

DOI: 10.1515/auom-2016-0043
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VERSITA
Vol. 24(2), 2016, 283–299

Extremal problem on $(2n, 2m - 1)$ -system points on the rays.

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Abstract

In this work derivation of accurate estimate the production inner radius non-overlapping domains and open set. The problems arise such type in the first time in work [1]. It is late result this work generalize and strengthen in works [2 – 13]. In works [7, 8, 10] introduce the general systems points, the name n -radial systems points. In this work a success the draw generalize some results the work [7].

Let \mathbb{N} , \mathbb{R} – the sets natural and real numbers conformity, \mathbb{C} – the plain complex numbers, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ – the Riemannian sphere, $\mathbb{R}_+ = (0, \infty)$.

For fix number $n \in \mathbb{N}$ system points

$$A_{2n, 2m-1} = \{a_{k,p} \in \mathbb{C} : k = \overline{1, 2n}, p = \overline{1, 2m-1}\},$$

we will called on the $(2n, 2m - 1)$ -system points on the rays, if at all $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$ the relations are executed:

$$\begin{aligned} 0 < |a_{k,1}| < \dots < |a_{k,2m-1}| < \infty; \\ \arg a_{k,1} = \arg a_{k,2} = \dots = \arg a_{k,2m-1} =: \theta_k; \\ 0 = \theta_1 < \theta_2 < \dots < \theta_n < \theta_{n+1} := 2\pi. \end{aligned} \quad (1)$$

For such systems of points we will consider the following sizes:

$$\alpha_k = \frac{1}{\pi} [\theta_{k+1} - \theta_k], \quad k = \overline{1, 2n}, \quad \alpha_{n+1} := \alpha_1, \quad \alpha_0 := \alpha_n, \quad \sum_{k=1}^{2n} \alpha_k = 2.$$

Key Words: inner radius of domain, quadratic differential, piecewise-separating transformation, the Green function, radial systems of points, logarithmic capacity, variational formula.

2010 Mathematics Subject Classification: Primary 30C70, 30C75.

Received: 25.01.2014

Accepted: 20.02.2015

Let's consider system of angular domains:

$$P_k = \{w \in \mathbb{C} : \theta_k < \arg w < \theta_{k+1}\}, \quad k = \overline{1, 2n}.$$

Let's consider the following "operating" functionalities for arbitrary $(2n, 2m - 1)$ -system points on the rays

$$M(A_{2n, 2m-1}^{(1)}) = \prod_{k=1}^n \prod_{p=1}^m \left[\chi \left(\left| a_{2k-1, 2p-1} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \cdot \chi \left(\left| a_{2k-1, 2p-1} \right|^{\frac{1}{\alpha_{2k-2}}} \right) \right]^{\frac{1}{2}} \cdot |a_{2k-1, 2p-1}|,$$

$$M(A_{2n, 2m-1}^{(2)}) = \prod_{k=1}^n \prod_{p=1}^{m-1} \left[\chi \left(\left| a_{2k-1, 2p} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \cdot \chi \left(\left| a_{2k-1, 2p} \right|^{\frac{1}{\alpha_{2k-2}}} \right) \right]^{\frac{1}{2}} \cdot |a_{2k-1, 2p}|,$$

$$M(A_{2n, 2m-1}^{(3)}) = \prod_{k=1}^n \prod_{p=1}^{m-1} \left[\chi \left(\left| a_{2k, 2p} \right|^{\frac{1}{\alpha_{2k}}} \right) \cdot \chi \left(\left| a_{2k, 2p} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \right]^{\frac{1}{2}} \cdot |a_{2k, 2p}|,$$

$$M(A_{2n, 2m-1}^{(4)}) = \prod_{k=1}^n \prod_{p=1}^m \left[\chi \left(\left| a_{2k, 2p-1} \right|^{\frac{1}{\alpha_{2k}}} \right) \cdot \chi \left(\left| a_{2k, 2p-1} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \right]^{\frac{1}{2}} \cdot |a_{2k, 2p-1}|,$$

where $\chi(t) = \frac{1}{2}(t + \frac{1}{t})$, $t \in \mathbb{R}_+$.

Let $\{B_0, B_{k,p}\}$, $\{B_{k,p}, B_\infty\}$ — arbitrary non-overlapping domains such that

$$a_{k,p} \in B_{k,p}, \quad B_{k,p} \subset \overline{\mathbb{C}}, \quad k = \overline{1, 2n}, \quad p = \overline{1, 2m-1}.$$

Let D , $D \subset \overline{\mathbb{C}}$ — arbitrary open set and $w = a \in D$, then $D(a)$ the define connected component D , the contain point a . For arbitrary $(2n, 2m - 1)$ -system points on the rays $A_{2n, 2m-1} = \{a_{k,p} \in \mathbb{C} : k = \overline{1, 2n}, p = \overline{1, 2m-1}\}$ and open set D , $A_{2n, 2m-1} \subset D$ the define $D_k(a_{s,p})$ connected component set $D(a_{s,p}) \cap \overline{P_k}$, the contain point $a_{s,p}$, $k = \overline{1, 2n}$, $s = k, k + 1$, $p = \overline{1, 2m-1}$, $a_{n+1,p} := a_{1,p}$.

The open set D , $A_{2n, 2m-1} \subset D$ satisfied condition meets the condition of unapplied in relation to the system of points $(2n, 2m - 1)$ -system points on the rays $A_{2n, 2m-1}$ if a condition is executed

$$D_k(a_{k,s}) \cap D_k(a_{k+1,p}) = \emptyset, \tag{2}$$

$k = \overline{1, 2n}$, $p, s = \overline{1, 2m-1}$ on all corners $\overline{P_k}$.

The define $r(B; a)$ inner radius domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ (look [4 - 7, 14]).

Subject of studying of our work are the following problems.

Problem 1. Let $n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $\alpha \in \mathbb{R}_+$. Maximum functional be found

$$J = \prod_{k=1}^n \prod_{p=1}^m r^\alpha(B_{2k-1, 2p-1}, a_{2k-1, 2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r(B_{2k-1, 2p}, a_{2k-1, 2p}) \times$$

$$\times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha (B_{2k,2p}, a_{2k,2p}) \cdot \prod_{k=1}^n \prod_{p=1}^m r (B_{2k,2p-1}, a_{2k,2p-1}),$$

where $A_{2n,2m-1}$ – arbitrary $(2n, 2m - 1)$ -system points on the rays, satisfied condition (1), $\{B_{k,p}\}$ – arbitrary set non-overlapping domains, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$, and all extremal the describe $(k = \overline{1, 2n}, p = \overline{1, 2m - 1})$.

Problem 2. Let $n, m \in \mathbb{N}, n \geq 2, m \geq 2, \alpha \in \mathbb{R}_+$. Maximum functional be found

$$I = \prod_{k=1}^n \prod_{p=1}^m r^\alpha (D, a_{2k-1,2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r (D, a_{2k-1,2p}) \times \\ \times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha (D, a_{2k,2p}) \cdot \prod_{k=1}^n \prod_{p=1}^m r (D, a_{2k,2p-1}),$$

where $A_{2n,2m-1}$ – arbitrary $(2n, 2m - 1)$ -system points on the rays, satisfied condition (1), D – arbitrary open set, the satisfied condition (2), $a_{k,p} \in D \subset \overline{\mathbb{C}}$, and all extremal the describe $(k = \overline{1, 2n}, p = \overline{1, 2m - 1})$.

Theorem 1. Let $n, m \in \mathbb{N}, n \geq 2, \alpha \in \mathbb{R}_+$. Then for all $(2n, 2m - 1)$ -system points on the rays $A_{2n,2m-1}$, and arbitrary set non-overlapping domains $\{B_{k,p}\}, a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}, k = \overline{1, 2n}, p = \overline{1, 2m - 1}$ be satisfied inequality

$$\prod_{k=1}^n \prod_{p=1}^m r^\alpha (B_{2k-1,2p-1}, a_{2k-1,2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r (B_{2k-1,2p}, a_{2k-1,2p}) \times \\ \times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha (B_{2k,2p}, a_{2k,2p}) \cdot \prod_{k=1}^n \prod_{p=1}^m r (B_{2k,2p-1}, a_{2k,2p-1}) \leq \\ \leq \left(\frac{4}{(2m - 1)n} \right)^{n(2m-1)(\alpha+1)} \cdot \left(M \left(A_{2n,2m-1}^{(1)} \right) \cdot M \left(A_{2n,2m-1}^{(3)} \right) \right)^\alpha \times \\ \times M \left(A_{2n,2m-1}^{(2)} \right) \cdot M \left(A_{2n,2m-1}^{(4)} \right) \cdot \left(\frac{\alpha^\alpha}{|\sqrt{\alpha} - 1| |\sqrt{\alpha - 1}|^2 |\sqrt{\alpha} + 1| |\sqrt{\alpha + 1}|^2} \right)^{n \frac{2m-1}{2}}.$$

The equality obtain in this inequality, when points $a_{k,p}$ and domains $B_{k,p}$ are, conformity, the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = w^{2n-2} (1 + w^{2n})^{2m-3} \times \\ \times \frac{i(\alpha - 1) \left((w^n + i)^{4m-2} - (w^n - i)^{4m-2} \right) - 2(1 + \alpha) (w^{2n} + 1)^{2m-1}}{\left((w^n + i)^{4m-2} + (w^n - i)^{4m-2} \right)^2} dw^2. \tag{3}$$

Theorem 2. *Let $n, m \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{R}_+$. Then for all $(2n, 2m - 1)$ -system points on the rays, arbitrary open set D , the satisfied condition (2), $a_{k,p} \in D \subset \overline{\mathbb{C}}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$ be satisfied inequality*

$$\begin{aligned} & \prod_{k=1}^n \prod_{p=1}^m r^\alpha(D, a_{2k-1, 2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r(D, a_{2k-1, 2p}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha(D, a_{2k, 2p}) \times \\ & \times \prod_{k=1}^n \prod_{p=1}^m r(D, a_{2k, 2p-1}) \leq \left(\frac{4}{(2m-1)n} \right)^{n(2m-1)(\alpha+1)} \times \\ & \times \left(M(A_{2n, 2m-1}^{(1)}) \cdot M(A_{2n, 2m-1}^{(3)}) \right)^\alpha \cdot M(A_{2n, 2m-1}^{(2)}) \cdot M(A_{2n, 2m-1}^{(4)}) \times \\ & \times \left(\frac{\alpha^\alpha}{|\sqrt{\alpha} - 1| |\sqrt{\alpha-1}|^2 |\sqrt{\alpha} + 1| |\sqrt{\alpha+1}|^2} \right)^{n \frac{2m-1}{2}}. \end{aligned}$$

The equality obtain in this inequality, when

$$D = \bigcup_{k,p} B_{k,p},$$

and points $a_{k,p}$ and domains $B_{k,p}$ are, conformity, the poles and the circular domains of the quadratic differential (3).

Proof theorem 1. The theorem of the proof leans on a method of the piece-dividing transformation developed by Dubinin (look [4 - 7]).

Function

$$z_k(w) = -i \left(e^{-i\theta_k w} \right)^{\frac{1}{\alpha_k}} \tag{4}$$

realizes univalent and conformal transformations of domain P_k to the right half-plane $\text{Re}z > 0$, for all $k = \overline{1, 2n}$.

Then function

$$\zeta_k(w) := \frac{1 - z_k(w)}{1 + z_k(w)} \tag{5}$$

maps univalent and conformal domain P_k on the unit circle $U = \{z : |z| \leq 1\}$, $k = \overline{1, 2n}$.

The define $\omega_{k,p}^{(1)} := \zeta_k(a_{k,p})$, $\omega_{k-1,p}^{(2)} := \zeta_{k-1}(a_{k,p})$, $a_{n+1,p} := a_{1,p}$, $\omega_{0,p}^{(2)} := \omega_{n,p}^{(2)}$, $\zeta_0 := \zeta_n$ ($k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$).

Family of functions $\{\zeta_k(w)\}_{k=1}^{2n}$, set by equality (5), it is possible for by piece-dividing transformation (look [4 - 7]) domains $\{B_{k,p} : k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}\}$ in relation to the system of corners $\{P_k\}_{k=1}^{2n}$. For any domain $\Delta \in \mathbb{C}$ the define $(\Delta)^* := \{w \in \overline{\mathbb{C}} : \frac{1}{\overline{w}} \in \Delta\}$. Let $\Omega_{k,p}^{(1)}$ the define connected

component $\zeta_k (B_{k,p} \cap \overline{P}_k) \cup (\zeta_k (B_{k,p} \cap \overline{P}_k))^*$, containing a point $\omega_{k,p}^{(1)}$, $\Omega_{k-1,p}^{(2)}$ - the define connected component $\zeta_{k-1} (B_{k,p} \cap \overline{P}_{k-1}) \cup (\zeta_{k-1} (B_{k,p} \cap \overline{P}_{k-1}))^*$, containing a point $\omega_{k-1,p}^{(2)}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$, $\overline{P}_0 := \overline{P}_n$, $\Omega_{0,p}^{(2)} := \Omega_{n,p}^{(2)}$. It is clear, that, $\Omega_{k,p}^{(s)}$ generally speaking, domains are multiconnected domains, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$, $s = 1, 2$. Pair of domains $\Omega_{k-1,p}^{(2)}$ and $\Omega_{k,p}^{(1)}$ grows out of piece-dividing transformation domains $B_{k,p}$ concerning families $\{P_{k-1}, P_k\}$, $\{\zeta_{k-1}, \zeta_k\}$ in point $a_{k,p}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$.

From a formula (5) we receive the following asymptotic expressions

$$\begin{aligned} |\zeta_k(w) - \zeta_k(a_{k,p})| &\sim \left[\alpha_k \cdot \chi \left(\left| a_{k,p} \right|^{\frac{1}{\alpha_k}} \right) \left| a_{k,p} \right| \right]^{-1} \cdot |w - a_{k,p}|, \quad w \rightarrow a_{k,p}, \quad w \in \overline{P}_k. \\ |\zeta_{k-1}(w) - \zeta_{k-1}(a_{k,p})| &\sim \left[\alpha_{k-1} \cdot \chi \left(\left| a_{k,p} \right|^{\frac{1}{\alpha_{k-1}}} \right) \left| a_{k,p} \right| \right]^{-1} \cdot |w - a_{k,p}|, \\ w &\rightarrow a_{k,p}, \quad w \in \overline{P}_{k-1}, \quad k = \overline{1, 2n}, p = \overline{1, 2m - 1}. \end{aligned} \tag{6}$$

From the theorem 1.9 [6] (look also [4, 5]) and formulas (6) we receive inequalities

$$\begin{aligned} r(B_{k,p}, a_{k,p}) &\leq \left\{ r \left(\Omega_{k,p}^{(1)}, \omega_{k,p}^{(1)} \right) \cdot r \left(\Omega_{k-1,p}^{(2)}, \omega_{k-1,p}^{(2)} \right) \cdot \left[\alpha_k \cdot \chi \left(\left| a_{k,p} \right|^{\frac{1}{\alpha_k}} \right) \left| a_{k,p} \right| \right] \right. \\ &\quad \left. \times \left[\alpha_{k-1} \cdot \chi \left(\left| a_{k,p} \right|^{\frac{1}{\alpha_{k-1}}} \right) \left| a_{k,p} \right| \right] \right\}^{\frac{1}{2}}, \quad k = \overline{1, 2n}, p = \overline{1, 2m - 1}. \end{aligned} \tag{7}$$

Using formulas (7), it is received the following ratio:

$$\begin{aligned} &\prod_{k=1}^n \prod_{p=1}^m r^\alpha (B_{2k-1, 2p-1}, a_{2k-1, 2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r (B_{2k-1, 2p}, a_{2k-1, 2p}) \times \\ &\quad \times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha (B_{2k, 2p}, a_{2k, 2p}) \cdot \prod_{k=1}^n \prod_{p=1}^m r (B_{2k, 2p-1}, a_{2k, 2p-1}) \leq \\ &\quad \leq \prod_{k=1}^n \prod_{p=1}^m \left[\alpha_{2k-1} \cdot \alpha_{2k-2} \cdot \left(\chi \left(\left| a_{2k-1, 2p-1} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \right) \right] \times \\ &\quad \times \left(\chi \left(\left| a_{2k-1, 2p-1} \right|^{\frac{1}{\alpha_{2k-2}}} \right) \right) \cdot |a_{2k-1, 2p-1}|^2 \right]^{\frac{\alpha}{2}} \times \\ &\quad \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[\alpha_{2k-1} \cdot \alpha_{2k-2} \cdot \left(\chi \left(\left| a_{2k-1, 2p} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \right) \right] \times \end{aligned}$$

$$\begin{aligned}
 & \times \left(\chi \left(\left| a_{2k-1,2p} \right|^{\frac{1}{\alpha_{2k-2}}} \right) \right) \cdot |a_{2k-1,2p}|^2 \Big]^{\frac{1}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[\alpha_{2k} \cdot \alpha_{2k-1} \cdot \chi \left(\left| a_{2k,2p} \right|^{\frac{1}{\alpha_{2k}}} \right) \cdot \chi \left(\left| a_{2k,2p} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \cdot |a_{2k,2p}|^2 \right]^{\frac{\alpha}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^m \left[\alpha_{2k} \cdot \alpha_{2k-1} \cdot \chi \left(\left| a_{2k,2p-1} \right|^{\frac{1}{\alpha_{2k}}} \right) \cdot \chi \left(\left| a_{2k,2p-1} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \cdot |a_{2k,2p-1}|^2 \right]^{\frac{1}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^m \left[r \left(\Omega_{2k-1,2p-1}^{(1)}, \omega_{2k-1,2p-1}^{(1)} \right) \cdot r \left(\Omega_{2k-2,2p-1}^{(2)}, \omega_{2k-2,2p-1}^{(2)} \right) \right]^{\frac{\alpha}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[r \left(\Omega_{2k-1,2p}^{(1)}, \omega_{2k-1,2p}^{(1)} \right) \cdot r \left(\Omega_{2k-2,2p}^{(2)}, \omega_{2k-2,2p}^{(2)} \right) \right]^{\frac{1}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[r \left(\Omega_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)} \right) \cdot r \left(\Omega_{2k-1,2p}^{(2)}, \omega_{2k-1,2p}^{(2)} \right) \right]^{\frac{\alpha}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^m \left[r \left(\Omega_{2k,2p-1}^{(1)}, \omega_{2k,2p-1}^{(1)} \right) \cdot r \left(\Omega_{2k-1,2p-1}^{(2)}, \omega_{2k-1,2p-1}^{(2)} \right) \right]^{\frac{1}{2}}. \quad (8)
 \end{aligned}$$

Let's note that

$$\begin{aligned}
 & \prod_{k=1}^n \prod_{p=1}^m \left[r \left(\Omega_{2k-1,2p-1}^{(1)}, \omega_{2k-1,2p-1}^{(1)} \right) \cdot r \left(\Omega_{2k-2,2p-1}^{(2)}, \omega_{2k-2,2p-1}^{(2)} \right) \right]^{\frac{\alpha}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[r \left(\Omega_{2k-1,2p}^{(1)}, \omega_{2k-1,2p}^{(1)} \right) \cdot r \left(\Omega_{2k-2,2p}^{(2)}, \omega_{2k-2,2p}^{(2)} \right) \right]^{\frac{1}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[r \left(\Omega_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)} \right) \cdot r \left(\Omega_{2k-1,2p}^{(2)}, \omega_{2k-1,2p}^{(2)} \right) \right]^{\frac{\alpha}{2}} \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^m \left[r \left(\Omega_{2k,2p-1}^{(1)}, \omega_{2k,2p-1}^{(1)} \right) \cdot r \left(\Omega_{2k-1,2p-1}^{(2)}, \omega_{2k-1,2p-1}^{(2)} \right) \right]^{\frac{1}{2}} = \\
 & = \prod_{k=1}^n \left\{ \prod_{p=1}^m r^\alpha \left(\Omega_{2k-1,2p-1}^{(1)}, \omega_{2k-1,2p-1}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k-1,2p}^{(1)}, \omega_{2k-1,2p}^{(1)} \right) \times \right. \\
 & \times \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k-1,2p}^{(2)}, \omega_{2k-1,2p}^{(2)} \right) \cdot \prod_{p=1}^m r \left(\Omega_{2k-1,2p-1}^{(2)}, \omega_{2k-1,2p-1}^{(2)} \right) \times
 \end{aligned}$$

$$\begin{aligned} & \times \prod_{p=1}^m r^\alpha \left(\Omega_{2k,2p-1}^{(2)}, \omega_{2k,2p-1}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k,2p}^{(2)}, \omega_{2k,2p}^{(2)} \right) \times \\ & \times \left. \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)} \right) \cdot \prod_{p=1}^m r \left(\Omega_{2k,2p-1}^{(1)}, \omega_{2k,2p-1}^{(1)} \right) \right\}^{\frac{1}{2}}, \end{aligned} \tag{9}$$

$$\begin{aligned} & \prod_{k=1}^n \prod_{p=1}^m [\alpha_{2k-1} \cdot \alpha_{2k-2}]^{\frac{\alpha}{2}} \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} [\alpha_{2k-1} \cdot \alpha_{2k-2}]^{\frac{1}{2}} \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} [\alpha_{2k} \cdot \alpha_{2k-1}]^{\frac{\alpha}{2}} \times \\ & \times \prod_{k=1}^n \prod_{p=1}^m [\alpha_{2k} \cdot \alpha_{2k-1}]^{\frac{1}{2}} = \prod_{k=1}^{2n} \alpha_k^{\frac{2m-1}{2}(\alpha+1)}, \end{aligned} \tag{10}$$

$$\begin{aligned} & \prod_{k=1}^n \prod_{p=1}^m \left[\left(\chi \left(\left| a_{2k-1,2p-1} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \right) \cdot \left(\chi \left(\left| a_{2k-1,2p-1} \right|^{\frac{1}{\alpha_{2k-2}}} \right) \right) \cdot |a_{2k-1,2p-1}|^2 \right]^{\frac{\alpha}{2}} \times \\ & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[\left(\chi \left(\left| a_{2k-1,2p} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \right) \cdot \left(\chi \left(\left| a_{2k-1,2p} \right|^{\frac{1}{\alpha_{2k-2}}} \right) \right) \cdot |a_{2k-1,2p}|^2 \right]^{\frac{1}{2}} \times \\ & \times \prod_{k=1}^n \prod_{p=1}^{m-1} \left[\chi \left(\left| a_{2k,2p} \right|^{\frac{1}{\alpha_{2k}}} \right) \cdot \chi \left(\left| a_{2k,2p} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \cdot |a_{2k,2p}|^2 \right]^{\frac{\alpha}{2}} \times \\ & \times \prod_{k=1}^n \prod_{p=1}^m \left[\chi \left(\left| a_{2k,2p-1} \right|^{\frac{1}{\alpha_{2k}}} \right) \cdot \chi \left(\left| a_{2k,2p-1} \right|^{\frac{1}{\alpha_{2k-1}}} \right) \cdot |a_{2k,2p-1}|^2 \right]^{\frac{1}{2}} = \\ & = \left(M \left(A_{2n,2m-1}^{(1)} \right) \cdot M \left(A_{2n,2m-1}^{(3)} \right) \right)^\alpha \cdot M \left(A_{2n,2m-1}^{(2)} \right) \cdot M \left(A_{2n,2m-1}^{(4)} \right). \end{aligned} \tag{11}$$

Then from (8) using formulas (9), (10), (11), it is received the following ratio

$$\begin{aligned} & \prod_{k=1}^n \prod_{p=1}^m r^\alpha (B_{2k-1,2p-1}, a_{2k-1,2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r (B_{2k-1,2p}, a_{2k-1,2p}) \times \\ & \times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha (B_{2k,2p}, a_{2k,2p}) \cdot \prod_{k=1}^n \prod_{p=1}^m r (B_{2k,2p-1}, a_{2k,2p-1}) \leq \\ & \leq \prod_{k=1}^{2n} \alpha_k^{\frac{2m-1}{2}(\alpha+1)} \cdot \left(M \left(A_{2n,2m-1}^{(1)} \right) \cdot M \left(A_{2n,2m-1}^{(3)} \right) \right)^\alpha \cdot M \left(A_{2n,2m-1}^{(2)} \right) \cdot M \left(A_{2n,2m-1}^{(4)} \right) \times \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{k=1}^n \left\{ \prod_{p=1}^m r^\alpha \left(\Omega_{2k-1, 2p-1}^{(1)}, \omega_{2k-1, 2p-1}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k-1, 2p}^{(1)}, \omega_{2k-1, 2p}^{(1)} \right) \times \right. \\
 & \times \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k-1, 2p}^{(2)}, \omega_{2k-1, 2p}^{(2)} \right) \cdot \prod_{p=1}^m r \left(\Omega_{2k-1, 2p-1}^{(2)}, \omega_{2k-1, 2p-1}^{(2)} \right) \times \\
 & \times \prod_{p=1}^m r^\alpha \left(\Omega_{2k, 2p-1}^{(2)}, \omega_{2k, 2p-1}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k, 2p}^{(2)}, \omega_{2k, 2p}^{(2)} \right) \times \\
 & \left. \times \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k, 2p}^{(1)}, \omega_{2k, 2p}^{(1)} \right) \cdot \prod_{p=1}^m r \left(\Omega_{2k, 2p-1}^{(1)}, \omega_{2k, 2p-1}^{(1)} \right) \right\}^{\frac{1}{2}}. \quad (12)
 \end{aligned}$$

Considering that

$$\prod_{k=1}^{2n} \alpha_k \leq \left(\frac{1}{n} \right)^{2n},$$

from the previous ratio we receive

$$\begin{aligned}
 & \prod_{k=1}^n \prod_{p=1}^m r^\alpha (B_{2k-1, 2p-1}, a_{2k-1, 2p-1}) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r (B_{2k-1, 2p}, a_{2k-1, 2p}) \times \\
 & \times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha (B_{2k, 2p}, a_{2k, 2p}) \cdot \prod_{k=1}^n \prod_{p=1}^m r (B_{2k, 2p-1}, a_{2k, 2p-1}) \leq \\
 & \leq \left(\frac{1}{n} \right)^{n(2m-1)(\alpha+1)} \cdot \left(M(A_{2n, 2m-1}^{(1)}) \cdot M(A_{2n, 2m-1}^{(3)}) \right)^\alpha \cdot M(A_{2n, 2m-1}^{(2)}) \cdot M(A_{2n, 2m-1}^{(4)}) \times \\
 & \times \prod_{k=1}^n \left\{ \prod_{p=1}^m r^\alpha \left(\Omega_{2k-1, 2p-1}^{(1)}, \omega_{2k-1, 2p-1}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k-1, 2p}^{(1)}, \omega_{2k-1, 2p}^{(1)} \right) \times \right. \\
 & \times \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k-1, 2p}^{(2)}, \omega_{2k-1, 2p}^{(2)} \right) \cdot \prod_{p=1}^m r \left(\Omega_{2k-1, 2p-1}^{(2)}, \omega_{2k-1, 2p-1}^{(2)} \right) \times \\
 & \times \prod_{p=1}^m r^\alpha \left(\Omega_{2k, 2p-1}^{(2)}, \omega_{2k, 2p-1}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k, 2p}^{(2)}, \omega_{2k, 2p}^{(2)} \right) \times \\
 & \left. \times \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k, 2p}^{(1)}, \omega_{2k, 2p}^{(1)} \right) \cdot \prod_{p=1}^m r \left(\Omega_{2k, 2p-1}^{(1)}, \omega_{2k, 2p-1}^{(1)} \right) \right\}^{\frac{1}{2}}. \quad (13)
 \end{aligned}$$

from the theorem 4.2.2 [7] inequalities follow

$$\begin{aligned} & \prod_{p=1}^m r^\alpha \left(\Omega_{2k-1, 2p-1}^{(1)}, \omega_{2k-1, 2p-1}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k-1, 2p}^{(1)}, \omega_{2k-1, 2p}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k-1, 2p}^{(2)}, \omega_{2k-1, 2p}^{(2)} \right) \times \\ & \times \prod_{p=1}^m r \left(\Omega_{2k-1, 2p-1}^{(2)}, \omega_{2k-1, 2p-1}^{(2)} \right) \leq \prod_{p=1}^{2m-1} r^\alpha \left(G_{2p-1}^{(1)}, g_{2p-1}^{(1)} \right) \cdot r \left(G_{2p}^{(1)}, g_{2p}^{(1)} \right), \\ & \prod_{p=1}^m r^\alpha \left(\Omega_{2k, 2p-1}^{(2)}, \omega_{2k, 2p-1}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k, 2p}^{(2)}, \omega_{2k, 2p}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r^\alpha \left(\Omega_{2k, 2p}^{(1)}, \omega_{2k, 2p}^{(1)} \right) \times \\ & \times \prod_{p=1}^m r \left(\Omega_{2k, 2p-1}^{(1)}, \omega_{2k, 2p-1}^{(1)} \right) \leq \prod_{p=1}^{2m-1} r \left(G_{2p-1}^{(2)}, g_{2p-1}^{(2)} \right) \cdot r^\alpha \left(G_{2p}^{(2)}, g_{2p}^{(2)} \right), \quad (14) \end{aligned}$$

where $G_{2p-1}^{(1)}, G_{2p}^{(1)}, G_{2p-1}^{(2)}, G_{2p}^{(2)}$ – system circular domains, $g_{2p-1}^{(1)}, g_{2p}^{(1)}, g_{2p-1}^{(2)}, g_{2p}^{(2)}$ – the poles of the quadratic differential

$$Q(\zeta_k) d\zeta_k^2 = \zeta_k^{2m-3} \cdot \frac{i(1-\alpha)\zeta_k^{4m-2} + 2(1+\alpha)\zeta_k^{2m-1} + i(\alpha-1)}{(\zeta_k^{4m-2} + 1)^2} \cdot d\zeta_k^2, \quad k = \overline{1, 2n} \quad (15)$$

From (14) we receive, using inequalities (13)

$$\begin{aligned} & \prod_{k=1}^n \prod_{p=1}^m r^\alpha \left(B_{2k-1, 2p-1}, a_{2k-1, 2p-1} \right) \cdot \prod_{k=1}^n \prod_{p=1}^{m-1} r \left(B_{2k-1, 2p}, a_{2k-1, 2p} \right) \times \\ & \times \prod_{k=1}^n \prod_{p=1}^{m-1} r^\alpha \left(B_{2k, 2p}, a_{2k, 2p} \right) \cdot \prod_{k=1}^n \prod_{p=1}^m r \left(B_{2k, 2p-1}, a_{2k, 2p-1} \right) \leq \\ & \leq \left(\frac{1}{n} \right)^{n(2m-1)(\alpha+1)} \cdot \left(M \left(A_{2n, 2m-1}^{(1)} \right) \cdot M \left(A_{2n, 2m-1}^{(3)} \right) \right)^\alpha \cdot M \left(A_{2n, 2m-1}^{(2)} \right) \cdot M \left(A_{2n, 2m-1}^{(4)} \right) \times \\ & \times \left(\prod_{p=1}^{2m-1} r^\alpha \left(G_{2p-1}, g_{2p-1} \right) \cdot r \left(G_{2p}, g_{2p} \right) \right)^n, \quad (16) \end{aligned}$$

where G_{2p-1}, G_{2p} – system circular domains, g_{2p-1}, g_{2p} – the poles of the quadratic differential (15).

From the last ratio, the approval of the theorem follows, using the theorem 4.1.2 [7]. **The theorem is proved.**

Proof theorem 2. At once we will note that from the condition of unapplying follows that $\text{cap } \overline{\mathbb{C}} \setminus D > 0$ and set D possesses Green’s generalized function

$$g_D(z, a), \text{ where } g_D(z, a) = \begin{cases} g_{D(a)}(z, a), & z \in D(a), \\ 0, & z \in \overline{\mathbb{C} \setminus D(a)}, \\ \lim_{\zeta \rightarrow z} g_{D(a)}(\zeta, a), & \zeta \in D(a), z \in \partial D(a) \end{cases} \quad \text{– Green’s}$$

generalized function open set D concerning a point $a \in D$, and $g_{D(a)}(z, a)$ – Green’s function domain $D(a)$ concerning a point $a \in D(a)$.

Further, we will use methods of works [6, 7]. Sets we will consider $E_0 = \overline{\mathbb{C}} \setminus D$; $E(a_{k,p}, t) = \{w \in \mathbb{C} : |w - a_{k,p}| \leq t\}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$, $n \geq 3$, $n, m \in \mathbb{N}$, $t \in \mathbb{R}_+$. The condenser we will enter into consideration for rather small $t > 0$

$$C(t, D, A_{2n,2m-1}) = \{E_0, E_1, E_2\},$$

where

$$E_1 = \bigcup_{k=1}^n \bigcup_{p=1}^m E(a_{2k-1,2p-1}, t) \bigcup_{k=1}^n \bigcup_{p=1}^{m-1} E(a_{2k,2p}, t),$$

$$E_2 = \bigcup_{k=1}^n \bigcup_{p=1}^m E(a_{2k,2p-1}, t) \bigcup_{k=1}^n \bigcup_{p=1}^{m-1} E(a_{2k-1,2p}, t)$$

. Capacity of the condenser $C(t, D, A_{2n,2m-1})$ is called as (look [5])

$$\text{cap}C(t, D, A_{2n,2m-1}) = \inf \int \int [(G'_x)^2 + (G'_y)^2] \, dx dy,$$

where an infimum undertakes on all continuous and to the lipschicevym in $\overline{\mathbb{C}}$ functions $G = G(z)$, such that $G|_{E_0} = 0$, $G|_{E_1} = \sqrt{\alpha}$, $G|_{E_2} = 1$

Let is named the module of condenser C , reverse the capacity of condenser

$$|C| = [\text{cap}C]^{-1}$$

From a theorem 1 [6] get

$$|C(t, D, A_{2n,2m-1})| = \frac{1}{2\pi} \cdot \frac{1}{n(2m - 1)(\alpha + 1)} \cdot \log \frac{1}{t} + M(D, A_{2n,2m-1}) + o(1), \quad t \rightarrow 0, \tag{17}$$

where

$$M(D, A_{2n,2m-1}) = \frac{1}{2\pi} \cdot \frac{1}{n^2(2m - 1)^2 \cdot (\alpha + 1)^2} \cdot \left[\alpha \sum_{k=1}^n \sum_{p=1}^{m-1} \log r(D, a_{2k,2p}) + \right.$$

$$\left. + \sum_{k=1}^n \sum_{p=1}^m \log r(D, a_{2k,2p-1}) + \sum_{k=1}^n \sum_{p=1}^{m-1} \log r(D, a_{2k-1,2p}) + \right.$$

$$\left. + \alpha \sum_{k=1}^n \sum_{p=1}^m \log r(D, a_{2k-1,2p-1}) + \right.$$

$$\begin{aligned}
 & +\alpha \sum_{(k,p) \neq (q,s)} (g_D(a_{2k,2p}, a_{2q,2s}) + g_D(a_{2k-1,2p-1}, a_{2q-1,2s-1})) + \\
 & +2\alpha \sum_{(k,p) \neq (q,s)} g_D(a_{2k-1,2p-1}, a_{2q,2s}) + 2\sqrt{\alpha} \sum_{(k,p) \neq (q,s)} (g_D(a_{2k,2p}, a_{2q-1,2s}) + \\
 & +g_D(a_{2k,2p}, a_{2q,2s-1}) + g_D(a_{2k-1,2p-1}, a_{2q-1,2s}) + g_D(a_{2k-1,2p-1}, a_{2q,2s-1})) + \\
 & + \sum_{(k,p) \neq (q,s)} (g_D(a_{2k-1,2p}, a_{2q-1,2s}) + g_D(a_{2k,2p-1}, a_{2q,2s-1})) + \\
 & +2 \sum_{(k,p) \neq (q,s)} g_D(a_{2k-1,2p}, a_{2q,2s-1}). \tag{18}
 \end{aligned}$$

Function (5) and definition $\omega_{k,p}^{(1)}$, $\omega_{k-1,p}^{(2)}$, $a_{n+1,p}$, $\omega_{0,p}^{(2)}$, ζ_0 , Δ , $(\Delta)^*$, using, by us the theorems entered at proof 1. Let, too, $\Omega_{k,p}^{(1)}$ define connected component $\zeta_k (D \cap \overline{P}_k) \cup (\zeta_k (D \cap \overline{P}_k))^*$, containing a point $\omega_{k,p}^{(1)}$, $\Omega_{k-1,p}^{(2)}$ - connected component $\zeta_{k-1} (D \cap \overline{P}_{k-1}) \cup (\zeta_{k-1} (D \cap \overline{P}_{k-1}))^*$, containing a point $\omega_{k-1,p}^{(2)}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$, $\overline{P}_0 := \overline{P}_{2n}$, $\Omega_{0,p}^{(2)} := \Omega_{n,p}^{(2)}$. It is clear, that $\Omega_{k,p}^{(s)}$ generally speaking, domains are multiconnected domains, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$, $s = 1, 2$. Pair of domains $\Omega_{k-1,p}^{(2)}$, $\Omega_{k,p}^{(1)}$ grows out of piece-dividing transformation open set D concerning families $\{P_{k-1}, P_k\}$, $\{\zeta_{k-1}, \zeta_k\}$ in point $a_{k,p}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m - 1}$.

Let's consider condensers

$$C_k(t, D, A_{2n,2m-1}) = (E_0^{(k)}, E_1^{(k)}, E_2^{(k)}),$$

where

$$E_s^{(k)} = \zeta_k (E_s \cap \overline{P}_k) \cup [\zeta_k (E_s \cap \overline{P}_k)]^*,$$

$k = \overline{1, 2n}$, $s = 0, 1, 2$, $\{P_k\}_{k=1}^{2n}$ - the system of corners corresponding to system of points $A_{2n,2m-1}$; operation $[A]^*$ compares to any the set $A \subset \overline{\mathbb{C}}$ a set, symmetric a set A is relative unit circle $|w| = 1$. From this it follows that to the condenser $C(t, D, A_{2n,2m-1})$, at dividing transformation is relative $\{P_k\}_{k=1}^{2n}$ and $\{\zeta_k\}_{k=1}^{2n}$, there corresponds a set of condensers the system of corners corresponding to system of points $A_{2n,2m-1}$; operation $[A]^*$ compares to any the set $A \subset \overline{\mathbb{C}}$ a set, symmetric a set A is relative unit circle $|w| = 1$. From this it follows that to the condenser $C(t, D, A_{2n,2m-1})$, at dividing transformation is relative $\{P_k\}_{k=1}^{2n}$ and $\{\zeta_k\}_{k=1}^{2n}$, there corresponds a set of condensers

$\{C_k(t, D, A_{2n, 2m-1})\}_{k=1}^{2n}$, symmetric relatively $\{z : |z| = 1\}$. According to works [6, 7], we will receive

$$\text{cap}C(t, D, A_{2n, 2m-1}) \geq \frac{1}{2} \sum_{k=1}^{2n} \text{cap}C_k(t, D, A_{2n, 2m-1}). \tag{19}$$

From here follows

$$|C(t, D, A_{2n, 2m-1})| \leq 2 \left(\sum_{k=1}^{2n} |C_k(t, D, A_{2n, 2m-1})|^{-1} \right)^{-1}. \tag{20}$$

The formula (17) gives a module asymptotics $C(t, D, A_{2n, 2m-1})$ at $t \rightarrow 0$, and $M(D, A_{2n, 2m-1})$ is the given module of a set D relatively $A_{2n, 2m-1}$. Using formulas (6) and that fact that a set D meets the condition of unapplied in relation to the system of points $A_{2n, 2m-1}$, for condensers we will receive similar asymptotic representations $C_k(t, D, A_{2n, 2m-1})$, $k = \overline{1, 2n}$

$$|C_k(t, D, A_{2n, 2m-1})| = \frac{1}{2\pi(2m-1)(\alpha+1)} \log \frac{1}{t} + M_k(D, A_{2n, 2m-1}) + o(1), \quad t \rightarrow 0, \tag{21}$$

where

$$\begin{aligned} M_{2k-1}(D, A_{2n, 2m-1}) &= \frac{1}{2\pi(2m-1)^2(\alpha+1)^2} \times \\ &\times \left[\alpha \sum_{p=1}^m \log \frac{r\left(\Omega_{2k-1, 2p-1}^{(1)}, \omega_{2k-1, 2p-1}^{(1)}\right)}{[\alpha_{2k-1} \cdot \chi(|a_{2k-1, 2p-1}|^{\alpha_{2k-1}}) |a_{2k-1, 2p-1}|]^{-1}} + \right. \\ &+ \alpha \sum_{p=1}^{m-1} \log \frac{r\left(\Omega_{2k-1, 2p}^{(2)}, \omega_{2k-1, 2p}^{(2)}\right)}{[\alpha_{2k-1} \cdot \chi(|a_{2k, 2p}|^{\alpha_{2k-1}}) |a_{2k, 2p}|]^{-1}} + \\ &+ \sum_{t=1}^m \log \frac{r\left(\Omega_{2k-1, 2t-1}^{(2)}, \omega_{2k-1, 2t-1}^{(2)}\right)}{[\alpha_{2k-1} \cdot \chi(|a_{2k, 2t-1}|^{\alpha_{2k-1}}) |a_{2k, 2t-1}|]^{-1}} + \\ &\left. + \sum_{t=1}^{m-1} \log \frac{r\left(\Omega_{2k-1, 2t}^{(1)}, \omega_{2k-1, 2t}^{(1)}\right)}{[\alpha_{2k-1} \cdot \chi(|a_{2k-1, 2t}|^{\alpha_{2k-1}}) |a_{2k-1, 2t}|]^{-1}} \right], \\ M_{2k}(D, A_{2n, 2m-1}) &= \\ &= \frac{1}{2\pi(2m-1)^2(\alpha+1)^2} \cdot \left[\alpha \sum_{p=1}^{m-1} \log \frac{r\left(\Omega_{2k, 2p}^{(1)}, \omega_{2k, 2p}^{(1)}\right)}{[\alpha_{2k} \cdot \chi(|a_{2k, 2p}|^{\alpha_{2k}}) |a_{2k, 2p}|]^{-1}} + \right. \end{aligned}$$

$$\begin{aligned}
 & + \alpha \sum_{p=1}^m \log \frac{r \left(\Omega_{2k, 2p-1}^{(2)}, \omega_{2k, 2p-1}^{(2)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k+1, 2p-1}|^{\alpha_{2k}}) |a_{2k+1, 2p-1}|]^{-1}} + \\
 & + \sum_{t=1}^{m-1} \log \frac{r \left(\Omega_{2k, 2t}^{(2)}, \omega_{2k, 2t}^{(2)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k+1, 2t}|^{\alpha_{2k}}) |a_{2k+1, 2t}|]^{-1}} + \\
 & + \left. \sum_{t=1}^m \log \frac{r \left(\Omega_{2k, 2t-1}^{(1)}, \omega_{2k, 2t-1}^{(1)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k, 2t-1}|^{\alpha_{2k}}) |a_{2k, 2t-1}|]^{-1}} \right], \quad k = \overline{1, n}.
 \end{aligned}$$

By means of (21), we receive

$$\begin{aligned}
 |C_k(t, D, A_{2n, 2m-1})|^{-1} &= \frac{2\pi(2m-1)(\alpha+1)}{\log \frac{1}{t}} \times \\
 & \times \left(1 + \frac{2\pi(2m-1)(\alpha+1)}{\log \frac{1}{t}} M_k(D, A_{2n, 2m}) + o\left(\frac{1}{\log \frac{1}{t}}\right) \right)^{-1} = \\
 & = \frac{2\pi(2m-1)(\alpha+1)}{\log \frac{1}{t}} - \left(\frac{2\pi(2m-1)(\alpha+1)}{\log \frac{1}{t}} \right)^2 M_k(D, A_{2n, 2m-1}) \quad (22)
 \end{aligned}$$

$$+ o\left(\left(\frac{1}{\log \frac{1}{t}}\right)^2\right), \quad t \rightarrow 0. \quad (23)$$

Further, from (22), follows that

$$\begin{aligned}
 \sum_{k=1}^{2n} |C_k(t, D, A_{2n, 2m-1})|^{-1} &= \frac{4\pi n(2m-1)(\alpha+1)}{\log \frac{1}{t}} - \\
 - \left(\frac{2\pi(2m-1)(\alpha+1)}{\log \frac{1}{t}} \right)^2 \cdot \sum_{k=1}^{2n} M_k(D, A_{2n, 2m-1}) &+ o\left(\left(\frac{1}{\log \frac{1}{t}}\right)^2\right), \quad t \rightarrow 0. \quad (24)
 \end{aligned}$$

In turn, allows (24) to receive the following asymptotic representation

$$\begin{aligned}
 \left(\sum_{k=1}^{2n} |C_k(t, D, A_{2n, 2m-1})|^{-1} \right)^{-1} &= \frac{\log \frac{1}{t}}{4\pi n(2m-1)(\alpha+1)} \times \\
 \times \left(1 - \frac{\pi(2m-1)(\alpha+1)}{n \log \frac{1}{t}} \cdot \sum_{k=1}^{2n} M_k(D, A_{2n, 2m-1}) + o\left(\frac{1}{\log \frac{1}{t}}\right) \right)^{-1} &=
 \end{aligned}$$

$$= \frac{\log \frac{1}{t}}{4\pi n(2m - 1)(\alpha + 1)} + \frac{1}{4n^2} \cdot \sum_{k=1}^{2n} M_k(D, A_{2n, 2m-1}) + o(1), \quad t \rightarrow 0. \quad (25)$$

Inequalities, (19) and (20) taking into (17) and (25) allow to notice that

$$\begin{aligned} & \frac{1}{2\pi} \cdot \frac{1}{n(2m - 1)(\alpha + 1)} \cdot \log \frac{1}{t} + M(D, A_{2n, 2m-1}) + o(1) \leq \\ & \leq \frac{\log \frac{1}{t}}{2\pi n(2m - 1)(\alpha + 1)} + \frac{1}{2n^2} \cdot \sum_{k=1}^{2n} M_k(D, A_{2n, 2m-1}) + o(1). \end{aligned} \quad (26)$$

From (26) at $t \rightarrow 0$ we receive that

$$M(D, A_{2n, 2m-1}) \leq \frac{1}{2n^2} \cdot \sum_{k=1}^{2n} M_k(D, A_{2n, 2m-1}). \quad (27)$$

Formulas (18), (21) and (27) lead to the following expression

$$\begin{aligned} & \frac{1}{2\pi} \cdot \frac{1}{n^2(2m - 1)^2 \cdot (\alpha + 1)^2} \cdot \left[\alpha \sum_{k=1}^n \sum_{p=1}^{m-1} \log r(D, a_{2k, 2p}) + \right. \\ & + \sum_{k=1}^n \sum_{p=1}^m \log r(D, a_{2k, 2p-1}) + \sum_{k=1}^n \sum_{p=1}^{m-1} \log r(D, a_{2k-1, 2p}) + \\ & \quad \left. + \alpha \sum_{k=1}^n \sum_{p=1}^m \log r(D, a_{2k-1, 2p-1}) + \right. \\ & + \alpha \sum_{(k,p) \neq (q,s)} (g_D(a_{2k, 2p}, a_{2q, 2s}) + g_D(a_{2k-1, 2p-1}, a_{2q-1, 2s-1})) + \\ & \quad + 2\alpha \sum_{(k,p) \neq (q,s)} g_D(a_{2k-1, 2p-1}, a_{2q, 2s}) + \\ & + 2\sqrt{\alpha} \sum_{(k,p) \neq (q,s)} (g_D(a_{2k, 2p}, a_{2q-1, 2s}) + g_D(a_{2k, 2p}, a_{2q, 2s-1}) + \\ & \quad + g_D(a_{2k-1, 2p-1}, a_{2q-1, 2s}) + g_D(a_{2k-1, 2p-1}, a_{2q, 2s-1})) + \\ & + \sum_{(k,p) \neq (q,s)} (g_D(a_{2k-1, 2p}, a_{2q-1, 2s}) + g_D(a_{2k, 2p-1}, a_{2q, 2s-1})) + \\ & \left. + 2 \sum_{(k,p) \neq (q,s)} g_D(a_{2k-1, 2p}, a_{2q, 2s-1}) \right] \leq \frac{1}{4\pi n^2(2m - 1)^2 (\alpha + 1)^2} \times \end{aligned}$$

$$\begin{aligned}
 & \times \left[\alpha \sum_{k=1}^n \sum_{p=1}^m \log \frac{r \left(\Omega_{2k-1, 2p-1}^{(1)}, \omega_{2k-1, 2p-1}^{(1)} \right)}{[\alpha_{2k-1} \cdot \chi (|a_{2k-1, 2p-1}|^{\alpha_{2k-1}}) |a_{2k-1, 2p-1}|]^{-1}} + \right. \\
 & + \alpha \sum_{k=1}^n \sum_{p=1}^{m-1} \log \frac{r \left(\Omega_{2k-1, 2p}^{(2)}, \omega_{2k-1, 2p}^{(2)} \right)}{[\alpha_{2k-1} \cdot \chi (|a_{2k, 2p}|^{\alpha_{2k-1}}) |a_{2k, 2p}|]^{-1}} + \\
 & + \sum_{k=1}^n \sum_{t=1}^m \log \frac{r \left(\Omega_{2k-1, 2t-1}^{(2)}, \omega_{2k-1, 2t-1}^{(2)} \right)}{[\alpha_{2k-1} \cdot \chi (|a_{2k, 2t-1}|^{\alpha_{2k-1}}) |a_{2k, 2t-1}|]^{-1}} + \\
 & + \sum_{k=1}^n \sum_{t=1}^{m-1} \log \frac{r \left(\Omega_{2k-1, 2t}^{(1)}, \omega_{2k-1, 2t}^{(1)} \right)}{[\alpha_{2k-1} \cdot \chi (|a_{2k-1, 2t}|^{\alpha_{2k-1}}) |a_{2k-1, 2t}|]^{-1}} + \\
 & + \alpha \sum_{k=1}^n \sum_{p=1}^{m-1} \log \frac{r \left(\Omega_{2k, 2p}^{(1)}, \omega_{2k, 2p}^{(1)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k, 2p}|^{\alpha_{2k}}) |a_{2k, 2p}|]^{-1}} + \\
 & + \alpha \sum_{k=1}^n \sum_{p=1}^m \log \frac{r \left(\Omega_{2k, 2p-1}^{(2)}, \omega_{2k, 2p-1}^{(2)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k+1, 2p-1}|^{\alpha_{2k}}) |a_{2k+1, 2p-1}|]^{-1}} + \\
 & + \sum_{k=1}^n \sum_{t=1}^{m-1} \log \frac{r \left(\Omega_{2k, 2t}^{(2)}, \omega_{2k, 2t}^{(2)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k+1, 2t}|^{\alpha_{2k}}) |a_{2k+1, 2t}|]^{-1}} + \\
 & \left. + \sum_{k=1}^n \sum_{t=1}^m \log \frac{r \left(\Omega_{2k, 2t-1}^{(1)}, \omega_{2k, 2t-1}^{(1)} \right)}{[\alpha_{2k} \cdot \chi (|a_{2k, 2t-1}|^{\alpha_{2k}}) |a_{2k, 2t-1}|]^{-1}} \right].
 \end{aligned}$$

Thus, we receive (12). Further, the proof of the theorem comes to an end in the same way, as well as the proof of the theorem 1. **The theorem is proved.**

Acknowledgement. The author is grateful to Prof. Bakhtin for suggesting problems and useful discussions.

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