SUMMATION INVARIANTS OF OBJECTS UNDER PROJECTIVE TRANSFORMATION GROUP WITH APPLICATION

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by

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LIST OF ABBREVIATIONS

CT	Computed Tomography Image
MRI	Magnetic Resonance Imaging Representation
2D	Two Dimensional
D	Dimension
DF	Degree of Freedom
SE	Standard Error
SD	Standard Deviation
m	Mean

LIST OF SYMBOLS

$\mathrm{A}(n)$	Affine group of degree n
$\mathrm{SE}(n)$	Special Euclidean group of degree n
$\mathrm{E}(n)$	Euclidean group of degree n
$\mathrm{GL}(n)$	General liner group of degree n
$\mathrm{PGL}(n)$	projective group of degree n
R	Real number
D_n	Dihedral group
S_n	Symmetric group
C	Complex number
R^n	Cartesian product of n copies of real numbers
P^n	Projective space of degree n
$\mathrm{O}(n)$	Orthogonal group
$\mathrm{SO}(n)$	Special Orthogonal group
$\mathrm{SL}(n, C)$	Complex Special linear group of degree n
J	Jet bundle
dim	Dimension
det	Determinant

Sym	Symmetric group or permutation group
$Diff$	Group of diffeomorphisms

PERJUMLAHAN TAK VARIAN OBJEK DI BAWAH KUMPULAN TRANSFORMASI UNJURAN DENGAN APLIKASI

ABSTRAK

Geometri tak varian adalah ciri geometri yang tidak berubah di bawah pelbagai jenis transformasi dan ia dapat digunakan dalam pemerihalan bentuk untuk mengatasi ber bagai kesukaran dalam pemasalahan pengecaman objek untuk visi komputer. Peranan parameter tak varian diiktiraf dalam beberapa aplikasi seperti perwakilan bentuk, pemadanan bentuk, pengecaman objek dan aplikasi robotik. Tesis ini mengetengahkan penyelesaian permasalahan berkaitan terbitan tak varian objek dua dimensi untuk kumpulan transformasi unjuran. Di dalam tesis ini, satu kaedah diberikan untuk menentukan perjumlahan tak varian objek planar pada kumpulan transformasi unjuran dan satu algoritma telah dibangunkan untuk menggunapakai tak varian yang diterbitkan sebagai penyelesaian beberapa permasalahan pengecaman objek pada kumpulan transformasi. Kaedah Cartan dengan rangka bergerak digunakan untuk menerbitkan tak varian ini. Kamiran potensi baru untuk lengkung 2*D* dicadangkan untuk memperolehi kamiran tak varian di bawah tindakan suatu subkumpulan bagi transformasi unjuran dengan 6 darjah kebebasan. Dalam kes data diskret, penjumlahan tak varian baru dicadangkan di bawah subkumpulan bagi transformasi unjuran objek planar dengan 6 darjah kebebasan. Selain itu, analisis perbandingan telah dilaksanakan untuk tak varian yang

diterbitkan dan tak varian sebelumnya untuk objek di bawah tindakan kumpulan Euclidean, afin dan unjuran. Disamping itu, prestasi kaedah ini telah dibincangkan untuk objek di bawah aras hingar putih bertaburan Gaussian. Ini dapat menyelesaikan permasalahan untuk menerbitkan tak varian untuk objek-objek di dalam kumpulan transformasi unjuran setempat dengan darjah enam kebebasan kerana adanya x dan y dalam penyebut pada ruangan Euclidean. Applikasi tak varian ini terhadap data diskrit, yang diperolehi daripada sampel bentuk geometri kereta, menghasilkan sesuatu corak yang diklasifikasikan serupa di bawah transformasi unjuran. Tambahan lagi, satu kaedah deduktif telah dicadangkan untuk menerbitkan tak varian transformasi unjuran dengan memecahkan kumpulan transformasi dengan 8 darjah kebebasan kepada subkumpulan kecil transformasi unjuran kurang dari 6 darjah kebebasan dengan tak variannya boleh diterbitkan dengan merujuk kepada potensi pembolehubah yang bertindak terhadap kumpulan piawai pada R^2. Oleh sebab imej objek di bawah penjelmaan unjuran dalam beberapa pandangan perspektif boleh cacat sebahagian besarnya, untuk menyelesaikan beberapa masalah yang berkaitan dengan pencarian data yang hilang pada objek planar yang menjalani dua kumpulan transformasi yang berbeza unjuran adalah diminati. Dengan itu suatu algoritma baru dibentangkan untuk mencari titik data dalam objek planar yang tidak ada tanpa apa-apa sebab manakala dua imej perspektif yang berbeza daripada objek adalah ada di bawah dua jenis kumpulan transformasi unjuran yang berbeza. Kaedah ini digunakan untuk menjana data yang hilang dari sempadan imej tomografi berkomputer tengkorak dan magnet perwakilan pengimejan resonans otak. Kemantapan algoritma ini dikaji dengan keadaan bunyi Gaussian putih dalam data sampel yang menunjukkan prestasi baik kaedah ini.

SUMMATION INVARIANTS OF OBJECTS UNDER PROJECTIVE TRANSFORMATION GROUP WITH APPLICATION

ABSTRACT

Geometric invariants are features which unchanged under a variety of transformations and they can be used as the shape descriptors to overcome many of problems of object recognition problems in computer vision. The role of invariants in computer vision has been advocated for various applications such as shape representation, shape matching, object recognition and robotic. This thesis solving problems associated with deriving invariants of two dimensional objects under projective transformation groups in Euclidean space. In this thesis, a method is given to determine projective invariants for planar objects under projective transformation groups and an algorithm is given to apply the derived invariants in order to solve some issues of object recognition under transformation groups. The Cartan's method of moving frame is applied to derive these invariants. Novel integral potentials for $2D$ curves are proposed to derive integral invariants under the action of a subgroup of projective transformation with 6 degrees of freedom. In case of discrete data, new summation invariants are proposed under subgroup of projective transformation of planar objects with 6 degrees of freedom. Besides, comparison analysis has been facilitated for the derived invariants and previous invariants for objects under Euclidean, affine and projective group actions. Moreover, the performance of the method has been discussed for objects under white Gaussian-distributed noise levels. This can solve the problem of deriving invariants for objects under local projective transformation groups happening because of the existing x and y in the dominator of this action in Euclidean space. Application of these invariants to discrete data, obtained from a sample of boundary of car contour, generates a pattern of similar classification under projective transformation. In addition, a deductive method is proposed to derive invariants for projective transformation by splitting this transformation group with 8 degrees of freedom to subgroups of the projective transformation with lower 6 degrees of freedom whose invariants can be derived based on the potential variables for the standard actions of projective groups on R^2. As image of an object under projective transformation in some perspective view may largely deformed, solving some problems associated with finding missing data in a planar object which undergoing two different projective transformation groups is of interest. Thereby a new algorithm is presented to find the data points in a planar object which are not available to any reason while two different perspective images of the object are available under two different types of projective transformation group. The method is applied to generate missing data from the boundary of a computed tomography image of a skull and a magnetic resonance imaging representation of the brain. The robustness of the algorithm is examined in the condition of white Gaussian noises to the sample data which shows the good performance of the method.

CHAPTER 1

INTRODUCTION

Geometric invariants have the most critical role in a wide variety of applications, especially in computer vision and object recognition. Given that geometric invariants have the property of stability under a variety of transformations, these shape descriptors can be used to overcome several difficulties in object recognition problems in computer vision. In fact, an invariant, which is a property of a class of mathematical objects that are unchanged under transformations applied to the object, arises in a wide variety of disciplines, such as mathematics, physics, and computer science.

An approach to geometry is to describe it as the study of invariants under certain allowed transformations which initially defined by Felix Klein. This involves considering our space as a set S and a subgroup G of the bijections of S. Two objects $A, B \subset S$ are said to be equivalent if there is an $f \in G$ such that $f(A) = B$. A property P of subsets of S is said to be a geometric property if it is invariant under the action of the group, which is to say that $P(S)$ is true (or false) if and only if $P(g(S))$ is true (or false) for every transformation $g \in G$. For example, the property of being a straight line is a geometric property in Euclidean geometry. Note that the question whether or not a certain property is geometric depends on the choice of group.

A typical object recognition issue to address is when an object is observed from different views, wherein the appearance of the same object varies. Another problem is when different images of the same object are taken from a moving camera, where one or more parts of the images have been corrupted and thus an entire part is missing. Regarding the application of invariants in classical geometry, one issue of concern is equivalence problems in determining whether or not two given sub-manifolds or two objects are congruent under a group transformation. In other words, two objects are congruent if one can be placed into the other by rotation, translation, or reflection. This determination helps in classifying these objects under the same class of undergoing transformation.

In particular, for objects undergoing projective transformations as a linear fractional action, the shape of the object may totally change in Cartesian coordinates and the classification of these objects is required to identify specific properties that do not change under these transformations. Invariant descriptors are required to solve the typical problems in this area, including the classification of objects under a certain transformation group or to construct missing data of the objects arising from different transformations.

The projective group plays an important role in computer vision and object recognition. The importance of this group lies in its ability to describe all possible viewing transformations on objects because it is the upper set of interest groups, including affine and Euclidean groups, in computer vision. Collinearity of three or more points, concurrency of three or more lines, conic section, and the cross ratio are examples of invariants of projective transformation as the subset of projective groups. Meanwhile, finding appropriate invariants of manifolds under transformation groups has created a large research space. The practical application of these invariants in solving different issues has recently received much attention.

Among all the methods proposed to find invariants under transformations, the moving frame method which firstly introduced by Gaston Darboux and are closely associated to Eli Cartan in the mid-1930s, appears to be completely simple, general, elegant, and applicable. This method has been modeled by Cartan as an algorithmic tool for studying invariants of manifolds under the action of a transformation group. Later, Fels and Olver (1998) formulated the systematic approach of Cartan's method for general transformation groups.

The last half of the 19th century witnessed the evolution in the domain of invariants, including algebraic and differential invariants. Previously, progress was made in finding invariants of general parameterized curves and surfaces. A differential invariant is an invariant for the action of the Lie group on a space, which involves the derivatives of graphs of functions in the space. One of the Euclidean invariants are curvatures which are most frequently used. Other differential invariants obtained by the Cartan's moving frame method have been applied in computer vision problems (Calabi et al., 1998; Olver, 1999, 2001b). These features are local to points on a shape, contrary to algebraic features that are global and can be used for arbitrary shapes.

From the literature regarding to limitations of differential invariants, a novel family of geometric invariants, also called summation invariants, has been introduced. This family of invariants utilizes potentials composed of summation of coordinates in a prolonged jet space. The generality and implementation of these invariants are better than the differential and integral invariants available in the literature. These invariants were derived for the Euclidean and affine transformation groups. In computer vision, very limited types of invariants have been used. For the case of projective transformation group, planar projective invariants were derived by innovation of using homogenous coordinates and applied for camera network.

1.1 Motivation

Summation potential of the Euclidean and affine group cannot be applied to derive invariants for the case of the projective group. Although in the case of projective transformation, planar projective invariants proposed for camera network by using homogenous coordinates for the transformed potentials instead of Cartesian Cartesian. However, derived invariants for objects in terms of Cartesian coordinates under fractional action can not be applied. Hence, in this thesis, it is motivated to derive invariant features for objects under action of projective transformation group in terms of Cartesian coordinates without using homogenous coordinates. Moreover, it is motivated to apply these invariants to solve the typical problems in the area of computer vision including classification of objects under projective transformations group on which are applied or to construct missing data of the objects arising from different view of transformations. These are required to find descriptors which are invariant due to the certain projective transformation group.

1.2 Problem statement

The existence of the x and y terms in the dominator of the projective group action causes problems in classically deriving invariants under the action of projective groups. The goal is to introduce a proper potentials to allow the derivation of the integral and summation invariants for the local action of projective group in terms of Cartesian coordinates based on the Cartan's moving frame method. These new potentials are in terms of integral for the continuous curves and summation for discrete data of objects which allow to make a jet space in terms of new variables for prolonged group action.

1.3 Objectives

In this thesis, a new method for deriving integral and summation invariants of planar objects under projective transformation groups is proposed, and the derived invariants are applied to solve certain issues in object recognition. The following three objectives are considered.

1. Derive integral and summation invariants of two-dimensional ($2D$) objects under projective transformation with six degree of freedom ($6DF$) and the new split-transformation method to derive invariants under projective transformation with more than $6DF$ (maximum of $8DF$).

2. Propose a new method for finding and reconstructing the data points of a planar object undergoing two different projective transformation group.

3. Apply the proposed method to find data of the boundary of a computed tomography image of a skull and a magnetic resonance imaging representation of the brain.

1.4 Organization of the thesis

The rest of this thesis is organized as follows. Chapter 2 presents the research background, including previous works and a detailed explanation on the integral and summation invariants under group actions. Chapter 3 discusses the method for deriving integral invariants under a special projective group for $2D$ curves. Chapter 4 explains the method for deriving summation invariants under a special projective group in the case of discrete data of $2D$ objects, and then compares these invariants with the previous summation invariants. Chapter 5 presents a deductive approach to derive the summation invariants for $2D$ objects under a projective group with a maximum of $8DF$, and applies this method in the classification of objects for the sample of car contours from a database. Chapter 6 proposes a novel method to generate the missing data of objects arising from two different views of transformation, and applies the proposed method for the boundary of a computed tomography image of a skull and a magnetic resonance imaging representation of the brain. Moreover, experimental results are provided to assess the robustness of our invariants under noise. Finally, Chapter 7 summarizes the drawn conclusions.

CHAPTER 2

LITERATURE REVIEW AND BACKGROUND THEORIES

2.1 Introduction

This thesis proposes a classical method for deriving the integral and summation invariants under the projective group, and then some application applies of these invariants will be presented including a new method for constructing the missing data of $2D$ objects in a typical object recognition problem.

The following background material is presented to provide a context for the remainder of the thesis. The group action and Cartan's moving frame method are first discussed, followed by the background of previous invariants, and finally, the summation invariants.

2.2 History of geometric invariants

An extensive study on invariants was conducted in the last half of the 19th century. Two types of invariants were developed, namely, algebraic and differential invariants. The algebraic invariant is associated with invariants of algebraic forms, namely, homogenous polynomials. Hilbert (1993) introduced a famous set of theorems for polynomial invariants that has become the foundation of algebraic geometry. Meanwhile, considerable research progress has been achieved in finding invariants of general parameterized curves and surface. Wilczynski (1906) and Weyl (1997) developed theories of invariants of general Lie group transformations. Among all the methods applied to derive invariants, the moving frame method appears to be superior because of its simplicity, integrity and generality (Faugeras, 1994). Theory of moving frame of normalization was introduced by Eli Cartan in 1935 and was formalized as a powerful framework by Fels and Olver (1998). This theory has good potential for various applications.

Some basic concepts of a group action is presented to describe generalization of Cartan's method of moving frame (Fels and Olver, 1999) which provide a powerful algorithm to construct a complete set of invariants that will be discussed through the following sections.

2.3 Transformation group and group action

In the 20th century, Felix Klein clarified the foundational role of groups in geometry, showed how each type of geometry such as Euclidean geometry, affine geometry, and projective geometry can be completely characterized by an underlying transformation group (Olver, 1999).

Definition 2.1. (Olver, 1999) A group is a set G admitting a binary multiplication operation $(G,.)$, denoted $g.h$ for group elements $g,h \in G$, which is subject to the following axioms:

1. Associativity: $g(hk) = (gh)k$ for $g,h,k \in G$.

2. Identity: The group contains a distinguished identity element, denoted e, satisfying $eg = ge = g$ for all $g \in G$.

3. Invertibility: Each group element g has an inverse $g^{-1} \in G$ satisfying $g.g^{-1} = g^{-1}.g = e$.

The simplest example of a group is the set R of real numbers, with addition being the group operation.

The collection of all permutations of the set $\{1,2,...,n\}$ under the operation of composition of functions is a group S_n called the symmetric group of degree n.

Definition 2.2. (Rose, 1978) If $(G,*)$ and $(H,\circ)$ are groups, then a function $f : G \longrightarrow H$ is a homomorphism if $f(x*y) = f(x) \circ f(y)$ for all $x,y \in G$.

Definition 2.3. (Olver, 1999) A map $\rho : G \longrightarrow H$ between groups G and H is called a group homomorphism if it satisfies $\rho(g.h) = \rho(g).\rho(h)$, $\rho(e) = e$, $\rho(g)^{-1} = \rho(g^{-1})$ for all $g,h \in G$.

As the fundamental concept of a manifold, which forms a differential geometric generalization of classical curves and surfaces, they look like open subsets of Euclidean space R^n. The formal definition of an n-dimensional (The number of local coordinates required to describe points thereon) manifold is a Hausdorf topological space X which is covered by a collection of open subsets $W_\alpha \subset X$, called coordinate charts, and one-to-one local coordinate maps $x_\alpha : W_\alpha \longrightarrow V_\alpha \subset R^n$ and $x_\beta : W_\beta \longrightarrow V_\beta \subset R^n$; the standard coordinates $x^\alpha = (x_1^\alpha,...,x_n^\alpha)$ and $x^\beta = (x_1^\beta,...,x_n^\beta)$ on V_α and V_β provide coordinates for the points on X. The maps $\chi_{\beta\alpha} = \chi_\beta \circ \chi_\alpha^{-1} : V_\alpha \longrightarrow V_\beta$, which must be continuous, define the changes of local coordinates $x_\beta = \chi_{\beta\alpha}(x_\alpha)$ on X. The manifold is smooth if the overlap maps are smooth where defined, and analytic if they are analytic. The simplest example of a manifold is an open subset $X \subset R^n$ of Euclidean space with dimension $dimX$.

Definition 2.4. (Feng et al., 2006) An action of group G on a manifold M is a map $\varphi : G \times M \mapsto M$ which satisfies in the following two properties:

1. $\varphi(e, x) = x$ where e is a identity element of G, $\forall x \in M$.

2.$\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$, $\forall g_1, g_2 \in G, \forall\ x \in M$. In other words a permutation $\varphi_g : M \mapsto M$ satisfies in $\varphi_e(x) = x$ and $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2}$.

Example 2.1. The group $\mathrm{GL}(n, R)$ of invertible $n \times n$ real matrices acts on R^n by matrix multiplication: $(M, x) \mapsto Mx$.

Example 2.2. The complex special linear group, $\mathrm{SL}(2, C)$ of 2×2 complex matrices with determinant 1, acts on the Rieman sphere by:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \frac{az + b}{cz + d}.$$

Definition 2.5. (Olver, 1999) A transformation group acting on a space M is defined by a group homomorphism $\rho : G \mapsto G(M)$ ($G(M)$ denotes the set of all one-to-one maps from M to M) mapping a given group G to the group of invertible maps on M. In other words for each g from G map $\rho(g) : M \mapsto M$ is induced so that the require to define a group homomorphism is:

$\rho(gh) = \rho(g) \circ \rho(h), \rho(e) = 1_{\mathrm{M}}, \rho(g^{-1}) = (\rho^{-1}(g))$, for each $g, h \in G$.

It is common to write $g.x$ instead $\rho(g)(x)$.

If ρ is an action of G on M, then, $\forall g \in G$, the map $\rho_g : M \mapsto M$ given by $\rho_g(x) = \rho(g, x)$ is a diffeomorphism of M, thus G is represented as a group of diffeomorphisms or ransformations of M. For this reason the Lie group G is also referred as a transformation group of the manifold M.

Theorem 2.1. (Olver, 1999) Actions of the group G on the set M are the same as group homomorphism from G to $Sym(M)$, the group of permutation (representation of G on M) of M.

Example 2.3. A transformation $\varphi : \mathrm{R}^n \mapsto \mathrm{R}^n$ is called an isometry if $||\varphi(x) - \varphi(y)|| = ||x - y||$ for all $x, y \in \mathrm{R}^n$. It preserves Euclidean distances. The group of linear isometries forms a subgroup $\mathrm{O}(n) \subset \mathrm{GL}(n,\mathrm{R})$ of the general linear group, known as orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{GL}(n,\mathrm{R}) | A^T A = I\}$.

Example 2.4. An affine transformation of the linear space R^n is a combination of a linear transformation and a translation, and hence has the general form $x \mapsto Ax + a$, where $A \in GL(n,\mathrm{R})$ is an invertible matrix and $a \in \mathrm{R}^n$ a fixed vector. The composition of two affine transformations is also affine, as is the inverse. Therefore the set $\mathrm{A}(n)$ of all affine transformations forms a group. The affine group often denoted by $\mathrm{A}(n) = \mathrm{GL}(n, R) \ltimes \mathrm{R}^n$.

Example 2.5. The group of symmetries of the cube acts on a variety of sets including: the set of eight vertices, the set of six faces, the set of twelve edges, and the set of four principal diagonals.

Example 2.6. Take a regular n-sides polygon in the plane. All rigid motions that send the polygon to itself form a dihedral group D_n. The group D_n ; $n \geq 3$ acts on a rigid motion of a regular n-gon in the plane, either rotation or a reflection.

Example 2.7. The symmetric group S_n acts on R^n by permuting the coordinates for $\sigma \in \mathrm{S}_n$ and $v = (C_1, ..., C_n) \in \mathrm{R}^n$; $\sigma v = (C_{\sigma(1)}, ..., C_{\sigma(n)})$.

2.4 Euclidean and affine transformation

Projective geometry is algebraically expressed by homogenous coordinates while Euclidean geometry is expressed by the Cartesian coordinate system. Cartesian coordinates are convenient to explain angles and lengths. They are simply transformed by matrix algebra to describe translations, rotations and change of scale. However, in the case of projections it can not be expressed simply in the algebraic form.

Some transformation that are non-linear on Euclidean space R^n, can be represented as linear transformation on the $n+1$-dimensional space R^{n+1}. Hence, it is possible to map points from Cartesian coordinates in to homogenous coordinates. The $2D$ point with Cartesian coordinates $x_c = (x\ y)^T$ is mapped into $x_h = (wx\ wy\ w)^T$ in homogenous coordinates, where w is an arbitrary scalar (Nixon, 2008).

Geometric transformation refers to the objects or the coordinate systems. affine, Euclidean and projective transformation are three important groups of transformations in computer vision. They are combination of translation, shearing, rotation, and scaling of an object.

Definition 2.6. (Nixon, 2008) A Euclidean transformation is defined as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} s_x \\ s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \nu_x \\ \nu_y \end{pmatrix}, \qquad (2.1)$$

where θ is rotation angle, $S = (s_x\ s_y)^T$ the scale and $t = (\nu_x\ \nu_y)^T$ the translation along each axis.

Definition 2.7. (Nixon, 2008) The standard affine transformation in $2D$ is defined as the following equation which is viewed as a combination of a translation, a rotation, a scaling and a shearing:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.2}$$

$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$. An affine transformation is any transformation that preserves collinearity and ratios of distances (the midpoint of a line segment remains the midpoint after transformation). In this sense, affine indicates a special class of projective transformations that do not move any objects from the affine space R^3 to the plane at infinity or conversely.

2.5 Projective transformation

In projective geometry, a homography is an isomorphism of projective spaces, is a bijection that maps lines to lines, and thus a collineation. In general, some collineations are not homographies, but the fundamental theorem of projective geometry states that is not so on account of real projective spaces of dimension at least two. Equivalent words incorporate projectivity, projective change, and projective collineation. In projective space, transformations are called homographes and they are more broad than similarity and affine transformations. Projective transformations are not defined on all of the plane, but only on the complement of a line. Any plane projective transformation can be expressed by an invertible 3×3 matrix in homogeneous coordinates. Conversely, any invertible 3×3 matrix defines a projective transformation of the planes. Projective transformation generally is one-to-one mapping of an n-dimensional projective space to m-dimensional space. In particular, an image plane is an $2D$ projective space P^2 residing in the $3D$ projective space P^3 such that a point in an image plane can be represented in the homogenous coordinates.

Projective transformations under composition form a group called projective linear group. In the projective plane P^2, projective linear group is denoted as $PGL(3)$. The elements of $PGL(3)$ is of the form

$$H_p = \begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & \nu \end{pmatrix}. \quad (2.3)$$

The block matrix form H_p will be

$$H_p = \begin{pmatrix} A & t \\ \mathbf{v}^T & \nu \end{pmatrix}, \quad (2.4)$$

where A is a non-singular matrix, t is a translation 2-vector, and $\mathbf{v} = (v_1\ v_2)^T$. This projective transformation has 8 degrees of freedom and can be computed from 4 point correspondence. These transformation preserve collinearity and cross ratio.

2.6 Lie group action and invariant

Sophus Lie introduced and developed the remarkable theory of continuous or Lie groups whose elements depend analytically on parameters.

Definition 2.8. (Gorbatsevich et al., 1994) A Lie group is a group G endowed with a structure of a differentiable manifold over M so that the maps $\varphi : G \times G \longrightarrow G$ where $\varphi : (x, y) \mapsto xy$ and $i : G \longrightarrow G$ where $i : x \mapsto x^{-1}$ are differentiable over M.

For instance, R^n under addition and inverses given by negatives $\varphi(g, h) = g + h, i(g) = -g$ for $g, h \in \mathrm{R}^n$ is an n-dimensional Lie group.

A left action of the group G on the manifold M is a smooth mapping $\varphi : G \times M \longmapsto M, \varphi : (g; x) \mapsto \varphi(g; x)$, such that $\varphi(e; x) = x$ and $\varphi(g'; \varphi(g; x)) = \varphi(g'g; x)$. Similarly, a right action is defined by a smooth mapping $\varphi : M \times G \mapsto M$ such that $\varphi(x; e) = x$ and $\varphi(\varphi(x; g); g') = \varphi(x; gg')$.

In both cases φ can define a mapping $g \mapsto \varphi_g$ by $\varphi_g = \varphi(g, .)$ or $\varphi_g = \varphi(., g)$. If φ is the mapping $\varphi_g : M \mapsto M$ associated with the action of g on M, it is seen that the left action satisfies the homomorphism property $\varphi_{g'} \circ f_g = \varphi_{g'g}$ and the right action satisfies the anti-homomorphism relation $\varphi_{g'} \circ \varphi_g = \varphi_{gg'}$. Since $\varphi_{g^{-1}} = (\varphi_g)^{-1}, \varphi_g$ is a diffeomorphism of M, so φ is a homomorphism $\varphi : G \mapsto \mathrm{Diff(M)}$ of G in to the group of diffeomorphisms of M. If φ_g is a right(left) action, $\varphi_{g^{-1}}$ is a left(right) action. In general, a transformation group is a symmetry group of an object if the object does not change under the action of group.

Definition 2.9. (Olver, 1999) Let $Y \subset M$. A symmetry of Y is an invertible transformation $\varphi : M \longrightarrow M$ that leaves Y fixed, so $\varphi(Y) = Y$.

In fact, starting with a given transformation group $\rho : G \mapsto G(M)$, the symmetry subgroup of a subset $Y \subset M$ is $G_Y = \rho^{-1}(S(Y)) = \{g \in G | g.Y = Y\}$. A transformation group G acting on M is a symmetry group of the subset Y if every group element is a symmetry, so that $G_Y = G$. In this case, Y is said to be a G-invariant subset of M.

Example 2.8. The square with vertices $S = \{(1,0),(0,1),(-1,0),(0,-1)\}$, the linear and affine symmetry groups coincide. They consist of eight linear isometries: the identity; rotations through 90, 180, and 270 degree; reflections through the two coordinate axes; and reflections through the two lines making 45 degree angles with the axes. However, there are some nonlinear transformation that fix two adjacent vertices and interchange the other two. For instance, the projective transformation

$$(x,y) \mapsto \left(\frac{x-y+1}{2x+2y}, \frac{-x+y+1}{2x+2y} \right)$$

fixes $(1,0)$ and $(0,1)$ while interchanging $(-1,0)$ and $(0,-1)$. Since the projective transformation preserve edge of square and collinearity (for example all points lying on a line initially still lie on a line after transformation), so there are 24 projective symmetries of a square form a group isomorphic to S^4.

An orbit of a transformation group is a minimal nonempty invariant subset. In particular, a fixed point is a G-invariant point $x_0 \in M$, so that $g.x_0 = x_0$ for all $g \in G$.

Proposition 2.1. (Olver, 1999) Given a transformation group acting on a space M, the orbit O_x through a point $x \in M$ is just the set of all images of x under arbitrary group transformation : $O_x = \{gx | g \in G\}$. A subset $S \subset M$ is G-invariant if and only if it is the union of orbits.

Definition 2.10. (Olver, 1999) An invariant of a transformation group is a real function $\mu : M \mapsto \mathrm{R}$ which satisfies:

$$\mu(g \circ x) = \mu(x), \forall x \in M, \forall g \in G. \tag{2.5}$$

Invariant functions are constant along each orbit and can be used to find equivalence classes of objets undergoing various types of transformations.

Example 2.9. In the case of rotation acting on R^2, consider the standard action

$$(x,y) \mapsto (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$$

of the rotation group $\mathrm{SO}(2) = \{\mathrm{A} \in \mathrm{O}(2) | \det(\mathrm{A}) = 1\}$ on R^2. The orbits are circles centered at the origin for any point $x = (0,0)$. Any circle $\{x^2 + y^2 = r^2\}$ centered at the origin is a rotationally invariant subset of the plane. These circles are the orbits of SO(2). The only fixed point is the origin. Every other invariant subset,e.g., $\{a < x^2 + y^2 = r^2 < b\}$, is a union of circles.
Also, an invariant function $\mu : \mathrm{R}^2 \to \mathrm{R}$ is $\mu(x,y) = \sqrt{x^2 + y^2}$.

Definition 2.11. (Steeb and Steeb, 2007) The group G is called a Lie transformation group of differential manifold M if there is a differential map $\varphi : G \times M \mapsto M$; $\varphi(g,x) = gx$ such that

1. $e.x = x$ for the identity element of G, $x \in M$.

2.$(g_1.g_2)x = g_1.(g_2x)$, $\forall g_1, g_2 \in G, x \in M$. are satisfied. Note that x is transformed to gx by the transformation φ.

Definition 2.12. (Olver, 1999) The isotropy subgroup of a point $x \in M$ is $G_x = \{g|g.x = x\} \subset G$ consisting of all group element g which fix x.

Definition 2.13. (Olver, 1999) The action is said to be free if the only element of G that fixes any element of M be the identity, $\varphi(g,x) = g.x = x$ for some $x \in M$ implies $g = e$, or equivalently every isotropy group is trivial. If the action is free, every point of M is transformed ($\varphi_g(x) \neq x$ if $g \neq e$) and the mapping $g \mapsto \varphi(g,x)$ is bijective for each (there is a one-to-one correspondence between G and each orbit in M). Thus if the action is free, then any given two points x', x are either unrelated or connected by a unique g, $\varphi_g(x) = x'$.

A group acts semi-regularly if all its orbits have the same dimension. The action is regular if each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is connected (Olver, 2003).

Theorem 2.2. (Olver, 1999) If G is an r-dimensional Lie group acting analytically on a manifold M, then each orbit is an analytic submanifold of M. Moreover, a point $x \in M$ belongs to an s-dimensional orbit if and only if its isotropy subgroup G_M is a closed Lie subgroup of dimension $r - s$.

Definition 2.14. (Olver, 1999) Let G be a Lie transformation group that acts regularly on the m-dimensional manifold M with s-dimensional orbits. A (local) cross-section is an $(m - s)$-dimensional submanifold $K \subset M$ such that K intersects each orbit transversally and at most once.

Proposition 2.2. (Olver, 1999) If a Lie group G acts regularly on a manifold M, then one can construct a local cross-section K passing through any point $x \in M$.

Example 2.10. Consider the standard action $x \mapsto \mathrm{Rx}$ of the rotation group $\mathrm{SO}(3)$, which is regular on $M = \mathrm{R}^3 - \{0\}$. The orbits are the spheres $||x|| = r$, and hence any ray $\mathrm{R_a} = \{\mathrm{ka}|\mathrm{ka} > 0\}, \mathrm{a} \neq 0$, provides a (global) cross-section. An open line segment parallel to one of the coordinate axes will be a coordinate cross-section provided it is not tangent to any orbit sphere. For example, the vertical half-lines $L = \{x = c_1, y = c_2, z > 0\}$ are local cross-sections. The most general local cross-section is given by a curve that transversally intersects each orbit sphere at most once.

2.7 Moving frame

Due to Olver (1999), Cartan's construction of moving frame through the normalization process was interpreted with the choice of cross section to the group orbits. The existence of a moving frame requires freeness of the underlying group action according to the following theorem.

Definition 2.15. (Olver, 1999) A moving frame is a G-equivariant map $\rho : M \to G$ which clearly depends on the choice of cross-section $K \subset M$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left equivalent, but, in many cases, the right versions are easier to compute.

Theorem 2.3. (Olver, 2003) A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Theorem 2.4. (Olver, 1999) Let G act freely and regularly on M, and let $K \subset M$ be a cross section, given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross section: $g.z = \rho(z).z \in K$ then ρ is a right moving frame for the group action.

Given local coordinates $z = (z_1, ..., z_m)$ on M, let $\varphi(g,z) = g.z$ be the explicit formulae for the group transformations. The right moving frame $g = \rho(z)$ associated with a coordinate cross section $k = \{z_1 = c_1, ..., z_r = c_r\}$ is obtained by solving the normalization equations $\varphi_1(g,z) = c_1, ..., \varphi_r(g,z) = c_r$ for the group parameters $g = (g_1, ..., g_r)$ in terms of coordinates $z = (z_1, ..., z_m)$.

2.8 Prolongation to jet space

Unfortunately, the dimension of an orbit is commonly greater than or equal to the dimension of the manifold, so the way to fix this problem is to create a larger space called jet space so that invariant function could be found there and group actions are prolonged so that the coordinates of jet space are appropriately transformed.

Moreover, most interesting group actions are not free, and therefore do not admit moving frame. There are two common method for solving this problem. The first is to look at the product action of G on several copies of M, leading to joint invariants (Olver, 2001b). The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants (Olver, 2009). Combining the two methods of prolongation and product lead to joint differential invariants (Olver, 2001a).

In differential geometry, the jet bundle is a specific construction that makes a new fiber bundle out of a given smooth fiber bundle. It helps to write differential equations on sections of a fiber bundle in an invariant form. Jets may also be seen as the coordinate free versions of Taylor expansions.

Historically, jet bundles are attributed to Ehresmann, and were an extension on the method of Elie Cartan, of dealing geometrically with higher derivatives, by imposing differential form conditions on newly introduced formal variables.

Given an r-dimensional Lie group G acting on an m-dimensional manifold M, for a sub-manifold $S \subset M$ of a prescribed dimension $p < m$, the group action is prolonged to the sub-manifold jet bundles $J^n = J^n(M, p)$ of order n.

2.9 Differential invariant

Olver (1999) proposed the jet space to generating the differential invariants which is coordinated as independent variables, dependent variables and derivatives of dependent variables called the derivative jet space. The coordinate of derivative jet space J^n for a smooth function $u = f(x)$ involves p independent variables $x = (x_1, ..., x_p)$ and q dependent variables $u = (u_1, ..., u_q)$ and all the partial derivatives of order up to n. A point in the derivative jet space J^n is denoted by $(x, u^{(n)})$, where $u^{(n)}$ contains dependent variables and partial derivatives up to order n. The action of group G on J^n is called the n^{th} prolongation and is denoted by $pr^{(n)}G$. This prolonged group action is defined so that the derivatives of function $\overline{u} = \overline{f}(\overline{x})$ are mapped to corresponding derivatives of transformed function $\overline{u} = \overline{f}(\overline{x})$.

Specifically, for any point $(x_0; u_0^{(n)}) \in J^n$, the prolonged group action is defined by

$$pr^{(n)}(x_0; u_0^{(n)}) = (\overline{x}_0; \overline{u}_0^{(n)}). \tag{2.6}$$

In other words, the transformed derivatives are found by evaluating the derivatives of transformed function $\overline{f}(\overline{x})$ at point x_0. Restriction of the prolonged transformation action to coordinates cross section components of ω^n will be normalized according to Theorem 2.4 to

$$\omega_1(g,z^n) = c_1, ..., \omega_r(g,z^n) = c_r, \tag{2.7}$$

where $r = dimG$. Solving the normalized equations for the group transformation leads to the right moving frame $g = pr^n(z^n)$.

Theorem 2.5. If $g = \rho(z)$ is the moving frame from the solution of normalized equation, then $I_1(z) = \omega_{r+1}(\rho(z),z), ..., I_{m-r}(z) = \omega_m(\rho(z),z)$ forms a complete system of functionally independent invariants.

Any finite dimensional Lie group action admits an infinite number of functionally independent differential invariants of progressively higher and higher order.

Example 2.11. (Olver and Sommer, 2005) Consider Euclidean group E(2) acts on $M = \mathrm{R}^2$ by mapping a point $z = (x,u)$ to

$$(y,v) = (x\cos(\theta) - u\sin(\theta) + a, x\sin(\theta) + u\cos(\theta) + b). \tag{2.8}$$

The first prolongation $pr^{(1)}G$ will act on the space coordinated by $\{x,u,u_x\}$. By equation (2.8) and the definition of prolongation, the transformed coordinates are given by

$$\begin{aligned}
\overline{x} &= x\cos\theta - u\sin\theta + a, && (2.9)\\
\overline{u} &= x\sin\theta + u\cos\theta + b, && (2.10)\\
\overline{u}_{\overline{x}} &= \frac{d\overline{u}}{dx}\frac{dx}{d\overline{x}} = \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta}. && (2.11)
\end{aligned}$$

In other words, the prolonged group action is

$$pr^{(1)}g \circ (x, u, u_x) = \tag{2.12}$$
$$\left(x\cos(\theta) - u\sin(\theta) + a, x\sin(\theta) + u\cos(\theta) + b, \frac{\sin(\theta) + u_x\cos(\theta)}{\cos(\theta) - u_x\sin(\theta)}\right)$$

given cross section normalized by

$$y = 0, v = 0, v_y = 0. \tag{2.13}$$

The classical Euclidean moving frame is obtained by solving equations (2.13):

$$\theta = -\tan^{-1}(\frac{\dot{u}}{\dot{x}}), a = \frac{x\dot{x} + u\dot{u}}{\sqrt{\dot{x}^2 + \dot{u}^2}}, b = \frac{x\dot{u} - u\dot{x}}{\sqrt{\dot{x}^2 + \dot{u}^2}}, \tag{2.14}$$

where $\dot{u} = \frac{du}{d\theta}$ and $\dot{x} = \frac{dx}{d\theta}$. Substituting the moving frame $\{\theta, a, b\}$ in the prolonged transformation (2.12) gives us

$$v_{yy} = \kappa = \frac{\dot{x}\ddot{u} - \ddot{x}\dot{u}}{(\dot{x}^2 + \dot{u}^2)^{\frac{3}{2}}}, v_{yyy} = \frac{d\kappa}{ds}, v_{yyyy} = \frac{d^2\kappa}{ds^2} + 3\kappa^3, \tag{2.15}$$

where $d/ds = \|\dot{z}\|^{-1}d/dt$.

Differential invariants are local so the occlusion problem will be less by choosing other points. Moreover they are applicable for any kind of curves.

Olver (1999) in the classical differential invariants proposed Euclidean curvature and torsion for space curves. However, the derivatives order of Euclidean curvature torsion are up to 2 and 3 order, respectively, and their affine analogs depended on derivatives of up to order 6.

The practical utilization of differential invariants is limited due to their high sensitivity to noise. Since in the condition of noisy original data, the numerical differentiation amplifies the effects of noise, This motivated the high interest in other types of invariants such as semi-differential (Van Gool et al., 1991), or joint invariants (Olver, 2001a) and various types of integral invariants (Sato and Cipolla, 1997; Manay et al., 2004).

Van Gool et al. (1991) tried to reduce high order differential invariants by joint invariant to lower order derivatives which was evaluated at several points on a curve. Weiss (1993) developed semi differential invariants which are applicable in matching shapes despite occlusion due locality of signature. However the fundamental problem of this type invariants as difficult ones still remains due to high order derivatives.

Statistical approach by using moment invariants was introduced (Hu, 1962). Moment invariants under affine transformations were derived from the classical moment invariants in Flusser and Suk (1993). Their limitation go back this fact that high order moments are sensitive to noise which results in high variances. Moreover, the error analytic related to these invariants is accessible in Liao and Pawlak (1996).