the i -th model.

Let $d_{DM,\tau,t}$ be the loss differential between the quantile forecasts from two competitive models, i and j (where i represents our proposal), that is, $d_{DM,\tau,t} = L_{\tau,t}^{(i)}(y_t, Q_{y_t}^{(i)}(\tau, \mathbf{x}_{t-1})) - L_{\tau,t}^{(j)}(y_t, Q_{y_t}^{(j)}(\tau, \mathbf{x}_{t-1}))$. After computing the quantities $d_{DM,\tau,t}$ for the forecasting sample, we compute the mean: $\overline{d_{DM,\tau}} = (T - ws)^{-1} \sum_{t=ws+1}^T d_{DM,\tau,t}$. We are interested in testing the null hypothesis $H_0 : E [\overline{d_{DM,\tau}}] = 0$ against the alternative $H_1 : E [\overline{d_{DM,\tau}}] \neq 0$. To that purpose, we compute for $\tau = \{0.1, 0.5, 0.9\}$ the test statistic Diebold and Mariano (2002)

¹⁶The test can be used in the presence of a rolling approach for the computation of density forecasts. The value ws indicates the size of the rolling window or the fact that time t forecasts depend, at maximum, on the last ws data points.

proposed:

$$DM_\tau = \frac{\overline{d_{DM,\tau}}}{\hat{\sigma}_{DM}/\sqrt{T - ws}}, \quad (3.15)$$

where $\hat{\sigma}_{DM}^2$ is a consistent estimate of $\sigma_{DM}^2 = \text{Var}(\sqrt{T - ws} \overline{d_{DM,\tau}})$, the asymptotic (long-run) variance. Diebold and Mariano (2002) showed that, under the null hypothesis of equal predictive accuracy, $DM_\tau \xrightarrow{d} \mathcal{N}(0, 1)$.

3.5 Results

3.5.1 Full sample analyses

First, we focus on Model (3.5) to analyze the full-sample estimated parameters and their p-values. This model was built to analyze and forecast the conditional quantiles of the first principal component of the realized range-based bias corrected bipower variations. For simplicity, Table 3.3 reports just the results associated with $\tau = \{0.1, 0.5, 0.9\}$.¹⁷ The standard errors are computed by means of a bootstrapping procedure using the *xy*-pair method, which provides accurate results without assuming any particular distribution for the error term.

When τ equals 0.1, only $sp500_{t-1}$ is not significant at the 5% level. At $\tau = \{0.2, 0.3, 0.4, 0.5\}$ all the coefficients have small p-values, while for $\tau = \{0.6, 0.7\}$ only $jump_{t-1}$ is not significant. Finally, when $\tau > 0.7$, only fpc_{t-1} and $sp500_{t-1}$ are highly significant. Therefore, the first important result is that the variables that significantly affect $Q_{fpc_t}(\tau|\mathbf{x}_{t-1})$ change according to the τ level. Notably, only fpc_{t-1} is always significant, whereas $sp500_{t-1}$ is not significant only at $\tau = 0.1$. It is important to highlight that only fpc_{t-1} and $sp500_{t-1}$ are significant in order to explain the high quantiles of volatility, which assume critical importance in finance. Moreover, the fact that $jump_{t-1}$ is not significant for high values of τ is a reasonable result since the volatility is already in a “high” state, and we might safely assume that the jump-risk is already incorporated in it.

The fact that fpc_{t-1} and $sp500_{t-1}$ are the most significant variables along the different quantiles levels finds a further confirmation when we compare two different models: a restricted model, that has fewer explanatory variables (just fpc_{t-1} and $sp500_{t-1}$), against an unrestricted one, that includes all the available covariates. The comparisons are made by means of the pseudo-coefficient of determination proposed by Koenker and Machado (1999), here denoted by $R^1(\tau)$, and the test statistic ξ_w proposed by Koenker and Bassett (1982b). $R^1(\tau)$ is a local goodness-of-fit measure, which ranges between 0 (when the covariates are

¹⁷Additional results for other quantiles are reported in Table C.0.2.

Table 3.3 Quantile regression results.

Variable	Coefficient Value	P-value
$\tau = 0.1$		
\overline{fpc}_{t-1}	0.24157	0.00548
\overline{fpc}_{5t-1}	0.15385	0.03927
vix_{t-1}	0.00020	0.00001
$sp500_{t-1}$	-0.00119	0.17431
$jump_{t-1}$	0.87638	0.03242
$\tau = 0.5$		
\overline{fpc}_{t-1}	0.44580	0.00000
\overline{fpc}_{5t-1}	0.28779	0.00013
vix_{t-1}	0.00019	0.00006
$sp500_{t-1}$	-0.00339	0.00002
$jump_{t-1}$	0.77113	0.01017
$\tau = 0.9$		
\overline{fpc}_{t-1}	1.44195	0.00000
\overline{fpc}_{5t-1}	0.15027	0.52722
vix_{t-1}	-0.00002	0.87801
$sp500_{t-1}$	-0.00919	0.00000
$jump_{t-1}$	-0.07240	0.90576

The table reports the coefficients and the p-values for Model (3.5); $\tau = \{0.1, 0.5, 0.9\}$. The standard errors are computed by means of the bootstrapping procedure, by employing the xy -pair method.

useless to predict the response quantiles) and 1 (in the case of a perfect fit); with ξ_w we aim to test the null hypothesis that the additional variables used in the unrestricted model do not significantly improve the goodness-of-fit with respect to the restricted model.¹⁸ Table 3.4 shows the values of the pseudo-coefficient of determination computed for the restricted and the unrestricted models at $\tau = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. We first observe that $R^1(\tau)$ is a positive function of τ for both the models and then note that the differences between the restricted and the unrestricted models decrease as τ increases, to substantially disappear at $\tau = 0.9$. Therefore, the contribution of \overline{fpc}_{5t-1} , vix_{t-1} , and $jump_{t-1}$ to the goodness-of-fit of Model (3.5) is largely irrelevant at $\tau = 0.9$. We obtain similar conclusions from the test proposed in Koenker and Bassett (1982b): the null hypothesis of the test is not rejected at $\tau = 0.9$, while, at lower quantiles, some of the additional variables provide sensible improvements in the model fit (see the fourth column of Table 3.4).

Returning to the coefficients' values, we note that the impact on the FPC conditional quantiles of \overline{fpc}_{t-1} , \overline{fpc}_{5t-1} , vix_{t-1} , and $jump_{t-1}$ is positive in all cases in which the coefficients are statistically significant. Therefore, there is a positive relationship between

¹⁸See B for the details about $R^1(\tau)$ and ξ_w .

Table 3.4 Restricted model against unrestricted model.

τ	$R^1(\tau)$ Unrestricted model	$R^1(\tau)$ Restricted model	ξ_w P-value
0.1	0.3655	0.2940	0.0000
0.2	0.4439	0.3785	0.0000
0.3	0.5005	0.4422	0.0000
0.4	0.5427	0.4952	0.0000
0.5	0.5801	0.5396	0.0000
0.6	0.6142	0.5803	0.0000
0.7	0.6474	0.6223	0.0000
0.8	0.6865	0.6755	0.0007
0.9	0.7492	0.7467	0.9123

The table reports some results coming from the comparison between the unrestricted model, that is Model (3.5), and the restricted model, in which the regressors are just fpc_{t-1} and $sp500_{t-1}$. The second and the third columns give the pseudo coefficients of determination, whereas the fourth one gives the p-values of the test statistic introduced by Koenker and Bassett (1982b).

these variables and $Q_{fpc_t}(\tau|\mathbf{x}_{t-1})$. With respect to the price jumps, the positive impact is a somewhat expected result, and it extends Corsi et al. (2010)'s findings on prices jumps' impact on realized volatility. However, since the coefficient of $sp500_{t-1}$, $\hat{\beta}_4(\tau)$, is always negative, in keeping with Black (1976), we find that an increasing market return implies greater stability and negative effects on $Q_{fpc_t}(\tau|\mathbf{x}_{t-1})$. We also checked on the persistence of volatility, measured by the sum of the HAR coefficients, that is, $\hat{\beta}_1(\tau) + \hat{\beta}_2(\tau)$, and noted that persistence is stronger at high levels of τ : $\hat{\beta}_1(0.1) + \hat{\beta}_2(0.1) = 0.396$, $\hat{\beta}_1(0.5) + \hat{\beta}_2(0.5) = 0.734$, and $\hat{\beta}_1(0.9) + \hat{\beta}_2(0.9) = 1.5923$. This evidence, which is coherent with the result on jumps, suggests that volatility in high regimes (upper quantiles) is more persistent as opposed to median or low regimes (lower quantiles); in addition, that unexpected movements/shocks (including jumps) may have a larger effect on lower volatility quantiles compared to their impact on higher volatility quantiles, as they convey relevant information. While these results, recovered from a full-sample analysis, provide an interesting interpretation, they do not take into account the possible structural changes in the relationship between covariates and volatility conditional quantiles. This problem is analyzed below.

Even if the coefficients' signs don't change over τ , it's important to determine whether changes in τ affect their magnitude. In other words, we want to check the so-called location-shift hypothesis, which states that the parameters in the conditional quantile equation are identical over τ . Important information can be drawn from Figure 3.3, which provides the coefficients' plots.

The impact of fpc_{t-1} on the conditional quantiles of volatility is constant up to $\tau = 0.7$, where it increases significantly. In the case of $\overline{fpc}_{5_{t-1}}$, we observe a slightly increasing trend

until $\tau = 0.7$, when the uncertainty level becomes noticeable. The impact of vix_{t-1} has a flat trend up to $\tau = 0.7$, when it begins a decreasing trend, reaching negative values in a region where the regressor is not significant. However, $sp500_{t-1}$ shows a negative effect on the volatility quantiles, and this relationship grows quickly at high values of τ . We associate this finding with the so-called *leverage effect*, as argued by Black (1976): increases in volatility are larger when previous returns are negative than when they have the same magnitude but are positive. To verify this claim, we divided the fpc_t series into deciles and, conditioning to those deciles, computed the mean of $sp500_{t-1}$ for the various groups. As expected, the mean of $sp500_{t-1}$, corresponding to the values of fpc_t in the first decile, is 0.19%, whereas the mean corresponding to the last decile, in which we have the highest fpc_t values, is -0.28%. As a consequence, the negative coefficients for $sp500_{t-1}$ can be seen as supporting the existence of the leverage effect. Finally, in the case of $jump_{t-1}$, we observe a wide band, where its impact grows at the beginning but takes negative turns from $\tau = 0.4$.

To summarize, we verify that the relationships between the regressors and the response variable are not constant over τ . In particular, the impact of fpc_{t-1} and $sp500_{t-1}$ grows considerably at high values of τ . Therefore, fpc_{t-1} and $sp500_{t-1}$ are critical indicators in the context of extreme events where volatility can reach high levels. The coefficients of the other explanatory variables do not exhibit particular trends at low-medium levels of τ , and when τ assumes high values, they become even more volatile, in a region of high uncertainty, given their wide confidence bands.

Analysis of the coefficients' plots shown in Figure 3.3 suggests that the hypothesis of equal slopes does not hold. In order to reach more accurate conclusions, we perform a variant of the Wald test introduced by Koenker and Bassett (1982a). The null hypothesis of the test is that the coefficient slopes are the same across quantiles. The test is performed taking into account three distant values of τ , $\tau = \{0.1, 0.5, 0.9\}$, to cover a wide interval.

When we compare the models estimated for $\tau = 0.1$ and $\tau = 0.5$, the null hypothesis is rejected at the 95% confidence level for fpc_{t-1} , $\overline{fpc}_{5_{t-1}}$, and $sp500_{t-1}$. When we consider $\tau = 0.5$ and $\tau = 0.9$, the null hypothesis is rejected at the 99% confidence level for fpc_{t-1} and $sp500_{t-1}$. We obtain the same result when we focus on $\tau = 0.1$ and $\tau = 0.9$, so we have evidence against the location shift hypothesis for those regressors. This finding confirms that the relationship between covariates and conditional quantiles varies across quantile values. This fundamentally relevant finding highlights that, when the interest lies on specific volatility quantiles, linear models can lead to inappropriate conclusions on whether there is a relationship between covariates and volatility measures, and if there is, on the strength of the relationship.

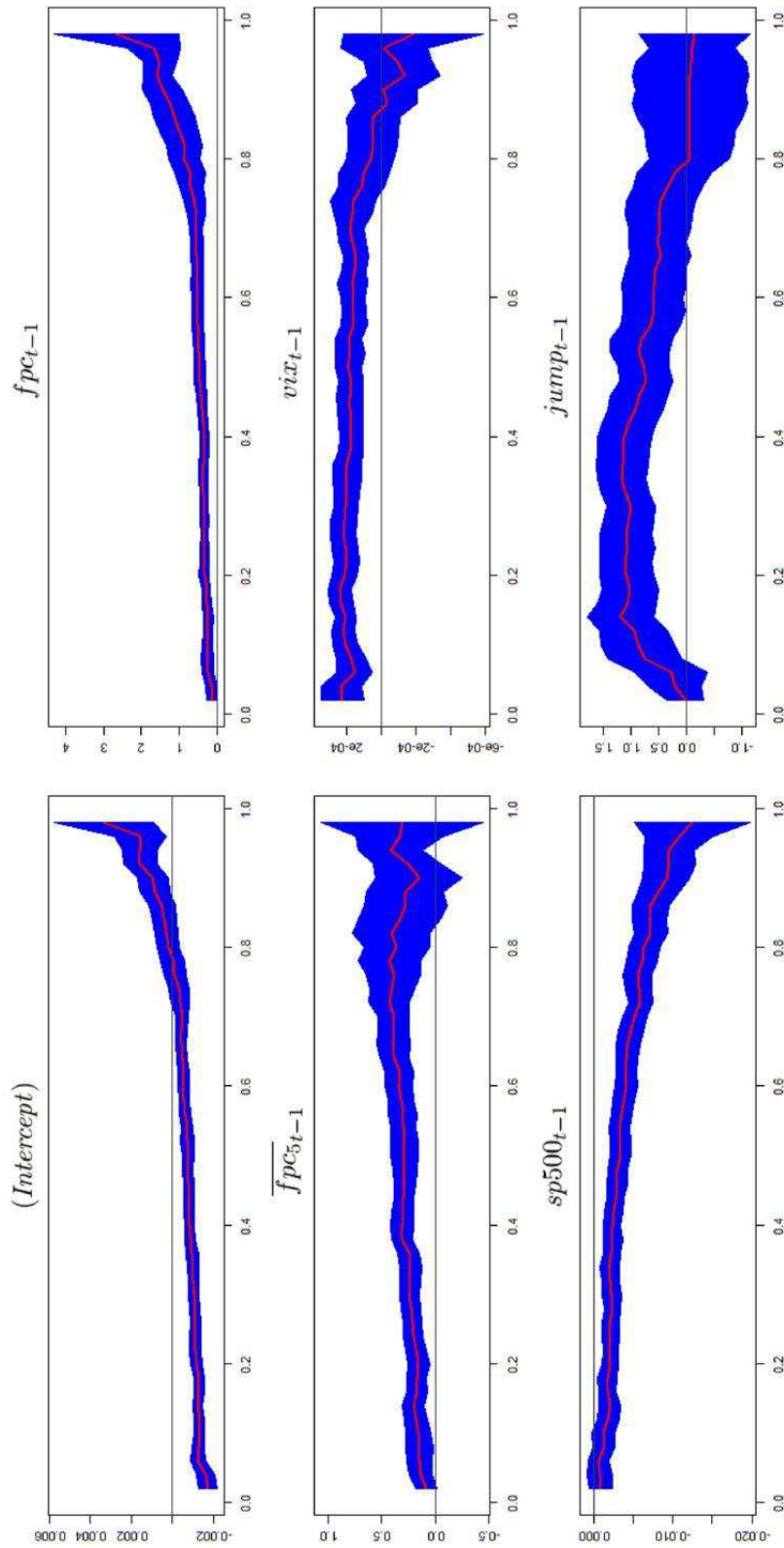


Fig. 3.3 Coefficients plots. The red lines represent the coefficients values over τ levels, while the blue areas are the associated 95% confidence intervals.

3.5.2 Rolling analysis

The U.S. subprime crisis and the European sovereign debt crisis have had noticeable effects on the financial system, with possible impacts also on the relationship between volatility and its determinants. Therefore, it is important to determine whether these events also affect the parameters of Model (3.5). To this end, we perform a rolling analysis with steps of one day, using a window size of 500 observations and τ ranging from 0.05 to 0.95 with steps of 0.05. Thus, we consider nineteen levels of τ and, for a given τ , obtain 2,133 estimates of a single coefficient. The finer grid adopted here allows us to recover a more accurate picture of the evolution of conditional quantiles. Nevertheless, the most relevant quantiles in this case are the upper quantiles, which are associated with the highest volatility levels. The estimated coefficients across time and quantiles are summarized in several figures.

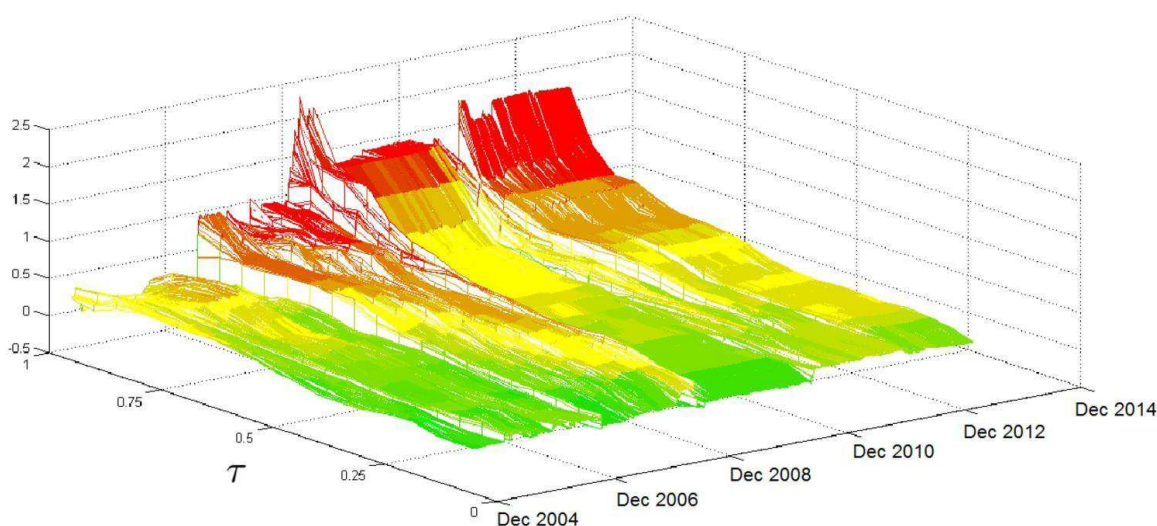


Fig. 3.4 Rolling analysis for fpc_{t-1} .

Figure 3.4 reports the evolution of the relationship between fpc_{t-1} and the conditional volatility quantiles over time and over τ . The first result that arises from Figure 3.4 is that the impact of fpc_{t-1} has a comparatively stable trend over time for medium-low τ levels; some jumps are recorded, mainly in the period of the subprime crisis, but their magnitude is negligible. The picture significantly changes in the region of high τ levels, where the surface is relatively flat and lies at low values in the beginning, but after the second half of 2007, when the effects of the subprime crisis start to be felt, there is a clear increase in the coefficient values, which reach their peak in the months between late 2008 and the beginning of 2009. Moreover, in this period we record the highest volatilities in the fpc_{t-1}

coefficients over τ levels. In the following months, the coefficient values decrease, but they remain at high levels until the end of the sample period. fp_{t-1} is a highly relevant variable for explaining the entire conditional distribution of fp_t since it is statistically significant over a large number of quantiles. Figure 3.4 verifies that the relationship between fp_t and fp_{t-1} is affected by particular events, such as the subprime crisis, mainly at medium-high τ levels. This finding confirms the change in the parameter across τ values, with an increasing pattern in τ and highlights that, during periods of market turbulence where the volatility stays at high levels, the volatility density overreacts to past movements of volatility, since the fp_{t-1} coefficient is larger than 1 for upper quantiles. Therefore, after a sudden increase in volatility at, say, time t , we have an increase in the conditional quantiles for time $t + 1$ and, therefore, an increase in the likelihood that we will observe additional volatility spikes (that is, volatility that exceeds a time-invariant threshold) at time $t + 1$.

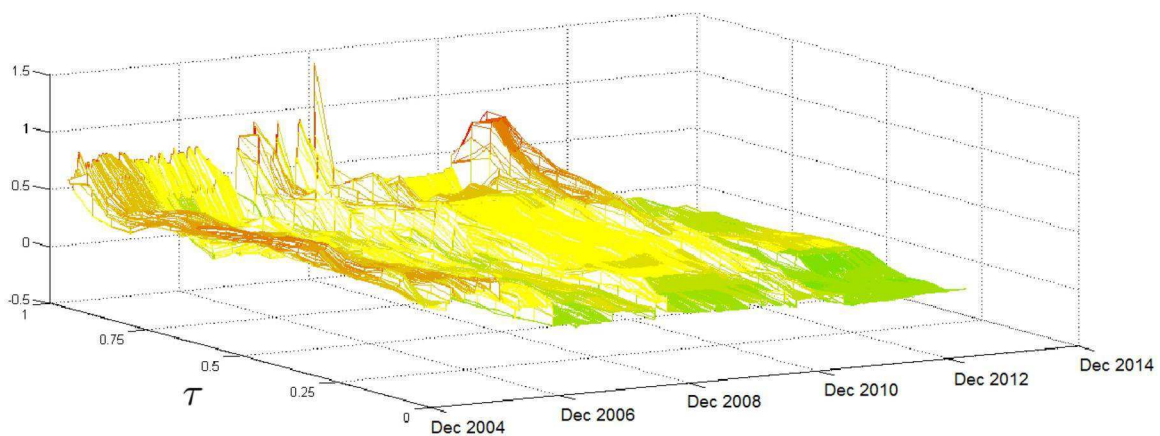


Fig. 3.5 Rolling analysis for $\overline{fp_{5t-1}}$.

Referring to $\overline{fp_{5t-1}}$, the HAR coefficient reported in Figure 3.5 has a volatile pattern until late 2008, when it reaches its peak. After that, the surface flattens, but another jump is recorded in mid-2011, mainly in the region of high τ values. Therefore, the relationship between the fp_t quantiles and $\overline{fp_{5t-1}}$, which reflects the perspectives of investors who have medium time horizons, is volatile over time, mainly in the region of high τ values. Again, this result can be associated with crises that affect the persistence and the probability that extreme volatilities will occur.

Section 3.5.1 pointed out that the persistence of volatility, measured by the sum of the HAR coefficients ($\hat{\beta}_1(\tau) + \hat{\beta}_2(\tau)$), is stronger at high levels of τ than it is at lower levels. Using the rolling analysis, we also determined how that persistence evolves over time. Figure 3.6, which focuses on just three τ values, $\tau = \{0.1, 0.5, 0.9\}$, shows that persistence is always

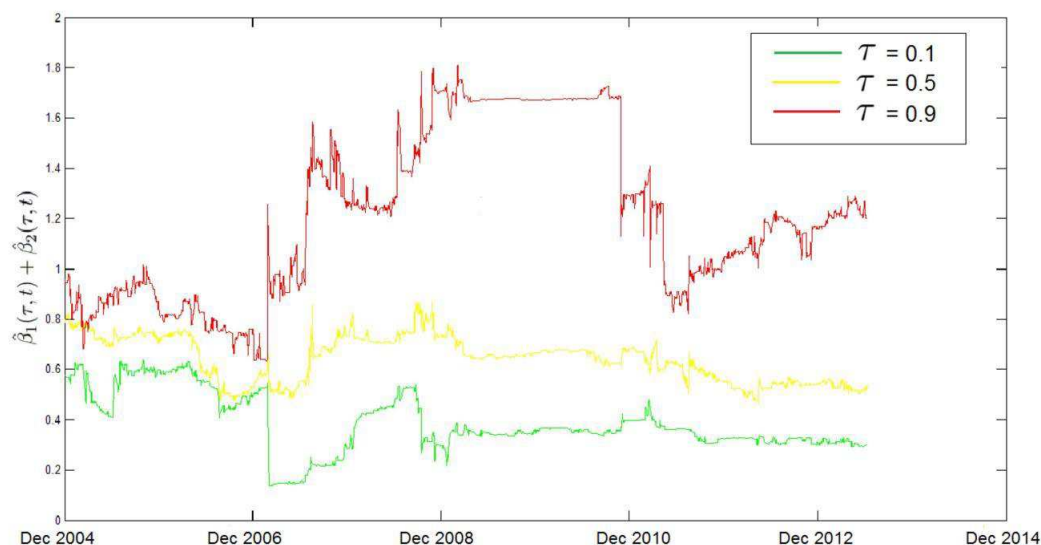


Fig. 3.6 The persistence of volatility over time.

positive, as one might expect, and that it has relevant differences across quantiles. Looking over time, we confirm the full-sample result that persistence increases with τ levels and note that the reaction of the persistence to the subprime crises is clear in all the three cases but is most pronounced for $\tau = 0.9$. However, the European sovereign debt crisis affects only the $\tau = 0.9$ case, where we note an increase in persistence in the last part of the sample.

Figure 3.7 shows two periods in which the vix_{t-1} coefficient has high values: between the end of 2008 and early 2010 and a shorter period from the end of 2011 to the first half of 2012. While in the first period the impact of vix_{t-1} significantly increases for all τ levels, in the second period the increase in the coefficient affects just the surface region in which τ takes high values. Unlike the HAR coefficients described above, the vix_{t-1} coefficient does not have a clear and stable increasing trend over τ . In addition, the relationship between the fpc_t conditional quantiles and vix_{t-1} is highly sensitive to the subprime crisis, when pessimism among financial operators, reflected in the implied volatility of the S&P 500 index options, was acute. This result is again somewhat expected since we focus on U.S.-based data and the subprime crisis had a high impact on the U.S. equity market. Our results are evidence that the perception of market risk has a great impact on the evolution of market volatility (as proxied by fpc_t), particularly during financial turmoil. The impact is not so clear-cut during the European sovereign crisis, which had less effect on the U.S. equity market.

Figure 3.8 shows how the impact of $sp500_{t-1}$ evolves over time and over τ . The surface given is almost always flat, the exception being the months between late 2008 and the end

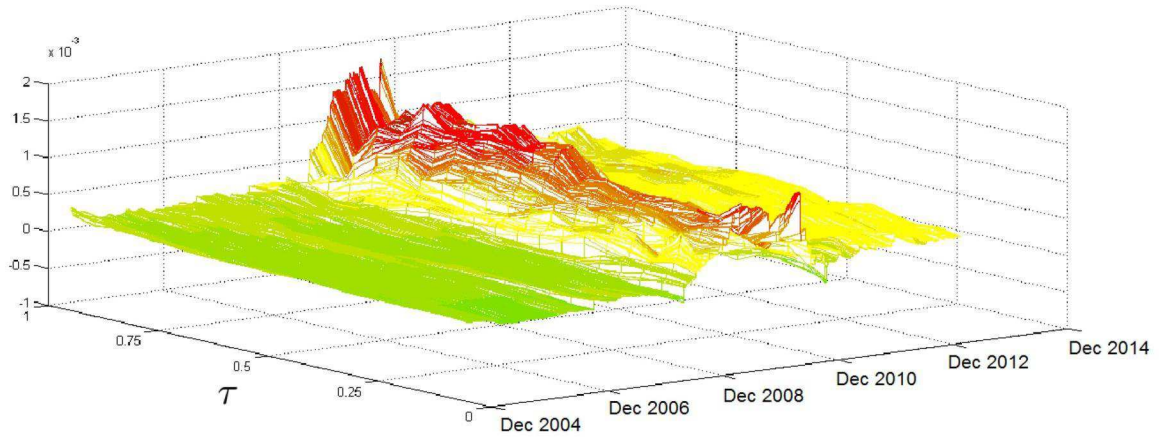


Fig. 3.7 Rolling analysis for vix_{t-1} .

of 2010, when the effects of the subprime crisis were particularly acute; during this time the coefficient values decrease as τ grows, mainly for values of τ above the median. The lagged value of the S&P 500 index return affects the entire conditional distribution of fpc_t and is statistically significant in almost all of the quantiles considered. Moreover, Figure 3.8 shows that the effect is negative and particularly pronounced during the subprime crisis, when negative returns exacerbated market risk, increasing the upper quantiles' volatility and increasing the likelihood of large and extreme volatility events.

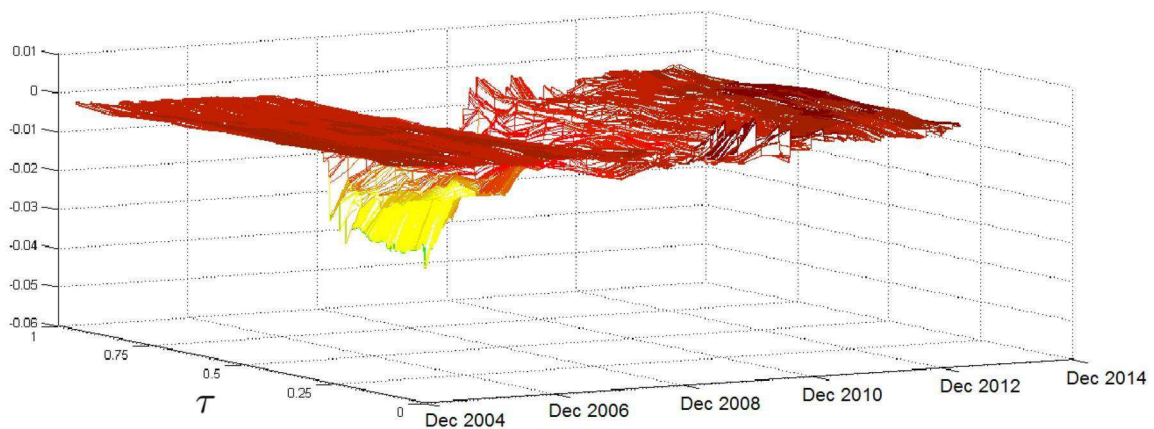


Fig. 3.8 Rolling analysis for $sp500_{t-1}$.

Finally, Figure 3.9 reports the surface associated to the jump component. At the beginning of the sample, the $jump_{t-1}$ coefficient takes on small values over the τ levels; however, it starts to grow in 2007, reaching high peaks at high τ levels during the subprime crisis. Al-

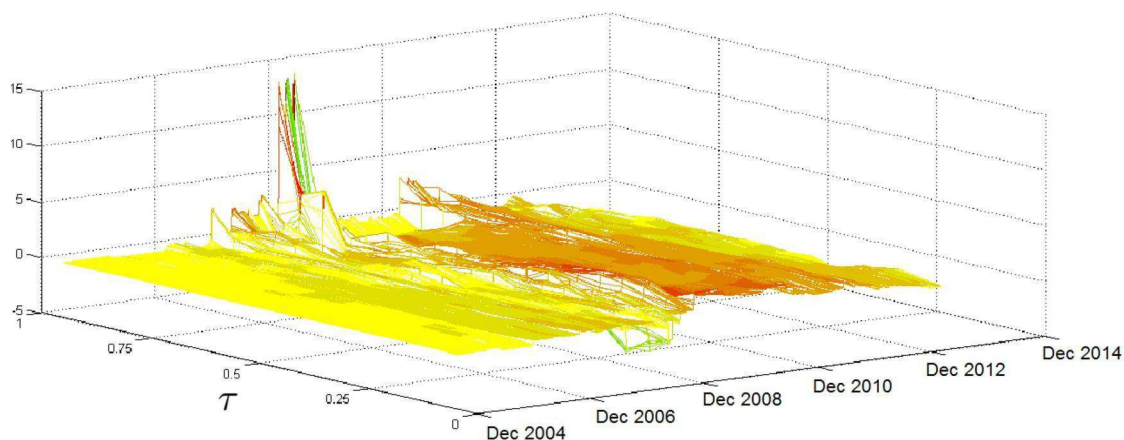


Fig. 3.9 Rolling analysis for $jump_{t-1}$.

though the coefficient reaches considerable values in this region, their statistical significance is limited. After the second half of 2009, the surface flattens out again until the end of the sample, with the exception of some peaks of moderate size that were recorded in 2011 during the sovereign debt crises.

To summarize, using the rolling analysis, we show that two special and extreme market events (the U.S. subprime crisis and the European sovereign debt crisis) affected the relationships between the realized volatility quantiles and a set of covariates. Our results show that coefficients can reasonably vary, with a potential and relevant impact on the forecasts of both the mean (or median) volatility and the volatility distribution (starting from the quantiles). The effects differ across quantiles and change with respect to the volatility upper tails, as compared to the median and the lower tail. Therefore, when volatility quantiles are modeled, the impacts of covariates might differ over time and over quantiles, being crucial during certain market phases. This result further supports the need for quantile-specific estimations when there is an interest in single volatility quantiles.

3.5.3 Evaluation of the predictive power

We evaluate the volatility density forecasts by means of the tests of Berkowitz (2001), Amisano and Giacomini (2007), and Diebold and Mariano (2002). We carry out the three tests by estimating a larger number of fpc_t conditional quantiles with respect to those analysed in Section 3.5.2. This allows recovering more precise volatility distribution functions. In particular, we consider forty-nine values of τ , ranging from 0.02 to 0.98, with steps of 0.02. Given the findings of the previous subsection, we must use a rolling procedure to build

density forecasts. However, we modify the rolling scheme previously adopted to keep a balance between the reliability of the estimated coefficients and computational times. In particular, we recover the forecasts on the basis of conditional quantiles estimated from subsamples of 100 observations and we roll over the sample every ten days. Within each 10 day window, we use fixed coefficients but produce one-step-ahead forecasts updating the conditioning information set.

With regard to the Berkowitz test, in the case of Model (3.5), z_t is normally distributed,¹⁹ as the likelihood ratio test LR_b equals 5.26 and the null hypothesis of the Berkowitz test, that is, $z_t \sim \mathcal{N}(0, 1)$ with no autocorrelation, is not rejected at the 5% significance level, validating the forecast goodness of Model (3.5). To determine whether the predictive power of our approach is affected by the U.S. subprime and the European sovereign debt crises, the series z_t is divided into two parts of equal length: the first referring to a period of relative calm, from the beginning of 2003 to the first half of 2007, and the second referring to a period of market turmoil, that was due to the two crises, between the second half of 2007 and the first half of 2013. In the first part LR_b equals 2.34, and in the second it equals 5.39. Nevertheless, the null hypothesis of the Berkowitz test is not rejected at the 5% level in both the cases. Therefore, the conditional quantile model and the approach we adopt to recover the conditional density forecasts are appropriate even during financial turbulence.

Analysis of the results reveals that, as in the analysis of the full sample, \overline{fpc}_{5t-1} , vix_{t-1} , and $jump_{t-1}$ are not significant to explain the volatility quantiles at high values of τ in many of the subsamples. Therefore, we must determine whether this result affects the output of the Berkowitz test using a restricted model in which the regressors are only fpc_{t-1} and $sp500_{t-1}$. LR_b equals 2.60, a smaller value than the previous cases. The last findings reported above suggest that the inclusion of non-significant explanatory variables penalizes the predictive power of Model (3.5). The restricted model gives the lowest value of LR_b , but the predictive power could be improved by selecting only those variables that are significant for each value of τ and for each subsample in order to forecast the conditional distribution of the volatility. Thus, the structure of Model (3.5) would change over time and over τ . However, this approach is not applied in the present work since it would require using only the significant variables in forty-nine models, one for each specific value of τ , while it should be considered across all of the rolling subsamples.

So far, we have focused on an absolute assessment of our approach. Now we compare it with a competing model that is fully parametric. We recover the predictive conditional distribution of the realized range volatility by means of a HARX model in which the mean dynamic is driven by a linear combination of the explanatory variables fpc_{t-1} and \overline{fpc}_{5t-1}

¹⁹See Figure C.0.4 for graphical evidences.

(the HAR terms) and the exogenous variables vix_{t-1} , $sp500_{t-1}$, and $jump_{t-1}$ (the X in the model's acronym). In addition, to capture the volatility-of-volatility effect argued by Corsi et al. (2008), a GJR-GARCH term (Glosten et al., 1993) is introduced on the innovation (we name the model HARX-GJR). The error term is also assumed to follow a normal-inverse Gaussian (\mathcal{NIG}) distribution with mean 0 and variance 1. Thus, the conditional variance is allowed to change over time, and the distribution of the error is flexible to features like fat tails and skewness.

We start by using the Berkowitz test to evaluate the density forecast performance of the HARX-GJR model. The likelihood ratio test LR_b equals 113.88, a high value that suggests a clear rejection of the null hypothesis. We also determined whether the HARX-GJR model works better when we use the logarithm of the volatility as a response variable, following the evidence in Corsi et al. (2010), and found that the likelihood ratio test LR_b provides a much lower value (20.27). Even so, the test signals a rejection of the null hypothesis with a low p-value. As a first finding, our approach provides more flexibility than the parametric HARX-GJR model does and is better in terms of the Berkowitz test.

To provide more accurate results, we move to a comparison of our approach to the HARX-GJR by means of the Amisano and Giacomini (2007) and the Diebold and Mariano (2002) tests. With regard to the Amisano and Giacomini (2007) test, we use four weights to compute the quantity given in Equation (4.23): $w_{CE}(fpc_t^{st})$, $w_{RT}(fpc_t^{st})$, $w_{LT}(fpc_t^{st})$, and $w_{NW}(fpc_t^{st})$. The associated likelihood ratio tests are denoted by AG_{CE} , AG_{RT} , AG_{LT} , and AG_{NW} , respectively. Overall, our approach provides better results than the HARX-GJR since \overline{WLR} is always positive; in fact, $AG_{CE} = 1.6521$, $AG_{RT} = 2.2766$, $AG_{LT} = 1.2961$, and $AG_{NW} = 1.8810$. Given that the critical values at the 5% level are equal to -1.96 and 1.96 (we are dealing with a two-sided test), the null hypothesis of equal performance is rejected only when we give a higher weight to the right tail of the volatility conditional distribution.

Similar results are obtained when we consider the single quantile loss function and the Diebold and Mariano (2002) test statistic. Our approach provides lower losses since $\overline{d_{DM,\tau}}$ is negative for all the τ levels we considered (0.1, 0.5, 0.9). However, the differences are statistically significant only at $\tau = 0.9$, given that $DM_{0.1} = -1.7283$ (0.0839), $DM_{0.5} = -1.7332$ (0.0831), and $DM_{0.9} = -2.6884$ (0.0072).

Summarizing, the results demonstrate the good performance of our approach, in particular when we focus on the right tail of the first principal component (a kind of market risk factor) distribution. We note that the right tail assumes critical importance in our framework, as it represents periods of extreme risks. Our findings indicate another relevant contribution of this study, as the quantile regression approach we propose can be used to recover density forecasts for a realized volatility measure. These forecasts improve on those of a traditional approach

because of the inclusion of quantile-specific coefficients. This feature of our approach might become particularly relevant in all empirical applications where predictive volatility density is required.

3.5.4 Single asset results

The results in Subsections 3.5.1-3.5.3 were based on a summary of the sixteen asset volatility movements, which was itself based on the first principal component. Now we search for confirmation of the main findings of Model (3.5), by running Model (3.6) at the single-asset level, for all the sixteen assets. For simplicity, only the results associated with $\tau = \{0.1, 0.5, 0.9\}$ are shown. Tables 3.5-3.7 provide the estimated parameters and their p-values.

We first focus on the relationships between $Q_{rrv_{i,t}}(\tau|\mathbf{x}_{i,t-1})$ and $rrv_{i,t-1}$, for $i = 1, \dots, 16$. When τ equals 0.1, $rrv_{i,t-1}$ is not significant, at the 5% level, to explain the conditional volatility quantiles of eight assets: *ATT*, *CAT*, *HON*, *IBM*, *PEP*, *PRG*, *TWX*, and *TXN*. At $\tau = 0.5$, $rrv_{i,t-1}$ is not significant only for *PRG*, whereas, at $\tau = 0.9$, $rrv_{i,t-1}$ is not significant for *PEP* and *PRG*. Compared with the other regressors and in line with the results obtained for the first principal component, $rrv_{i,t-1}$ is one of the most significant explanatory variables at high levels of τ , so it assumes critical importance in the context of extreme events. The $rrv_{i,t-1}$ coefficient takes a negative value only for *PEP* at $\tau = 0.1$, but here it is not statistically significant.

In all the other cases, it is always positive. Moreover, the magnitude of the impact that $rrv_{i,t-1}$ has on $Q_{rrv_{i,t}}(\tau|\mathbf{x}_{i,t-1})$ is positive function of τ , for all sixteen assets.²⁰

The differences among the assets increase as τ grows, and, at $\tau = 0.9$, the financial companies (*BAC*, *CTG*, *JPM* and *WFC*) record the highest coefficient values, highlighting the crucial importance of the extreme events in the financial system. fpc_{t-1} , the homologous regressor included in Model (3.5), has a weaker impact on the conditional volatility quantiles than $rrv_{i,t-1}$ does only for the financial companies *BAC* ($\tau = 0.1$), *JPM* ($\tau = 0.5$), and *CTG* ($\tau = 0.9$). Therefore, the relationships between the conditional volatility quantiles and the lagged value of the response variable are stronger for the first principal component than for the single assets. $\overline{rrv}_{5i,t-1}$ is not significant (significance level of 0.05) at $\tau = 0.1$ for *CTG*, *JPM*, and *PRG*, as the p-values of its coefficient indicate. At $\tau = 0.5$ it is not significant only for *PRG*, whereas, when τ equals 0.9, it is not significant for *CTG*, *JPM*, and *PRG*. The $\overline{rrv}_{5i,t-1}$ coefficient is always positive and, with the exception of *CTG* and *JPM*, it is positive function of τ . Moreover, the differences among the $\overline{rrv}_{5i,t-1}$ coefficient values are more marked at high τ levels for all the sixteen assets.

²⁰See Figures C.0.5-C.0.9 for further graphical evidences on single asset estimates.

Table 3.5 Quantile regressions for each asset ($\tau = 0.1$).

Stocks	Intercept	$rrv_{i,t-1}$	$\overline{rrv}_{5i,t-1}$	$vik_{i,t-1}$	$sp500_{i,t-1}$	$jump_{i,t-1}$
ATT	-0.00002 (0.11239)	0.04589 (0.70992)	0.28691 (0.00124)	0.00002 (0.00265)	0.00055 (0.00090)	0.03515 (0.21811)
BAC	-0.00012 (0.00019)	0.25007 (0.00001)	0.14869 (0.01091)	0.00005 (0.00009)	-0.00033 (0.18180)	-0.00202 (0.99308)
BOI	-0.00008 (0.00093)	0.15852 (0.00963)	0.21132 (0.00008)	0.00004 (0.00012)	-0.00081 (0.00006)	0.04509 (0.37152)
CAT	-0.00012 (0.00001)	0.10649 (0.19309)	0.25142 (0.00000)	0.00006 (0.00000)	-0.00067 (0.00695)	-0.00001 (0.99972)
CTG	-0.00030 (0.00003)	0.18252 (0.03744)	0.12645 (0.08613)	0.00012 (0.00002)	-0.00081 (0.08236)	0.95128 (0.00090)
FDX	-0.00013 (0.00000)	0.18287 (0.00461)	0.23422 (0.00000)	0.00006 (0.00000)	-0.00065 (0.00340)	-0.07955 (0.11994)
HON	-0.00012 (0.00000)	0.07855 (0.36506)	0.26336 (0.00005)	0.00005 (0.00000)	-0.00097 (0.00053)	0.01054 (0.70667)
HPQ	-0.00004 (0.01747)	0.16799 (0.00848)	0.15327 (0.00325)	0.00003 (0.00004)	-0.00082 (0.00209)	-0.00039 (0.99120)
IBM	-0.00003 (0.00089)	0.13709 (0.19222)	0.22922 (0.00097)	0.00002 (0.00007)	-0.00050 (0.00396)	0.23369 (0.03116)
JPM	-0.00019 (0.00018)	0.23775 (0.02594)	0.14460 (0.10227)	0.00008 (0.00009)	-0.00047 (0.13311)	0.02273 (0.89037)
MIDZ	-0.00005 (0.00000)	0.08280 (0.01171)	0.19490 (0.00000)	0.00002 (0.00000)	-0.00027 (0.00492)	-0.00297 (0.82097)
PEP	-0.00005 (0.00032)	-0.02564 (0.76780)	0.26404 (0.00000)	0.00003 (0.00001)	-0.00036 (0.06062)	0.06751 (0.32140)
PRG	-0.00006 (0.00083)	0.01883 (0.86893)	0.00483 (0.91483)	0.00003 (0.00003)	-0.00025 (0.00724)	0.10243 (0.49121)
TWX	-0.00005 (0.01154)	0.13005 (0.19936)	0.29034 (0.00003)	0.00003 (0.00160)	-0.00072 (0.00032)	0.18752 (0.00854)
TXN	-0.00003 (0.19320)	0.09522 (0.16872)	0.32663 (0.00000)	0.00002 (0.00382)	-0.00137 (0.00003)	0.00907 (0.82139)
WFC	-0.00012 (0.00001)	0.20878 (0.00269)	0.20321 (0.00021)	0.00005 (0.00000)	-0.00064 (0.00248)	-0.00044 (0.99768)

The table reports for each stock (the ticker is given in the first column) the coefficients and the p-values for Model (3.6); $\tau = 0.1$.

Table 3.6 Quantile regressions for each asset ($\tau = 0.5$).

Stocks	Intercept	$rrv_{i,t-1}$	$\overline{rrv}_{i,t-1}$	vlx_{t-1}	$sp500_{t-1}$	$jump_{i,t-1}$
ATT	-0.00002 (0.12244)	0.30163 (0.00673)	0.52921 (0.00000)	0.00001 (0.04070)	-0.00079 (0.00000)	-0.00400 (0.91673)
BAC	-0.00012 (0.00196)	0.38505 (0.00000)	0.40227 (0.00000)	0.00005 (0.00135)	-0.00167 (0.00000)	0.14596 (0.66459)
BOI	-0.00011 (0.00005)	0.36842 (0.00000)	0.36566 (0.00000)	0.00005 (0.00002)	-0.00112 (0.00000)	0.07570 (0.35862)
CAT	-0.00013 (0.00002)	0.40863 (0.00000)	0.40806 (0.00000)	0.00005 (0.00002)	-0.00141 (0.00000)	-0.00043 (0.99254)
CTG	-0.00031 (0.00058)	0.44363 (0.00003)	0.25602 (0.00040)	0.00013 (0.00051)	-0.00109 (0.00405)	0.71026 (0.02396)
FDX	-0.00011 (0.00000)	0.33461 (0.00000)	0.44110 (0.00000)	0.00005 (0.00000)	-0.00133 (0.00000)	-0.04139 (0.11644)
HON	-0.00009 (0.00007)	0.33249 (0.00017)	0.42692 (0.00000)	0.00004 (0.00002)	-0.00126 (0.00000)	-0.02821 (0.34547)
HPQ	-0.00006 (0.00069)	0.33324 (0.00000)	0.39040 (0.00000)	0.00003 (0.00000)	-0.00135 (0.00000)	-0.01217 (0.69966)
IBM	-0.00004 (0.00138)	0.33790 (0.00005)	0.42321 (0.00000)	0.00002 (0.00033)	-0.00084 (0.00000)	0.39259 (0.15630)
JPM	-0.00016 (0.00009)	0.50524 (0.00000)	0.26286 (0.00003)	0.00007 (0.00006)	-0.00161 (0.00000)	-0.02738 (0.92411)
MDZ	-0.00006 (0.00000)	0.30852 (0.00000)	0.36733 (0.00000)	0.00003 (0.00000)	-0.00050 (0.00000)	-0.01370 (0.70265)
PEP	-0.00004 (0.00009)	0.31618 (0.03741)	0.39554 (0.00566)	0.00002 (0.00003)	-0.00047 (0.00004)	0.03569 (0.60587)
PRG	-0.00010 (0.00219)	0.11543 (0.65757)	0.21804 (0.20756)	0.00005 (0.00132)	-0.00050 (0.00548)	1.20000 (0.10113)
TWX	-0.00007 (0.00013)	0.37437 (0.00000)	0.39468 (0.00000)	0.00003 (0.00001)	-0.00098 (0.00000)	0.18763 (0.02508)
TXN	-0.00002 (0.24687)	0.27555 (0.00000)	0.49860 (0.00000)	0.00002 (0.00631)	-0.00168 (0.00000)	-0.00497 (0.91838)
WFC	-0.00011 (0.00595)	0.37207 (0.00000)	0.43713 (0.00000)	0.00005 (0.00524)	-0.00133 (0.00000)	-0.00109 (0.99176)

The table reports for each stock (the ticker is given in the first column) the coefficients and the p-values for Model (3.6); $\tau = 0.5$.

Table 3.7 Quantile regressions for each asset ($\tau = 0.9$).

Stocks	Intercept	r_{t-1}^{VIX}	\overline{r}_{t-1}^{VIX}	vix_{t-1}	$sp500_{t-1}$	$jump_{t-1}$
ATT	-0.00003 (0.32728)	0.70584 (0.00000)	0.69582 (0.00000)	0.00002 (0.14326)	-0.00179 (0.00001)	0.00282 (0.98094)
BAC	-0.00005 (0.60023)	1.04077 (0.00000)	0.48227 (0.00390)	0.00002 (0.60645)	-0.00339 (0.00017)	1.56280 (0.21587)
BOI	-0.00008 (0.17547)	0.60555 (0.00014)	0.79933 (0.00000)	0.00004 (0.14547)	-0.00254 (0.00000)	-0.00471 (0.97864)
CAT	-0.00005 (0.40021)	0.79303 (0.00000)	0.61374 (0.00002)	0.00003 (0.22975)	-0.00321 (0.00001)	-0.00103 (0.99147)
CTG	-0.00003 (0.86813)	1.58940 (0.00110)	0.19051 (0.54878)	0.00001 (0.83966)	-0.00376 (0.00025)	-0.73284 (0.15626)
FDX	-0.00006 (0.16058)	0.57441 (0.00000)	0.78243 (0.00000)	0.00004 (0.06126)	-0.00290 (0.00000)	-0.12050 (0.44618)
HON	-0.00009 (0.09866)	0.69621 (0.00000)	0.60597 (0.00001)	0.00005 (0.03832)	-0.00321 (0.00000)	-0.00630 (0.92379)
HPQ	-0.00008 (0.12832)	0.72014 (0.00001)	0.66113 (0.00012)	0.00004 (0.04617)	-0.00344 (0.00000)	-0.03320 (0.47355)
IBM	-0.00001 (0.84229)	0.55602 (0.02084)	1.01283 (0.00018)	0.00000 (0.89257)	-0.00218 (0.00000)	0.41659 (0.55832)
JPM	-0.00014 (0.06252)	1.34345 (0.00000)	0.17516 (0.05766)	0.00006 (0.06809)	-0.00349 (0.00001)	1.55650 (0.08945)
MDZ	0.00000 (0.97148)	0.86207 (0.00000)	0.53696 (0.00188)	0.00001 (0.50692)	-0.00155 (0.00001)	-0.03639 (0.23018)
PEP	0.00000 (0.91957)	0.36885 (0.36232)	1.17046 (0.02804)	0.00000 (0.86108)	-0.00107 (0.00280)	-0.00578 (0.96858)
PRG	-0.00010 (0.02213)	0.59766 (0.16982)	0.47298 (0.13898)	0.00004 (0.01355)	-0.00154 (0.00017)	1.52132 (0.37350)
TWX	-0.00010 (0.00486)	0.86695 (0.00000)	0.52144 (0.00001)	0.00004 (0.00418)	-0.00232 (0.00002)	-0.01217 (0.94692)
TXN	-0.00015 (0.00221)	0.56407 (0.00000)	0.80214 (0.00000)	0.00007 (0.00025)	-0.00381 (0.00000)	0.04170 (0.60829)
WFC	-0.00004 (0.57678)	0.94470 (0.00000)	0.64679 (0.00000)	0.00002 (0.53260)	-0.00309 (0.00001)	-0.00292 (0.99593)

The table reports for each stock (the ticker is given in the first column) the coefficients and the p-values for Model (3.6); $\tau = 0.9$.

As for the comparison with the results obtained from Model (3.5), at $\tau = \{0.1, 0.5\}$ $\overline{rrv}_{5i,t-1}$ has a stronger impact on the conditional volatility quantiles with respect to \overline{fpc}_{5t-1} for most of the sixteen assets. When τ equals 0.9, it is pointless to compare the coefficients' values because \overline{fpc}_{5t-1} is not statistically significant at that level. Therefore, when we take into account the explanatory variables \overline{fpc}_{5t-1} and $\overline{rrv}_{5i,t-1}$, we record stronger relationships for Model (3.6) than for Model (3.5).

The p-values of the vix_{t-1} coefficient are less than 0.05 at $\tau = \{0.1, 0.5\}$ for all the sixteen assets. When τ equals 0.9, vix_{t-1} is significant in the cases of *HON*, *HPQ*, *PRG*, *TWX* and *TXN*. Unlike the coefficients previously mentioned, that of vix_{t-1} does not have a particular trend over τ ; it takes positive values for all the 16 assets, so, as in the context of the first principal component, it has a positive impact on the conditional volatility quantiles. In addition, at $\tau = \{0.1, 0.5\}$, the vix_{t-1} coefficient is larger in Model (3.5) than it is in Model (3.6) for all the sixteen companies. Therefore, vix_{t-1} has a more marked impact on the conditional volatility quantiles of the first principal component. The comparisons made at $\tau = 0.9$ are useless given the high p-values of the coefficients of interest.

$sp500_{t-1}$ is always significant ($\alpha = 0.05$) for the sixteen assets, with the exception of *BAC*, *CTG* and *JPM* at $\tau = 0.1$. Its coefficient is always negative, as expected, so $sp500_{t-1}$ has a negative impact on the conditional volatilities' quantiles. In addition, the magnitude of the impact is negative function of τ for all the assets, and those relationships become more marked at high τ levels. With the exception of one case (*TXN* at $\tau = 0.1$), the impact of $sp500_{t-1}$ at $\tau = \{0.1, 0.5, 0.9\}$ is more pronounced on the conditional volatility quantiles of the first principal component than it is when the assets are considered individually, as comparing the absolute values of the related coefficients shows.

At $\tau = 0.1$, $jump_{i,t-1}$ is significant, at the 5% level, for *CTG*, *IBM* and *TWX*. It is significant only for *CTG* and *TWX* at $\tau = 0.5$, whereas it is never significant when τ equals 0.9. $jump_{i,t-1}$'s coefficient takes both negative and positive values and, with the exception of a few assets, it does not have a particular trend over τ . Comparing these results with those obtained for the fpc_t conditional quantiles, we find that, with the exception of *CTG* ($\tau = 0.1$) and *PRG* ($\tau = 0.5$), the lagged value of the component associated to jumps has more impact in Model (3.5) than in Model (3.6) at $\tau = \{0.1, 0.5\}$. It is pointless to compare the coefficients at $\tau = 0.9$ given their high p-values.

To summarize, the explanatory variables $rrv_{i,t-1}$, $\overline{rrv}_{5i,t-1}$, vix_{t-1} , and $sp500_{t-1}$ are sufficient to explain the conditional volatility quantiles of the sixteen assets in most of the cases studied. Their coefficients tend to take the same sign for the sixteen assets: positive in the cases of $rrv_{i,t-1}$, $\overline{rrv}_{5i,t-1}$ and vix_{t-1} , negative in the case of $sp500_{t-1}$. Moreover, the coefficients of $rrv_{i,t-1}$, $\overline{rrv}_{5i,t-1}$, and $sp500_{t-1}$ have a clear trend over τ , providing evidence

against the location-shift hypothesis, which assumes homogeneous impacts of the regressors across quantiles. However, $jump_{i,t-1}$ is significant in only a few cases. Furthermore, with the exception of $\overline{rrv}_{5i,t-1}$, we find that, in most of the studied cases, the relationships between the explanatory variables and the conditional volatility quantiles are more pronounced in the context of the first principal component than when the assets are individually considered. This result shows that the first principal component captures a kind of systematic effect, where the relationship between macro and finance covariates and volatility quantiles is clearer. At the single-asset level the impact of covariates is more heterogeneous than for the first principal component, perhaps suggesting the need for company- (or sector-) specific covariates.

The last point of our analysis refers to the assessment of Model (3.6)'s predictive power, which we apply for each of the sixteen assets. As in the case of the model for the first principal component, we use the tests Berkowitz (2001), Amisano and Giacomini (2007), and Diebold and Mariano (2002) proposed. With regard to the Berkowitz test, Table 3.8 provides the values of the likelihood ratio defined by Equation (4.18) and the results generated by Model (3.6), showing that the null hypothesis of the test, that is, $z_t \sim \mathcal{N}(0, 1)$ with no autocorrelation, is not rejected for ten assets: *BAC*, *CTG*, *HON*, *HPQ*, *IBM*, *JPM*, *MDZ*, *PRG*, *TXN*, and *WFC*. The results from the other six cases stem from the fact that some variables, mainly $jump_{i,t-1}$, are not significant in many subsamples for several τ levels. As indicated in Section 3.5.3, which focused on the first principal component, the predictive power of our approach could be improved by selecting, for each subsample and each τ , only the regressors that are significant in order to explain the individually evaluated conditional quantiles. Thus, the structure of Model (3.6) would change over time. Now we compare our approach with the HARX-GJR model. The results that arise from using the HARX-GJR model are given in Table 3.8. As in the first principal component context, the likelihood ratio test (4.18), denoted by $LR_{b,HARX-GJR}$, takes high values for all sixteen assets, suggesting that the null hypothesis is rejected with low p-values.

Table 3.9 reports the results of the Amisano and Giacomini (2007) test and shows that our model provides, overall, better results: the test statistic values (4.24) are in most cases positive and the null hypothesis of equal performance is almost always rejected at the 5% level. Similar results are obtained for the Diebold and Mariano (2002) test, as we can see from Table 3.10.

The sign of the test statistic (4.22) is always negative, suggesting that our approach implies a lower loss. Moreover, the performances are in most cases statistically different, given that the null hypothesis is often rejected at the 5% level. Therefore, the three tests provide clear evidence of a better performance of our method with respect to the benchmark also in the case of single asset realized volatilities.

Table 3.8 Berkowitz test for the single asset analysis.

Asset	LR_b	$LR_{b,HARX-GJR}$
ATT	10.23 (0.0167)	150.26 (0.0000)
BAC	3.00 (0.3916)	125.11 (0.0000)
BOI	7.90 (0.04812)	93.09 (0.0000)
CAT	9.31 (0.0254)	88.11 (0.0000)
CTG	4.28 (0.2327)	142.99 (0.0000)
FDX	12.62 (0.0055)	95.63 (0.0000)
HON	5.95 (0.1140)	115.59 (0.0000)
HPQ	5.33 (0.1491)	83.80 (0.0000)
IBM	4.25 (0.2357)	157.06 (0.0000)
JPM	3.26 (0.3532)	124.01 (0.0000)
MDZ	5.49 (0.1392)	159.86 (0.0000)
PEP	9.97 (0.0188)	164.45 (0.0000)
PRG	2.25 (0.5222)	130.88 (0.0000)
TWX	11.09 (0.0112)	99.10 (0.0000)
TXN	1.90 (0.5934)	79.09 (0.0000)
WFC	7.04 (0.0706)	143.04 (0.0000)

The table reports for each stock (the ticker is given in the first column) the values of the likelihood ratio test (the p-values are given in brackets) proposed by Berkowitz (2001), generated by Model (3.6), LR_b , and the HARX-GJR model, $LR_{b,HARX-GJR}$.

To conclude, we found similar results between Models (3.5) and (3.6). In particular, for both models the lagged value of the response variable and the lagged value of the S&P 500 return were fundamental explanatory variables at high τ levels, which are the most critical. In contrast, the lagged value of the jump component is significant in a few cases. We determined that the relationships between four explanatory variables ($rrv_{i,t-1}$, vix_{t-1} , $sp500_{t-1}$, and $jump_{i,t-1}$) and the conditional volatility quantiles are almost always stronger in Model (3.5) than in Model (3.6). However, in the case of $\overline{rrv}_{5i,t-1}$, the relationships are stronger in Model (3.6) than in Model (3.5). Finally, even in the single-asset analysis, the goodness of the predicted power of our approach is validated by means of the three tests.

3.6 Conclusions

We proposed a semi-parametric approach to model and forecast the conditional distribution of asset returns' volatility. We used the quantile regression approach, considering as predictors variables built from the lagged values of the estimated volatility, following the HAR structure Corsi (2009) developed, and macroeconomic and financial variables that reflect the overall market behavior.

Table 3.9 Amisano-Giacomini test for the single asset analysis.

Asset	AG_{NW}	AG_{CE}	AG_{RT}	AG_{LT}
ATT	3.8957 (0.0001)	4.3336 (0.0000)	2.8878 (0.0039)	4.6406 (0.0000)
BAC	4.0325 (0.0001)	4.3822 (0.0000)	3.0110 (0.0026)	5.4706 (0.0000)
BOI	13.9648 (0.0000)	9.3975 (0.0000)	18.9700 (0.0000)	9.4658 (0.0000)
CAT	4.3298 (0.0000)	4.5230 (0.0000)	2.9384 (0.0033)	5.1361 (0.0000)
CTG	3.6318 (0.0003)	3.7586 (0.0002)	2.8238 (0.0047)	4.1108 (0.0000)
FDX	3.0102 (0.0026)	2.9183 (0.0035)	3.9842 (0.0001)	2.0103 (0.0444)
HON	2.9055 (0.0037)	3.0381 (0.0024)	2.4728 (0.0134)	2.8784 (0.0040)
HPQ	16.6222 (0.0000)	7.4219 (0.0000)	2.7958 (0.0052)	5.3210 (0.0000)
IBM	-0.2210 (0.8251)	0.3105 (0.7562)	-0.6036 (0.5461)	0.5044 (0.6140)
JPM	0.7630 (0.4455)	0.6326 (0.5270)	1.0247 (0.3055)	0.4642 (0.6425)
MDZ	-0.3915 (0.6954)	-0.3848 (0.7004)	-0.3545 (0.7230)	-0.4181 (0.6759)
PEP	1.0614 (0.2885)	2.1311 (0.0331)	0.0232 (0.9815)	2.8849 (0.0039)
PRG	3.2625 (0.0011)	3.2615 (0.0011)	3.3250 (0.0009)	3.2064 (0.0013)
TWX	0.4999 (0.6171)	0.0756 (0.9397)	1.7995 (0.0719)	0.1434 (0.8860)
TXN	3.1661 (0.0015)	3.1672 (0.0015)	2.4961 (0.0126)	3.0988 (0.0019)
WFC	3.0723 (0.0021)	3.1620 (0.0016)	2.4755 (0.0133)	3.2976 (0.0010)

The table reports, for each stock (the ticker is given in the first column), the values of the likelihood ratio test (the p-values are given in brackets) proposed by Amisano and Giacomini (2007), for different weights. Each weight places greater emphasis on particular regions of the distribution: center (AG_{CE}), right tail (AG_{RT}) and left tail (AG_{LT}). AG_{NW} coincides with the unweighted likelihood ratio test.

We estimated volatility using the realized range-based bias corrected bipower variation introduced by Christensen et al. (2009), which is a consistent estimator of the integrated variance in the presence of microstructure noise and jumps in the context of high-frequency data. Our analyses considered sixteen companies that operate in a variety of sectors in the U.S. market, and the results provide evidence of relevant impacts by the explanatory variables. In particular, the lagged values of the estimated volatility and the S&P 500 return were critical indicators in the context of extreme events, where volatility can reach considerably high levels. These two regressors were highly significant in terms of their ability to explain the high quantiles of volatility. Moreover, the test Koenker and Bassett (1982a) introduced allowed us to reject the location-shift hypothesis, highlighting the heterogeneous impacts of the regressors across quantiles.

In order to assess the evolution of the relationships among the variables over time, we carried out a rolling analysis with steps of one day and subsamples consisting of 500 observations. Thus, verified that two special events, the U.S. subprime crisis and the European sovereign debt crisis, have affected those relationships. In particular, acute sensitivity was recorded at high levels of quantiles. Finally, the tests developed by Berkowitz (2001), Amisano and Giacomini (2007) and Diebold and Mariano (2002) validated the forecast

Table 3.10 Diebold-Mariano test for the single asset analysis.

Asset	$DM_{0.1}$	$DM_{0.5}$	$DM_{0.9}$
ATT	-1.9227 (0.0545)	-2.9818 (0.0029)	-3.0612 (0.0022)
BAC	-1.6373 (0.1016)	-1.9346 (0.0530)	-1.8357 (0.0664)
BOI	-1.8368 (0.0662)	-3.2551 (0.0011)	-3.3817 (0.0007)
CAT	-2.2598 (0.0238)	-3.2398 (0.0012)	-2.8698 (0.0041)
CTG	-1.8642 (0.0623)	-1.9876 (0.0469)	-1.7358 (0.0826)
FDX	-2.0123 (0.0442)	-2.6834 (0.0073)	-3.1223 (0.0018)
HON	-1.8906 (0.0587)	-2.3123 (0.0208)	-3.2590 (0.0011)
HPQ	-1.4229 (0.1548)	-3.5429 (0.0004)	-2.5960 (0.0094)
IBM	-2.6321 (0.0085)	-4.2343 (0.0000)	-5.7195 (0.0000)
JPM	-1.9990 (0.0456)	-2.0265 (0.0427)	-2.3821 (0.0172)
MDZ	-2.8359 (0.0046)	-4.2537 (0.0000)	-4.6886 (0.0000)
PEP	-2.4810 (0.0131)	-4.7986 (0.0000)	-4.1165 (0.0000)
PRG	-3.3287 (0.0009)	-5.0239 (0.0000)	-5.9189 (0.0000)
TWX	-1.6401 (0.1010)	-2.7771 (0.0055)	-2.0985 (0.0359)
TXN	-1.4331 (0.1518)	-3.0835 (0.0020)	-2.0368 (0.0417)
WFC	-1.6855 (0.0919)	-1.7348 (0.0828)	-2.3336 (0.0196)

The table reports, for each stock (the ticker is given in the first column), the values of the likelihood ratio test (the p-values are given in brackets) proposed by Diebold and Mariano (2002), at $\tau = \{0.1, 0.5, 0.9\}$.

performances, even in periods of financial turmoil. We compared our approach with a HARX-GJR model, which combines a HAR structure on the realized volatility mean with additional exogenous variables, and a GJR-GARCH (Glosten et al., 1993) for the mean innovation variances. The results give evidence of a superior performance of our approach.

Our findings provide supporting evidence for the use of quantile regression methods for the quantile forecasts and for the density forecast of the realized range volatility. The improvement over traditional methods is marked and will be relevant in all areas where volatility quantile values and volatility density forecast play a role. The use of predicted volatility levels is central, for instance, in the pricing of equity derivatives, in the development of equity derivative trading strategies, and in risk measurement when risk is associated with volatility, while the volatility distribution is of interest for trading/pricing volatility derivatives, for designing volatility hedges for generic portfolios, and for accounting for the uncertainty on volatility point forecasts.

In the present paper we worked with multiple assets volatilities, inside an univariate framework. We left to future works the generalization of our approach to a multivariate version and its integration in returns-based volatility models where realized volatility measures represent further conditioning information sources.

Chapter 4

The Dynamic Impact of Uncertainty in Causing and Forecasting the Distribution of Oil Returns and Risk

4.1 Oil movements and uncertainty: an introductory discussion

Following the seminal work of Hamilton (1983), a large literature exists that connects movements in oil returns and its volatility with recessions and inflationary episodes in the US economy (e.g., see Elder and Serletis (2010), Kang and Ratti (2013a,b), Antonakakis et al. (2014) for detailed reviews). Hamilton (2008) indicates that nine of ten recessions in the US since World War II have been preceded by an increase in oil prices (Hamilton, 2008). Interestingly, Hamilton (2009) even goes as far as arguing that a large proportion of the recent downturn in the US GDP during the “Great Recession” can also be attributed to the oil price shock of 2007-2008.

In turn, this implies that it is of paramount importance to determine the variables that drives the oil market to properly model and forecast, both returns and volatility of oil spot prices. In this regard, a recently growing literature emphasizes the role of economic policy uncertainty on real activity (e.g., see Bloom (2009), Colombo (2013), Jones and Olson (2013), Mumtaz and Zanetti (2013), Karnizova and Li (2014), Jurado et al. (2015) for detailed reviews), which, in turn, affects oil-price movements (Aloui et al., 2015; Antonakakis et al., 2014; Kang and Ratti, 2013a,b). Equity-market uncertainty also feeds into oil-price movements because, as Bloom (2009)’s firm-based theoretical framework notes, equity-

market uncertainty affects hiring and investment and, hence, production decisions of firms. In this regard, empirical evidence relating oil price movements and stock market volatility can be found in Kang et al. (2015).

Against this backdrop, the objective of this paper is to analyse whether recently developed news-based measures of economic policy uncertainty (EPU) and equity market uncertainty (EMU) by Baker et al. (2013) can predict, both in- and out-of-sample, returns and volatility of oil. Realizing the possibility that the oil market is also likely to drive these uncertainties (see e.g. Kang and Ratti (2013a,b), Antonakakis et al. (2014)), we employ a modified bi-variate quantile causality-based model (for prediction and forecasting, as developed by Balcilar et al. (2015)), which combines the causality in quantile test of Jeong et al. (2012), with the k -th order nonparametric Granger causality test of Nishiyama et al. (2011) for our purpose, using daily data on oil returns, EPU and EMU, covering the period 02/01/1986-23/04/2015.

Conditional mean-based evidence of EPU (mildly and negatively), affecting oil price from structural vector autoregressive (SVAR) models, can be found in Kang and Ratti (2013a,b) and Antonakakis et al. (2014), and confirmed using copula models by Aloui et al. (2015). To the best of our knowledge, our paper is the first attempt to analyse the importance of both EPU and EMU in forecasting both in- and out-of-sample oil returns and its volatility over the entire conditional distribution of oil returns and volatility. The nonparametric causality in quantile test employed in our study for both in-sample and out-of-sample forecasting has the following novelties: first, the test is robust to functional misspecification errors and can detect general dependence between time series. This is particularly important in our application, since it is well known that high-frequency data display nonlinear dynamics.¹ Second, the test statistic does not only test for causality in the mean, it also tests for causality that may exist in the tail area of the joint distribution of the series.² Third, the test easily lends itself to test for causality in variance. Testing for causality in variance allows us to test for the volatility spillover phenomenon, since, at times, causality in conditional mean (first moment) may not exist, but there may be second or higher order causality. Moreover, the use of quantile-based methods, allows analysing the causality structure depending on the volatility state (high versus low). Understandably, given the structure of the model employed, it is easily tenable to forecasting the entire conditional distribution out-of-sample for both oil returns and volatility,

¹The Brock et al. (1996) test applied to the residuals recovered from autoregressive models fitted to oil returns and natural logarithms of EPU and EMU, as well as to the vector autoregressive models comprising of oil returns and logarithms of EPU or EMU, reject the null hypothesis of serial dependence at 1% level of significance across various dimensions. These results provide strong evidence of nonlinearity in the data. Complete details of these tests are available upon request from the authors.

²Our data showed that oil returns is skewed to the left while, EPU and EMU are skewed to the right, with all the three variables having non-normal distributions. Complete details on the summary statistics of the three variables are available upon request from the authors.

using a recursive or rolling estimation of the model. Besides, the evaluation of asymmetric effects produced by EPU and EPU on the oil movements is important in revealing the states where uncertainty assumes critical relevance. At this stage, however, two related papers require mentioning. First, Bekiros et al. (2015) analysed the importance of the EPU in forecasting oil returns over the in- and post crisis periods, using a wide variety of constant parameter and time-varying parameter VAR models. The authors depict that EPU matters in point forecasts of oil returns, but only when one allows for time-varying parameters (with stochastic volatility in the error structure) in the VAR model. And second, Balcilar et al. (2015) who develops the framework we use in this paper, used the model to analyse in-sample causality running from EPU and EMU to oil returns and volatility. They concluded that, for oil returns, EPU and EMU has strong predictive power over the entire distribution barring regions around the median, but for volatility, the predictability virtually covers the entire distribution, with some exceptions in the tails. Our contribution primarily involves extending the paper by Balcilar et al. (2015) to out-of-sample density forecasting of oil returns and its volatility using a rolling window scheme. Note that, in the process, we are also able to provide a time-varying approach to the in-sample quantile causality for both oil returns and its volatility. This is important, given that we detect breaks in their respective conditional distributions, and hence, full-sample quantile causality could be possibly misleading. Further, unlike Bekiros et al. (2015), where the authors only concentrate on point forecast of oil returns, we are able to analyse density forecast for both returns and volatility of oil returns. This again is more informative than point forecasts, since we are able to understand the role of EPU and EMU in forecasting oil returns and volatility at different phases (bearish, normal and bullish) of the oil market. Hence, our contribution primarily involves looking at out of-sample density forecasts for oil returns and its volatility using the information content of measures of policy and equity market uncertainties at the highest possible (daily) frequency. The importance of our contribution can be justified by the suggestion made by Campbell (2008): “The ultimate test of any predictive model is its out-of-sample performance”. The rest of the paper is organized as follows: Section 4.2 presents the details of the methodologies pursued, while in Section 4.3 we describe the data and the rolling window procedure. Section 4.4 presents the results and Section 4.5 concludes, with an economic discussion of the results obtained.

4.2 Causality and forecasting methods

4.2.1 Causality in quantiles

Let $\{y_t\}_{t \in T}$ be the time series of the oil returns. Differently, the logarithm of the Economic Policy Uncertainty (EPU) and the logarithm of the Equity Market Uncertainty (EMU) are denoted, respectively, as $\{x_{1,t}\}_{t \in T}$ and $\{x_{2,t}\}_{t \in T}$.³ In the present section we describe a method which studies the causality relations in a bivariate framework. For simplicity of notation, in the following we use x_t in place of $x_{1,t}$ ($x_{2,t}$) when we study the causality implications of EPU (EMU) on y_t . First, we make use of the following notations: $\mathcal{Y}_{t-1} \equiv (y_{t-1}, \dots, y_{t-p})$, $\mathcal{X}_{t-1} \equiv (x_{t-1}, \dots, x_{t-q})$, $\mathcal{Z}_{t-1} \equiv (y_{t-1}, \dots, y_{t-p}, x_{t-1}, \dots, x_{t-q})$, for $(p, q) > 1$; moreover, we denote by $F_{y_t|\mathcal{Z}_{t-1}}(y_t|\mathcal{Z}_{t-1})$ and $F_{y_t|\mathcal{Y}_{t-1}}(y_t|\mathcal{Y}_{t-1})$ the distributions of y_t , conditional on \mathcal{Z}_{t-1} and \mathcal{Y}_{t-1} , respectively. We assume the distribution of y_t is absolutely continuous in y for almost all $v = (\mathcal{Y}, \mathcal{Z})$. To simplify the notation, for any $\tau \in (0, 1)$, we denote by $Q_\tau(\mathcal{Z}_{t-1}) \equiv Q_\tau(y_t|\mathcal{Z}_{t-1})$ and $Q_\tau(\mathcal{Y}_{t-1}) \equiv Q_\tau(y_t|\mathcal{Y}_{t-1})$ the τ -th quantiles of y_t conditional to \mathcal{Z}_{t-1} and \mathcal{Y}_{t-1} , respectively.

Granger (1989) defines the causality in mean (the well-known Granger causality) by means of a comparison between expected values computed conditioning to two different sets. We thus say that x_t does not cause y_t in mean with respect to \mathcal{Z}_{t-1} if $\mathbb{E}[y_t|\mathcal{Z}_{t-1}] = \mathbb{E}[y_t|\mathcal{Y}_{t-1}]$. Differently, x_t is a prima facie cause in mean of y_t with respect to \mathcal{Z}_{t-1} if $\mathbb{E}[y_t|\mathcal{Z}_{t-1}] \neq \mathbb{E}[y_t|\mathcal{Y}_{t-1}]$.

To define the Granger causality in quantiles, we follow Jeong et al. (2012) that defines the causality as follows: x_t does not cause y_t in its τ -th quantile, with respect to \mathcal{Z}_{t-1} , if $Q_\tau(\mathcal{Z}_{t-1}) = Q_\tau(\mathcal{Y}_{t-1})$. On the other hand, x_t is a prima facie cause in the τ -th quantile of y_t , with respect to \mathcal{Z}_{t-1} , if $Q_\tau(\mathcal{Z}_{t-1}) \neq Q_\tau(\mathcal{Y}_{t-1})$.

The quantile causality definition leads to the identification of hypotheses we could test. Our interest lies in the detection of casuality, and, similarly to the tests for Granger causality, we associate the null hypothesis to the absence of causality. As a result, the system hypotheses to be tested is:

$$\begin{cases} H_0 : P[F_{y_t|\mathcal{Z}_{t-1}}(Q_\tau(\mathcal{Y}_{t-1})|\mathcal{Z}_{t-1}) = \tau] = 1 \\ H_1 : P[F_{y_t|\mathcal{Z}_{t-1}}(Q_\tau(\mathcal{Y}_{t-1})|\mathcal{Z}_{t-1}) = \tau] < 1 \end{cases} \quad (4.1)$$

In order to test the hypotheses given in (4.1), Jeong et al. (2012) suggest the use of the a specific distance measure:

³Detailed descriptions of the oil returns, EPU and EMU are given in Section 4.3.1.

$$J_T = \mathbb{E} \left[[F_{y_t|\mathcal{Z}_{t-1}}(Q_\tau(\mathcal{Y}_{t-1})|\mathcal{Z}_{t-1}) - \tau]^2 g_{\mathcal{Z}_{t-1}}(\mathcal{Z}_{t-1}) \right], \quad (4.2)$$

where $g_{\mathcal{Z}_{t-1}}(\mathcal{Z}_{t-1})$ denotes the marginal density function of \mathcal{Z}_{t-1} . Notably, $J_T \geq 0$ with equality holding if H_0 in (4.1) is true, while under the alternative H_1 we have a strict inequality.

Jeong et al. (2012) proposed the evaluation of the distance function by the following feasible kernel-based estimator:

$$\hat{J}_T = \frac{1}{T(T-1)h^m} \sum_{t=1}^T \sum_{s \neq t} K \left(\frac{\mathcal{Z}_{t-1} - \mathcal{Z}_{s-1}}{h} \right) \tilde{\varepsilon}_t \tilde{\varepsilon}_s, \quad (4.3)$$

where $m = p + q$, $K(\cdot)$ is the kernel function with bandwidth h , whereas $\tilde{\varepsilon}_t$ is defined as

$$\tilde{\varepsilon}_t = \mathbf{1}_{\{y_t \leq \tilde{Q}_\tau(\mathcal{Y}_{t-1})\}} - \tau, \quad (4.4)$$

with $\mathbf{1}_{\{\cdot\}}$ denoting the indicator function taking value 1 if the condition in $\{\cdot\}$ is true and zero otherwise.

Jeong et al. (2012), set $\tilde{Q}_\tau(\mathcal{Y}_{t-1})$ equal to $\tilde{F}_{y_t|\mathcal{Y}_{t-1}}^{-1}(\tau|\mathcal{Y}_{t-1})$, where

$$\tilde{F}_{y_t|\mathcal{Y}_{t-1}}(y_t|\mathcal{Y}_{t-1}) = \frac{\sum_{s \neq t} C_{t-1,s-1} \mathbf{1}_{\{y_s \leq y_t\}}}{\sum_{s \neq t} C_{t-1,s-1}} \quad (4.5)$$

is the Nadaraya-Watson kernel estimator of $F_{y_t|\mathcal{Y}_{t-1}}(y_t|\mathcal{Y}_{t-1})$, with the kernel function $C_{t-1,s-1} = C(\mathcal{Y}_{t-1} - \mathcal{Y}_{s-1})/a$, and a is the bandwidth.

Finally, Jeong et al. (2012) prove that, given $\sigma_\varepsilon^2(\mathcal{Z}_{t-1}) = \tau(1 - \tau)$ and a set of additional assumptions, $Th^{m/2} \hat{J}_T \xrightarrow{L} \mathcal{N}(0, \sigma_0^2)$, where

$$\sigma_0^2 = 2\mathbb{E} \left[\sigma_\varepsilon^4(\mathcal{Z}_{t-1}) g_{\mathcal{Z}_{t-1}}(\mathcal{Z}_{t-1}) \right] \left(\int K^2(u) du \right). \quad (4.6)$$

Therefore, the test for the presence of causality in a given quantile corresponds to a significance test of the quantity in (4.3) whose standard error depends on the sample estimator of the variance reported in equation (4.6).

More recently, Balcilar et al. (2015) extended the approach introduced by Jeong et al. (2012), and developed a test for the causality in quantiles but with a focus on the second moment of y_t . This novel approach allows testing for the presence of quantile causality when considering the density of the risk or of the dispersion characterising the variable y_t . We refer to this type of causality as quantile causality in variance.

Balcilar et al. (2015) start from the work Nishiyama et al. (2011), where the process governing $\{y_t\}_{t \in T}$ takes the following form

$$y_t = \gamma(\mathcal{Y}_{t-1}) + \rho(\mathcal{X}_{t-1}) + \zeta_t, \quad (4.7)$$

with ζ_t being a white noise process, whereas $\gamma(\cdot)$ and $\rho(\cdot)$ are unknown functions satisfying conditions ensuring stationarity of y_t .

Balcilar et al. (2015) noticed that the specification in (4.7) does not allow for Granger-type causality testing from x_t to y_t , but could possibly detect the predictive power from x_t to y_t^2 , when $\rho(\cdot)$ is a general nonlinear function. Therefore, the model allows deriving a general test for quantile causality in variance, where the impact does not need to come from squared values of \mathcal{X}_{t-1} . Notably, the test could detect an impact from levels or non-linear transformations of \mathcal{X}_{t-1} to the squared values of y_t , where the squared of y_t is just a proxy of the conditional variance of y_t . Clearly, this corresponds to an implicit assumption of heteroskedasticity with the variance driven by (transformed) lagged values of x_t (and y_t).

To introduce a test for quantile causality in variance, Balcilar et al. (2015) reformulate Equation (4.7) as follows:

$$\begin{cases} H_0 : P[F_{y_t^2 | \mathcal{X}_{t-1}}(Q_\tau(\mathcal{Y}_{t-1}) | \mathcal{X}_{t-1}) = \tau] = 1 \\ H_1 : P[F_{y_t^2 | \mathcal{X}_{t-1}}(Q_\tau(\mathcal{Y}_{t-1}) | \mathcal{X}_{t-1}) = \tau] < 1 \end{cases} \quad (4.8)$$

In order to test the hypotheses in (4.8), Balcilar et al. (2015) proposed using the test statistic \widehat{J}_T , replacing y_t by y_t^2 , and preserving the same asymptotic distribution.

In our empirical analyses, for both quantile causality in mean and variance, we computed \widehat{J}_T making use of the Gaussian kernel for both $K(\cdot)$ and $C(\cdot)$, with bandwidths obtained through the least squares cross-validation method (Balcilar et al., 2015; Jeong et al., 2012).

4.2.2 Quantiles and density forecasting

While Section 4.2.1 focuses on the presence of quantile causality, the present Section introduces methods aiming at the forecasting implications of EMU and EPU. As mentioned in the introduction, one of the research questions points at the evaluation of the potentially different impact of the two uncertainty indexes. A forecasting exercise allows for a direct comparison of the two, which can be introduced jointly in a model, allowing for testing on their statistical impact as well as for their forecasting impact. We thus introduce in the forecasting exercise both EMU and EPU, whose joint impact could provide important implications given that they quantify two different sources of uncertainty. In particular,

we aim to forecast both the conditional quantiles and distributions of y_t and y_t^2 taking into account the information associated with the two indexes.

The first step consists in estimating the conditional quantiles and, for this purpose, we make use of the quantile regression approach introduced by Koenker and Bassett (1978). We first remind that $\mathbf{1}_{\{\cdot\}}$ is the indicator function taking value 1 if the condition in $\{\cdot\}$ is true, 0 otherwise. The approach introduced by Koenker and Bassett (1978) makes use of the asymmetric loss function

$$\rho_\tau(\varepsilon) = \varepsilon[\tau - \mathbf{1}_{\{\varepsilon < 0\}}]. \quad (4.9)$$

Starting from the case of y_t and given $\mathscr{W}_{t-1} \equiv (y_{t-1}, \dots, y_{t-p}, x_{1,t-1}, \dots, x_{1,t-q}, x_{2,t-1}, \dots, x_{2,t-r})$, for $(p, q, r) > 1$, Koenker and Bassett (1978) showed that the minimizer, $Q_\tau(y_t | \mathscr{W}_{t-1})$, of the expected loss

$$\mathbb{E}[\rho_\tau(y_t - Q_\tau(y_t | \mathscr{W}_{t-1}))] \quad (4.10)$$

satisfies $F_Y(Q_\tau(y_t | \mathscr{W}_{t-1})) - \tau = 0$, where $Q_\tau(y_t | \mathscr{W}_{t-1})$, the conditional τ -th quantile of y_t , is equal to

$$\begin{aligned} Q_\tau(y_t | \mathscr{W}_{t-1}) &= \alpha_0(\tau) + \beta_1(\tau)y_{t-1} + \dots + \beta_p(\tau)y_{t-p} + \delta_1(\tau)x_{1,t-1} \\ &+ \dots + \delta_q(\tau)x_{1,t-q} + \lambda_1(\tau)x_{2,t-1} + \dots + \lambda_r(\tau)x_{2,t-r}. \end{aligned} \quad (4.11)$$

The unknown parameters in Equation (4.11) are estimated by minimizing equation (4.10). Then, with a horizon of one period ahead, the forecast of the τ -th conditional quantile of y_t is computed as

$$\begin{aligned} \widehat{Q}_\tau(y_{t+1} | \mathscr{W}_t) &= \widehat{\alpha}_0(\tau) + \widehat{\beta}_1(\tau)y_t + \dots + \widehat{\beta}_p(\tau)y_{t-p+1} + \widehat{\delta}_1(\tau)x_{1,t} \\ &+ \dots + \widehat{\delta}_q(\tau)x_{1,t-q+1} + \widehat{\lambda}_1(\tau)x_{2,t} + \dots + \widehat{\lambda}_r(\tau)x_{2,t-r+1}. \end{aligned} \quad (4.12)$$

After obtaining a grid of forecasted quantiles, computed at different τ values, the second step consists in forecasting the conditional distribution of the oil returns. The standard quantile regression approach allows estimating individual quantiles, but it does not guarantee their coherence, i.e. their increasing monotonicity in $\tau \in (0, 1)$. For instance, it might occur that the predicted 95-th percentile of the response variable is lower than the 90-th percentile. If quantiles cross, corrections must be applied in order to obtain a valid conditional distribution of volatility. For instance, in order to cope with the crossing problem, Koenker (1984) applied parallel quantile planes, whereas Bondell et al. (2010) estimated the quantile regression coefficients with a constrained optimization method.

Here we follow a different approach, proposed by Zhao (2011). Given a collection of ϑ predicted conditional quantiles ($\widehat{Q}_{\tau_1}(y_{t+1} | \mathscr{W}_t), \dots, \widehat{Q}_{\tau_\vartheta}(y_{t+1} | \mathscr{W}_t)$), for $0 < \tau_j < \tau_{j+1} < 1$,

$j = 1, \dots, \vartheta - 1$, we first rearrange them into ascending order, by making use of the quantile bootstrap method proposed by Chernozhukov et al. (2010). Then, starting from the rearranged quantiles, denoted by $(Q_{\tau_1}^*(y_{t+1}|\mathscr{W}_t), \dots, Q_{\tau_\vartheta}^*(y_{t+1}|\mathscr{W}_t))$, we estimate the entire conditional distribution with a nonparametric kernel method. The predicted density equals

$$\widehat{f}_{y_{t+1}|\mathscr{W}_t}(y^*|\mathscr{W}_t) = \frac{1}{\vartheta h_\vartheta} \sum_{i=1}^{\vartheta} K_e \left(\frac{y^* - Q_{\tau_i}^*(y_{t+1}|\mathscr{W}_t)}{h_\vartheta} \right), \quad (4.13)$$

where y^* are evenly interpolated points that generates the support of the estimated distribution, h_ϑ is the bandwidth, $K_e(\cdot)$ is the kernel function, and $\widehat{f}(\cdot|\mathscr{W}_t) \equiv \widehat{f}_{y_{t+1}|\mathscr{W}_t}(\cdot|\mathscr{W}_{t-1})$ is the one-period ahead forecasted density, given the information set available in t . Following Gaglianone and Lima (2012), we use as $K_e(\cdot)$ the Epanechnikov kernel.

Given $\mathscr{W}_{2,t-1} \equiv (y_{t-1}^2, \dots, y_{t-p}^2, x_{1,t-1}, \dots, x_{1,t-q}, x_{2,t-1}, \dots, x_{2,t-r})$, for $(p, q, r) > 1$, the conditional distribution of y_t^2 , denoted as $\widehat{f}_{y_{t+1}^2|\mathscr{W}_{2,t}}(\cdot|\mathscr{W}_{2,t})$, is obtained by applying the same methodology described above, by replacing $Q_{\tau_i}^*(y_{t+1}|\mathscr{W}_t)$ by $Q_{\tau_i}^*(y_{t+1}^2|\mathscr{W}_{2,t})$. Specifically, $Q_{\tau}^*(y_{t+1}^2|\mathscr{W}_{2,t})$ is the conditional τ -th quantile of y_t^2 , adjusted for the crossing quantiles issue, arising from the original one $Q_{\tau}(y_{t+1}^2|\mathscr{W}_{2,t})$. We estimated the latter as

$$\begin{aligned} \widehat{Q}_{\tau}(y_{t+1}^2|\mathscr{W}_{2,t}) &= \widehat{\alpha}_0(\tau) + \widehat{\beta}_1(\tau)y_t^2 + \dots + \widehat{\beta}_p(\tau)y_{t-p+1}^2 + \widehat{\delta}_1(\tau)x_{1,t} \\ &+ \dots + \widehat{\delta}_q(\tau)x_{1,t-q+1} + \widehat{\lambda}_1(\tau)x_{2,t} + \dots + \widehat{\lambda}_r(\tau)x_{2,t-r+1}. \end{aligned} \quad (4.14)$$

We compute the coefficients standard errors through the bootstrap method (Efron, 1979), whose advantages are well-known: it assumes no particular distribution of the errors, it is not based on asymptotic model properties and it is available regardless of the statistic of interest's complexity. Among all the available bootstrapping methods, we make use of the xy -pair method (Kocherginsky, 2003), whose advantages for quantile regression problems are highlighted in Davino et al. (2014).

4.2.3 Evaluation of the predictive accuracy

We evaluate the predictive accuracy of the method described in Section 4.2.2 by using five testing approaches, introduced, respectively, by Berkowitz (2001), Diebold and Mariano (2002), Amisano and Giacomini (2007), Diks et al. (2011), Gneiting and Ranjan (2011). In the following, we give the main details about the five tests below, focusing on the conditional quantiles and distribution of y_t . The same methodology applies to y_t^2 with $Q_{\tau}^*(y_t^2|\mathscr{W}_{2,t})$ replacing $Q_{\tau}^*(y_t|\mathscr{W}_t)$. We also remind that the forecast evaluation takes as input a collection of one-step-ahead forecasts.

As regards the Berkowitz (2001) test, we first compute the variable

$$\psi_{t+1} = \int_{-\infty}^{y_{t+1}} \hat{f}(u|\mathcal{W}_t) du = \hat{F}(y_{t+1}|\mathcal{W}_t), \quad (4.15)$$

where $\hat{F}(y_{t+1}|\mathcal{W}_t)$ is the distribution function corresponding to the density $\hat{f}(y_{t+1}|\mathcal{W}_t)$; ψ_{t+1} is computed sequentially M times, where $M < T$ is the number of periods included in the interval spanning the forecasting evaluation.

Rosenblatt (1952) showed that, if the model is correctly specified, ψ_{t+1} is i.i.d. and uniformly distributed on $(0, 1)$; that result holds regardless of the y_t distribution, even if $\hat{F}(\cdot|\mathcal{W}_t)$ changes over time. Berkowitz (2001) observed that, if $\psi_{t+1} \sim \mathcal{U}(0, 1)$, then

$$z_{t+1} = \Phi^{-1}(\psi_{t+1}) \sim \mathcal{N}(0, 1), \quad (4.16)$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal distribution function.

Given that, under correct model specification, z_{t+1} should be independent and identically distributed as standard normal, an alternative hypothesis is that the mean and the variance differ from 0 and 1, respectively, with a first-order autoregressive structure. In particular, Berkowitz (2001) considered the model

$$z_{t+1} - \mu_b = \rho_b(z_t - \mu_b) + e_{t+1} \quad (4.17)$$

to test the null hypothesis $H_0 : \mu_b = 0, \rho_b = 0, \text{var}(e_{t+1}) = \sigma_b^2 = 1$, which corresponds to the appropriate specification of the density forecasting model. The test built on (4.17) is based on the likelihood-ratio statistic

$$LR_b = -2 [L_b(0, 1, 0; z_{t+1}) - L_b(\hat{\mu}_b, \hat{\sigma}_b, \hat{\rho}_b; z_{t+1})], \quad (4.18)$$

where $L_b(\hat{\mu}_b, \hat{\sigma}_b, \hat{\rho}_b; z_{t+1})$ is the likelihood function associated with Equation (4.17) and computed from the maximum-likelihood estimates of the unknown parameters μ_b , σ_b and ρ_b . Under the null hypothesis H_0 , the test statistic is distributed as $\chi^2(3)$.

First of all, we use the Berkowitz test for an absolute assessment of the density forecasts recovered from (4.13). Then, we also implement the test on a restricted model, i.e. the one which uses in Equation (4.11) just $\mathcal{Y}_{t-1} \equiv (y_{t-1}, \dots, y_{t-p})$ as predictors; we denote by $\hat{f}(\cdot|\mathcal{Y}_t)$ the density we forecast on the basis of the restricted model. Hence, we can assess the joint contribution of EPU and EMU in predicting the distribution of the oil returns by comparing the LR_b values arising from the unrestricted and the restricted models. We also evaluate the contribution of each uncertainty index separately, by adding to the restricted model just the

lagged values of $x_{1,t}$ when we focus on EPU, or the lagged values of $x_{2,t}$, when we consider EMU.

The approach proposed by Berkowitz (2001) evaluates the goodness of a specific sequence of density forecasts, relative to the unknown data-generating process. However, given a certain model, the Berkowitz test has power only for misspecifications of the first two moments, but in practice, that model could be misspecified at higher-order moments. In that case, a valid solution consists in comparing density forecasts, i.e. performing a relative comparison given a specific measure of accuracy. Hence, in addition to the approach proposed by Berkowitz (2001), we also consider the tests introduced by Diebold and Mariano (2002), Amisano and Giacomini (2007), Diks et al. (2011), Gneiting and Ranjan (2011).

We implement the test developed by Diebold and Mariano (2002) on the basis of the losses generated by the unrestricted and the restricted models, denoted by $L_{\tau,t+1}(y_{t+1}|\mathcal{W}_t)$ and $L_{\tau,t+1}(y_{t+1}|\mathcal{Y}_t)$, respectively. Among the various loss functions adopted in the literature, following Giglio et al. (2012), we make use of those defined as follows:

$$L_{\tau,t+1}(y_{t+1}|\mathcal{W}_t) = (\tau - \mathbf{1}_{\{y_{t+1} - Q_{\tau}^*(y_{t+1}|\mathcal{W}_t) < 0\}}) [y_{t+1} - Q_{\tau}^*(y_{t+1}|\mathcal{W}_t)], \quad (4.19)$$

$$L_{\tau,t+1}(y_{t+1}|\mathcal{Y}_t) = (\tau - \mathbf{1}_{\{y_{t+1} - Q_{\tau}^*(y_{t+1}|\mathcal{Y}_t) < 0\}}) [y_{t+1} - Q_{\tau}^*(y_{t+1}|\mathcal{Y}_t)]. \quad (4.20)$$

Given the loss differential,

$$d_{DM,\tau,t+1} = L_{\tau,t+1}(y_{t+1}|\mathcal{W}_t) - L_{\tau,t+1}(y_{t+1}|\mathcal{Y}_t), \quad (4.21)$$

which we evaluate for all the periods included in $[t + 1, t + M]$, we compute its average value, denoted by $\overline{d_{DM,\tau}}$.

We are interested in testing the null hypothesis $H_0 : E[\overline{d_{DM,\tau}}] = 0$ against the alternative $H_1 : E[\overline{d_{DM,\tau}}] \neq 0$. For that purpose, Diebold and Mariano (2002) proposed the test statistic:

$$DM_{\tau} = \frac{\overline{d_{DM,\tau}}}{\hat{\sigma}_{DM}/\sqrt{M}}, \quad (4.22)$$

where $\hat{\sigma}_{DM}^2$ is a consistent estimate of $\sigma_{DM}^2 = Var(\sqrt{M} \overline{d_{DM,\tau}})$, the asymptotic (long-run) variance. Diebold and Mariano (2002) showed that, under the null hypothesis of equal predictive accuracy, $DM_{\tau} \xrightarrow{d} \mathcal{N}(0, 1)$; in case the null hypothesis is rejected and the (4.22) takes negative values, we have evidence for the unrestricted model having better performance. To evaluate the DM_{τ} test statistic, we focus on selected quantiles, setting $\tau = \{0.05, 0.5, 0.95\}$. In this way, we do not evaluate the entire density forecast, but only the impact of uncertainty indexes on selected quantiles.

The next tests focus on the entire density forecast, thus allowing for a much broader evaluation of the uncertainty indexes relevance. The approach introduced by Amisano and Giacomini (2007) compares two different competing models on the basis of their *log-scores*. In particular, the *log-scores* of the unrestricted and the restricted models are denoted by $\log(\hat{f}(y_{t+1}|\mathcal{W}_t))$ and $\log(\hat{f}(y_{t+1}|\mathcal{Y}_t))$, respectively. Given a sequences of density forecasts, it is possible to compute the quantity defined as

$$WLR_{t+1} = w(y_{t+1}^{st}) \left[\log(\hat{f}(y_{t+1}|\mathcal{W}_t)) - \log(\hat{f}(y_{t+1}|\mathcal{Y}_t)) \right], \quad (4.23)$$

where y_{t+1}^{st} is the standardized oil return in $t + 1$, whereas $w(y_{t+1}^{st})$ is the weight the forecaster arbitrarily chooses to emphasize particular regions of the distribution's support.

After computing WLR_{t+1} for the M periods included in the interval spanning the forecast evaluation, we evaluate its mean, which we denoted by \overline{WLR} . In order to test the null hypothesis of equal performance, that is, $H_0 : E[\overline{WLR}] = 0$, against the alternative of a different predictive ability $H_1 : E[\overline{WLR}] \neq 0$, Amisano and Giacomini (2007) suggest the use of a weighted likelihood ratio test:

$$AG = \frac{\overline{WLR}}{\hat{\sigma}_{AG}/\sqrt{M}}, \quad (4.24)$$

where $\hat{\sigma}_{AG}^2$ is a heteroskedasticity- and autocorrelation-consistent (HAC) Newey and West (1987) estimator of $\sigma_{AG}^2 = \text{Var}(\sqrt{M} \overline{WLR})$. AG is positive in case of a better performance of the unrestricted model, otherwise it takes negative values. Amisano and Giacomini (2007) showed that, under the null hypothesis, $AG \xrightarrow{d} \mathcal{N}(0, 1)$.

We applied the Amisano and Giacomini (2007) test by using four designs for the weights entering Equation (4.23), in order to verify how the results change according to the particular regions of the distribution's support on which we are focusing. We set $w_{CE}(y_{t+1}^{st}) = \phi(y_{t+1}^{st})$ to give an higher weight to the center of the distribution, $w_{RT}(y_{t+1}^{st}) = \Phi(y_{t+1}^{st})$ when we focus more on the right tail, $w_{LT}(y_{t+1}^{st}) = 1 - \Phi(y_{t+1}^{st})$ for the left tail, and $w_{NW}(y_{t+1}^{st}) = 1$ when giving equal importance to the entire support.⁴

As noticed by Diks et al. (2011) and Gneiting and Ranjan (2011), the weighted logarithmic scoring rule as in Amisano and Giacomini (2007) favors density forecasts with more probability mass in the region of interest and, as a result, the resulting test of equal predictive ability is biased toward such density forecasts; hence, they proposed different scores to solve this shortcoming of the Amisano and Giacomini (2007) test.

⁴Note that $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density function and the standard normal distribution function, respectively.

In the case of the unrestricted model, the score introduced by Diks et al. (2011) is defined as

$$S^{csl}(y_{t+1}|\mathcal{W}_t) = w_{csl,t}(y_{t+1}) \log \hat{f}(y_{t+1}|\mathcal{W}_t) + (1 - w_{csl,t}(y_{t+1})) \log \left[1 - \int w_{csl,t}(s) \hat{f}(s|\mathcal{W}_t) ds \right], \quad (4.25)$$

where $w_{csl,t}(\cdot)$ is the weighting function, by which we focus on the density's region of interest, whereas the second addend in (4.25) avoids the mistake of attaching comparable scores to density forecasts that have similar tail shapes but may have completely different tail probabilities (Diks et al., 2011).

Let \bar{y}_1 and \bar{y}_3 be the in-sample first and third quartile of y_t , respectively, we set $w_{csl,t}(y_{t+1}) = \mathbf{1}_{\{y_{t+1} \leq \bar{y}_1\}}$ when we focus on the left tail, $w_{csl,t}(y_{t+1}) = \mathbf{1}_{\{\bar{y}_1 \leq y_{t+1} \leq \bar{y}_3\}}$ when we place the attention on the center of the distribution, $w_{csl,t}(y_{t+1}) = \mathbf{1}_{\{y_{t+1} \geq \bar{y}_3\}}$ when we consider the right tail.

Similarly, we denote the score function of the restricted model as $S^{csl}(y_{t+1}|\mathcal{B}_t)$, obtained by replacing \mathcal{W}_t by \mathcal{B}_t in (4.25).

Let \bar{S}^{csl} be the mean of the differences $S^{csl}(y_{t+1}|\mathcal{W}_t) - S^{csl}(y_{t+1}|\mathcal{B}_t)$, computed for all the periods included in the interval $[t + 1, t + M]$. The test statistic proposed by Diks et al. (2011) equals:

$$DPD = \frac{\bar{S}^{csl}}{\hat{\sigma}_{DPD} / \sqrt{M}}, \quad (4.26)$$

where $\hat{\sigma}_{DPD}^2$ is a heteroskedasticity- and autocorrelation-consistent (HAC) Newey and West (1987) estimator of $\sigma_{DPD}^2 = Var(\sqrt{M} \bar{S}^{csl})$. We have evidence of a better/worse performance of the unrestricted model when DPD takes positive/negative values. Diks et al. (2011) showed that, under the null hypothesis of equal performance, $DPD \xrightarrow{d} \mathcal{N}(0, 1)$.

Finally, again focusing on the unrestricted model, the score proposed by Gneiting and Ranjan (2011) is defined as follows:

$$S^{gr}(y_{t+1}|\mathcal{W}_t) = \frac{1}{I-1} \sum_{i=1}^I w(\tau_i) QS_{\tau_i} \left[\hat{F}^{-1}(\tau_i|\mathcal{W}_t), y_{t+1} \right], \quad (4.27)$$

where $\tau_i = i/I$ and

$$QS_{\tau_i} \left[\hat{F}^{-1}(\tau_i|\mathcal{W}_t), y_{t+1} \right] = 2 \left[\mathbf{1}_{\{y_{t+1} < \hat{F}^{-1}(\tau_i|\mathcal{W}_t)\}} - \tau_i \right] (\hat{F}^{-1}(\tau_i|\mathcal{W}_t) - y_{t+1}). \quad (4.28)$$

It is interesting to observe that the quantity defined in (4.28) is similar to the one in (4.19); nevertheless, the loss given in (4.27) is more informative than $L_{\tau,t+1}(y_{t+1}|\mathcal{W}_t)$, since

it is equal to the weighted average of several $QS_{\tau_i} \left[\widehat{F}^{-1}(\tau_i | \mathcal{W}_t), y_{t+1} \right]$ values computed for a sufficiently large grid of probabilities levels.

As for the weight function, as suggested by Gneiting and Ranjan (2011), we set $w(\tau_i) = \tau_i(1 - \tau_i)$, $w(\tau_i) = \tau_i^2$, $w(\tau_i) = (1 - \tau_i)^2$ to assign greater importance to the center, the right tail and the left tail of the distribution, respectively. Similarly, we denote the score arising from the restricted model as $S^{gr}(y_{t+1} | \mathcal{Y}_t)$; we stress we obtain the score by replacing \mathcal{W}_t by \mathcal{Y}_t in (4.28). Let \bar{S}^{gr} be the average value of the differences $S^{gr}(y_{t+1} | \mathcal{W}_t) - S^{gr}(y_{t+1} | \mathcal{Y}_t)$ computed for all the periods included in $[t + 1, t + M]$, the null hypothesis of equal performance is tested through the statistic

$$GR = \frac{\bar{S}^{gr}}{\hat{\sigma}_{GR} / \sqrt{M}}, \quad (4.29)$$

where $\hat{\sigma}_{GR}^2$ is a heteroskedasticity- and autocorrelation-consistent (HAC) Newey and West (1987) estimator of $\sigma_{GR}^2 = Var(\sqrt{M} \bar{S}^{gr})$. We have evidence of a better/worse performance of the unrestricted model when GR takes negative/positive values. Gneiting and Ranjan (2011) showed that, under the null hypothesis, $GR \xrightarrow{d} \mathcal{N}(0, 1)$.

4.3 Dataset and rolling analysis

4.3.1 Data description

In our analyses we make use of three series: the oil prices and two uncertainty indexes, EPU and EMU. The series are sampled at daily frequency and cover the period between January 2, 1986 and April 23, 2015, for a total of 7646 days.

We denote by $\{y_t\}_{t \in T}$ the series of the oil returns, that is $y_t = \log(oil_t) - \log(oil_{t-1})$, where oil_t is the spot price of the West Texas Intermediate (WTI) crude oil at day t .⁵ $\{oil_t\}_{t \in T}$ is not stationary: both the augmented Dickey and Fuller (1981) and the Phillips and Perron (1988) tests don't reject the null hypothesis of unit root with p-values of 0.2623 and 0.2112, respectively; differently, the p-values of the two tests are less than 0.01 for both $\{y_t\}_{t \in T}$ and $\{y_t^2\}_{t \in T}$.

EPU and *EMU* are two indices measuring the US economic policy and equity market uncertainty.⁶ *EPU* is built from newspaper archives of the Access World News's News-Bank service, by restricting the attention on United States and taking into account the number of articles containing at least one of the terms belonging to 3 sets. The first set is "economic/economy", the second is "uncertain/uncertainty" and the third set is "legisla-

⁵The series of the oil prices is recovered from Thomson Reuters Datastream.

⁶The data and the details about EPU and EMU are available on <http://www.policyuncertainty.com/>.

tion/deficit/regulation/congress/federal reserve/white house”. Using the same news source, *EMU* is built from articles containing the terms previously mentioned and one or more of the following: “equity market/equity price/stock market”. From *EPU* and *EMU* we compute $\{x_{1,t}\}_{t \in T}$ and $\{x_{2,t}\}_{t \in T}$, which are not affected by unit root: in both the cases the p-values of the augmented Dickey and Fuller (1981) and of the Phillips and Perron (1988) tests are less than 0.01.

Table 4.1 Descriptive statistics.

Variable	Mean	St. Deviation	Min	Max	skewness	kurtosis
y_t	0.0001	0.0248	-0.4069	0.1924	-0.7639	18.3329
y_t^2	0.0006	0.0026	0.0000	0.1655	37.9947	2294.8710
$x_{1,t}$	4.3665	0.6776	1.2185	6.5780	-0.2679	3.2726
$x_{2,t}$	3.8459	1.0575	1.5688	7.8655	0.2718	2.7157

The table reports some descriptive statistics computed for y_t , y_t^2 , $x_{1,t}$ and $x_{2,t}$. From left to right we report the mean, the standard deviation, the minimum and maximum values, the skewness and the kurtosis indices.

We report in Table 4.1 some descriptive statistics computed for the variables above described. y_t and y_t^2 have average values close to zero, with standard deviations equal to 0.0248 and 0.0026, respectively; y_t ranges from -0.4069 to 0.1924 and its distribution is affected by negative skewness and leptokurtosis. y_t^2 has strong positive skewness and leptokurtosis, due to the presence of relevant extreme values in its right tail. The uncertainty indexes, $x_{1,t}$ and $x_{2,t}$, are centered around 4.366 and 3.8459, with standard deviations equal to 0.6776 and 1.0575, respectively. Their distributions are slightly skewed, quite mesocurtic and affected by the presence of a few extreme values in the tails. The explorative analysis highlights the presence of extreme values for the variables of interest, mainly y_t and y_t^2 , suggesting the wisdom of using the quantile regression (Koenker and Bassett, 1978) in the forecasting exercise, rather than the ordinary least squares approach, because the latter does not guarantee robust results in the presence of outliers.

We now move to the most relevant empirical analyses, and investigate the ability of $x_{1,t} = \log(EPU_t)$ and $x_{2,t} = \log(EMU_t)$ in causing and predicting both the oil returns (y_t) and the squared oil returns (y_t^2) series, the latter being a measure of risk or dispersion of oil returns. We first describe the approach we follow in deriving the out-of-sample density forecasts.

4.3.2 Dynamic analysis and rolling window procedure

As noticed by Balcilar et al. (2015), the relationships among y_t or y_t^2 and the uncertainty indices are not stable over time. They applied the Bai and Perron (2003) test, detecting the presence of multiple structural breaks in the oil returns series for the EPU- and EMU-based VARs.⁷ Here, we follow a different approach, by implementing the DQ and the SQ tests introduced by Qu (2008), which reveal structural changes with unknown timing in regression quantiles. Following Tillmann and Wolters (2015), whose study focuses on the US inflation persistence, we proceed in two stages.

First, we use the DQ test in order to capture possible changes in the entire conditional distribution of the response variable. Given that we do not have any prior information as to which part of the conditional distribution is affected by breaks, we take into account a large range of quantiles levels, namely $\tau = \{0.05, 0.1, 0.15, \dots, 0.95\}$.

By setting $p = q = r = 2$ in Models (4.12)-(4.14), given $\xi(\tau) = [\alpha_0(\tau), \beta_1(\tau), \beta_2(\tau), \delta_1(\tau), \delta_2(\tau), \lambda_1(\tau), \lambda_2(\tau)]$ and $1 \leq T_1 < T^* < T_2 \leq T$, the hypotheses of the DQ tests are defined as follows:

$$\begin{cases} H_0 : \xi_t(\tau) = \xi(\tau), \text{ for all } t \text{ and for all } \tau \in \{0.05, 0.1, 0.15, \dots, 0.95\} \\ H_1 : \xi_t(\tau) = \begin{cases} \xi_1(\tau), \text{ for } t = T_1, \dots, T^* \\ \xi_2(\tau), \text{ for } t = T^*, \dots, T_2 \end{cases}, \text{ for some } \tau \in \{0.05, 0.1, 0.15, \dots, 0.95\} \end{cases} \quad (4.30)$$

In a second step, we implement, at the dates where the null hypothesis of the DQ test is rejected at the level of 0.01, the SQ test; in this way we detect structural changes in prespecified quantiles, in order to identify the specific regions of the distribution affected by breaks; for simplicity, we implemented the SQ test at three quantiles levels, i.e. $\tau = \{0.1, 0.5, 0.9\}$. In the present work, the hypotheses of the SQ tests are defined as follows:

$$\begin{cases} H_0 : \xi_t(\tau) = \xi(\tau), \text{ for all } t \text{ and for a given } \tau \in \{0.1, 0.5, 0.9\} \\ H_1 : \xi_t(\tau) = \begin{cases} \xi_1(\tau), \text{ for } t = T_1, \dots, T^* \\ \xi_2(\tau), \text{ for } t = T^*, \dots, T_2 \end{cases}, \text{ for a given } \tau \in \{0.1, 0.5, 0.9\} \end{cases} \quad (4.31)$$

The tests proposed by Qu (2008) are subgradient and have good properties also in small samples. The tables containing the critical values of the DQ and the SQ tests are available in Qu (2008).

⁷We were also able to detect four (18/01/1991, 26/03/2003, 02/12/2008, and 05/11/2011) and five (18/02/1999, 24/03/2003, 31/05/2007, 11/12/2008 and 05/11/2011) breaks with EPU and EMU being the independent variables respectively, in relation to oil returns.

The output of the DQ test, applied to the conditional quantiles and distribution of y_t is given in the left panel of Table 4.2. The number of breaks, detected at the level of 0.01, is equal to 7. The results of the SQ test are given in the right panel of Table 4.2; here, we can see that the breaks mainly affect the extreme conditional quantiles of y_t , rather than the central ones.

Table 4.2 Structural breaks in the conditional distribution and quantiles of y_t .

Structural breaks in the conditional distribution		Structural breaks at specific quantiles		
Dates of breaks	<i>DQ</i>	<i>SQ</i> ($\tau = 0.1$)	<i>SQ</i> ($\tau = 0.5$)	<i>SQ</i> ($\tau = 0.9$)
20/03/1987	1.0733	2.7402 ***	1.2612	2.4531 ***
11/05/1989	1.0852	1.6406 **	1.5053	2.2918 ***
21/09/1990	1.0644	1.2656	1.5886 *	1.7786 **
05/11/1991	1.0826	2.3462 ***	1.2124	2.2366 ***
04/09/2000	1.0522	3.0317 ***	1.2306	2.6934 ***
16/08/2013	1.0598	2.1177 ***	1.4521	2.3642 ***
27/01/2015	1.1041	2.2104 ***	1.4634	1.3431

The table reports the output of the DQ and the SQ tests, introduced by Qu (2008). The former detects the presence of structural breaks in the conditional distribution of y_t at the level of 0.01, whereas the latter detects the presence of structural breaks at specific quantiles, namely at $\tau = \{0.1, 0.5, 0.9\}$; *, ** and *** refer, respectively, to the 10%, 5% and 1% significance level.

Similarly, we show the results of the two tests, arising from the estimation of the y_t^2 conditional quantiles and distribution, in Table 4.3. The number of breaks is equal to 13 and the structural changes mainly occur at medium-high levels of τ .

The results discussed above highlight the presence of structural breaks over time and, as a result, the conclusions drawn from the full sample analysis might not be consistent. In order to capture the dynamics in the relations among the variables of interest, we differ from Balcilar et al. (2015) by implementing a rolling window procedure for causality testing, model estimation and forecast computation. The window used for the estimation of the model has a width of 500 observations. Moreover, to make a balance between flexibility, efficiency, and computational burden, we re-estimated the model with step of 5 days. In details, and focusing on causality testing at quantiles, the first window we consider includes the observations recorded between the first and the 500-th day of the sample. At time $t = 500$, we compute, for the first time, \hat{J}_T at different quantiles levels, with τ ranging from 0.05 to 0.95 and step of 0.05, for a total of 19 \hat{J}_T values.

At $t = 500$, we also estimate, for the first time, the parameters of the models defined, respectively, in (4.12) and (4.14), by setting τ from 0.01 to 0.99, with step of 0.01, to obtain quantiles vectors of length 99. The finer grid of quantiles used in the forecasting exercise,

Table 4.3 Structural breaks in the conditional distribution and quantiles of y_t^2 .

Structural breaks in the conditional distribution		Structural breaks at specific quantiles		
Dates of breaks	DQ	$SQ(\tau = 0.1)$	$SQ(\tau = 0.5)$	$SQ(\tau = 0.9)$
08/09/1986	1.0669	1.4449	1.7797 **	1.4096
08/12/1988	1.0630	1.3808	2.0024 ***	1.5430 **
09/10/1989	1.0599	1.2835	1.7168 **	1.5511 *
26/09/1990	1.1024	0.9704	1.3059	2.1648 ***
03/07/1991	1.0663	0.9263	2.0506 ***	1.9799 ***
27/01/1993	1.1142	0.9083	2.0441 ***	1.4157
16/03/1994	1.0746	1.4506	1.3358	1.7598 **
13/12/1994	1.0769	0.8698	1.8874 ***	0.8569
24/06/1996	1.0593	1.5555 *	1.7563 **	1.8971 ***
26/03/1999	1.0827	1.2611	1.0853	1.6581 *
18/05/2009	1.0926	1.3832	1.8667 ***	2.6657
04/12/2012	1.1001	0.9915	1.4124	1.4504
24/02/2015	1.0721	1.6582 **	1.4334	1.5175 *

The table reports the output of the DQ and the SQ tests, introduced by Qu (2008). The former detects the presence of structural breaks in the conditional distribution of y_t^2 at the level of 0.01, whereas the latter detects the presence of structural breaks at specific quantiles, namely at $\tau = \{0.1, 0.5, 0.9\}$; *, ** and *** refer, respectively, to the 10%, 5% and 1% significance level.

with respect to the causality analysis, is due to the need of estimating with adequate precision the conditional distributions of y_t and y_t^2 . Given the parameter estimates obtained at time $t = 500$, we compute the forecasts of the conditional quantiles and distributions of y_t and y_t^2 for $t = 501, \dots, 505$. Note we are not making a 5-step-ahead forecast, but simply fix the model parameters for 5 days, and compute five one-step-ahead forecasts. For instance, to recover the quantile forecasts we multiply the values of the predictors observed in $t = 500, \dots, 504$ by the coefficients estimated in $t = 500$.

The second window includes the observations between the 6-th and the 505-th day. Hence, at $t = 505$, we compute for the second time, updating the previous output obtained in $t = 500$, both \hat{J}_T and the estimated parameters by which we forecast, for $t = 506, \dots, 510$, the conditional quantiles and distributions of y_t and y_t^2 . The procedure goes on until the entire dataset is completely exploited.

As for the implementation of the tests proposed by Berkowitz (2001), Diebold and Mariano (2002), Amisano and Giacomini (2007), Diks et al. (2011) and Gneiting and Ranjan (2011), described in Section 4.2.3, starting from $t + 1 = 501$, we compare the forecasts formulated in t with the out-of-sample observations y_{t+1} and y_{t+1}^2 . Therefore, on the basis of those comparisons, we compute the quantity in (4.16) along with the scores characterizing each of the tests defined, respectively, in (4.22), (4.24), (4.26), (4.29). In our analysis, the

forecasting evaluation is carried out on rolling intervals consisting of $M = 500$ periods. Therefore, in $t = 1000$, we compute for the first time the five test statistics mentioned above. By updating the (4.16) and the scores by one period ahead, we compute the statistics for the second time in $t = 1001$, and the procedure goes on until the entire dataset is completely exploited. Note that we make use of two windows: the first refers to the model estimation, while the latter defines the range over which we evaluate the density forecast performances of the restricted and unrestricted models.

In applying the test introduced by Jeong et al. (2012), in Balcilar et al. (2015) the lag order q is determined on the basis of the Schwarz Information Criterion computed on the VAR comprising oil returns and EPU or EMU. With our data and sample, we obtain $q = 9$ in the case of EPU, whereas $q = 5$ for EMU. Differently to Balcilar et al. (2015), where the test is applied on the full sample, our analysis is carried out through the rolling window procedure above described. Consequently, on the one side, a large q would imply huge computational costs and, on the other side, we have, most likely, that q would change from one window to another. For that reason, as rule of thumb, and again to make a balance between precision of the analyses and computational burden, we set $q = 2$ in applying the causality test in quantiles. Likewise, for the models defined in (4.12)-(4.14), we set $p = q = r = 2$.

4.4 Empirical findings

First of all, we analyze the causality in quantiles. In Figure 4.1 we report the values of the test statistic \widehat{J}_T , defined in (4.3), in case we study the causality implications of $x_{1,t}$ on y_t ; differently, Figure 4.2 displays the output of the test applied for $x_{2,t}$. The results in Figures 4.1-4.2 are very similar: periods in which \widehat{J}_T takes low values (pointing out the low or inexistent power of the two uncertainty indices in causing the oil returns) are followed by periods of relevant peaks, such as in the second half of the 1980s, at the beginning and at the end of the 1990s, between the years 2006-2008. Moreover, we can see that the causality relations are stronger at the central τ levels. Despite the regimes change over time, the periods in which the uncertainty indices are significant in causing the oil returns are less persistent than the ones characterized by no causality.

In Figures 4.3-4.4 we study the causality relations of $x_{1,t}$ and $x_{2,t}$, respectively, on y_t^2 . Here, we observe a stronger causality impact with respect to the case of y_t , since the periods in which $x_{1,t}$ and $x_{2,t}$ are significant in causing y_t^2 are more persistent. Once again, the causality relations are stronger at the central levels of τ .

We now define the dummy variable $D_{y_t, x_{1,t}}(\tau)$, taking value 1 if the test statistic \widehat{J}_T , defined in (4.3) and applied on the pair $(x_{1,t}, y_t)$, at the τ level, as discussed above, is greater

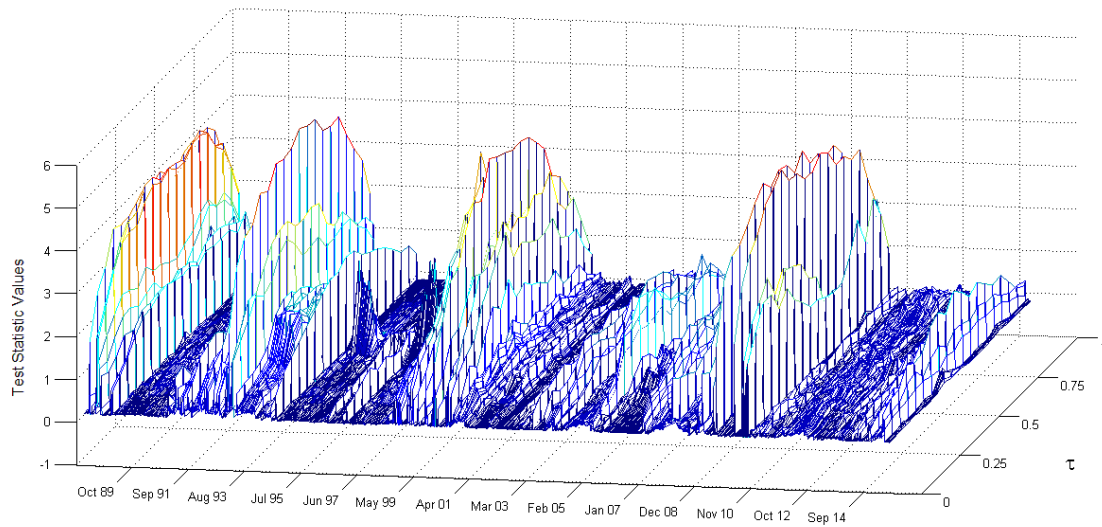


Fig. 4.1 The causality of the Economic Policy Uncertainty (in logarithm) on the oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.

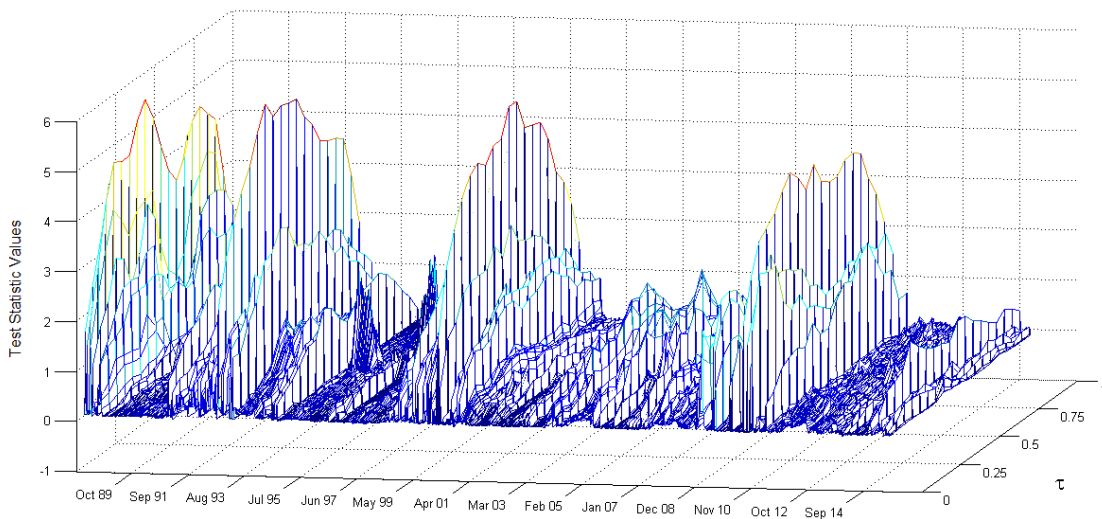


Fig. 4.2 The causality of the Market Equity Uncertainty (in logarithm) on the oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.

that 1.96, 0 otherwise. Likewise, $D_{y_t, x_{2,t}}(\tau)$, $D_{y_t^2, x_{1,t}}(\tau)$ and $D_{y_t^2, x_{2,t}}(\tau)$ are computed by using the same methodology and are obtained from the pairs $(x_{2,t}, y_t)$, $(x_{1,t}, y_t^2)$ and $(x_{2,t}, y_t^2)$, respectively. Figure 4.5 reports the linear correlation coefficients of those dummy variables,

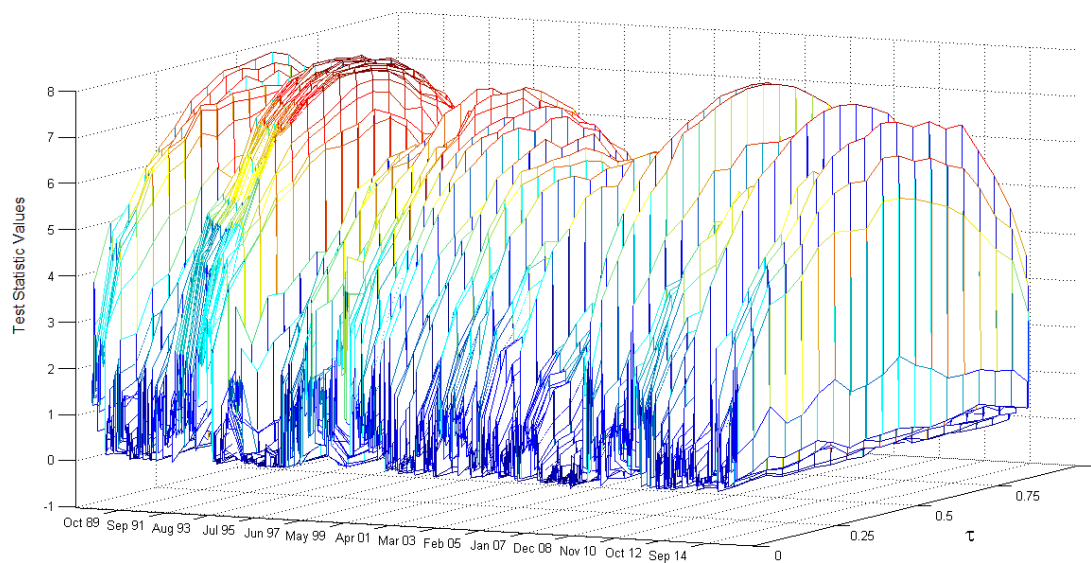


Fig. 4.3 The causality of Economic Policy Uncertainty (in logarithm) on the squared oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.

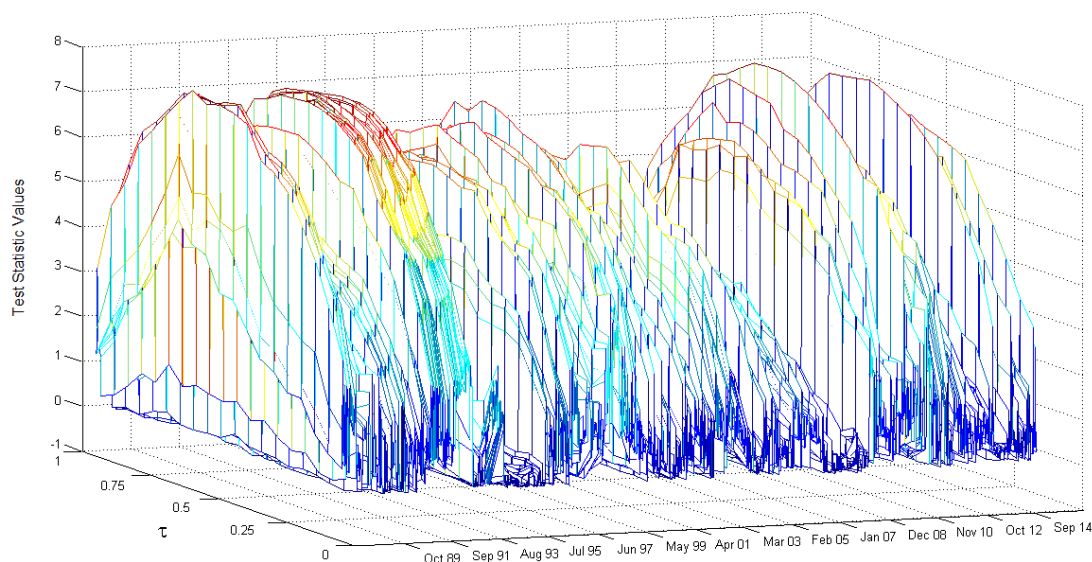


Fig. 4.4 The causality of the Market Equity Uncertainty (in logarithm) on the squared oil returns quantiles. The figure reports the values of the test statistic (4.3), computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead.

denoted as $\rho_{D_{y_t}}(\tau) = \rho(D_{y_t, x_{1,t}}(\tau), D_{y_t, x_{2,t}}(\tau))$ and $\rho_{D_{y_t^2}}(\tau) = \rho(D_{y_t^2(\tau), x_{1,t}}, D_{y_t^2(\tau), x_{2,t}}(\tau))$; we can see that the correlations, computed at different τ levels, are not negligible, mainly in the case of y_t^2 .

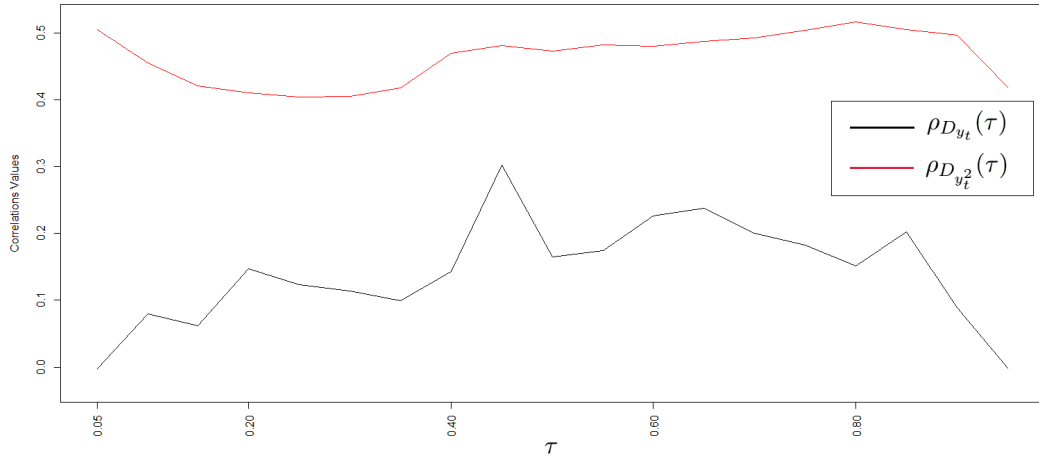


Fig. 4.5 The figure reports the linear correlation coefficients $\rho_{D_{y_t}}(\tau) = \rho(D_{y_t, x_{1,t}}(\tau), D_{y_t, x_{2,t}}(\tau))$ and $\rho_{D_{y_t^2}}(\tau) = \rho(D_{y_t^2(\tau), x_{1,t}}, D_{y_t^2(\tau), x_{2,t}}(\tau))$. $D_{y_t, x_{1,t}}(\tau)$ is a dummy variable taking value 1 if the test statistic (4.3), applied for the pair $(x_{1,t}, y_t)$ at the τ level, and computed through a rolling procedure with window size of 500 observations and step of 5 periods ahead, is greater than 1.96, 0 otherwise. $D_{y_t, x_{2,t}}(\tau)$, $D_{y_t^2, x_{1,t}}(\tau)$ and $D_{y_t^2, x_{2,t}}(\tau)$ are computed in the same way.

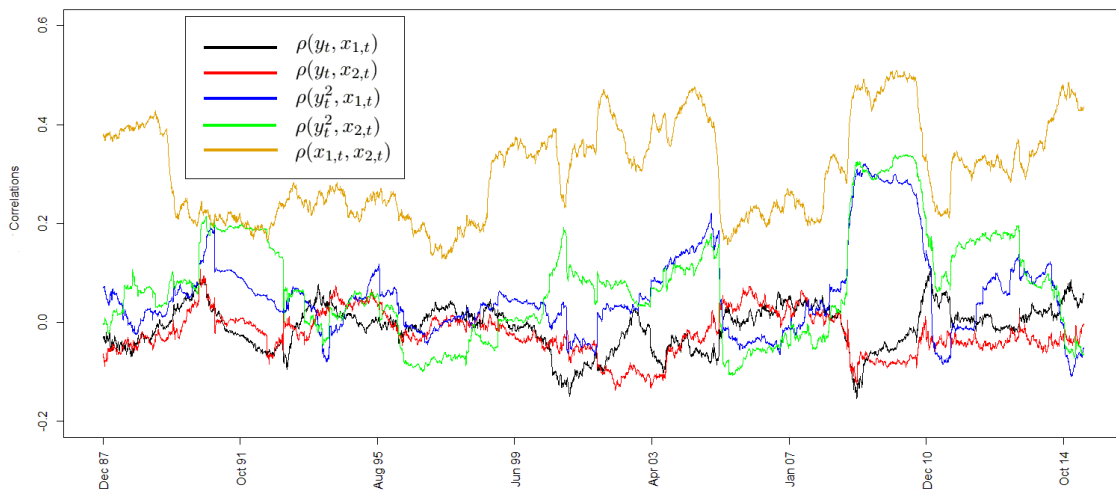


Fig. 4.6 The dynamic correlations, $\rho(\cdot)$, of the variables $y_t, y_t^2, x_{1,t}, x_{2,t}$. The linear correlations coefficients are computed through a rolling window procedure with window size of 500 observations and step of one period ahead.

It is also important to evaluate the correlations among the variables of interest. In Figure 4.6 we show the trend of the linear correlation coefficients, computed through a rolling

window procedure, with window size of 500 observations and step of one period ahead. It is important to see that all the correlations are not constant over time. $\rho(x_{1,t}, x_{2,t})$ always records the highest values; furthermore, $x_{1,t}$ and $x_{2,t}$ are more correlated with y_t^2 than y_t . Moreover, given the correlation coefficients obtained through the rolling window procedure, we compute their conditional average values. In particular, the correlation coefficients are computed by conditioning each pair of variables on the values of y_t and y_t^2 , respectively, such that y_t and y_t^2 are lower or greater than their τ -th in-sample quantiles, for $\tau = \{0.1, 0.5, 0.9\}$. The results are given in Table 4.4. We can see that, for each pair of variables and for each τ , the results deeply differ depending on whether we condition the correlations coefficients on the values of y_t or y_t^2 being greater or lower than their respective in-sample τ quantiles. The results discussed above highlight the importance of considering the joint impact of EPU and EMU in forecasting the oil movements, as well as the need of using the quantile regression method, given the asymmetric relations among the variables at different τ levels.

After estimating the parameters of Models (4.12)-(4.14) for each of the subsamples determined by the rolling window procedure, we computed their respective average values and standard deviations. We checked that all the coefficients' p-values, on average, are greater than 0.05, pointing out that, over time, the explanatory variables are not always statistically significant in explaining the conditional quantiles of y_t and y_t^2 . For that reason, we report in Table 4.5 the mean (columns 2-7) and the standard deviation (columns 8-13) of the coefficients conditional on the fact that their respective p-values are less or equal than 0.05; those average values are denoted by $\bar{\beta}_j(\tau)$, $\bar{\delta}_j(\tau)$, $\bar{\lambda}_j(\tau)$, whereas the standard deviations are denoted as $\sigma_{\beta_j(\tau)}$, $\sigma_{\delta_j(\tau)}$, $\sigma_{\lambda_j(\tau)}$, for $j = \{1, 2\}$. For simplicity, we display the results obtained at $\tau = \{0.1, 0.5, 0.9\}$. Starting with the estimation of Model (4.12), on average, the impact of the explanatory variables changes according to the τ levels, an evidence against the so-called location-shift hypothesis, which assumes homogeneous effects of the covariates across the conditional quantiles of the response variable. It is possible to observe a precise trend of the coefficients values over τ : negative for $\bar{\beta}_j(\tau)$, positive for $\bar{\delta}_j(\tau)$, $\bar{\lambda}_j(\tau)$, $j = \{1, 2\}$. On average, the lags of y_t have a positive impact on the left tail of the response variable conditional distribution; on the other hand, their effects become negative at medium-high τ levels. This is expected as past negative returns lead to the increase of the series dispersion and thus move the 0.1 (0.9) quantile further on the left (right), with an additional effect on the median. On the contrary, positive returns shrink the density toward the median, which is also moving to the right. We interpret these evidences as a form of asymmetry, where the sign of the shocks lead to opposite effects on the quantiles, and thus on the distribution, of the target variable.

Table 4.4 Conditional average correlations.

	$\tau = 0.1$		$\tau = 0.5$		$\tau = 0.9$	
	lower	greater	lower	greater	lower	greater
$\bar{\rho}(y_t, x_{1,t})$	-0.1415	0.0286	-0.0837	0.0785	-0.0474	0.1428
$\bar{\rho}(y_t, x_{2,t})$	-0.1750	0.0272	-0.1023	0.0941	-0.0716	0.1535
$\bar{\rho}(y_t^2, x_{1,t})$	-0.0600	0.0610	0.0064	0.0853	0.0179	0.1277
$\bar{\rho}(y_t^2, x_{2,t})$	-0.0649	0.0810	-0.0176	0.1140	0.0190	0.1993

The table reports the average correlations among the variables $y_t, y_t^2, x_{1,t}$ and $x_{2,t}$. The correlation coefficients are computed by conditioning the pairs of the variables on the values of y_t and y_t^2 , respectively, such that y_t and y_t^2 are lower or greater than their τ -th quantiles, for $\tau = \{0.1, 0.5, 0.9\}$.

Table 4.5 Quantile regression output.

τ	$\bar{\beta}_1(\tau)$	$\bar{\beta}_2(\tau)$	$\bar{\delta}_1(\tau)$	$\bar{\delta}_2(\tau)$	$\bar{\lambda}_1(\tau)$	$\bar{\lambda}_2(\tau)$	$\sigma_{\beta_1(\tau)}$	$\sigma_{\beta_2(\tau)}$	$\sigma_{\delta_1(\tau)}$	$\sigma_{\delta_2(\tau)}$	$\sigma_{\lambda_1(\tau)}$	$\sigma_{\lambda_2(\tau)}$
	Estimates of Model (4.12)											
0.1	0.024	17.893	-0.835	-0.040	-0.529	-0.020	19.122	5.997	0.399	0.694	0.207	0.683
0.5	-11.531	-7.864	-0.464	0.291	0.011	0.130	1.613	0.543	0.139	0.332	0.257	0.260
0.9	-16.625	-18.386	0.559	0.829	0.555	0.765	15.213	6.581	0.146	0.414	0.401	0.119
	Estimates of Model (4.14)											
0.1	0.467	1.027	0.001	-4e-04	-2e-04	-1e-05	0.444	0.903	3e-05	1e-03	2e-05	3e-04
0.5	6.711	7.977	0.009	0.011	0.003	0.005	5.032	2.895	0.004	0.007	0.005	0.003
0.9	67.069	85.650	0.038	0.020	0.050	0.004	27.312	34.422	0.016	0.065	0.031	0.046

The table reports the average values (%), in columns 2-7, and the standard deviations (%), in columns 8-13, computed over the subsamples determined by the rolling window procedure, of the estimated parameters, conditional to the fact that they are statistically significant at the level of 0.05. The rolling window procedure is applied by using a window size of 500 observations and step of 5 days ahead.

The opposite phenomenon is observed for $x_{1,t-j}$ and $x_{2,t-j}$, $j = \{1, 2\}$; for the uncertainty indexes, we were expecting those signs. In fact, an increase in the uncertainty, moves the lower quantiles to the left and the upper quantiles to the right, with the impact on the median being smaller than that on other quantiles for $j = 1$.

With the exception of $x_{2,t-j}$, $j = \{1, 2\}$, the coefficients of the other explanatory variables are less volatile at the central levels of τ . In Table 4.6 we report the number of subsamples in which each coefficient turns out to be statistically significant at the level of 0.05. It is possible to see that, at τ equal to 0.1, 0.5 and 0.9, $x_{2,t-2}$, $x_{2,t-1}$ and y_{t-1} record, respectively, the highest number of periods in which their coefficients are statistically significant.

Moving to the estimation of Model (4.14), just $\bar{\delta}_2(0.1)$, $\bar{\lambda}_1(0.1)$ and $\bar{\lambda}_2(0.1)$ are negative; nevertheless those coefficients take very low values. With the exception of $\bar{\lambda}_2(\tau)$, all the other coefficients exhibit, on average, an increasing trend over τ . At τ equal to 0.1, 0.5 and 0.9, y_{t-2} , $x_{1,t-1}$ and y_{t-1} , record, respectively, the highest number of periods in which their coefficients are statistically significant at the 5% level, as it is possible to see from Table 4.6.

Table 4.6 Persistence of significance over the rolled windows.

τ	$n_{\beta_1(\tau)}$	$n_{\beta_2(\tau)}$	$n_{\delta_1(\tau)}$	$n_{\delta_2(\tau)}$	$n_{\lambda_1(\tau)}$	$n_{\lambda_2(\tau)}$
Estimates of Model (4.12)						
0.1	112	37	165	114	54	190
0.5	102	9	30	64	113	35
0.9	322	156	93	47	105	53
Estimates of Model (4.14)						
0.1	43	63	50	13	35	24
0.5	142	50	150	57	145	103
0.9	174	27	126	117	69	90

The table reports the number of subsamples, determined by the rolling window procedure, in which each coefficient turns out to be statistically significant at the level of 0.05. The rolling window procedure is applied by using a window size of 500 observations and step of 5 days ahead.

The larger impact of the squared lagged returns on upper quantiles is again expected, signaling that large movements (either positive or negative) lead to a huge increase of the risk. We observe a similar pattern on the uncertainty indexes, then increase has a large impact on the upper quantiles of the squared returns.

Subfigures 4.7(a)-4.7(c) show the conditional distributions of y_t and y_t^2 , respectively, estimated from the first of the rolled windows. We clearly note the problem of crossing in quantiles vanishes by applying the quantile bootstrap method proposed by Chernozhukov et al. (2010). Moreover, the Epanechnikov kernel method allows to obtain smoother distributions. Subfigures 4.7(b)-4.7(d) show the conditional densities of y_t and y_t^2 , respectively, estimated

from the first and the last windows of the rolling window procedure, by applying the Epanechnikov kernel method. Notably, the shape of each density changes over time, thus supporting the need for a rolling evaluation.

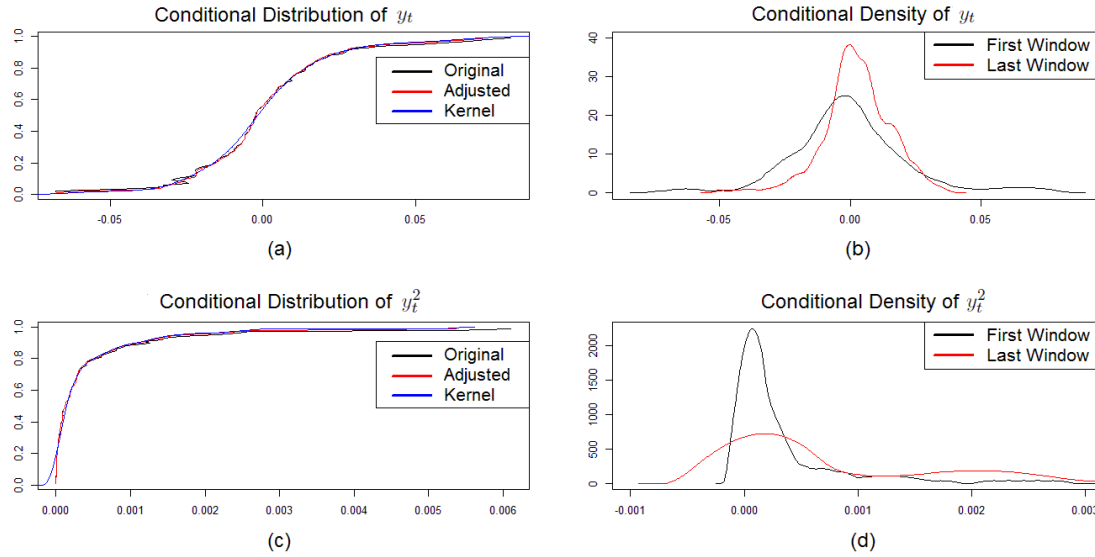


Fig. 4.7 Conditional distributions and densities of y_t and y_t^2 . Subfigures (a) and (c) display, respectively, the conditional distributions of y_t and y_t^2 , estimated from the first subsample determined through the rolling window procedure. “Original”, “Adjusted” and “Kernel” stand for the distributions arising directly from Models (4.12)-(4.14), the ones obtained by adjusting the original estimates through the quantile bootstrap method proposed by Chernozhukov et al. (2010), and the ones built by means of the Epanechnikov kernel, respectively. Subfigures (b) and (d) show the conditional densities of y_t and y_t^2 , respectively, estimated from the first and the last windows determined by the rolling window procedure, by applying the Epanechnikov kernel method.

We now evaluate the possible asymmetric effects of the uncertainty indices on the oil movements. In doing that, we slightly modify Models (4.12) and (4.14); first of all, we center to zero both $x_{1,t-j}$ and $x_{2,t-j}$, $j = \{1, 2\}$, by subtracting from them their respective average values. Those new variables, which now can take both positive and negative values, are denoted by $x_{1,t-j}^*$ and $x_{2,t-j}^*$, $j = \{1, 2\}$, respectively. Secondly, we make use of the following indicator functions: $\mathbf{1}_{\{x_{1,t-j}^* < 0\}}$ and $\mathbf{1}_{\{x_{2,t-j}^* < 0\}}$, $j = \{1, 2\}$, which take value 1 if the condition in $\{\cdot\}$ is true, 0 otherwise. Specifically, the new models are defined as follows:

$$\begin{aligned}
 Q_\tau(y_t | \mathscr{Y}_{t-1}) &= \alpha_0(\tau) + \beta_1(\tau)y_{t-1} + \beta_2(\tau)y_{t-2} + \delta_1^d(\tau)x_{1,t-1}^* + \delta_2^d(\tau)x_{1,t-2}^* \quad (4.32) \\
 &+ \lambda_1^d(\tau)x_{2,t-1}^* + \lambda_2^d(\tau)x_{2,t-2}^* + \delta_1^*(\tau)\mathbf{1}_{\{x_{1,t-1}^* < 0\}}x_{1,t-1}^* \\
 &+ \delta_2^*(\tau)\mathbf{1}_{\{x_{1,t-2}^* < 0\}}x_{1,t-2}^* + \lambda_1^*(\tau)\mathbf{1}_{\{x_{2,t-1}^* < 0\}}x_{2,t-1}^* \\
 &+ \lambda_2^*(\tau)\mathbf{1}_{\{x_{2,t-2}^* < 0\}}x_{2,t-2}^*,
 \end{aligned}$$

$$\begin{aligned}
Q_\tau(y_t^2 | \mathcal{W}_{2,t-1}) &= \alpha_0(\tau) + \beta_1(\tau)y_{t-1}^2 + \beta_2(\tau)y_{t-2}^2 + \delta_1^d(\tau)x_{1,t-1}^* & (4.33) \\
&+ \delta_2^d(\tau)x_{1,t-2}^* + \lambda_1^d(\tau)x_{2,t-1}^* + \lambda_2^d(\tau)x_{2,t-2}^* \\
&+ \delta_1^*(\tau)\mathbf{1}_{\{x_{1,t-1}^* < 0\}}x_{1,t-1}^* + \delta_2^*(\tau)\mathbf{1}_{\{x_{1,t-2}^* < 0\}}x_{1,t-2}^* \\
&+ \lambda_1^*(\tau)\mathbf{1}_{\{x_{2,t-1}^* < 0\}}x_{2,t-1}^* + \lambda_2^*(\tau)\mathbf{1}_{\{x_{2,t-2}^* < 0\}}x_{2,t-2}^*.
\end{aligned}$$

In evaluating the asymmetric effects of EPU (EMU) on the oil movements, it is important to notice that the impact of $x_{1,t-j}^*$ ($x_{2,t-j}^*$), $j = \{1, 2\}$, is quantified by $\delta_j^d(\tau)$ ($\lambda_j^d(\tau)$) if $x_{1,t-j}^* \geq 0$ ($x_{2,t-j}^* \geq 0$); differently, its impact is equal to $\delta_j^d(\tau) + \delta_j^*(\tau)$ ($\lambda_j^d(\tau) + \lambda_j^*(\tau)$) if $x_{1,t-j}^* < 0$ ($x_{2,t-j}^* < 0$).

We report the results arising from the estimation of Models (4.32)-(4.33) in Table 4.7. Here, we display the average values of the coefficients over the rolled subsamples (window size of 500 observations and steps of 5 days ahead), conditioned to the fact that they are statistically significant at the level of 5%; we also report their standard deviations. More precisely, for instance, in order to evaluate correctly the asymmetric effects of $x_{1,t-1}^*$, for each window, we considered the cases where all the coefficients $[\widehat{\delta}_1^d(\tau), \widehat{\delta}_1^{**}(\tau) = \widehat{\delta}_1^d(\tau) + \widehat{\delta}_1^*(\tau)]$ are simultaneously significant, and then we computed their average values. Likewise, we applied the same methodology for the other coefficients estimated from Models (4.32)-(4.33).

From Table 4.7, it is possible to see that, on average, both the uncertainty indices have asymmetric effects on the oil movements, and that the impact is stronger in the states where they take high values, i.e. when $x_{1,t-j}^*$ and $x_{2,t-j}^*$ take positive values. Indeed, the means of $\widehat{\delta}_j^d(\tau)$ and $\widehat{\lambda}_j^d(\tau)$ are almost always greater, in absolute value, than the means of $(\widehat{\delta}_j^d(\tau) + \widehat{\delta}_j^*(\tau))$ and $(\widehat{\lambda}_j^d(\tau) + \widehat{\lambda}_j^*(\tau))$, $j = \{1, 2\}$, respectively. This is a somewhat expected result suggesting that increases in uncertainty do have a larger impact on oil movements compared to decreases in uncertainty.

The last point concerns the evaluation of the Models (4.12)-(4.14) predictive power, placing particular emphasis on the contribution of the two uncertainty indices ($x_{1,t}$ and $x_{2,t}$) in forecasting the y_t and y_t^2 quantiles and distributions. The results arising from the Berkowitz (2001) test are given in Subfigures 4.8(a), which displays the case of y_t , and 4.8(b), where we focus on y_t^2 . In both the cases we display the p-values obtained from 4 different models: Model 1 includes all the available predictors, i.e. y_{t-j} , $x_{1,t-j}$, $x_{2,t-j}$ in Subfigure 4.8(a), y_{t-j}^2 , $x_{1,t-j}$, $x_{2,t-j}$ in Subfigure 4.8(b); Model 2 has just y_{t-j} in Subfigure 4.8(a), y_{t-j}^2 in Subfigure 4.8(b); Model 3 comprises y_{t-j} , $x_{1,t-j}$ in Subfigure 4.8(a), y_{t-j}^2 , $x_{1,t-j}$ in Subfigure 4.8(b); finally, Model 4 includes y_{t-j} , $x_{2,t-j}$ in Subfigure 4.8(a), y_{t-j}^2 , $x_{2,t-j}$ in Subfigure 4.8(b). In all the cases we set $j = \{1, 2\}$.

Table 4.7 The asymmetric impact of uncertainty on the oil movements.

τ	$\delta_1^d(\tau)$	$\delta_1^{**}(\tau)$	$\delta_2^d(\tau)$	$\delta_2^{**}(\tau)$	$\lambda_1^d(\tau)$	$\lambda_1^{**}(\tau)$	$\lambda_2^d(\tau)$	$\lambda_2^{**}(\tau)$
Estimates of Model (4.32)								
0.1	-2.44 (1.9)	-0.12 (0.4)	2.42 (1.2)	-1.16 (0.6)	-1.07 (1.0)	0.21 (0.2)	-1.74 (0.5)	0.33 (1.1)
0.5	0.95 (0.4)	-0.33 (0.2)	0.33 (1.1)	-0.38 (0.2)	-0.92 (0.8)	0.32 (0.1)	-1.53 (0.1)	0.29 (1.1)
0.9	2.20 (1.9)	-0.92 (0.8)	1.45 (1.1)	-0.50 (0.7)	-0.65 (1.9)	0.54 (0.4)	-1.00 (0.1)	0.33 (0.1)
Estimates of Model (4.33)								
0.1	1.00 (8.2)	-0.10 (2.2)	-2.92 (10.2)	0.19 (3.1)	2.23 (2.9)	-0.01 (3.2)	0.06 (0.5)	-0.02 (0.6)
0.5	2.68 (0.8)	-0.30 (1.4)	-2.07 (2.6)	0.17 (0.3)	3.13 (1.0)	-0.12 (0.2)	1.90 (1.2)	-0.09 (1.3)
0.9	14.14 (12.0)	-0.52 (3.1)	29.51 (15.2)	-2.90 (3.9)	5.56 (3.7)	-1.28 (1.8)	4.34 (4.8)	-1.89 (2.5)

The table reports the average values, computed over the subsamples determined by the rolling window procedure, of the uncertainty indices' coefficients (in brackets we report their standard deviations) estimated for Models (4.32) and (4.33), conditional to the fact that they are statistically significant at the level of 0.05. The rolling window procedure is applied by using a window size of 500 observations and step of 5 days ahead. $\delta_j^{**}(\tau)$ is the conditional average value of the sums $\delta_j^d(\tau) + \delta_j^*(\tau)$, computed from the windows where the two coefficients are simultaneously significant; similarly, $\lambda_j^{**}(\tau)$ is the conditional average value $\lambda_j^d(\tau) + \lambda_j^*(\tau)$, $j = \{1, 2\}$.

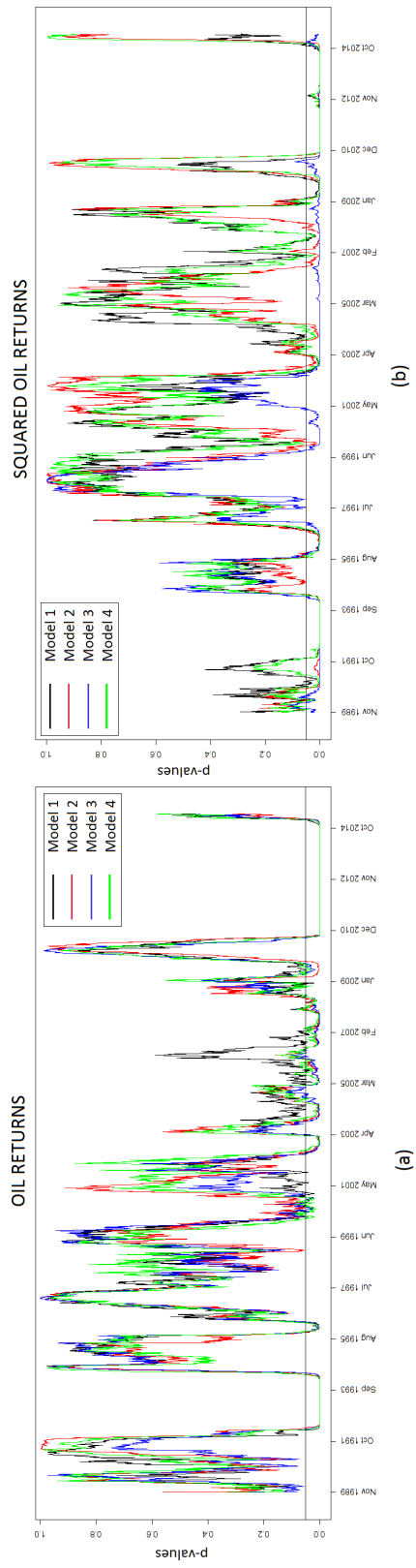


Fig. 4.8 The p-values of the Berkowitz (2001) test over the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions. Model 1 includes all the predictors in (4.12) and (4.14), respectively. Model 2 includes just y_{t-1} and y_{t-2} , in “OIL RETURNS”, y_{t-1}^2 and y_{t-2}^2 , in “SQUARED OIL RETURNS”. Model 3 includes $y_{t-1}, y_{t-2}, x_{1,t-1}$ and $x_{1,t-2}$, in “OIL RETURNS”, $y_{t-1}^2, y_{t-2}^2, x_{1,t-1}$ and $x_{1,t-2}$ in “SQUARED OIL RETURNS”. Finally, Model 4 includes $y_{t-1}, y_{t-2}, x_{2,t-1}$ and $x_{2,t-2}$, in “OIL RETURNS”, $y_{t-1}^2, y_{t-2}^2, x_{2,t-1}$ and $x_{2,t-2}$ in “SQUARED OIL RETURNS”.

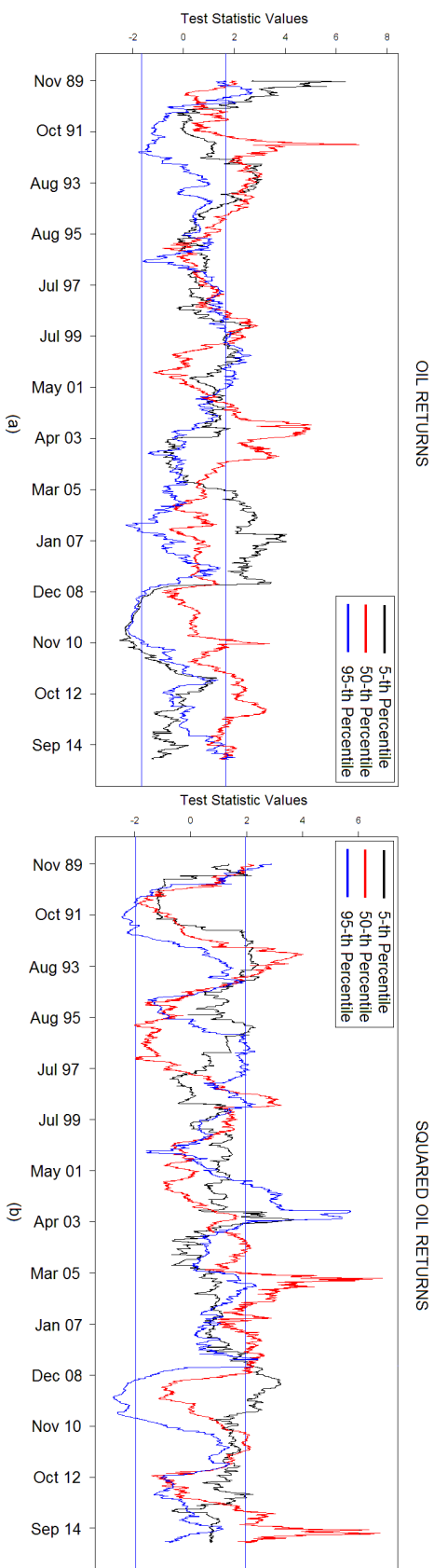


Fig. 4.9 The Diebold and Mariano (2002) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied at three different τ values: 0.05 (black lines), 0.50 (red lines) and 0.95 (blue lines).

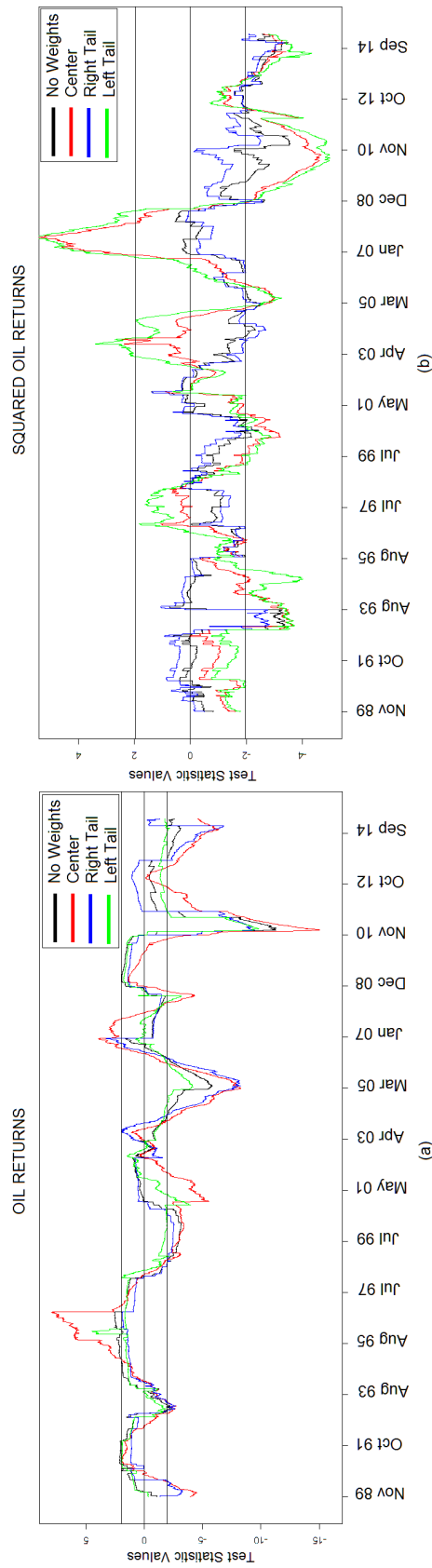


Fig. 4.10 The Amisano and Giacomini (2007) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied by placing greater emphasis on the center (red lines), on the right tail (blue lines) and on the left tail (green lines) of the conditional distributions. It is also applied by placing equal weight to the different regions of the distributions (black lines).

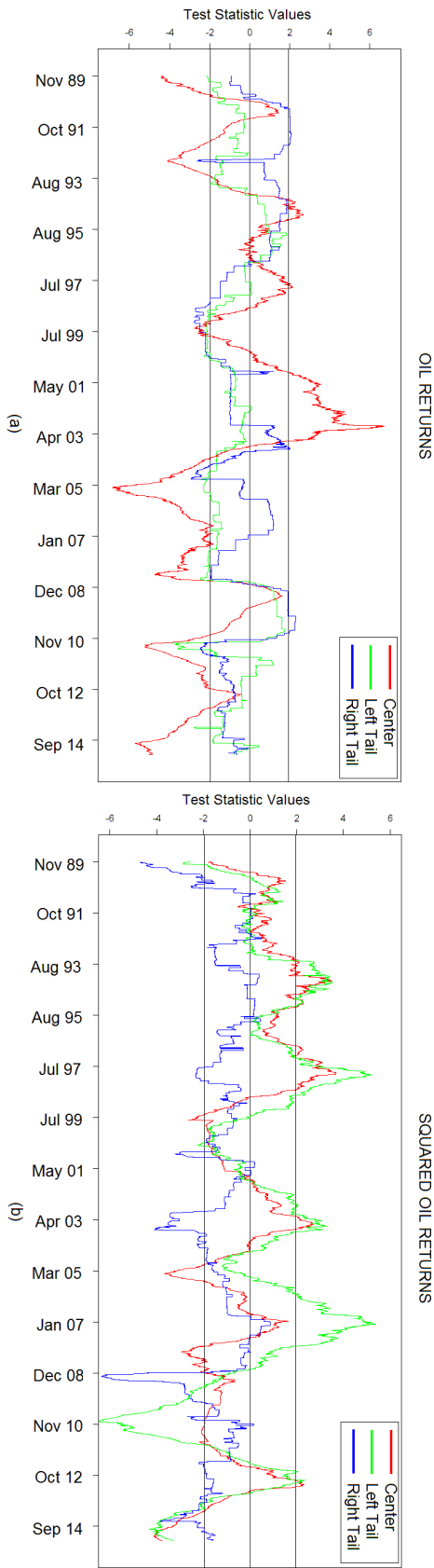


Fig. 4.11 The Diks et al. (2011) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied by placing greater emphasis on the center (red lines), on the right tail (blue lines) and on the left tail (green lines) of the conditional distributions.

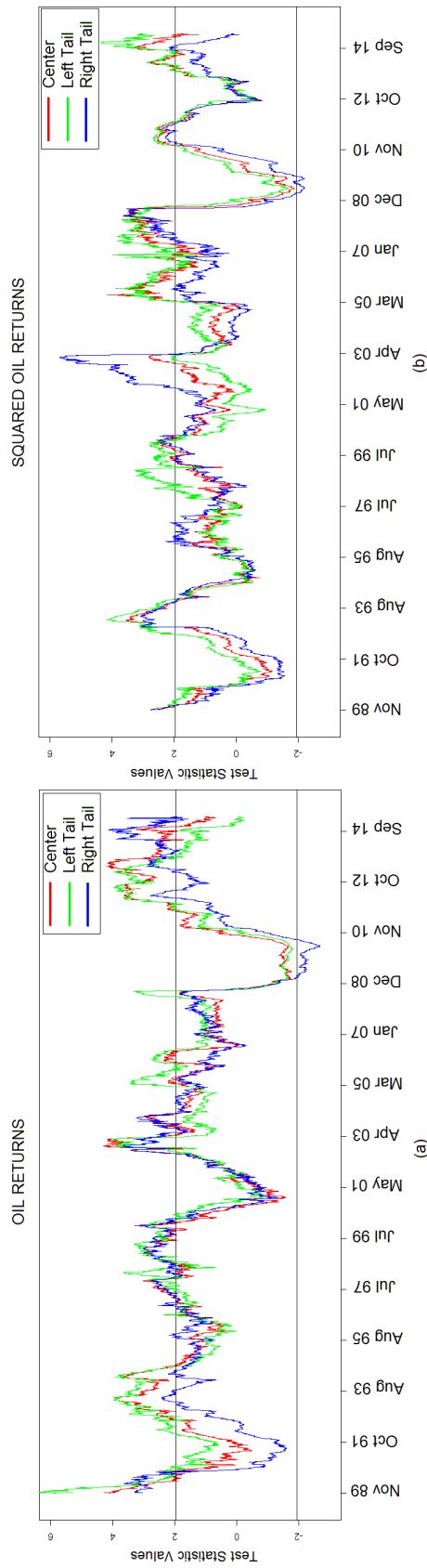


Fig. 4.12 The Gneiting and Ranjan (2011) test statistic values over the the rolled windows. Subfigure (a) and (b) refer, respectively, to the forecasts of the y_t and y_t^2 distributions, where we compare the restricted (which contain just the lags of y_t and y_t^2) and the unrestricted models (which include also the lags of $x_{1,t}$ and $x_{2,t}$). The test is applied by placing greater emphasis on the center (red lines), on the right tail (blue lines) and on the left tail (green lines) of the conditional distributions.

For all the 4 models, there are periods in which the null hypothesis of correct specification is not rejected, and others where the null hypothesis is rejected. In some periods, the inclusion of the uncertainty indices implies evident benefits: here, the p-values generated by Models 1-3-4 are greater than 0.05, whereas those of Model 2 are less than 0.05. We can also observe that during the years 2003-2007 and 2009 (in Subfigure 4.8(a)), 1990-1991, 2004, 2009-2010 and 2014 (in Subfigure 4.8(b)), the null hypothesis is not rejected just for Model 1, highlighting the importance of exploiting the joint predictive power of EPU and EMU.

The direct comparisons between the restricted (the one including just the lags of y_t) and the unrestricted (which includes also the lags of $x_{1,t}$ and $x_{2,t}$) models, based on the tests proposed by Diebold and Mariano (2002), Amisano and Giacomini (2007), Diks et al. (2011) and Gneiting and Ranjan (2011), are evaluated by means of Figures 4.9, 4.10, 4.11 and 4.12, respectively. As for the forecasting of the y_t quantiles and distribution (Subfigures 4.9(a), 4.10(a), 4.11(a) 4.12(a)), it is possible to see that the test statistics change their sign over time, pointing out periods in which the unrestricted model works better, providing the highest scores, followed by others where the best performance is recorded by the restricted model. Nevertheless, the null hypothesis of equal performance is not always rejected at the level of 5% and the periods in which the unrestricted model records the best performance, statistically significant, are less frequent than the ones where it is outperformed by the restricted model.

In general, all the tests give evidence of the best performance of the unrestricted model in the second half of the 2000s and at the end of the 2000s, when we place emphasis on the right tail of the y_t conditional distribution. The tests proposed by Amisano and Giacomini (2007) and Diks et al. (2011) detect further periods, namely in the middle of the 1990s and at the beginning of the 2000s, where the unrestricted model outperforms the restricted one, mainly when we focus on the center of the distribution.

Now we collect the information coming from the five tests to highlight the periods where EPU and EMU turn out to be crucial in forecasting the y_t conditional distribution and quantiles. For this purpose, we computed, for each of the applied tests (i.e. those developed by Berkowitz (2001), Diebold and Mariano (2002), Amisano and Giacomini (2007), Diks et al. (2011) and Gneiting and Ranjan (2011)) a dummy variable, D_t^{pred} , taking value 1 if the unrestricted model records a (statistically) better performance, at the level of 0.05, than the restricted one at t , 0 otherwise. In case of the Berkowitz (2001) test, D_t^{pred} takes value 1 if the null hypothesis is not rejected for the unrestricted model (which includes all the available covariates) and rejected for the restricted one (which includes just the lagged values of y_t or y_t^2) at t . In order to clean the series from the periods where the better performance of one of the two models lasts for a few day, being negligible, we compute, for each test, the following

moving average:

$$SD_t^{pred} = \frac{1}{2M_s + 1} \sum_{t-M_s}^{t+M_s} D_t^{pred}. \quad (4.34)$$

In our work we set $M_s = 10$, hence the moving averages span 21 days, and identify the periods where the unrestricted model provides the best performance as those where $SD_t^{pred} \geq 0.5$. In Figure 4.13 we display those periods with different colours for each test, linking them with the y_t , $x_{1,t}$ and $x_{2,t}$ series. It is possible to see that there exists a relevant evidence of a crucial role of EPU and EMU in forecasting the y_t conditional quantiles and distributions, as highlighted at least by 3 of the implemented tests, during the years 2005-2007 and 2008-2010. Those periods are close to two special events: the “2008 oil price bubble”, which spans the years 2007-2008, and the US subprime crisis marked by the Lehman Brothers’s default in September 2008.

As regards the case of y_t^2 , (Subfigures 4.9(b), 4.10(b), 4.11(b) 4.12(b)), the tests proposed by Diebold and Mariano (2002) and Gneiting and Ranjan (2011) lead to conclusions similar to those obtained for y_t ; indeed there is evidence of a better performance of the unrestricted model in the years 2009-2010, when the emphasis is placed on the right tail. The Diebold and Mariano (2002)’s test detects such an evidence also at the beginning of the 1990s. Differently, according to the the tests developed by Amisano and Giacomini (2007) and Diks et al. (2011), the unrestricted model works better, with respect to the y_t case, when we focus on the center and on the left tail of the y_t^2 distribution. Furthermore, on the basis of the Diks et al. (2011)’s test, the periods characterized by the best performance, statistically significant, of the unrestricted model, become more persistent with respect to the y_t case.

Like the y_t case, we collect the information coming from the several implemented tests and highlight the periods of the enduring best performance of the unrestricted model, by computing the quantity in (4.34). We display the results in Figure 4.14. Similarly to y_t , we have evidence of a crucial role of EPU and EMU in forecasting the y_t^2 conditional quantiles and distributions during the years 2005-2007 and 2008-2010.

To summarize, we checked that the relationships among the variables included in Models (4.12) and (4.14) change over time. As a result, the logarithms of EPU and EMU turn out to be relevant variables, in causing and forecasting the conditional quantiles and distributions of y_t and y_t^2 , just in some periods. Such an evidence supports the use of the rolling window procedure to capture those dynamics, instead of carrying out a full sample analysis. The periods in which the two uncertainty indices are significant in causing and in forecasting y_t^2 , at different regions of its distribution, are more persistent with respect to the ones recorded in the case of y_t ; as for the forecasting exercise, this phenomenon is observed through the tests proposed by Amisano and Giacomini (2007) and Diks et al. (2011).

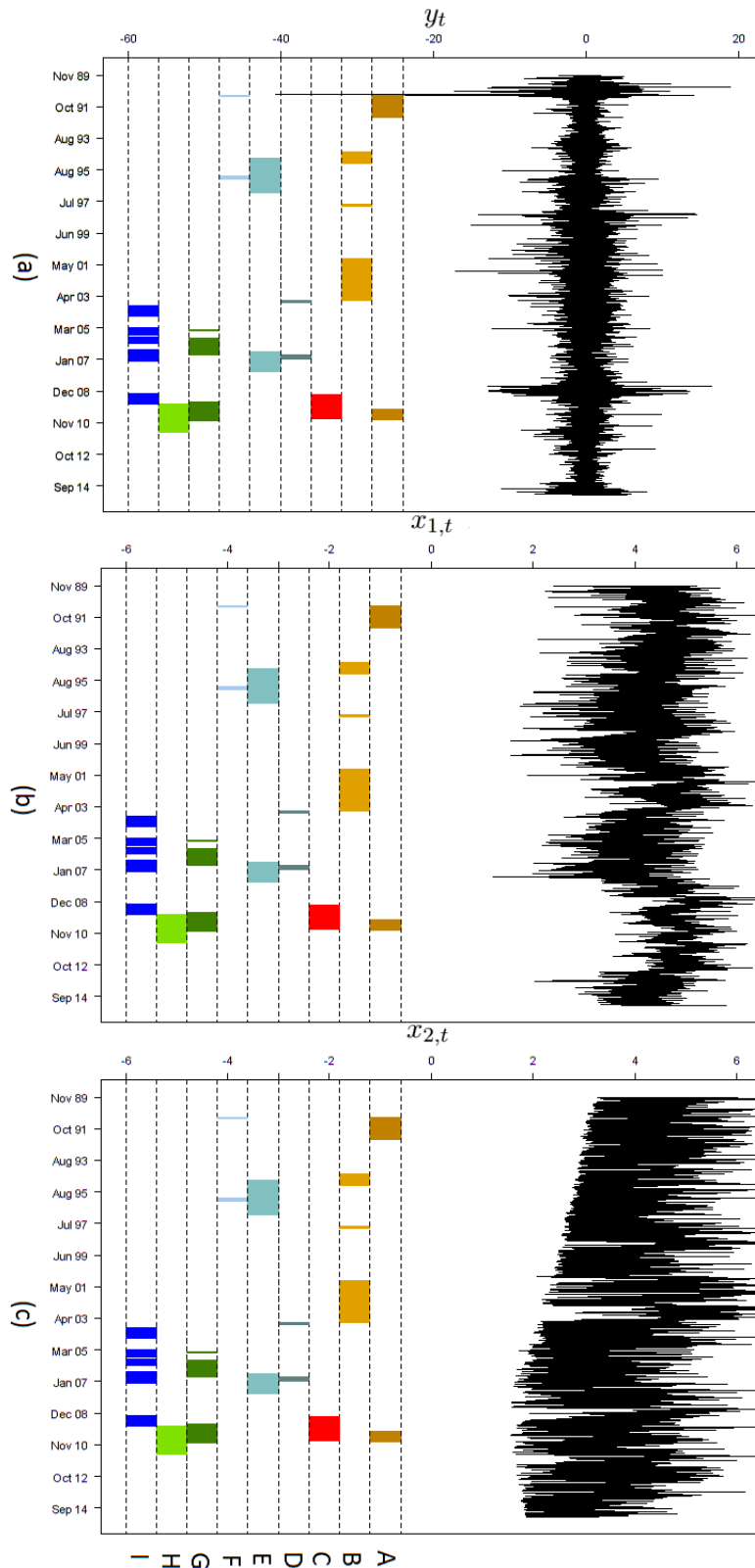


Fig. 4.13 Evidence of the crucial impact of EPU and EMU in forecasting the y_t conditional distribution and quantiles over time, detected by several tests and linked to the y_t (a), $x_{1,t}$ (b), and $x_{2,t}$ (c) series. A, B, C, D, E, F, G, H, I stand for the tests proposed by Diks et al. (2011), with focus on the right and the center parts of the distribution, Gneiting and Ranjan (2011), with focus on the right tail of the distribution, Amisano and Giacomini (2007), with focus on the right, the center and the left parts of the distribution, Diebold and Mariano (2002), with focus on the 95-th and the 5-th percentile, Berkowitz (2001), respectively.

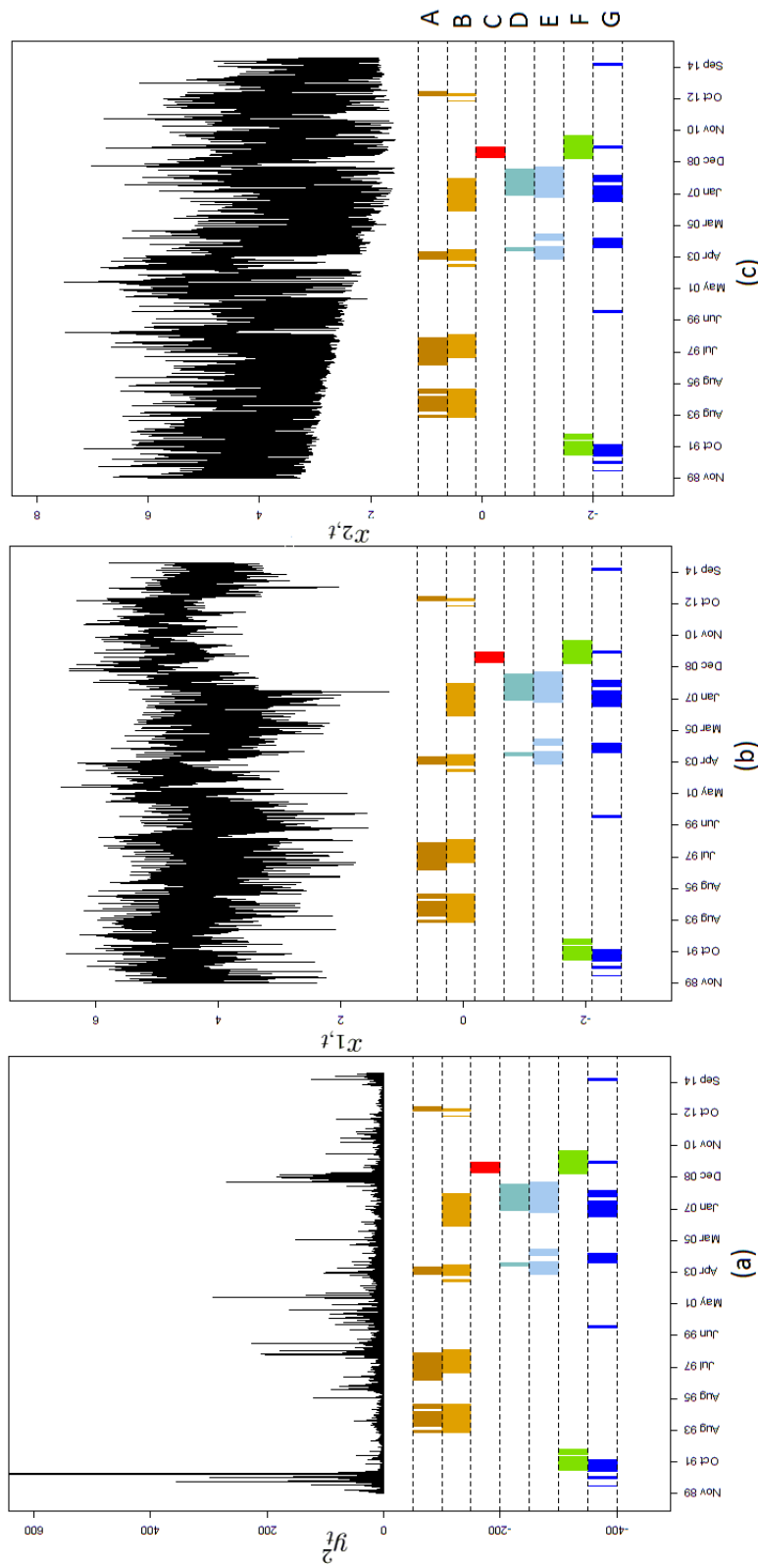


Fig. 4.14 Evidence of the crucial impact of EPU and EMU in forecasting the y_t^2 conditional distribution and quantiles over time, detected by several tests and linked to the y_t (a), $x_{1,t}$ (b), and $x_{2,t}$ (c) series. A, B, C, D, E, F, G stand for the tests proposed by Diks et al. (2011), with focus on the center and the left parts of the distribution, Gneiting and Ranjan (2011), with focus on the right tail of the distribution, Amisano and Giacomini (2007), with focus on the center and the left parts of the distribution, Diebold and Mariano (2002), with focus on the 95-th percentile, Berkowitz (2001), respectively.

The reason might be the following: with the squared returns of oil, we focus on the y_t volatility, a measure of dispersion (and thus uncertainty) which fits better the nature of EPU and EMU, which are themselves uncertainty indicators.

4.5 Concluding comments

In the present work we checked that the relations among the oil movements, defined as returns (y_t) and squared returns (y_t^2), and the uncertainty indices (EPU and EMU) are affected by structural breaks. The conclusions drawn from a full sample analysis would be misleading and, therefore, we implemented a rolling window procedure in order to capture the dynamics among the involved variables.

We first showed that the impact of EPU and EMU in causing the quantiles of the oil returns changes over time. Indeed, periods characterized by low or inexistent power of the two uncertainty indices in causing the oil returns are followed by periods of relevant causality evidence. Nevertheless, despite the changing in regimes over time, the periods in which the uncertainty indices are significant in causing the oil returns are less persistent than the ones characterized by no causality. Differently, when we focused on the y_t^2 case, then considering a quantile causality in variance, we observed stronger causality impacts, since the periods in which $x_{1,t}$ and $x_{2,t}$ are significant in causing the y_t^2 quantiles are more persistent. In both the cases, the causality relations are stronger at central quantiles levels.

Similarly, EPU and EMU turned out to be important drivers in forecasting the y_t and the y_t^2 conditional distributions just in some periods. Indeed, their coefficients are not always statistically significant, at the 5% level; as a result, the predictive power of the model we propose, evaluated ex post, from the out-of-sample realizations, is significantly improved by EPU and EMU just in some periods, as showed by several tests, namely those introduced by Berkowitz (2001), Diebold and Mariano (2002), Amisano and Giacomini (2007), Diks et al. (2011), Gneiting and Ranjan (2011). Moreover, those tests reveal different impacts according to the different regions of the response variables conditional distributions. In particular, through the Amisano and Giacomini (2007) and the Diks et al. (2011) testing approaches, we checked that, consistently to the causality analysis, the periods in which the two uncertainty indices are significant in forecasting y_t^2 , at different regions of its distribution, are more persistent with respect to the ones recorded in the case of y_t . The reason might be the following: with the squared returns of oil, we focus on the y_t volatility, a measure of dispersion (and thus uncertainty) which fits better the nature of EPU and EMU, which are themselves uncertainty indicators.

Chapter 5

Further research

5.1 The Conditional Quantile-Dependent Autoregressive Value-at-Risk

“The Conditional Quantile-Dependent Autoregressive VaR”, not yet completed, is a work I’m developing jointly with Prof. Massimiliano Caporin and Prof. Sandra Paterlini. Here we propose a novel methodology to evaluate the tail dependence between a single institution and the entire system and vice versa. In this way, it is possible to evaluate, on the one hand, the sensitivity of a single institution to the system and, on the other hand, the relevance of the same firm inside the entire system. Obviously, it is also possible to measure the tail dependence between two different institutions.

The method we propose modifies the approach developed in Adrian and Brunnermeier (2011), where a measure of systemic risk, the Conditional Value-at-Risk (CoVaR), is introduced. In particular, Adrian and Brunnermeier (2011) defines an institution’s contribution to systemic risk as the difference between CoVaR conditional on the institution being under distress and the CoVaR in the median state of the institution.

Our first contribution consists in enriching the $\Delta CoVaR$ model (Adrian and Brunnermeier, 2011) by introducing the Quantile-on-Quantile dependence, in the spirit of Sim and Zhou (2015). In this way, we go further the traditional quantile regression approach (Koenker and Bassett, 1978), where a conditional quantile of the response variable is estimated as function of one or more explanatory variables. For instance, in the financial environment, and in particular in the context of extreme risk, it is possible to find many works where the Value-at-Risk of a certain variable is modelled as function of different explanatory variables, on the basis of the approach introduced by Koenker and Bassett (1978), see, e.g., Engle and

Manganelli (2004), White et al. (2008) White et al. (2010). Differently, our approach allows to estimate the conditional quantiles of the response variable as function of the quantiles of the covariates, following the approach proposed by Sim and Zhou (2015). Hence, for instance, we are able to estimate the Conditional Value-at-Risk of a certain variable according to different states (e.g. distress or normal) of the regressors. In doing that, we slightly modify the classic asymmetric loss function on which the Koenker and Bassett (1978)'s work is based. Moreover, we found in Sim and Zhou (2015) some critical gaps, mainly related to the inferential process, that we seek to bridge.

Our second contribution consists in adding the latent autoregressive component in the CoVaR model, whose importance is highlighted in Engle and Manganelli (2004). The inclusion of the CoVaR lags implies challenging estimation issues, such that we cannot estimate our model by resorting to the traditional optimization tools. For this reason, we employ the Genetic Algorithm, with well-calibrated inputs, to obtain the point estimates.

Our dataset includes the daily returns recorded for all the firms operating in the U.S. banking sector in the period between the years 2000 and 2015. The data are downloaded from Thomson Reuters Datastream. Furthermore, we build an U.S. banking sector index, from both the returns and the market capitalizations of the single banking companies. The preliminar results show interesting improvements coming from the approach we propose, and will be developed in the next months.

5.2 The Role of Causality in Quantiles and Expectiles in Networks Combinations

“The Role of Causality in Quantiles and Expectiles in Networks Combinations”, not yet completed, is a work I’m developing jointly with Prof. Massimiliano Caporin and Dr. Roberto Panzica. Here, the focus is placed on the so-called systematic risk, i.e. the risk an investor of a well-diversified portfolio is exposed to, due to the dependence of financial returns to common factors (Lintner, 1965a,b; Markowitz, 1952; Mossin, 1966; Ross, 1976). Moreover, in the last decades, interconnections between different firms and sectors, a potential propagation mechanism of shocks throughout the economy, have attracted an increasing attention in the literature, see, e.g., Acemoglu et al. (2012) and Hautsch et al. (2013).

In this paper, we link systematic risk and network connections following the approach introduced in Billio et al. (2015). In particular, Billio et al. (2015) developed a framework where network interconnections and common factors risks co-exist, by means of a model, a variation of the traditional Capital Asset Pricing Model, where networks are used to infer the exogenous and contemporaneous links across assets. The authors showed that it is possible

to disentangle direct common factors exposures of a single stock to the indirect exposure to the common factors that arises from network interconnections.

Our main contribution consists in building a network structure from the combination of different network metrics. In particular, we make use of networks built on the basis of the causality relations in quantiles/expectiles among the different firms and sectors, starting from the method introduced by Jeong et al. (2012). To the best of our knowledge, such as method is completely novel in the financial econometrics literature. Preliminary analyses, applied to financial data, show interesting results, that will be developed in the next months.

5.3 The US Real GNP is Trend-Stationary After All

“The US Real GNP is Trend-Stationary After All”, submitted, has been written jointly with Prof. Rangan Gupta and Prof. Tolga Omay. The work applies the Fractional Frequency Flexible Fourier Form (FFFFF) Dickey-Fuller (DF)-type unit root test on the natural logarithm of US real GNP over the quarterly period of 1875:1-2015:2, to determine whether the same is trend- or difference-stationary. Here, we show that, despite the standard and the Integer Frequency Flexible Fourier Form (IFFFF) DF-type test fails to reject the null of unit root, the relatively more powerful FFFFF DF-type test provides strong evidence of the real GNP as being trend-stationary, i.e., US output returns to a deterministic non linear log-trend in the long run.

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Appendix A

Volatility estimation in the presence of noise and jumps

Let p_t be the logarithmic price of a financial asset at time t . We assume that it follows the Brownian semi-martingale process:

$$p_t = p_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u, \quad t \geq 0, \quad (\text{A.1})$$

where the drift $\mu = (\mu_t)_{t \geq 0}$ is locally bounded and predictable, and $\sigma = (\sigma_t)_{t \geq 0}$ is a strictly positive process, independent of the standard Brownian Motion $W = (W_t)_{t \geq 0}$, and càdlàg.

In the high-frequency context, the quadratic variation assumes an important role. If a trading day equals the interval $[0, 1]$ and is divided into n subintervals with same width, that is, $0 = t_0 < t_1 < \dots < t_n = 1$, the quadratic variation is defined as

$$QV = \text{p-lim}_{n \rightarrow \infty} \sum_{i=1}^n (p_{t_i} - p_{t_{i-1}})^2, \quad (\text{A.2})$$

with $\max_{1 \leq i \leq n} \{t_i - t_{i-1}\} \rightarrow 0$. If the price evolution is described by Equation (A.1), and μ and σ satisfy certain regularity conditions, the quadratic variation equals the integrated volatility (Hull and White, 1987):

$$IV = \int_0^t \sigma_u^2 du. \quad (\text{A.3})$$

Let m be the number of prices recorded at each subinterval and N the total number of observations for a trading day, that is, $N = mn$. The daily volatility can be estimated through

the realized variance:

$$RV^N = \sum_{i=1}^N r_{i\Delta,\Delta}^2, \quad (\text{A.4})$$

where $r_{i\Delta,\Delta} = p_{i/N} - p_{(i-1)/N}$ is the intraday return recorded at the i -th discrete point for $i = 1, \dots, N$, and $\Delta = 1/N$. If microstructure noise is absent, RV^N is a consistent estimator of IV as $N \rightarrow \infty$. In particular, Jacod (1994), Jacod and Protter (1998), and Barndorff-Nielsen and Shephard (2002) obtained the asymptotic distribution of RV^N :

$$N^{1/2} \left(RV^N - \int_0^1 \sigma_u^2 du \right) \xrightarrow{d} \mathcal{MN} \left(0, 2 \int_0^1 \sigma_u^4 du \right), \quad (\text{A.5})$$

where \mathcal{MN} denotes the mixed normal distribution. In (A.5) $\int_0^1 \sigma_u^4 du$ is the integrated quarticity (IQ), which can be estimated through the realized quarticity, denoted as RQ^N . The realized quarticity is given as

$$RQ^N = \frac{N}{3} \sum_{i=1}^N r_{i\Delta,\Delta}^4 \xrightarrow{p} \int_0^1 \sigma_u^4 du. \quad (\text{A.6})$$

We stress that RV^N is computed by considering just the last price of each subinterval. In order to reduce this information loss (within interval prices are completely disregarded) Martens and van Dijk (2007) and Christensen and Podolskij (2007) proposed the realized range-based variance ($RRV^{n,m}$), a modified version of the quantity given in Equation (A.4). Thus, more information is used, as the maximum and the minimum prices are both taken into account in each subinterval. Let $s_{p_{i\Delta,\Delta},m} = \max_{0 \leq s,t \leq m} \left(p_{\frac{i-1}{n} + \frac{t}{N}} - p_{\frac{i-1}{n} + \frac{s}{N}} \right)$ be the range for $i = 1, \dots, n$; the estimator of interest is defined as:

$$RRV^{n,m} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^n s_{p_{i\Delta,\Delta},m}^2, \quad (\text{A.7})$$

where $\lambda_{r,m} = E \left[\left| \max_{0 \leq s,t \leq m} \left(W_{\frac{t}{m}} - W_{\frac{s}{m}} \right) \right|^\zeta \right]$ is the ζ -th moment of the range of a standard Brownian Motion (W) over a unit interval; $\lambda_{\zeta,m}$ is computed through numerical simulation and $\lambda_{2,m} \rightarrow \lambda_2 = 4 \log(2)$ as $m \rightarrow \infty$. Christensen et al. (2009) showed that, without microstructure noise, $RRV^{n,m} \xrightarrow{p} \int_0^1 \sigma_u^2 du$ as $n \rightarrow \infty$ and

$$\sqrt{n} \left(RRV^{n,m} - \int_0^1 \sigma_u^2 du \right) \xrightarrow{d} \mathcal{MN} \left(0, \Lambda_c \int_0^1 \sigma_u^4 du \right), \quad (\text{A.8})$$

where $\Lambda_c = \lim_{m \rightarrow c} \Lambda_m$ and $\Lambda_m = (\lambda_{4,m} - \lambda_{2,m}^2) / \lambda_{2,m}^2$; Λ_m is decreasing in m and takes values between 2 ($m = 1$) and about 0.4 ($m \rightarrow \infty$). Therefore, comparing (A.5) and (A.8) shows that $RRV^{n,m}$ is more efficient than RV^N if $m > 1$.

So far, we have not considered microstructure noise and the effects of price jumps, although they might have a significant impact. As a consequence, the estimators described above might be biased because of the effects of measurement errors. A solution to the jumps is given by the range-based bipower variation ($RBV^{n,m}$), defined as

$$RBV^{n,m} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^{n-1} S_{p_{i\Delta,\Delta,m}} S_{p_{(i+1)\Delta,\Delta,m}}. \quad (\text{A.9})$$

Notably, $RRV^{n,m}$ and $RBV^{n,m}$ are jump non-robust and jump robust estimators of the integrated variance, respectively. As a result, Christensen and Podolskij (2006) used the two estimators in order to test the null hypothesis of no jumps at day t , by computing the following test statistic:

$$Z_{TP,t} = \frac{\sqrt{n} (1 - RBV_t^{n,m} / RRV_t^{n,m})}{\sqrt{v_m \max \left\{ RQQ_t^{n,m} / (RBV_t^{n,m})^2, 1 \right\}}} \xrightarrow{d} \mathcal{N}(0, 1), \quad (\text{A.10})$$

where $RQQ^{n,m} = \frac{n}{\lambda_{1,m}^4} \sum_{i=1}^{n-3} S_{p_{i\Delta,\Delta,m}} S_{p_{(i+1)\Delta,\Delta,m}} S_{p_{(i+2)\Delta,\Delta,m}} S_{p_{(i+3)\Delta,\Delta,m}}$ is the range-based quad-power quarticity, and $v_m = \lambda_{2,m}^2 (\Lambda_m^R + \Lambda_m^B - 2\Lambda_m^{RB})$, $\Lambda_m^R = (\lambda_{4,m} - \lambda_{2,m}^2) / \lambda_{2,m}^2$, $\Lambda_m^B = (\lambda_{2,m}^2 + 2\lambda_{1,m}^2 \lambda_{2,m} - 3\lambda_{1,m}^4) / \lambda_{1,m}^4$, $\Lambda_m^{RB} = (2\lambda_{3,m} \lambda_{1,m} - 2\lambda_{2,m} \lambda_{1,m}^2) / (\lambda_{2,m} \lambda_{1,m}^2)$.

Now we also consider the presence of microstructure noise. Suppose that p_t satisfies Equation (A.1) and that the new price process is equal to $p_t^* = p_t + \eta_t$, with η denoting the microstructure noise. Christensen et al. (2009) modeled the noise $\eta = (\eta_t)_{t \geq 0}$ as a sequence of i.i.d. random variables, such that $\mathbb{E}[\eta_t] = 0$, $\mathbb{E}[\eta_t^2] = \omega^2$, and $\eta \perp p$, showing that

$$\hat{\omega}_N^2 = \frac{RV^N}{2N} \xrightarrow{p} \omega^2, \quad (\text{A.11})$$

and

$$N^{1/2} (\hat{\omega}_N^2 - \omega^2) \xrightarrow{d} \mathcal{N}(0, \omega^4). \quad (\text{A.12})$$

Moreover, it is assumed that the jump component affects the price process through the relationship $\tilde{p}_t = p_t^* + \sum_{i=1}^{N_t} J_i$, where $J = (J_i)_{i=1, \dots, N_t}$ is the jump size component and $N = (N_t)_{t \geq 0}$ is a finite activity-counting process. In this context, Christensen et al. (2009) proposed the realized range-based bias corrected bipower variation ($RRV_{BVC}^{n,m}$), a consistent estimator of the integrated variance in the presence of price jumps and noise; the estimator is

defined as

$$RRV_{BVBC}^{n,m} = \frac{1}{\tilde{\lambda}_{1,m}^2} \sum_{i=1}^{n-1} \left| s_{p_{i\Delta,\Delta},m} - 2\hat{\omega}_N \right| \left| s_{p_{(i+1)\Delta,\Delta},m} - 2\hat{\omega}_N \right|, \quad (\text{A.13})$$

where $\tilde{\lambda}_{\zeta,m} = \mathbb{E} \left[\left| \max_{t:\eta_{\frac{t}{m}}=\omega, s:\eta_{\frac{s}{m}}=-\omega} \left(W_{\frac{t}{m}} - W_{\frac{s}{m}} \right) \right|^{\zeta} \right]$, with $1 \leq s, t \leq m$. $RRV_{BVBC}^{n,m}$ assumes an important role in the present work since it will be used to estimate the volatility in the model presented in Section 3.3.

Appendix B

Quantile regression: estimation and testing

The method of least squares is a widely used tool in statistics, given its attractive computational tractability, its simplicity, and the optimal results it guarantees given certain assumptions. Nevertheless, this approach might lead to erroneous conclusions when the model's hypotheses are violated. Here we concentrate on one relevant assumption, that of linearity.

In this case, Koenker and Bassett (1978) proposed an alternative method, regression quantiles, that ensures robust results. Regression quantiles allows one quantile of the conditional distribution of a variable to be estimated instead of restricting attention to the conditional mean. Let Y be a real-valued random variable with distribution function $F_Y(y) = P(Y \leq y)$. For any $0 < \tau < 1$, the τ -th quantile of Y is equal to $F_Y^{-1}(\tau) = \inf\{y : F_Y(y) \geq \tau\}$.

Let $I(\cdot)$ be the indicator function taking value 1 if the condition in (\cdot) is true, 0 otherwise, the approach introduced by Koenker and Bassett (1978) makes use of the asymmetric loss function

$$\rho_\tau(\varepsilon) = \varepsilon[\tau - I(\varepsilon < 0)], \quad (\text{B.1})$$

showing that the minimizer \tilde{y}_τ of the expected loss function $\mathbb{E}[\rho_\tau(Y - \tilde{y}_\tau)]$ satisfies $F_Y(\tilde{y}_\tau) - \tau = 0$. In particular, \tilde{y}_τ is the conditional quantile function $Q_Y(\tau|X_1, X_2, \dots, X_\delta)$ in the linear quantile regression:

$$\tilde{y}_\tau = Q_Y(\tau|X_1, X_2, \dots, X_\delta) = \beta_0(\tau) + \beta_1(\tau)X_1 + \dots + \beta_\delta(\tau)X_\delta, \quad (\text{B.2})$$

where $\mathbf{X} = (X_1, X_2, \dots, X_\delta)$ is the vector of δ explanatory variables.

Given the time index $t = 1, \dots, T$, let y_t and $x_{j,t}$, for $j = 1, \dots, \delta$, be, respectively, the realizations of Y and X_j at t . Then the parameter vector $\hat{\boldsymbol{\beta}}(\tau) = (\hat{\beta}_0(\tau), \dots, \hat{\beta}_\delta(\tau))$ in (B.2) is estimated as a solution of the quantile regression problem:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^{\delta+1}} \sum_{t=1}^T \rho_\tau(y_t - \beta_0 - \beta_1 x_{1,t} - \dots - \beta_\delta x_{\delta,t}). \quad (\text{B.3})$$

Let \mathbf{x}_t be the row vector that includes the t -th observation of \mathbf{X} , if the errors $\varepsilon_t = y_t - \mathbf{x}_t' \boldsymbol{\beta}(\tau)$ are i.i.d., with distribution F_ε and density f_ε , where $f_\varepsilon(F_\varepsilon^{-1}(\tau)) > 0$ is in the neighborhood of τ , the quantile estimator $\hat{\boldsymbol{\beta}}(\tau)$ is asymptotically distributed as

$$\sqrt{T} [\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau)] \xrightarrow{d} \mathcal{N}(0, \tilde{k}^2(\tau) \mathbf{D}^{-1}), \quad (\text{B.4})$$

where $\tilde{k}^2(\tau) = \frac{\tau(1-\tau)}{f_\varepsilon(F_\varepsilon^{-1}(\tau))^2}$ and $\mathbf{D} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_t \mathbf{x}_t' \mathbf{x}_t$ is a positive definite matrix.

When the errors are independent but non-identically distributed, the error density $f_{\varepsilon,t}$ changes in the sample. In this context, the asymptotic distribution becomes

$$\sqrt{T} [\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau)] \xrightarrow{d} \mathcal{N}(0, \tau(1-\tau) \mathbf{D}_1(\tau)^{-1} \mathbf{D} \mathbf{D}_1(\tau)^{-1}), \quad (\text{B.5})$$

where $\mathbf{D}_1(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_t f_{\varepsilon,t}(F_\varepsilon^{-1}(\tau)) \mathbf{x}_t' \mathbf{x}_t$ is a positive definite matrix (Koenker and Bassett, 1982a).

Equations (B.4) and (B.5) show that the asymptotic covariance matrix of the quantile regression coefficients depends on the error density, so the matrix could be difficult to estimate. This problem can be avoided by developing the inferential procedure in a different way. Resampling methods to estimate the parameters' standard errors are a valid alternative to the asymptotic results discussed above, and several studies recommend using the bootstrap method in the quantile regression framework (e.g., Buchinsky (1995)). Efron (1979) introduced the computer-based bootstrap method to estimate the variance and distribution of an estimate and, more generally, of a statistic. The main advantages of the bootstrapping approach are well-known: it assumes no particular distribution of the errors, it is not based on asymptotic model properties, and it is available regardless of the statistic of interest's complexity. Efron (1979) showed that this approach works well on a variety of estimation problems. Nevertheless, the bootstrap estimates are influenced by the sample's variability and the bootstrap resampling's variability, the former from the fact that the estimates are based on just one sample for a certain population and the latter from the finite number of replications (Davino et al., 2014).

Several types of bootstrapping methods are discussed in the literature and applied in quantile regression estimation. Three of them are the xy -pair method (Kocherginsky, 2003), the method based on pivotal estimating functions (Parzen et al., 1994), and the Markov chain marginal bootstrap (He and Hu, 2002). Davino et al. (2014) compared these three techniques by using two kinds of models: a quantile regression homogeneous error model, in which the error term does not depend on the explanatory variables, and a quantile regression heterogeneous error model, in which the error term is a function of the regressors. Using simulated data, the authors show that the three methods produce similar results for a homogeneous model in terms of estimated coefficients and standard errors. However, when the heterogeneous model is considered, the xy -pair method has the best results, while the worst results come from the Markov chain marginal bootstrap. We use the xy -pair method to estimate the parameters' standard errors.

When time series are used, the problem of serial correlation becomes critical, a relevant issue in the present work, whose aim is to model the conditional quantiles of volatility, a variable whose current value is typically affected by past values. If the dynamic of the model is neglected, the errors become serially correlated, with consequences in inference. Several studies have applied the quantile regression, taking into account models with autoregressive structure. For instance, Koenker and Xiao (2004) proposed the Quantile Autoregression (*QAR*) model, in which the τ -th conditional quantile of the response variable is explained by the lagged values of the same dependent variable. The authors focused on the first-order autoregression (although the analysis could be extended to the general case) and estimated the parameters of the model by solving the problem given in Equation (B.3), using as regressor the lagged value of the response variable. Engle and Manganelli (2004) proposed the *CAViaR* model to estimate the conditional Value-at-Risk of a financial institution. The model has an autoregressive structure and is estimated following the approach proposed by Koenker and Bassett (1978). Weiss (1990) performed a median regression for a model with serial correlation and found that the estimates are unbiased but the inference is wrong. Therefore, it is important to check for residual serial correlation and to take into account some possible solutions if such correlations are present, such as adding the lagged values of the variables or computing the parameters' standard errors by means of the bootstrapping methods mentioned above.

A further aspect of the evaluation of the quantile regression output refers to the goodness-of-fit assessment. Koenker and Machado (1999) introduced a goodness-of-fit quantity for quantile regression that is analogous to the coefficient of determination for the least squares regression. For simplicity, we introduce the approach in a quantile regression model for y_t

with just one regressor, x_t . At a given quantile τ , we evaluated two quantities, the residual absolute sum of weighted differences, denoted as

$$\begin{aligned} RASW_\tau &= \sum_{y_t \geq \hat{\beta}_0(\tau) + \hat{\beta}_1(\tau)x_t} \tau \left| y_t - \hat{\beta}_0(\tau) - \hat{\beta}_1(\tau)x_t \right| \\ &+ \sum_{y_t < \hat{\beta}_0(\tau) + \hat{\beta}_1(\tau)x_t} (1 - \tau) \left| y_t - \hat{\beta}_0(\tau) - \hat{\beta}_1(\tau)x_t \right|, \end{aligned} \quad (\text{B.6})$$

and the total absolute sum of the weighted differences, which reads

$$TASW_\tau = \sum_{y_t \geq \hat{q}_y(\tau)} \tau \left| y_t - \hat{q}_y(\tau) \right| + \sum_{y_t < \hat{q}_y(\tau)} (1 - \tau) \left| y_t - \hat{q}_y(\tau) \right|, \quad (\text{B.7})$$

where $\hat{q}_y(\tau)$ is the estimated unconditional τ th quantile of Y . Given these two quantities, a pseudo R^2 is defined as:

$$R^1(\tau) = 1 - \frac{RASW_\tau}{TASW_\tau}. \quad (\text{B.8})$$

Notably, $R^1(\tau)$ ranges between 0 and 1, like the coefficient of determination. Nevertheless, it is a local measure of fit, so, unlike the R^2 in the least squares regression context, it can't be used as a global goodness-of-fit measure because it quantifies the relative success of two models, restricted and unrestricted, at a given quantile (Koenker and Machado, 1999).

Another approach to evaluating the goodness of fit at a specified quantile is that proposed by Koenker and Bassett (1982b). In that study's testing framework, the null hypothesis is that a set of explanatory variables used to specify a conditional quantile in a general model does not improve the fit with respect to a restricted model (where those variables are not included). Therefore, the test reads like the F-test for the significance of a subset of coefficients. In the quantile regression framework, the test is of a Wald-type, and the associated test statistic, ξ_w , is based on the estimated coefficients of the unrestricted model.

In a quantile regression approach, several hypotheses can be tested on the conditional quantile parameters or on the innovations. One of these hypotheses is of particular interest for the analysis of realized variance series. We refer to the so-called *location shift hypothesis*, which requires the parameters that multiply the explanatory variables in (B.2) to be identical over τ . Thus, changes among the conditional quantiles occur only in the intercepts. If this null hypothesis is rejected, (B.2) would be a location-shift and scale-shift model. In the present work, the location-shift hypothesis is tested by means of a variant of the Wald test

that was introduced by Koenker and Bassett (1982a). The null hypothesis of the test is that the coefficient slopes are the same across quantiles. The test statistic is asymptotically distributed as \mathcal{F} , and the numerator's degrees of freedom are equal to the rank of the null hypothesis, while those related to the denominator are determined by subtracting the sample size by the number of parameters that characterize the model of interest. Clearly, if the location-shift hypothesis is accepted, the potential advantages of quantile regressions over linear regressions would be limited, as the covariates would have constant impact across the response quantiles.

Appendix C

Additional tables and figures

Table C.0.1 Jump testing results.

Stock	Mean of $Z_{TP,t}$	Median of $Z_{TP,t}$	St. Dev. of $Z_{TP,t}$	Percentage of rejections
ATT	3.9481	3.5437	2.4303	80.87%
BAC	3.0540	2.4240	2.6882	51.67%
BOI	4.0296	3.5337	2.4356	73.75%
CAT	3.7747	2.9523	2.8838	61.63%
CTG	2.1973	2.1218	2.5268	44.89%
FDX	5.6218	4.5568	3.6689	85.72%
HON	4.5356	4.1011	2.5571	83.41%
HPQ	3.5559	3.3228	1.9524	75.57%
IBM	3.3777	3.0468	2.0277	69.43%
JPM	2.8338	2.4880	2.0705	53.86%
MDZ	7.8518	5.7816	5.6762	91.97%
PEP	4.6374	4.1476	2.6429	84.28%
PRG	4.3166	3.9646	2.3446	82.05%
TWX	3.7861	3.3687	2.4973	75.04%
TXN	3.5967	3.1302	2.4412	70.87%
WFC	4.0498	3.4154	3.1662	67.61%

The table reports for each stock (the ticker is given in the first column) some results obtained from the ratio-statistic Z_{TP} , computed for each day t . The percentage of rejections is computed dividing the number of days in which the null hypothesis of no jumps is rejected, at the 5% level, by the total number of considered days.

Table C.0.2 Quantile regression results.

Variable	Coefficient Value	P-value
$\tau = 0.2$		
\overline{fpc}_{t-1}	0.33668	0.00001
\overline{fpc}_{5t-1}	0.15472	0.01578
vix_{t-1}	0.00021	0.00000
$sp500_{t-1}$	-0.00203	0.00139
$jump_{t-1}$	1.08254	0.00020
$\tau = 0.3$		
\overline{fpc}_{t-1}	0.32993	0.00000
\overline{fpc}_{5t-1}	0.23897	0.00003
vix_{t-1}	0.00020	0.00002
$sp500_{t-1}$	-0.00227	0.00071
$jump_{t-1}$	1.00748	0.00042
$\tau = 0.4$		
\overline{fpc}_{t-1}	0.33576	0.00004
\overline{fpc}_{5t-1}	0.30622	0.00000
vix_{t-1}	0.00017	0.00006
$sp500_{t-1}$	-0.00230	0.00080
$jump_{t-1}$	1.12156	0.00014
$\tau = 0.6$		
\overline{fpc}_{t-1}	0.49959	0.00000
\overline{fpc}_{5t-1}	0.34008	0.00001
vix_{t-1}	0.00016	0.00033
$sp500_{t-1}$	-0.00415	0.00000
$jump_{t-1}$	0.60128	0.09212
$\tau = 0.7$		
\overline{fpc}_{t-1}	0.54495	0.00001
\overline{fpc}_{5t-1}	0.38919	0.00006
vix_{t-1}	0.00018	0.00178
$sp500_{t-1}$	-0.00520	0.00000
$jump_{t-1}$	0.47827	0.14515
$\tau = 0.8$		
\overline{fpc}_{t-1}	0.86765	0.00060
\overline{fpc}_{5t-1}	0.35868	0.04776
vix_{t-1}	0.00006	0.50600
$sp500_{t-1}$	-0.00635	0.00000
$jump_{t-1}$	-0.05593	0.89486

The table reports the coefficients and the p-values for Model (3.5); $\tau = \{0.2, 0.3, 0.4, 0.6, 0.7, 0.8\}$. The standard errors are computed by means of the bootstrapping procedure, by employing the xy -pair method.

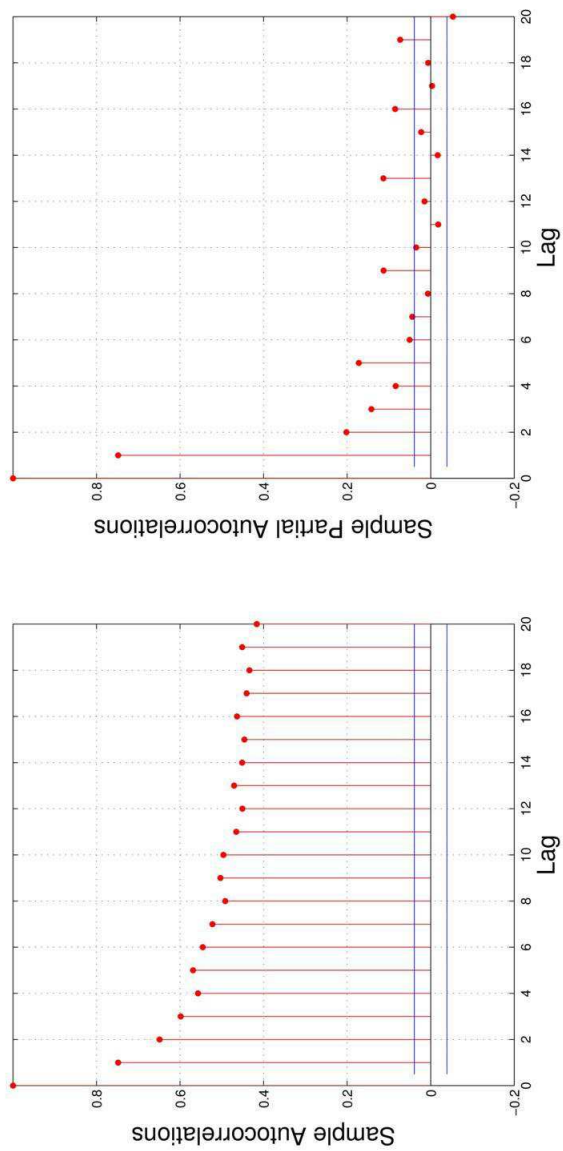


Fig. C.0.1 Autocorrelation and partial autocorrelation functions for the *fpc* series.

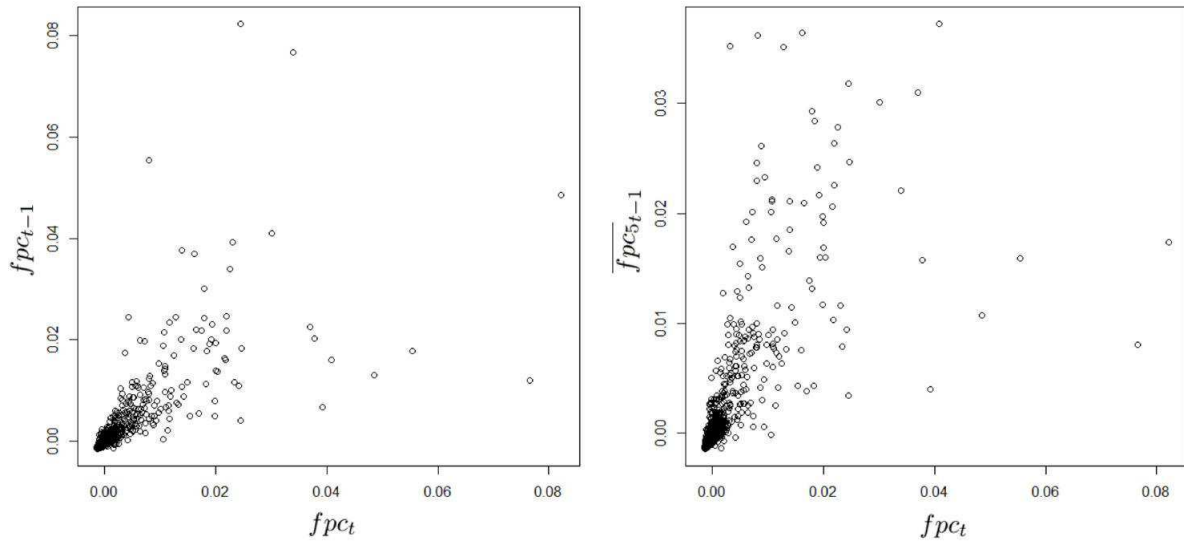
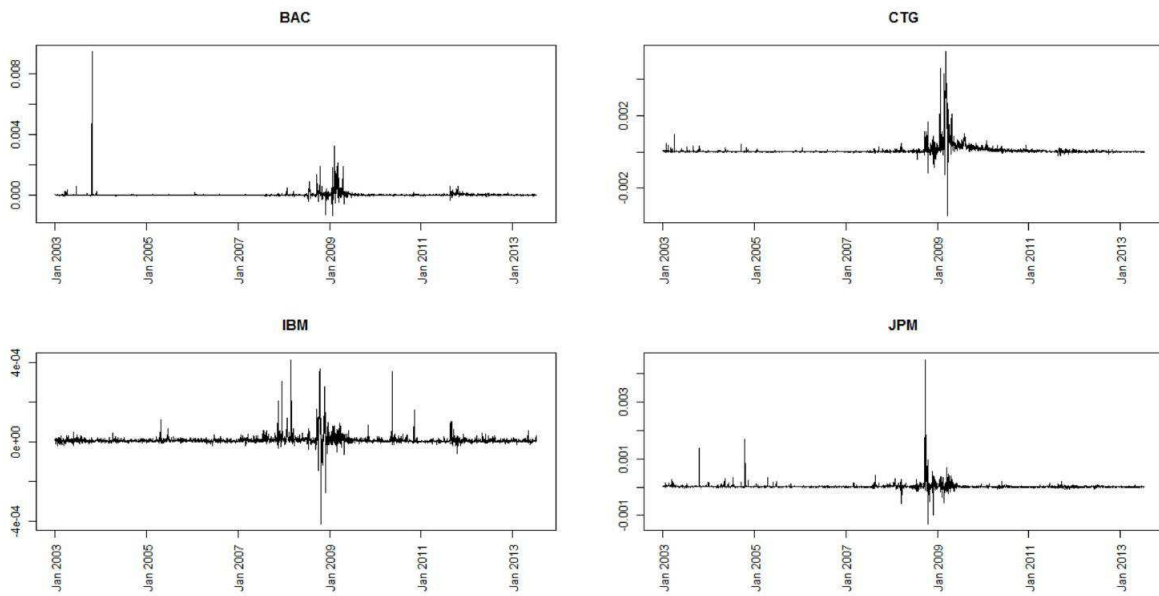
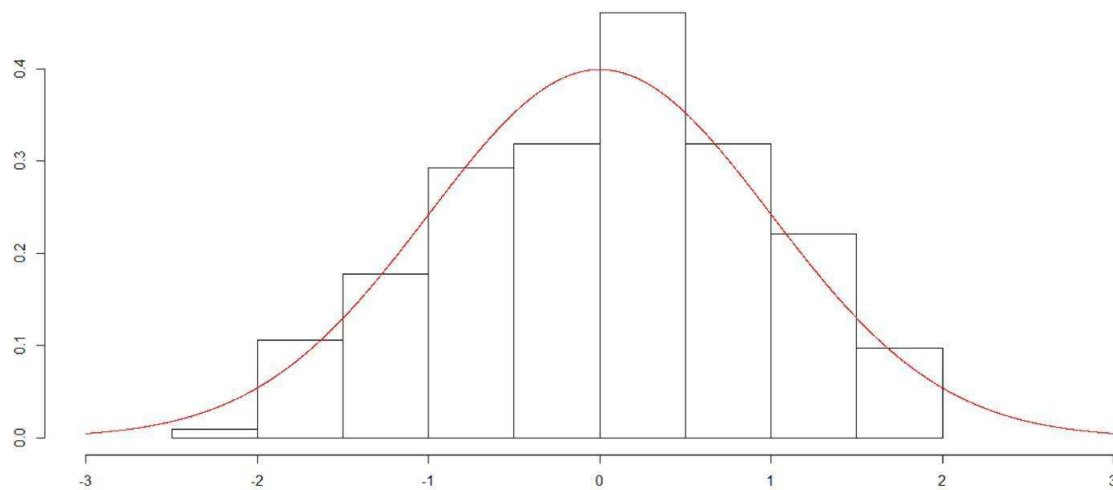
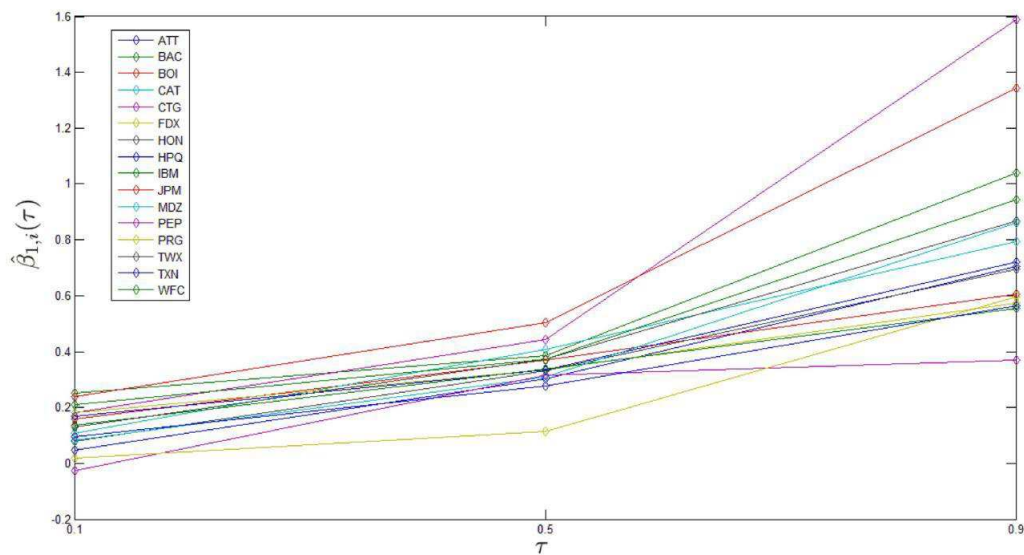


Fig. C.0.2 Scatter plots.

Fig. C.0.3 The trend over time of the jump component, measured by *JUMP*, for the companies *BAC*, *CTG*, *IBM* and *JPM*.

Fig. C.0.4 Distribution of z_t .Fig. C.0.5 The impacts of $rrv_{i,t-1}$ on the conditional volatility quantiles over τ for each asset.

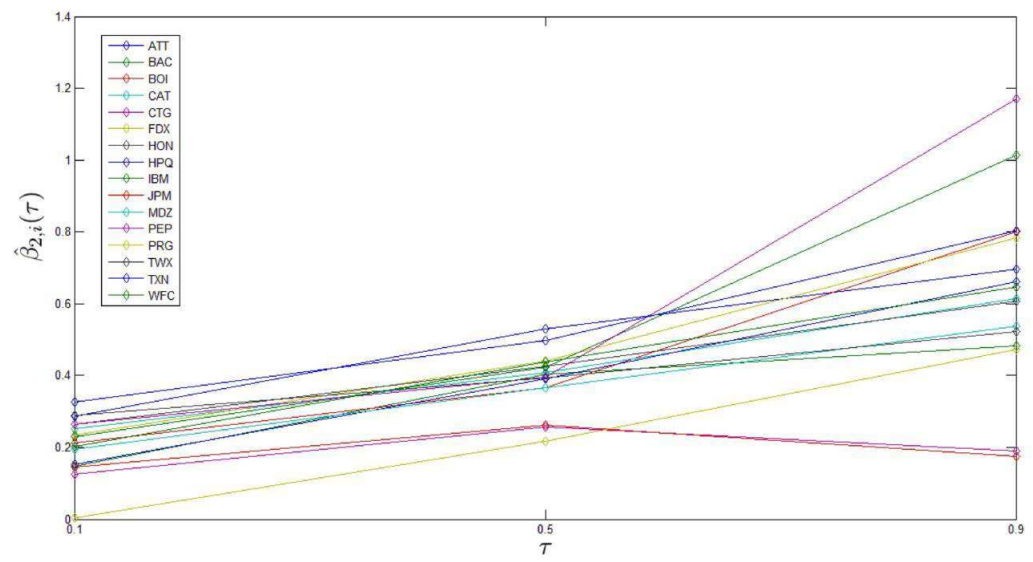


Fig. C.0.6 The impacts of $\overline{rrv}_{5,i,t-1}$ on the conditional volatility quantiles over τ for each asset.

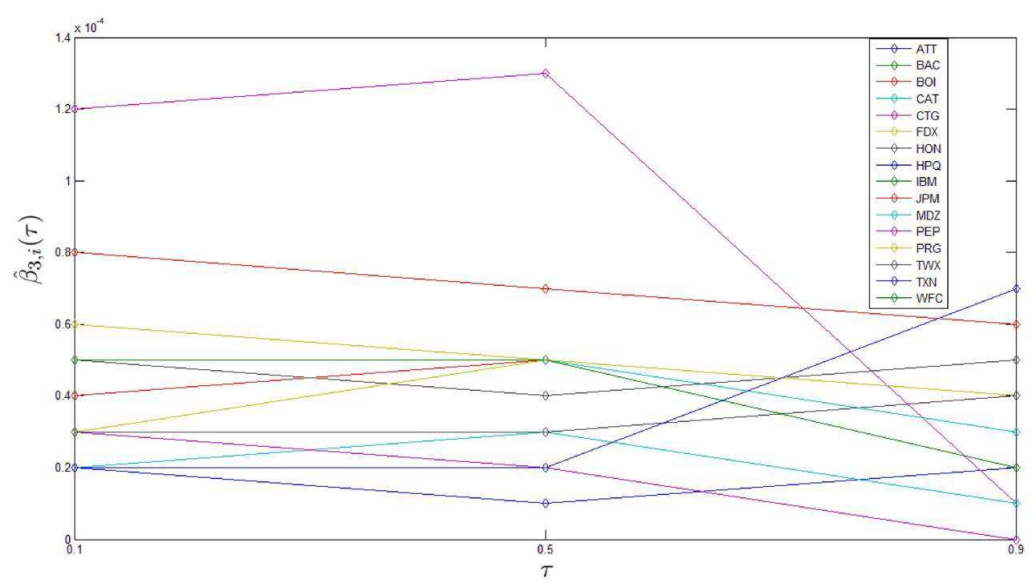


Fig. C.0.7 The impacts of vix_{t-1} on the conditional volatility quantiles over τ for each asset.

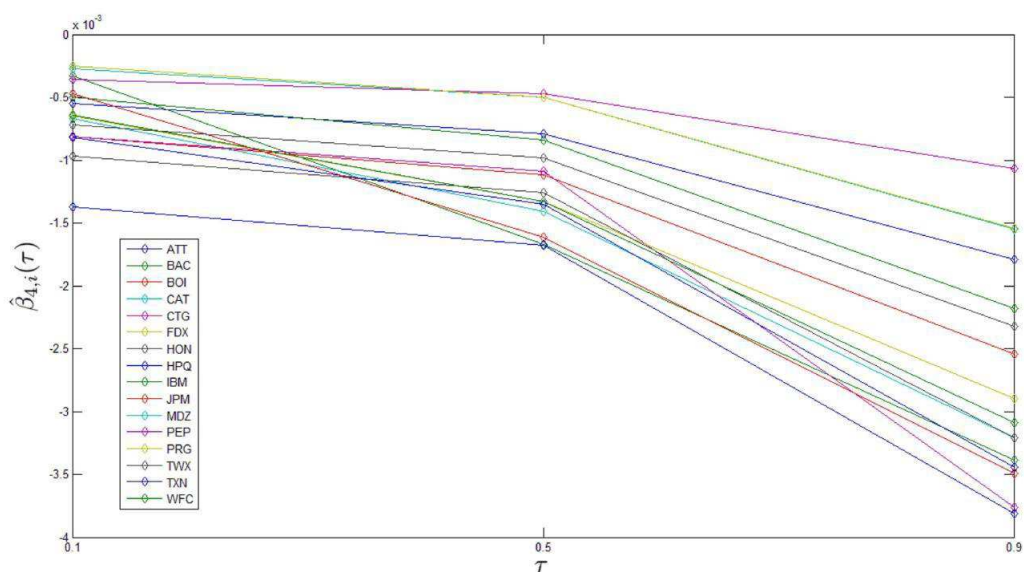


Fig. C.0.8 The impacts of $sp500_{t-1}$ on the conditional volatility quantiles over τ for each asset.

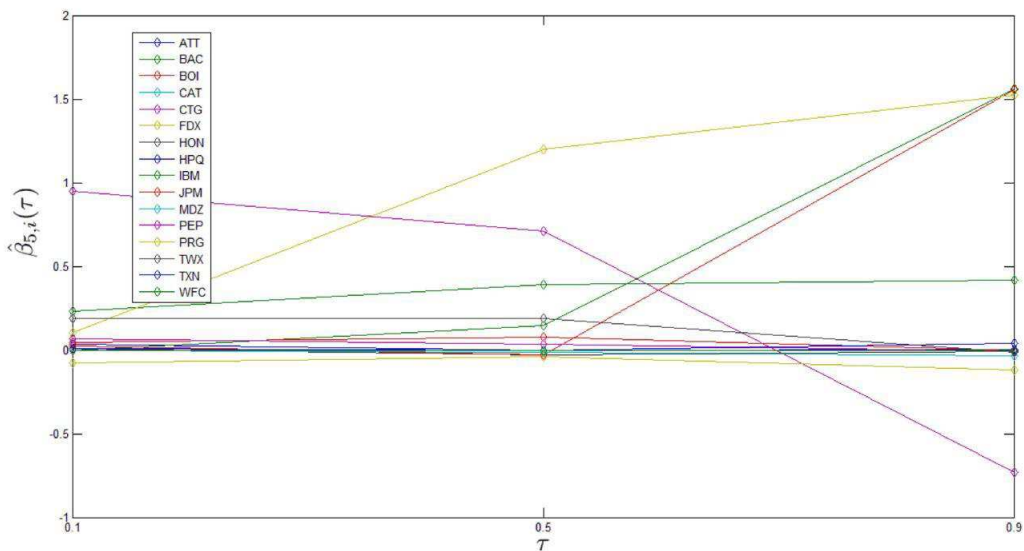


Fig. C.0.9 The impacts of $jump_{i,t-1}$ on the conditional volatility quantiles over τ for each asset.

