B-SPLINE COLLOCATION APPROACH FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

by

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LIST OF ABBREVIATIONS

CAGD	Computer Aided Graphic Design
KdV	Korteweg de Vries
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
CuBS	Cubic B-spline
KdVB	Korteweg de Vries – Burgers’
MFDCM	Mixed Finite Difference and Collocation Method
CM	Collocation Method
BBQI	Blended B-spline Quasi-Interpolation
MBQI	Multilevel B-spline Quasi-Interpolation
QuBS	Quartic B-spline
CuTBS	Cubic Trigonometric B-spline
QuTBS	Quartic Trigonometric B-spline
CuHBS	Cubic Hybrid B-spline
QuHBS	Quartic Hybrid B-spline

LIST OF SYMBOLS

u	Dependent variable
x	Independent variable
t	Independent variable
C^m	Continuity of order m
x_j	Grid point in x-direction
t_k	Grid point in t-direction
h	Step size in x-dimension
Δt	Size size in t-dimension
n	Number of interval in x-dimension
N	Number of interval in t-dimension
a	Starting value of x
b	Ending value of x
T	Ending value of t
j	Index for space
k	Index for time
$B_{m,j}(x)$	j-th B-spline basis function of order m
$T_{m,j}(x)$	j-th trigonometric B-spline basis function of order m
$H_{m,j}(x)$	j-th hybrid B-spline basis function of order m
ω	Given function
ϕ	Given function
f	Given function
q	Given function

G	Given function
$\theta,\ \gamma$	Free parameter in between 0 to 1
$\alpha,\ \beta,\ \varepsilon,\ \mu$	Constant
$C(t)$	Time dependent unknown
$Q(t)$	Time dependent unknown
$\mathbf{C}^k$	Vector of $C(t)$ at time t_k
$\mathbf{Q}^k$	Vector of $Q(t)$ at time t_k
L_∞	Maximum error
L_2	Euclidean error
η	Mode number
$\bar{u}(x,t)$	Analytical solution

PENDEKATAN KOLOKASI SPLIN-B UNTUK MENYELESAIKAN PERSAMAAN-PERSAMAAN PEMBEZAAN SEPARA

ABSTRAK

Fungsi-fungsi splin-B dan trigonometri splin-B telah digunakan secara meluas dalam Rekabentuk Geometri Berbantu Komputer *(RGBK)* sebagai alat untuk menjana lengkung dan permukaan. Kelebihan fungsi-fungsi secara sepotong ini ialah ciri sokongan setempat dimana fungsi-fungsi ini dikatakan mempunyai sokongan dalam selang tertentu. Disebabkan oleh ciri ini, splin-B telah digunakan untuk menjana penyelesaian-penyelesaian berangka bagi persamaan pembezaan separa linear dan tak linear. Dalam tesis ini, dua jenis fungsi asas splin-B dipertimbangkan. Ianya adalah fungsi asas splin-B dan fungsi asas trigonometri splin-B. Pembangunan fungsi-fungsi ini untuk peringkat-peringkat yang berbeza dilaksanakan. Satu fungsi baru dipanggil fungsi asas hibrid splin-B dibangunkan dimana satu parameter digabungkan bersama fungsi-fungsi asas splin-B dan trigonometri splin-B diperkenalkan. Kaedah-kaedah kolokasi berdasarkan fungsi-fungsi asas tersebut dan hampiran beza terhingga dibangunkan. Splin-B digunakan untuk interpolasi penyelesaian pada dimensi-x dan hampiran beza terhingga digunakan untuk mendiskrit pembeza-pembeza masa. Secara umum, masalah nilai awal-sempadan yang melibatkan persamaan gelombang satu dimensi, persamaan tak linear Klein-Gordon dan persamaan Korteweg de Vries diselesaikan menggunakan kaedah-kaedah kolokasi ini. Dalam usaha untuk menunjukkan keupayaan skim-skim tersebut, beberapa masalah diselesaikan dan dibandingkan dengan penyelesaian-penyelesaian tepat dan keputusan-keputusan daripada literatur. Satu lagi penemuan

baru tesis ini ialah kaedah-kaedah kolokasi yang boleh diterima pakai untuk menyelesaikan persamaan-persamaan pembezaan separa tak linear dengan keputusan yang tepat. Kestabilan skim-skim dianalisis menggunakan analisis kestabilan Von Neumann dan ralat pemangkasan diteliti. Kaedah-kaedah yang dicadangkan telah dibuktikan stabil tidak bersyarat. Kaedah kolokasi splin-B kubik dan kuartik telah disahkan sebagai $O(\Delta t) + O(h^2)$ tepat. Sumbangan dan inovasi utama tesis ini adalah pembangunan fungsi asas hibrid splin-B dan kebolehgunaan kaedah-kaedah kolokasi yang dicadangkan untuk menyelesaikan persamaan-persamaan pembezaan separa tak linear.

B-SPLINE COLLOCATION APPROACH FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

ABSTRACT

The B-spline and trigonometric B-spline functions were used extensively in Computer Aided Geometric Design (CAGD) as tools to generate curves and surfaces. An advantage of these piecewise functions is its local support properties where the functions are said to have support in specific interval. Due to this properties, B-splines have been used to generate the numerical solutions of linear and nonlinear partial differential equations. In this thesis, two types of B-spline basis function are considered. These are B-spline basis function and trigonometric B-spline basis function. The development of these functions for different orders is carried out. A new function called hybrid B-spline basis function is developed where a new parameter incorporated with B-spline and trigonometric B-spline basis functions is introduced. Collocation methods based on the proposed basis functions and finite difference approximation are developed. B-splines are used to interpolate the solution in x-dimension and finite difference approximations are used to discretize the time derivatives. In general, initial-boundary value problems involving one-dimensional wave equation, nonlinear Klein-Gordon equation and Korteweg de Vries equation are solved using the collocation methods. In order to demonstrate the capability of the schemes, some problems are solved and compared with the analytical solutions and the results from literature. Another new finding of this thesis is the collocation methods that are applicable to solve nonlinear partial differential equations with accurate result. The stability of the schemes is analysed using Von

Neumann stability analysis and the truncation errors are examined. The proposed methods have been proved to be unconditionally stable. Cubic and quartic B-spline collocation methods have been verified as $O(\Delta t)+O(h^2)$ accurate. The main contribution and innovation of this thesis are the development of hybrid B-spline basis function and the applicability of the proposed collocation methods to solve nonlinear partial differential equations.

CHAPTER 1

INTRODUCTION

1.1 Introduction to travelling waves and their modelling

A variety of physical phenomena involve waves. For example, a simple phenomena is when a pebble is dropped into a pool. There is a disturbance created which moves outward until it finally reaches the edge of the pool. This disturbance is called wave. Wave is defined as a disturbance that propagates through space or medium and time which transfers energy from one point to another (Faughn, 2003).

According to Faughn (2003), waves are divided into three types:

(a) Travelling wave where the wave transfers energy from one point to another by vibration. Demonstration of this wave motion is to flip one end of a long rope with the opposite end fixed (as shown in Figure 1.1). Then, the end rope that is not fixed is moved, there is a pulse that travels to the other end of the fixed rope with definite speed.

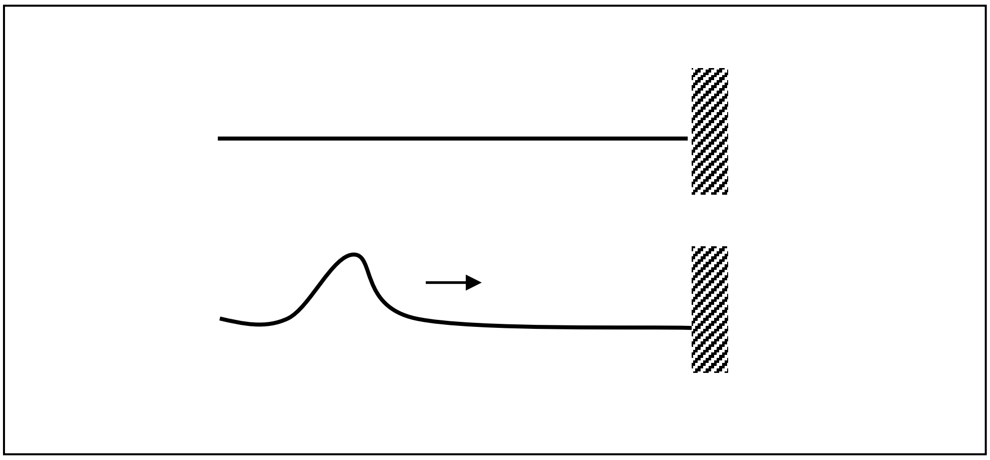

Figure 1.1: Demonstration of travelling wave (Faughn, 2003)

(b) Transverse wave where a disturbance moves perpendicular to the wave motion. The stretched spring is usually used to explain the propagation of transverse wave. The end of the spring is pumped up and down (as shown in Figure 1.2). It will produce a hump which propagates or moves perpendicular to the direction of the movement of the spring.

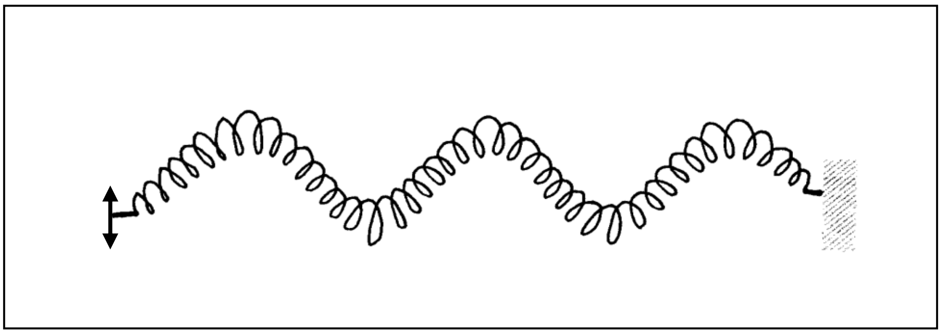

Figure 1.2: Demonstration of tranverse wave (Faughn, 2003)

(c) Longitudinal wave is the disturbance move parallel to the wave motion. To demonstrate this wave, a stretched spring is pumped back and forth. This action produces compressed and stretched regions of the spring where parallel of wave motion is seen as in Figure 1.3.

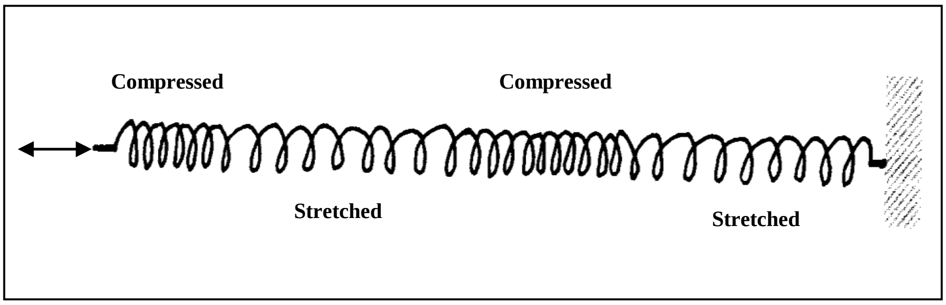

Figure 1.3: Demonstration of longitudinal wave (Faughn, 2003)

In this thesis, three types of travelling wave are considered. These are the one-dimensional wave equation, nonlinear Klein-Gordon equation and Korteweg de Vries (KdV) equation. To gain in depth understanding of these phenomena, these equations need to be solved efficiently and accurately. Various mathematical methods are available. However, the use of B-spline has been rather limited. This thesis will focus on the use of B-splines for solving equations which describe travelling waves.

1.2 Introduction to partial differential equation

Mathematical models have been developed to represent some problems that arise in real life phenomenon. The model usually contains derivatives of an unknown function which lead to a type of equation called differential equation. There are two basic types of differential equations: ordinary differential equations (ODE) and partial differential equations (PDE).

PDEs are used to describe a wide variety of physical phenomenon especially in physics and engineering problems. The problems inherently bring the need for partial derivatives in the description of their behaviour. A PDE is a differential equation which is expresses a relation among the partial derivatives of the function that contains two or more independent variables (Vvedensky, 1993). The general form of PDE for a function *u* with two independent variables, *x* and *t* is

$$F\left(x, t, u(x,t), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}, \ldots\right) = 0 \tag{1.1}$$

where $u(x,t)$ is the solution of (1.1).

The order of PDE is equal to the order of the highest derivative appearing in (1.1). A differential equation is said to be linear if

i) the function F is algebraically linear in each variables and

ii) the coefficients of the dependent variable and its derivatives are function of the independent variables.

Otherwise, it is called nonlinear PDE (DuChateau & Zachmann, 2011).

The solution of PDE has to satisfy the initial and boundary conditions which appear in the problem formulation. There are four main types of boundary conditions:

i) Dirichlet condition where the solution, u has to satisfy the given value on the boundary, s.

$$u\big|_s = g$$

ii) Neumann condition where the solution, u has to satisfy the value of derivatives on the boundary, s.

$$\left.\frac{\partial u}{\partial \eta}\right|_s = g$$

iii) Robin condition where the solution u has to satisfy a combination of u and its derivatives on the boundary, s.

$$\left.\left(\alpha u + \beta \frac{\partial u}{\partial \eta}\right)\right|_s = g$$

iv) Mixed condition where the solution u has to satisfy the following condition

$$\left.\begin{matrix} u\big|_{s_1} \\ \left.\dfrac{\partial u}{\partial \eta}\right|_{s_2} \end{matrix}\right\} = g$$

where $s_1 \cup s_2 = s$.

Initial conditions define the solution, u at initial time, t. Differential equation with the only initial condition or boundary condition is called initial value problem or boundary value problem, respectively. A problem consisting differential equation with initial and boundary conditions is called an initial-boundary value problem.

1.3 Spline approach for solving partial differential equation

Spline functions can be used to approximate the solution of differential equation using piecewise-polynomial approximation. Bickley (1968) began the investigation by applying cubic spline on two-point boundary value problems involving a linear ODE. The method was then improved and the same mathematical problems were solved by Fyfe (1969) and Albasiny and Hoskins (1969). The spline approximations were then developed for PDEs. Papamichael and Whiteman (1973) approximated the heat conduction equation using spline function in the space direction and finite difference approximation in the time direction.

B-spline functions are piecewise functions with local support properties where the functions are said to have support in specific interval. Due to these properties, B-splines have been used to generate the numerical solution of linear problems. For example, Dehghan and Lakestani (2007) generated the solution of one-dimensional hyperbolic equation using cubic B-spline scaling functions. Çağlar et al. (2008) solved the boundary value problem for one-dimensional heat equation by third degree B-spline functions. Subsequently, Goh et al. (2011) presented cubic B-spline method incorporated with finite difference approximation for solving one-dimensional heat and wave equation.

Nonlinear wave involve nonlinear partial differential equations which can also be solved using B-spline approximations. Gardner and Gardner (1990) produced a finite element solution of the regularised long wave equation using Galerkin's method based on cubic B-spline. The proposed method was also tested by them on the equal width wave equation (Gardner & Gardner, 1992). The numerical solution of Burger's equation were developed by Dağ et al. (2005) using a method based on collocation of cubic B-spline over finite elements. New types of spline called Blended B-spline and multilevel B-spline were proposed by Yu et al. (2013a, 2013b) for solving KdV equation. In term of errors, the methods cited above indicated that B-spline approximations were easy to implement.

1.4 Motivation of research

The finite difference approximation is the oldest method used to approximate the derivatives in differential equation. In the last five decades, theoretical results have been obtained regarding the accuracy, stability and convergence of the finite difference method for various PDEs. The spline approximation method is now also used as a tool to approximate the solution of partial differential equation. The spline method has flexibility to generate the accurate approximation at any point in the domain compared to the finite difference method which yields approximation only at grid points.

Caglar et al. (2006) introduced the cubic B-spline interpolation method to solve two-point boundary value problem of ODE. Evidently, B-spline demonstrates better approximation compared to the finite difference method, finite element method and finite volume method. Due to the advantages of finite difference approximations and B-spline functions, Goh et al. (2011) and Abbas et al. (2014) presented a combination of both methods to solve linear PDE. The method performed very well compared to the use of only finite difference scheme. But the question is, does this method perform equally well for nonlinear PDEs? And, how accurate is the approximation obtained using a hybrid of B-spline function and the finite difference method?

In order to answer the questions, the nonlinear Klein-Gordon equation and KdV equation will be considered as initial-boundary value problems in this thesis. The numerical scheme proposed by Goh et al. (2011) and Abbas et al. (2014) will be applied to the problems. A new hybrid B-spline function will be introduced and applied to the same problems. Our new hybrid B-spline function will also used to solve the linear one-dimensional wave equation as a preliminary case study. The stability and error analysis of the methods will be discussed for each schemes.

1.5 Objective of research

The objectives of this study are:

1. to develop hybrid B-spline basis function for solving PDEs.
2. to develop and apply B-spline collocation method, trigonometric B-spline collocation method and hybrid B-spline collocation method in solving one-dimensional partial differential equation i.e. one-dimensional wave equation, nonlinear Klein-Gordon equation and KdV equation.
3. to investigate the stability and error analysis of the proposed methods when applied to the selected PDEs.
4. to make a comparative study between the proposed methods and the current methods available in literature.

1.6 Scope and methodology

The recent literatures on spline-based methods for solving ODEs and PDEs will be summarized. Two types of B-spline basis function will be considered. These are B-spline basis function and trigonometric B-spline basis function. The development of these functions for different orders will be carried out. Based on these two functions, hybrid B-spline basis function will be developed.

Subsequently, collocation methods based on the proposed B-spline basis functions incorporated with finite difference approximations will be developed. Initial-boundary value problems involving one-dimensional wave equation, nonlinear Klein-Gordon equation and KdV equation will be solved, in general, using the collocation methods. In order to demonstrate the capability of the schemes, some problems will be solved and compared with the analytical solution and the results from literatures. MATLAB R2012a has been used as software to solve the problems. The stability of the schemes will be analysed using Von Neumann stability analysis. Lastly, the truncation errors of selected schemes will be examined.

1.7 Thesis organization

This thesis contains eight chapters. Chapter 1 gives a brief introduction of wave and partial differential equations. The motivation, objective, scope and methodology of the research are also described in this chapter. A review on B-spline and trigonometric B-spline basis function of some order are discussed in Chapter 2. The formation of hybrid B-spline basis function is included in this chapter. In this chapter, an existing cubic B-spline collocation method on solving one-dimensional wave equation is also briefly discussed.

Chapter 3 provides the review of related literatures. The histories of spline application on solving differential equation are covered. A survey of analytical and numerical methods which has been used for solving one-dimensional wave equation, nonlinear Klein-gordon equation and KdV equation are revealed in this chapter.

Selected literatures on B-spline collocation method applied to linear and nonlinear PDEs are also presented.

B-spline and trigonometric B-spline collocation method applied to initial-boundary value problems of nonlinear Klein-Gordon equation and KdV equation are discussed in Chapter 4 and 5, respectively. Chapter 6 presents the application of hybrid B-spline collocation method to initial-boundary value problems of one-dimensional wave equation, nonlinear Klein-Gordon equation and KdV equation. For comparison purpose, the proposed methods are tested on several problems.

The stability of the linearized scheme obtained from Chapter 4 to 6 are analysed using Von Neumann stability analysis in Chapter 7. The local truncation errors of selected schemes are examined in the same chapter. Finally, the conclusions on whole of this thesis with the suggestions toward future research are presented in Chapter 8.

CHAPTER 2

REVIEW ON BASIS FUNCTIONS

2.1 Introduction

B-spline function were first formulated in the 1940s but were only seriously developed in the 1970s by several researchers (De Boor, 2001). The designation "B" stands for Basis so the full name of this function is basis spline. Trigonometric B-spline functions were introduced by I. J. Schoenberg in 1964 (De Boor, 2001). As suggested by the name, the functions were constructed from sine functions instead of polynomial functions.

The B-spline and trigonometric B-spline functions were used extensively in Computer Aided Geometric Design (CAGD) as tools to generate curves and surfaces. The following sections will discuss the theory to generate the B-spline and trigonometric B-spline basis functions. A new basis function called hybrid B-spline basis function will be introduced. The theory of hybrid B-spline basis function of order four will be developed. At the end of this chapter, a collocation method based on B-spline basis function will be described.

2.2 B-spline basis function

Generally, there are two ways to generate B-spline basis functions. These are by divided difference of truncated power function and by recursive formula. Since it is most useful for computer implementation, recursive formula is used in this thesis. Generally, B-spline basis function of order m satisfies the following properties (De Boor, 2001):

i. Non-negativity

$B_{m,j}(x)$ are positive for all *m*, *j* and *x*

ii. Local support

$B_{m,j}(x) > 0$ in knot span $[x_j, x_{j+m})$ and $B_{m,j}(x) = 0$ in elsewhere. Hence, the function $B_{m,j}(x)$ is said to have support on interval $[x_j, x_{j+m})$.

iii. Partition of unity

$\sum_{j=0}^{n} B_{m,j}(x) = 1$ for knot span $[x_{m-1}, x_{n+1},]$.

iv. Continuity at join

$B_{m,j}(x)$ has continuity C^{m-2} at each knot x_j. C^{m-2} is the degree of derivative for $B_{m,j}(x)$ at each knot x_j equal to $m-2$.

v. Translation invariance

$B_{m,j}(x) = B_{m,0}(x - x_j)$ which is B-spline basis function with same order are translated to each other.

Initially, the non-decreasing sequence of knot is considered as $\{x_0, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n\}$ where $x_{j-1} \le x_j \le x_{j+1}$ for $j = 0,1,\ldots,n.$ The *j*-th B-spline basis function of order *m* (degree $m-1$) can be defined recursively as

$$B_{m,j}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} B_{m-1,j}(x) + \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} B_{m-1,j+1}(x) \tag{2.1}$$

where B-spline basis function of order 1 is defined as

$$B_{1,j}(x) = \begin{cases} 1 & [x_j, x_{j+1}) \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

for $m = 2,3,\ldots$ and $j = 0,1,\ldots,n.$ This recursive formula is known as The Cox-de Boor recursion formula (De Boor, 2001).

Let us consider the knots as $x_0 = 0,$ $x_1 = 1$ and $x_2 = 2.$ The basis function of order 1, $B_{1,0}(x) = 1$ in the knot span $[0,1)$ but $B_{1,0}(x) = 0$ elsewhere, $B_{1,1}(x) = 1$ in the knot span $[1,2)$ but $B_{1,1}(x) = 0$ in elsewhere and so on. Each basis function is not continuous at the knots. In other words, this basis function has C^{-1} continuity on the knots. Figure 2.1 show the plot of B-spline basis function of order 1. It can be seen that the function is a step function.

In order to generate B-spline basis function of order 2, $m = 2$ is substituted into Eq. (2.1) to obtain

$$B_{2,j}(x) = \frac{1}{h}\begin{cases} x - x_j & [x_j, x_{j+1}) \\ x_{j+2} - x & [x_{j+1}, x_{j+2}) \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

The function, $B_{2,j}(x) = \frac{1}{h}(x - x_j)$ in knot span $[x_j, x_{j+1})$, $B_{2,j}(x) = \frac{1}{h}(x_{j+2} - x)$ in knot span $[x_{j+1}, x_{j+2})$ and $B_{2,j}(x) = 0$ in elsewhere. Figure 2.2 depicts the B-spline basis function of order 2. The figure shows that the function is linear. Each of two consecutive knots is joined and has C^0 continuity.

The quadratic b-spline basis function is obtained by substituting $m = 3$ into Eq. (2.1). After some simplification, the following basis function of order 3 is obtained.

$$B_{3,j}(x) = \frac{1}{2h^2}\begin{cases} (x - x_j)^2 & [x_j, x_{j+1}) \\ h^2 + 2h(x - x_{j+1}) - 2(x - x_{j+1})^2 & [x_{j+1}, x_{j+2}) \\ (x_{j+3} - x)^2 & [x_{j+2}, x_{j+3}) \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

Each knot spans contain quadratic polynomial except for x_{j+3} onward. Figure 2.3 shows that the three polynomials are joined to each other smoothly. These functions have C^1 continuity at the knots.

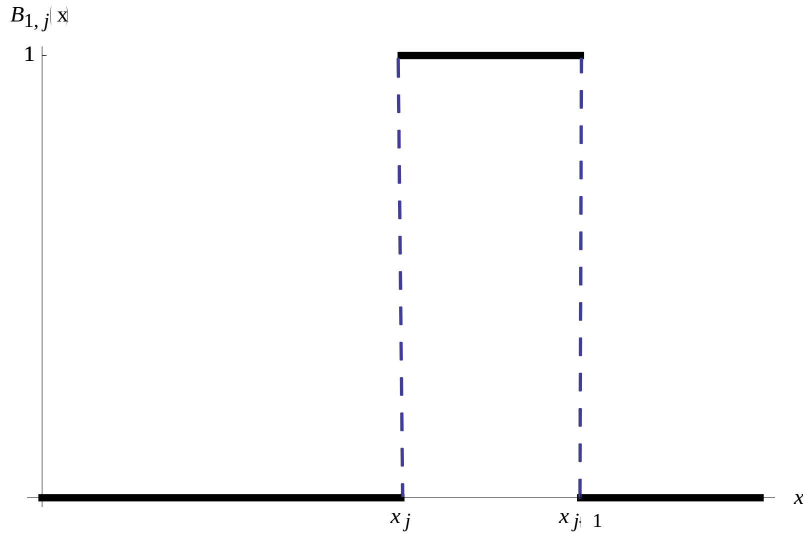

Figure 2.1: B-spline basis function of order 1.

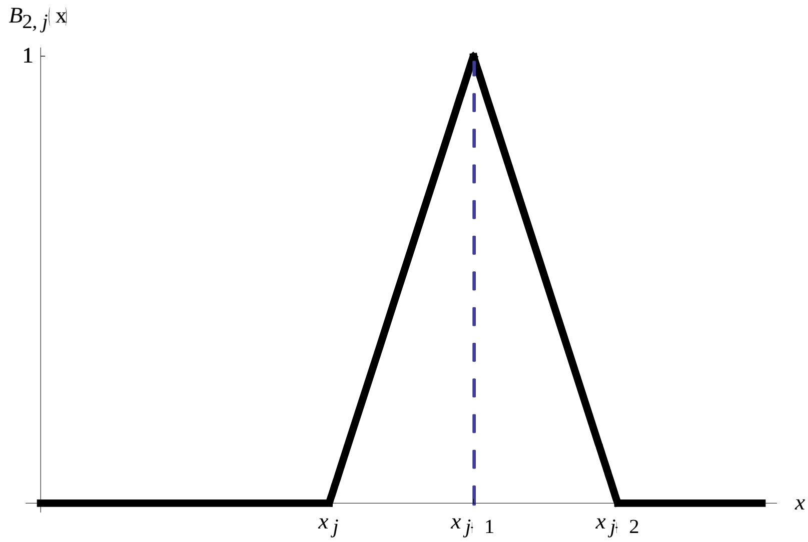

Figure 2.2: B-spline basis function of order 2.

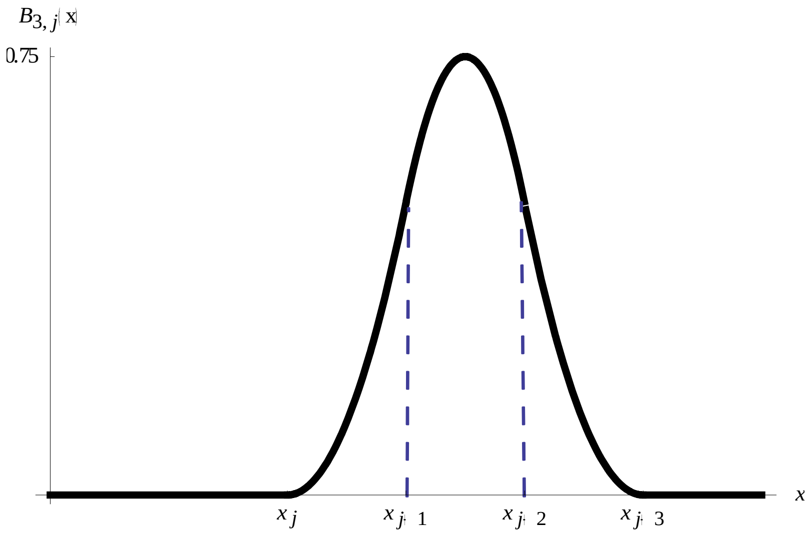

Figure 2.3: B-spline basis function of order 3.

The cubic B-spline basis function is generated from Eq. (2.1) by substituting $m = 4.$ It can be written as

$$B_{4,j}(x) = \frac{1}{6h^3} \begin{cases} (x - x_j)^3 & [x_j, x_{j+1}) \\ h^3 + 3h^2(x - x_{j+1}) + 3h(x - x_{j+1})^2 - 3(x - x_{j+1})^3 & [x_{j+1}, x_{j+2}) \\ h^3 + 3h^2(x_{j+3} - x) + 3h(x_{j+3} - x)^2 - 3(x_{j+3} - x)^3 & [x_{j+2}, x_{j+3}) \\ (x_{j+4} - x)^3 & [x_{j+3}, x_{j+4}) \\ 0 & \text{otherwise} \end{cases} \quad (2.5)$$

According to Eq. (2.5), each knot spans have cubic polynomial except for knot x_{j+4} onward. Figure 2.4 depicts the B-spline basis function of order 4. Each of the polynomial are joined smoothly with derivatives up to second order continuous, C^2.

The B-spline basis function of order 5 is also generated from Eq. (2.1). This function also called quartic B-spline basis function which is can be written as

$$B_{5,j}(x) = \frac{1}{24h^4} \begin{cases} \left(x - x_j\right)^4 & \left[x_j, x_{j+1}\right) \\ h^4 + 4h^3\left(x - x_{j+1}\right) + 6h^2\left(x - x_{j+1}\right)^2 & \\ +4h\left(x - x_{j+1}\right)^3 - 4\left(x - x_{j+1}\right)^4 & \left[x_{j+1}, x_{j+2}\right) \\ 11h^4 + 12h^3\left(x - x_{j+2}\right) - 6h^2\left(x - x_{j+2}\right)^2 & \\ -12h\left(x - x_{j+2}\right)^3 + 6\left(x - x_{j+2}\right)^4 & \left[x_{j+2}, x_{j+3}\right) \\ h^4 + 4h^3\left(x_{j+4} - x\right) + 6h^2\left(x_{j+4} - x\right)^2 & \\ +4h\left(x_{j+4} - x\right)^3 - 4\left(x_{j+4} - x\right)^4 & \left[x_{j+3}, x_{j+4}\right) \\ \left(x_{j+5} - x\right)^4 & \left[x_{j+4}, x_{j+5}\right) \\ 0 & \text{otherwise} \end{cases} \qquad (2.6)$$

Eq. (2.6) is a piecewise function constituted by five segments of quartic polynomials. The plot of this function is shown in Figure 2.5. All segments are joined to each other smoothly with C^3 continuity at the knots.

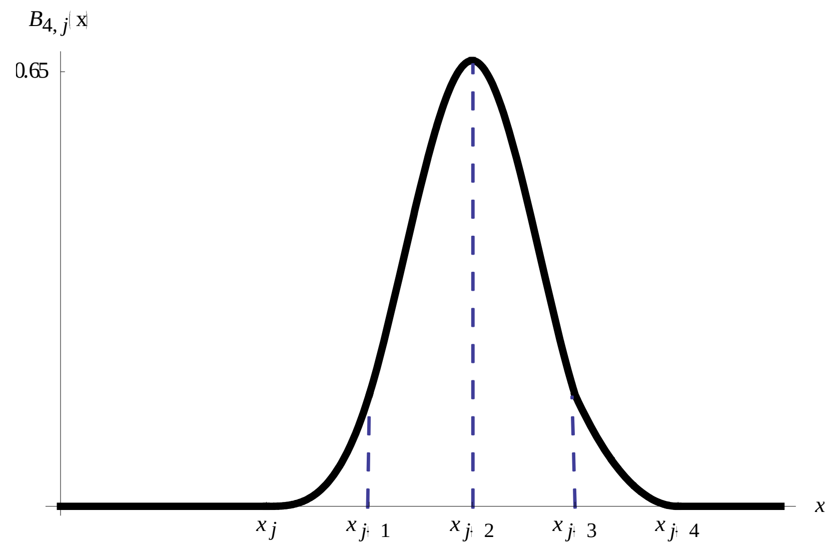

Figure 2.4: B-spline basis function of order 4

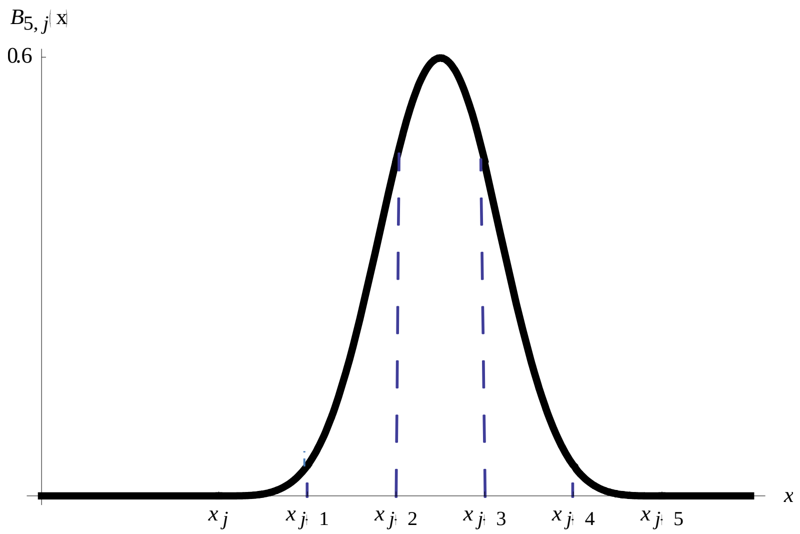

Figure 2.5: B-spline basis function of order 5

2.3 Trigonometric B-spline basis function

Let’s consider a knot vector $\mathbf{x} = (x_0, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$ with step size $h = x_{j+1} - x_j$ where $x_{j-1} \le x_j \le x_{j+1}$ for $j = 0, 1, \ldots, n.$ Thus, the *j*-th trigonometric B-spline basis function of order *m* (degree $m-1$) is defined recursively as (Walz, 1997)

$$T_{m,j}(x) = \frac{\sin\left(\dfrac{x - x_j}{2}\right)}{\sin\left(\dfrac{x_{j+m-1} - x_j}{2}\right)} T_{m-1,j}(x) + \frac{\sin\left(\dfrac{x_{j+m} - x}{2}\right)}{\sin\left(\dfrac{x_{j+m} - x_{j+1}}{2}\right)} T_{m-1,j+1}(x) \qquad (2.7)$$

for $m = 2, 3, \ldots$ where the first order trigonometric B-spline basis function is given as

$$T_{1,j}(x) = \begin{cases} 1 & \left[x_j, x_{j+1}\right) \\ 0 & \text{otherwise} \end{cases} \qquad (2.8)$$

According to previous section, the trigonometric B-spline function of order 1 is a step size function. Figure 2.6 shows the function in details.

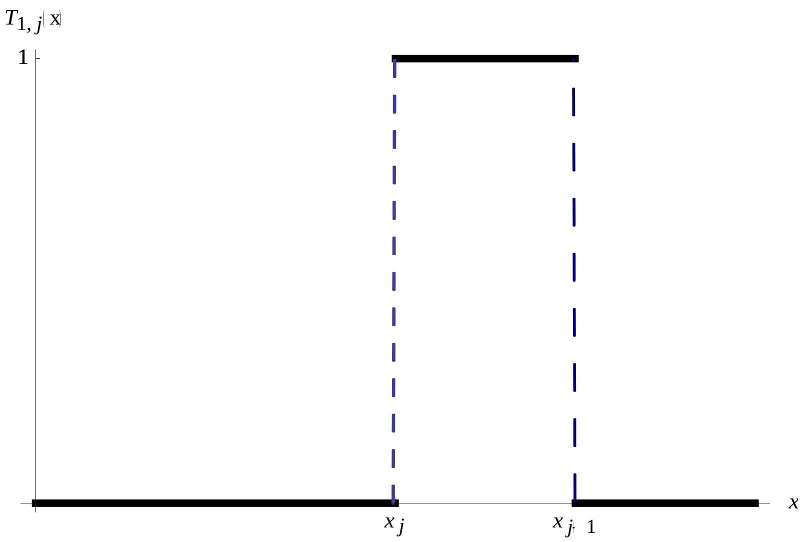

Figure 2.6: Trigonometric B-spline basis function of order 1

By using similar approach as section 2.2, the following trigonometric B-spline basis function of order two to five are obtained:

i. Second order $(m = 2)$

$$T_{2,j}(x) = \frac{1}{\kappa_1}\begin{cases} p\left(x_j\right) & \left[x_j, x_{j+1}\right) \\ q\left(x_{j+2}\right) & \left[x_{j+1}, x_{j+2}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

ii. Third order $(m = 3)$

$$T_{3,j}\left(x\right) = \frac{1}{\kappa_2}\begin{cases} p^2\left(x_j\right) & \left[x_j, x_{j+1}\right) \\ p\left(x_j\right)q\left(x_{j+2}\right) + p\left(x_{j+1}\right)q\left(x_{j+3}\right) & \left[x_{j+1}, x_{j+2}\right) \\ q^2\left(x_{j+3}\right) & \left[x_{j+2}, x_{j+3}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

iii. Fourth order $(m = 4)$

$$
T_{4,j}(x)=\frac{1}{\kappa_3}\begin{cases}
p^3\left(x_j\right) & \left[x_j,x_{j+1}\right) \\
p^2\left(x_j\right)q\left(x_{j+2}\right)+p\left(x_j\right)q\left(x_{j+3}\right)p\left(x_{j+1}\right) & \\
+q\left(x_{j+4}\right)p^2\left(x_{j+1}\right) & \left[x_{j+1},x_{j+2}\right) \\
p\left(x_j\right)q^2\left(x_{j+3}\right)+q\left(x_{j+4}\right)p\left(x_{j+1}\right)q\left(x_{j+3}\right) & \\
+q^2\left(x_{j+4}\right)p\left(x_{j+2}\right) & \left[x_{j+2},x_{j+3}\right) \\
q^3\left(x_{j+4}\right) & \left[x_{j+3},x_{j+4}\right) \\
o & \text{otherwise}
\end{cases} \tag{2.11}
$$

iv. Fifth order $(m=5)$

$$
T_{5,j}(x)=\frac{1}{\kappa_4}\begin{cases}
p^4\left(x_j\right) & \left[x_j,x_{j+1}\right) \\
p^3\left(x_j\right)q\left(x_{j+2}\right)+p^2\left(x_j\right)q\left(x_{j+3}\right)p\left(x_{j+1}\right) & \\
+p\left(x_j\right)q\left(x_{j+4}\right)p^2\left(x_{j+1}\right)+q\left(x_{j+5}\right)p^3\left(x_{j+1}\right) & \left[x_{j+1},x_{j+2}\right) \\
p^2\left(x_j\right)q^2\left(x_{j+3}\right) & \\
+p\left(x_j\right)q\left(x_{j+4}\right)p\left(x_{j+1}\right)q\left(x_{j+3}\right) & \\
+p\left(x_j\right)q^2\left(x_{j+4}\right)p\left(x_{j+2}\right)+q\left(x_{j+5}\right)p^2\left(x_{j+1}\right)q\left(x_{j+3}\right) & \left[x_{j+2},x_{j+3}\right) \\
+q\left(x_{j+5}\right)p\left(x_{j+1}\right)q\left(x_{j+4}\right)p\left(x_{j+2}\right) & \\
+q^2\left(x_{j+5}\right)p^2\left(x_{j+2}\right) & \\
p\left(x_j\right)q^3\left(x_{j+4}\right)+q\left(x_{j+5}\right)p\left(x_{j+1}\right)q^2\left(x_{j+4}\right) & \\
+q^2\left(x_{j+5}\right)p\left(x_{j+2}\right)q\left(x_{j+4}\right)+q^3\left(x_{j+5}\right)p\left(x_{j+3}\right) & \left[x_{j+3},x_{j+4}\right) \\
q^4\left(x_{j+5}\right) & \left[x_{j+4},x_{j+5}\right) \\
0 & \text{otherwise}
\end{cases} \tag{2.12}
$$

where $p(x_j)=\sin\left(\frac{x-x_j}{2}\right)$, $q(x_j)=\sin\left(\frac{x_j-x}{2}\right)$, $\kappa_1=\sin\left(\frac{h}{2}\right)$, $\kappa_2=\sin\left(\frac{h}{2}\right)\sin(h)$, $\kappa_3=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)$ and $\kappa_4=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)$. The plots of each functions are shown in Figure 2.7 to Figure 2.10

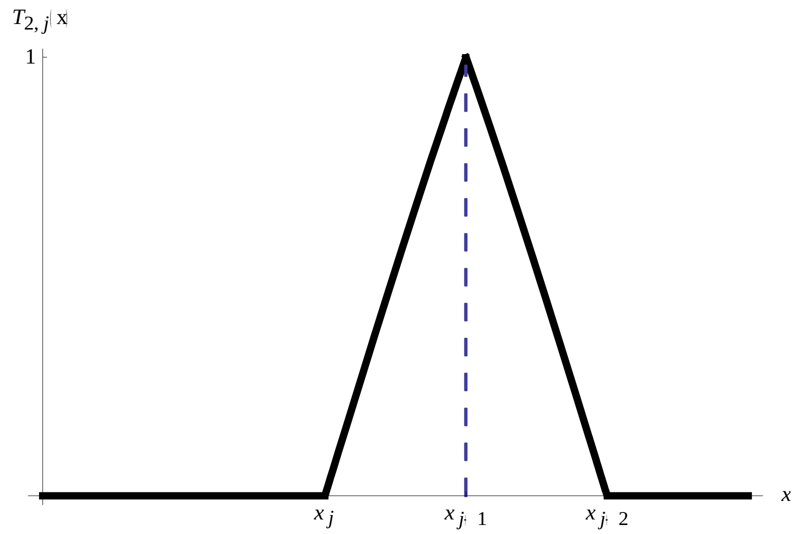

Figure 2.7: Trigonometric B-spline basis function of order 2

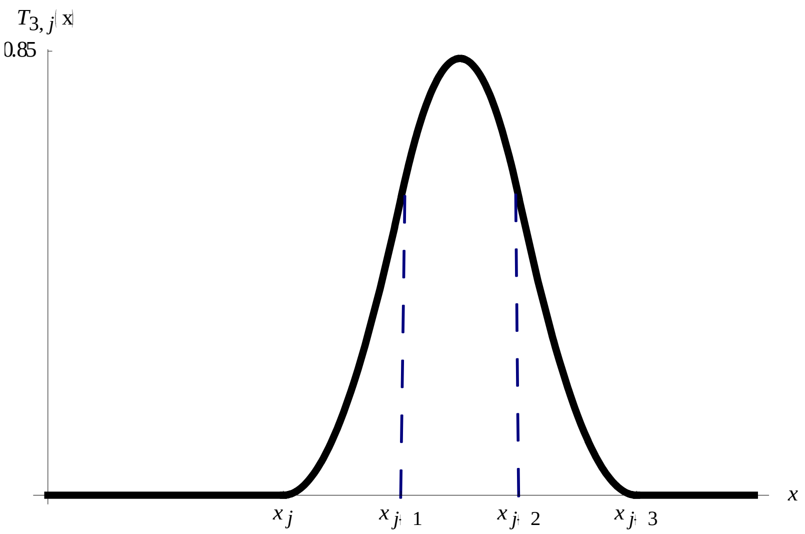

Figure 2.8: Trigonometric B-spline basis function of order 3

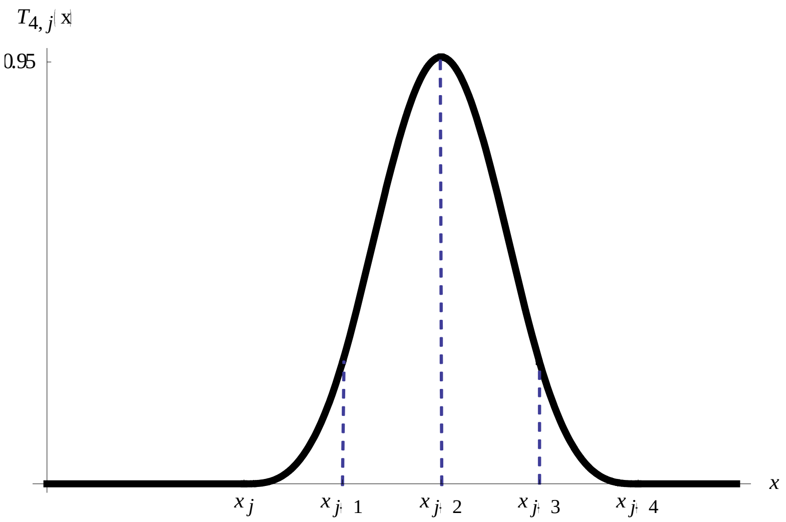

Figure 2.9: Trigonometric B-spline basis function of order 4

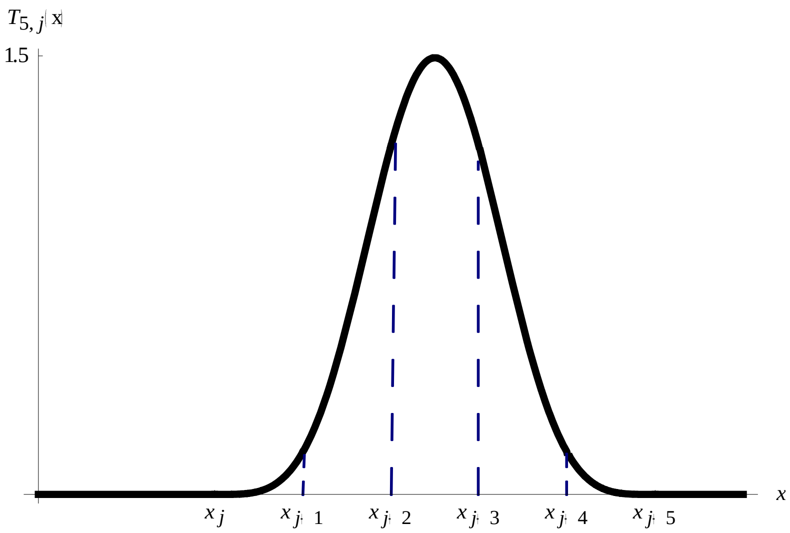

Figure 2.10: Trigonometric B-spline basis function of order 5

2.4 Hybrid B-spline basis function

One of the main contributions of this thesis is a new form of basis functions have been developed. Functions that combine the B-spline and trigonometric B-spline basis functions are called hybrid B-spline basis functions. The *m*-th order of hybrid B-spline basis function is given as

$$H_{m,j}(x) = \gamma B_{m,j}(x) + (1-\gamma) T_{m,j}(x) \tag{2.13}$$

where $B_{m,j}(x)$, B-spline basis function, $T_{m,j}(x)$, trigonometric B-spline basis function and $0 < \gamma < 1.$ The value of γ plays an important role in the hybrid B-spline basis function. If $\gamma = 0,$ the basis function is equal to trigonometric B-spline basis function and if $\gamma = 1,$ the basis function is equal to B-spline basis function.

Hybrid B-spline basis function of order 4 is constructed by substituting $m = 4$ into Eq. (2.13). Thus,

$$H_{4,j}(x) = \gamma B_{4,j}(x) + (1-\gamma) T_{4,j}(x) \tag{2.14}$$

where $B_{4,j}(x)$ is given in Eq. (2.5) and $T_{4,j}(x)$ is given in Eq. (2.11). Figure 2.11-2.13 depict the basis functions for different values of $\gamma.$ It can be seen that Figure

2.11 with $\gamma = 0.1$ is close to Figure 2.9 and Figure 2.13 with $\gamma = 0.9$ is close to Figure 2.4. In this thesis, fourth and fifth order of hybrid B-spline basis function will be used to solve selected partial differential equations.

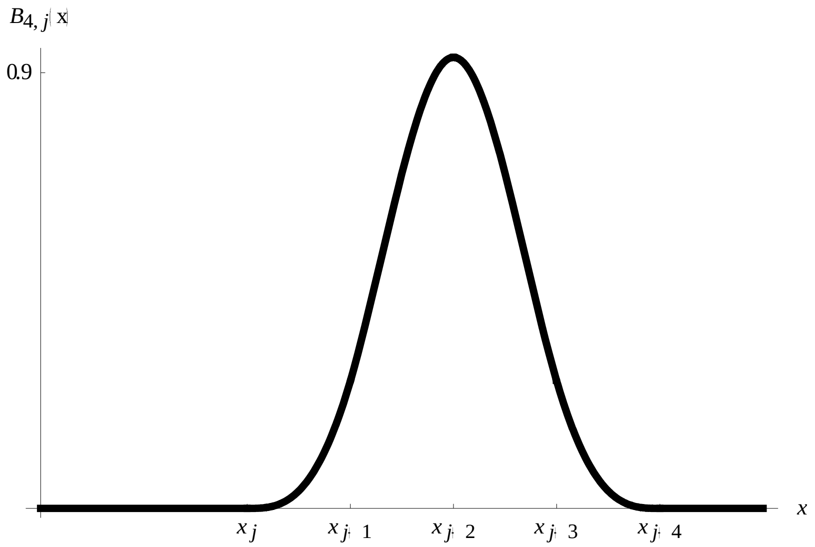

Figure 2.11: Hybrid B-spline basis function of order 4 with $\gamma = 0.1$

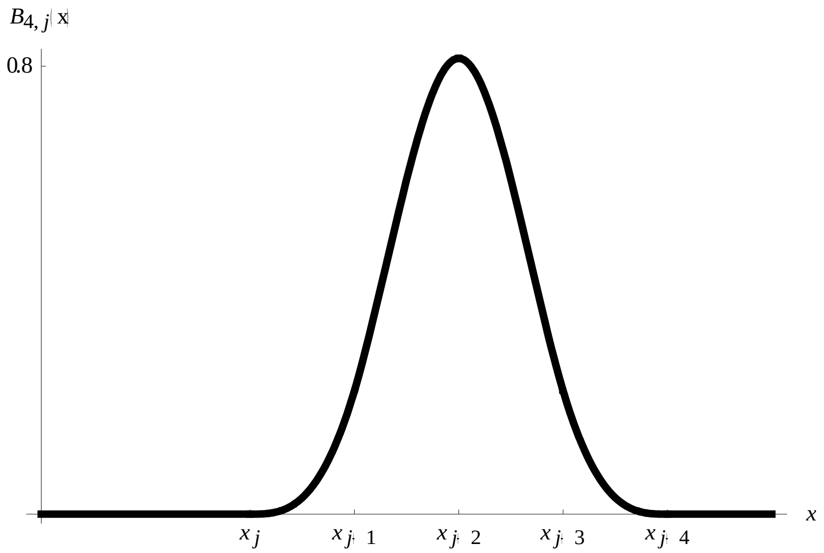

Figure 2.12: Hybrid B-spline basis function of order 4 with $\gamma = 0.5$

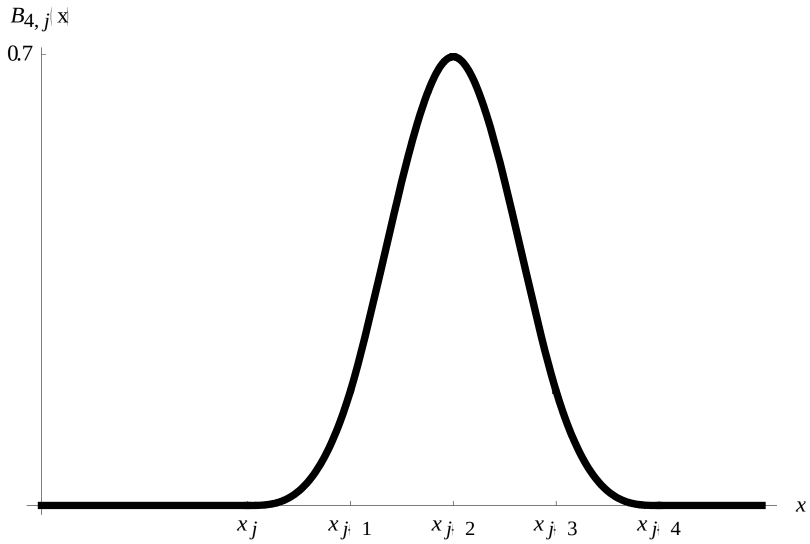

Figure 2.13: Hybrid B-spline basis function of order 4 with $\gamma = 0.9$

2.5 Collocation method

In this section, cubic B-spline (CuBS) collocation method is described. The following one-dimensional wave equation is considered:

$$u_{tt} - u_{xx} = 0 \tag{2.15}$$

with the given initial and boundary conditions

$$u(x,0) = \omega_1(x), \;\; u_t(x,0) = \omega_2(x), \;\; a \le x \le b$$

$$u(a,t) = \phi_1(t), \;\; u(b,t) = \phi_2(t), \;\; 0 \le t \le T$$

A uniform mesh with grid points (x_j, t_k) is considered to discretize the grid region $\Omega = [a,b] \times [0,T]$ with $x_j = a + jh$ and $t_k = k\Delta t$ where $j = 0,1,2,\ldots,n$ and $k = 0,1,2,\ldots,N$. The value of h and Δt denote mesh space size and time step size, respectively. An approximation of wave equation by θ-weighted scheme is given as follows (Dağ et al., 2005)

$$(u_{tt})_j^k - (1-\theta)(u_{xx})_j^k - \theta(u_{xx})_j^{k+1} = 0 \tag{2.16}$$

In order to solve the equation, time derivative term is discretized by central difference approach. The value of θ is chosen to be 0.5 (Dağ et al., 2005). Hence, the following semi implicit scheme is produced

$$u_j^{k+1} - 0.5(\Delta t)^2 (u_{xx})_j^{k+1} = 2u_j^k + 0.5(\Delta t)^2 (u_{xx})_j^k - u_j^{k-1} \tag{2.17}$$

which is evaluated for $j = 0,1,\cdots,n$ at each time level *k*. According to the method, an approximate solution of this equation is

$$u(x,t) = \sum_{j=-3}^{n-1} C_j(t) B_{4,j}(x) \tag{2.18}$$

where $C_j(t)$ are time dependent unknowns to be determined and $B_{4,j}(x)$ is cubic B-spline basis function of order 4 given in Eq. (2.5).

Due to the local support properties of the basis function, there are only three nonzero basis functions included for evaluation at each x_j. Thus, the approximate solution, $u(x_j, t_k)$ and the derivatives with respect to x can be obtained as follows

$$\begin{aligned} u_j^k &= C_{j-3}^k(t_k) B_{4,j-3}(x_j) + C_{j-2}^k(t_k) B_{4,j-2}(x_j) + C_{j-1}^k(t_k) B_{4,j-1}(x_j) \\ &= \left(\frac{1}{6}\right) C_{j-3}^k + \left(\frac{4}{6}\right) C_{j-2}^k + \left(\frac{1}{6}\right) C_{j-1}^k \end{aligned} \tag{2.19}$$

$$\begin{aligned} (u_x)_j^k &= C_{j-3}^k(t_k) B'_{4,j-3}(x_j) + C_{j-2}^k(t_k) B'_{4,j-2}(x_j) + C_{j-1}^k(t_k) B'_{4,j-1}(x_j) \\ &= \left(\frac{-1}{2h}\right) C_{j-3}^k + \left(\frac{1}{2h}\right) C_{j-1}^k \end{aligned} \tag{2.20}$$

$$\begin{aligned} (u_{xx})_j^k &= C_{j-3}^k(t) B''_{4,j-3}(x_j) + C_{j-2}^k(t) B''_{4,j-2}(x_j) + C_{j-1}^k(t) B''_{4,j-1}(x_j) \\ &= \left(\frac{1}{h^2}\right) C_{j-3}^k + \left(\frac{-2}{h^2}\right) C_{j-2}^k + \left(\frac{1}{h^2}\right) C_{j-1}^k \end{aligned} \tag{2.21}$$

for $j = 0,1,\ldots n$ (derivation in Appendix A).

Initially, time dependent unknown $\mathbf{C}^0$ is calculated by using the following initial condition and boundary values of the derivatives of the initial condition (Caglar et al., 2006; Dağ et al., 2005):

i. $(u_x)_j^0 = \omega_1'(x_j)$ for $j = 0$

$$\left(\frac{-1}{2h}\right)C_{-3}^0 + \left(\frac{1}{2h}\right)C_{-1}^0 = \omega_1'(x_0) \tag{2.22}$$

ii. $u_j^0 = \omega_1(x_j)$ for $j = 0,1,2,...,n$

$$\left(\frac{1}{6}\right)C_{j-3}^0 + \left(\frac{4}{6}\right)C_{j-2}^0 + \left(\frac{1}{6}\right)C_{j-1}^0 = \omega_1(x_j) \tag{2.23}$$

iii. $(u_x)_j^0 = \omega_1'(x_j)$ for $j = n$

$$\left(\frac{-1}{2h}\right)C_{n-3}^0 + \left(\frac{1}{2h}\right)C_{n-1}^0 = \omega_1'(x_n) \tag{2.24}$$

This yields a $(n+3)\times(n+3)$ matrix system, $\mathbf{AC}^0 = \mathbf{B}$. The solution of the system can be obtained by using the Thomas Algorithm. Then, the calculation is continued to generate the time dependent unknowns, $\mathbf{C}^k$ for $k \geq 1$ using (2.17) scheme. However, the system consists only $(n+1)$ linear equations with $(n+3)$ unknowns. Thus, the boundary conditions are approximated as follows and substituted to the system.

i. $\frac{1}{6}C_{-3}^{k+1} + \frac{4}{6}C_{-2}^{k+1} + \frac{1}{6}C_{-1}^{k+1} = \phi_1(t_{k+1})$

ii. $\frac{1}{6}C_{n-3}^{k+1} + \frac{4}{6}C_{n-2}^{k+1} + \frac{1}{6}C_{n-1}^{k+1} = \phi_2(t_{k+1})$

This operation produce a $(n+3)\times(n+3)$ tridiagonal matrix system, $\mathbf{MC}^{k+1} = \mathbf{NC}^k - \mathbf{PC}^{k-1} + \mathbf{Q}$. This system can be solved using the Thomas Algorithm repeatedly for $k = 0,1,\cdots,N$.