NUMERICAL AND APPROXIMATE-ANALYTICAL SOLUTION OF FUZZY INITIAL VALUE PROBLEMS

By

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Thesis Submitted in fulfillment of the requirements for the Degree of (Doctor of Philosophy)

April 2015

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my supervisor, Professor Ahmad Izani Md. Ismail. Through his many roles as a supervisor and mentor, he has had a tremendous influence on both my professional and personal development. Without his guidance and encouragement, this thesis would not have been possible. Most importantly, he has dedicated precious time and energy to shaping this thesis. I am very grateful for his support and patience throughout my PhD work. I am also grateful to my friend Dr Mohammad Ghoreishi for many useful discussion and advice that made this thesis possible. I am grateful to my wife Lubna and my father in law Dr. Hussein Fatlawi for their support and patience. I am also very grateful to my mother for her affection and to my father soul . Without them, this work would never have come into existence.

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for the upper solution of Eq. (5.45) for all ∈ [0,1] .

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
FDEs	Fuzzy Differential Equations
FIVP	Fuzzy Initial Value Problem
FIVPs	Fuzzy Initial Value Problems
IVP	Crisp Initial Value Problem
IVPs	Crisp Initial Value Problems
ADM	Adomian Decomposition Method
HPM	Homotopy Perturbation Method
VIM	Variational Iteration Method
HAM	Homotopy Analysis Method
OHAM	Optimal Homotopy Asymptotic Method
RK5	Fifth order Runge-Kutta Method with 6 stages
RK4	Fourth order Runge-Kutta Method
RK3	Third order Runge-Kutta Method
LKM	Linear K-step Method
$\hbar$	Convergence-control parameter (HAM)
$\mathcal{H}(t)$	Auxiliary function (HAM)
$\mathcal{L}$	Linear operator
$\mathcal{R}$	Auxiliary function (HAM)
p	Embedding parameter
$\lambda(t,\eta)$	Lagrange multiplier (VIM)
r	r-level fuzzy set

t	Crisp variable
$\tilde{y}$	Fuzzy variable
μ	Fuzzy membership function
$\mathbb{R}$	Real numbers domain
$\mathcal{A}$	Adomian polynomials
Eq	Equation
Eqs	Equations
CPU	Central Processing Unit

PENYELESAIAN BERANGKA DAN ANALISIS HAMPIRAN BAGI MASALAH NILAI AWAL KABUR

ABSTRAK

Persamaan pembezaan kabur (FDEs) digunakan untuk memodel masalah tertentu dalam bidang sains dan kejuruteraan dan telah dikaji oleh ramai penyelidik . Masalah tertentu memerlukan penyelesaian FDEs yang memenuhi keadaan awal kabur menimbulkan masalah awal kabur (FIVPs). Contoh masalah seperti ini boleh didapati dalam fizik, kejuruteraan, model penduduk, dinamik reaktor nuklear, masalah perubatan, rangkaian neural dan teori kawalan. Walau bagaimanapun, kebanyakan masalah nilai awal kabur tidak boleh diselesaikan dengan tepat. Tambahan pula, penyelesaian analisis tepat yang diperoleh juga mungkin begitu sukar untuk dinilai dan oleh itu kaedah berangka dan analisis hampiran perlu untuk memperoleh penyelesaian. Dalam dua dekad yang lalu, pembangunan teknik-teknik berangka dan analisis hampiran untuk menyelesaikan persamaan ini merupakan satu bidang yang penting dalam penyelidikan. Terdapat keperluan untuk merumuskan teknik baru, cekap serta lebih tepat dan ini adalah bidang tumpuan tesis ini. Dalam tesis ini , kami mencadangkan satu kaedah berangka baru berdasarkan kaedah Runge – Kutta peringkat lima dengan enam tahap untuk menyelesaikan sistem linear dan tak linear peringkat pertama dan tinggi yang melibatkan persamaan pembezaan biasa. Kami juga menjalankan analisis ralat dan penumpuan kaedah. Selain itu, kami juga telah mengkaji beberapa kaedah hampiran kaedah analisis variasi, kaedah pengusikan homotopi, dan telah merumuskan dan menggunakan kaedah- kaedah ini untuk menyelesaikan FIVPs linear dan tak linear peringkat pertama yang melibatkan persamaan pembezaan biasa. Kami menggubal dan menggunakan kaedah- kaedah ini untuk menyelesaikan FIVP linear dan tak linear peringkat secara langsung tanpa menurunkan kepada sistem peringkat pertama sebagaimana yang telah dilakukan oleh kebanyakan penyelidik lain. Kami juga telah mencadangkan dua kaedah analisis hampiran- kaedah analisis homotopi dan kaedah asimptot homotopi optimum - untuk mendapatkan penyelesaian hampiran FIVPs linear dan tak linear peringkat pertama dan tinggi dengan satu kajian penumpuan empirikal. Beberapa contoh ujian diberi untuk menggambarkan kebersanankaedah yang dicadangkan dan ketepatan nya .

NUMERICAL AND APPROXIMATE- ANALYTICAL SOLUTION OF FUZZY INITIAL VALUE PROBLEMS

ABSTRACT

Fuzzy differential equations (FDEs) are used for the modeling of some problems in science and engineering and have been studied by many researchers. Certain problems require the solution of FDEs which satisfy fuzzy initial conditions giving rise to fuzzy initial problems (FIVPs). Examples of such problems can be found in physics, engineering, population models, nuclear reactor dynamics, medical problems, neural networks and control theory. However, most fuzzy initial value problems cannot be solved exactly. Furthermore, exact analytical solutions obtained may also be so difficult to evaluate and therefore numerical and approximate- analytical methods may be necessary to evaluate the solution. In the last two decades, the development of numerical and approximate -analytical techniques to solve these equations has been an important area of research. There is a need to formulate new, efficient, more accurate techniques and this is the area of focus of this thesis. In this thesis, we propose a new numerical method based on fifth order Runge-Kutta method with six stages to solve first and high order linear and nonlinear FIVPs involving ordinary differential equations. We also conduct the error and convergence analysis of the method. In addition, we have also studied several approximate-analytical methods- Variation Iterative Method, Homotopy Perturbation Method have been formulated and applied to solve linear and nonlinear first order FIVPs involving ordinary differential equations. Also, we formulated and employed these methods to solve linear and nonlinear high order FIVPs directly without reducing into a first order system as was done by most other researchers. We have also

proposed another two approximate–analytical methods - Homotopy Analysis Method and Optimal Homotopy Asymptotic Method - to obtain an approximate solution of linear and nonlinear first and high order FIVPs together with an empirical study of the convergence. Some test examples are given to illustrate the proposed methods to show their feasibility and accuracy.

CHAPTER 1

INTRODUCTION TO THESIS

1.1 General Introduction

The field of fuzzy set theory [209] permits the gradual assessment of the membership of elements in an ordinary or crisp set; this is described with the aid of a membership function valued in the real interval between 0 and 1. Fuzzy sets generalize crisp or classical sets, since the indicator functions of crisp sets are special cases of the membership functions of the fuzzy sets, if the latter only takes the values 0 or 1. Fuzzy set theory can be applied in a wide range of domains of real world problems in which information is incomplete or imprecise. Furthermore, according to [127, 208] most of our traditional tools for formal modeling and computing are non-fuzzy (crisp), deterministic and precise in character. By crisp, we mean dichotomies that are yes-or-no-types rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false-and nothing in between.

However, the characteristic features of real world system problems are such that real situations are very often vague in a number of ways. Due to lack of information, the future state of the system might not be known completely. This type of vagueness has long been appropriately handled by probability theory and statistics [208]. This type of vagueness or uncertainty is called “stochastic uncertainty” as opposed to something vague concerning the description of the semantic meaning of the events, phenomena or statements themselves. This is called fuzziness [205]. Uncertainty can be found in many areas of our daily lives, such as in engineering [110, 173], in medicine [28, 69], in meteorology [87] in manufacturing [139] and others [63, 111, 128]. The first research publication on fuzzy set theory by Zadeh [204] was followed by [205] that showed the intention of these pioneering authors to generalize the classical notion of a set and a proposition to accommodate fuzziness.

Many dynamical real life problems may be formulated as a mathematical model in the form of systems of ordinary or partial differential equations. Differential equations have proved to be a successful modeling paradigm. In this thesis, we focus on ordinary differential equations (ODEs) in a wide range of disciplines. The behavior of an idealized version of a problem under study is usually adequately described by one or more ODEs. Real-world problems, however, contain uncertainty. According to [203-205] the fuzziness can arise in the experimental part, the data collection, the measurement process, as well as the time when determining the initial conditions. These are obvious when dealing with "living" material such as soil, water, microbial populations, etc. In order to handle these problems, the use of fuzzy sets can be an effective tool for a better understanding of the studied phenomena.

Fuzzy set theory is a powerful tool for the modeling of vagueness, and for processing uncertainty or subjective information on mathematical models. For such mathematical modeling, the use of fuzzy differential equations may be necessary.

Fuzzy differential equations (FDEs) are utilized to analyze the behavior of phenomena that are subject to imprecise or uncertain factors. FDEs [126, 164, 168, 198, 207] takes into account the information about the behavior of a dynamical system which is uncertain in order to obtain a more realistic and flexible model. FDE models have a wide range of applications in many branches of engineering and in the field of medicine. These models are used in various applications including population models [39, 163, 191, 206], quantum optics gravity [65], control design [153], and medicine [8, 38, 90, 154, 185] and other applications [77, 183]. In [153] a genetic fuzzy system an approach to control a nonlinear dynamic model of the HIV infection has been presented. This system was conceived to find the fuzzy controllers that are capable of boosting the immune response while reducing the impact on the body because of the use of potentially toxic medicaments. Genetic fuzzy system problems, both optimal control solutions and rule based and for medical treatment, could be obtained at the same time. Another problem of FDEs was introduced in [48] in the case of hydraulic differential electric servo cylinders. The actual models are strongly nonlinear, coupled with systems of differential equations. The key point is how to fuzzify the classical model in order to get meaningful conclusions and the key term in this fuzzification is that of the frictional term.

This thesis deals with the formulation, analysis and approximation of the numerical and approximate analytical methods by using the properties of fuzzy set theory to obtain reliable solutions of the first and high order (or n^{th} order, $n \geq 2$, see [54,78]) linear and nonlinear FIVPs involving ODEs.

1.2 Background and Motivation

One of the aims for studying fuzzy set theory is to develop the methodology of formulations and to find solutions of problems that are too complicated or ill-defined to be acceptable for analysis by conventional techniques. Therefore, fuzziness may be considered as a type of imprecision that stems from a grouping of elements into classes that do not have exact defined boundaries [205].

When a real world problem is transformed into a deterministic IVP of ODEs, namely $y'(t) = f(t, y(t)), t_0 \leq t \leq T$ subject to the initial condition $y(t_0) = y_0$, we cannot usually be sure that the model is perfect. For example, the initial value may not be known exactly and the function f may contain uncertain parameters. If they are estimated through certain measurements, they are necessarily subject to errors [177]. The analysis of the effect of these errors leads to the study of the qualitative behavior of the solutions of these problems. Thus, it would be natural to employ FIVPs [26, 46, 47, 78]. Most of these FIVPs have crisp or fuzzy coefficients and initial conditions described by a vector of fuzzy numbers.

The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Chang and Zadeh [56]. There are various definitions for the concept of fuzzy number that will be discussed in detail in Chapter (2). FDEs were discussed in, amongst others, [37, 53] and used in some mathematical models in [30,194]. Kandel [128] employed the concept of FDEs for the analysis of fuzzy dynamical systems, but the initial value problem was treated by Kaleva in [121-123], Seikkala in [177], Ouyang and Wu in [164] and Nieto in [155]. Chang and Zadeh in [56] were the first to introduce the concept of fuzzy derivative. It was followed by Dubois and Prade in [62, 63] who used the extension principle [79, 147, 204]. In [202, 204], Zadeh proposed the extension principle which later becomes an important tool in fuzzy set theory and its applications that any crisp function can be extended to take the fuzzy set as arguments by applying this extension principle. Puri and Ralescu in [169] have discussed other methods and they proposed two definitions of derivatives with one based on the H-difference and another dentition with more general concepts of fuzzy function [45]. There are many suggestions on how to define a fuzzy derivative and in consequence to study FDEs [19, 175]. One is the Hukuhara derivative of a set-valued function [121]. First and higher order FDEs with Hukuhara differentiability is considered in [60] and it became fuzziness as time goes on. Hence, the fuzzy solution behaves quite differently from the crisp solution. To alleviate the situation, Hullermeier [111] interpreted fuzzy differential equation as a family of differential inclusions. The main shortcoming of using differential inclusions is that fuzzy number-valued function without a derivative.

Two analytical methods were introduced by Buckley and Feuring [54] for solving n^{th}-order linear differential equations with fuzzy initial conditions. One is the classical method and the other is the extension principle method. The first method of solution was to fuzzify the crisp solution and then checks to see if it satisfies the differential equation with fuzzy initial conditions and the second method was the reverse of the first method in that they first solved the fuzzy initial value problem and then checked to see if it defined a fuzzy function.

In many cases, the analytical solution may be so difficult to obtain and therefore numerical and approximate-analytical methods may be necessary to evaluate the solution so as to analyze the physical problem. There are many numerical methods proposed to obtain the numerical solution of first order linear or nonlinear FIVP.

Numerical solution of first order FIVPs was introduced by Friedman and Kandel [74] through the Euler method and by Abbasbandy and Allahviranloo [3] through the Taylor methods. Also second order generalized Runge – Kutta methods have also implemented to obtain the numerical solution of first order FIVPs by Akbarzadeh and Mohseni in [17]. Furthermore the third order Runge–Kutta method was introduced to new concepts by Kanagarajan and Sambath [124, 125]. Also the two-step and multi-step numerical methods were used to solve the first order FIVP Modified Euler method [180] and Simpson method [59] or multi-step methods [71]. Some of these numerical methods were proposed to obtain the numerical solution of high order FIVP. Examples of the methods are the second, third and fourth Runge-Kutta methods [70, 81, 116,117]. It is clear that there should be more investigation of the numerical methods to solve FIVPs as many existing methods have not been explored.

Several approximate-analytical methods have been proposed to obtain the approximate solution of linear and nonlinear FIVPs which are mostly first order problems. The variational iteration method (VIM) was used to solve first order linear and nonlinear FIVPs [23, 83]. The Adomian decomposition method (ADM) was also proposed to obtain an approximate solution of first order linear and nonlinear FIVPs [21, 35, 83]. The Homotopy Perturbation Method (HPM) and differential transform method were used to solve the first order linear FIVPs [84]. Abbasbandy and Allahviranloo used VIM to solve high order linear FIVPs by reducing the high order FIVP to first order system [7]. It is clear that there should be more investigation of approximate-analytical methods for FIVPs that they have yet to be applied.

Furthermore, there is a need to focus on methods which do not depend on reduction to a first order system.

The focus of this thesis is the study of numerical and approximate-analytical techniques for the solution of first and high order linear and nonlinear FIVPs. The use of approximate-analytical methods in these equations is not without their difficulties. Our purpose of this research is to study the techniques to obtain accurate numerical and approximate solutions for linear and nonlinear first and higher order FIVPs without reducing them to a first order system.

1.3 Objectives

The objectives of this study are:

1. To develop, analyze and apply a new and accurate numerical method based on fifth order Runge-Kutta method with six stages (RK5) to solve linear and nonlinear first and high order FIVPs.
2. To formulate approximate-analytical techniques– Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) to obtain an approximate solution of first order FIVPs and for higher order FIVPs directly without reducing to a first order system.
3. To conduct a comparative study of numerical and approximate- analytical methods HPM and VIM for solving linear and nonlinear first and high order FIVPs involving ordinary differential equations.
4. To develop, analyze and apply the approximate-analytical Homotopy Analysis Method (HAM) and Optimal Homotopy Asymptotic Method

(OHAM) in order to solve linear and nonlinear FIVPs for the first and high orders.

5. To implement and compare these two methods (HAM and OHAM) for the first and the high orders FIVPs directly without reducing to the first order system.

   Note that the both methods are relatively new approximate-analytical techniques.

1.4 Methodology

The methodology of this study is as follows:

1. The fifth order Runge-Kutta method with six stages will be studied for crisp IVPs involving ordinary differential equations. Then, a new fifth order Runge-Kutta method with six stages (RK5) will be constructed and formulated to solve first ad high order linear and nonlinear FIVPs. The convergence and error analysis of this method will be studied and analysed. Computational experiments will be conducted for this method using Mathematica.
2. The HPM and VIM will be studied for crisp IVPs involving ordinary differential equations. Then, HPM and VIM will be constructed and formulated to obtain an approximate-analytical solution of first and high order linear and nonlinear FIVPs. The high order FIVPs will be solved directly without reducing it to a first order system. A comparative study of

these methods will be conducted via computational experiments using Mathematica codes.

3. The general structures of HAM and OHAM will be studied for crisp IVPs involving ordinary differential equations. Then, HAM and OHAM will be constructed and formulated to obtain an approximate- analytical solution of first and high order linear and nonlinear FIVPs. The high order FIVPs by HAM and OHAM will be solved directly without reducing it into first order system. Next, the convergence of these methods will be studied also. Computational experiments for problems with known and unknown solutions will be conducted using Mathematica 8, 9 and Maple 15, 16 software's code.

1.5 Structure of Thesis

This thesis comprises nine chapters. Chapter 1 is the introduction followed by: Chapter 2, where some basic fuzzy set concepts will be presented, including some propositions, properties and definitions of fuzzy set and numbers and fuzzy differential equation that will be used later in our study. Chapter 3 is a study of previous research that has been conducted. Chapter 4 presents a numerical method based on fifth order Runge Kutta method with six stages (RK5). It is discussed in detail, and this is followed by a complete error and convergence analysis. The algorithm is illustrated by solving several linear and nonlinear FIVP. Chapter 5 provides details of approximate–analytical approach for FIVPs based on Homotopy Perturbation Method (HPM) that have been formulated and presented. The method is illustrated by solving several linear and nonlinear first and high orders FIVP. Chapter 6 introduces the formulation of another approximate- analytical method

Variational Iteration Method (VIM) to obtain an approximate solution of first and high order FIVP. The method is illustrated by solving several linear and nonlinear first and high orders FIVP are also compared with the HPM results in Chapter 5. In Chapter 7, a relatively new approximate–analytical approach based on Homotopy Analysis Method (HAM) will be formulated and presented and illustrated by solving several linear and nonlinear first and high orders FIVP. In Chapter 8, another Optimal Homotopy Asymptotic Method (OHAM) is formulated and presented in detail in order to solve first and high orders linear and nonlinear FIVPs. The method is illustrated by solving some linear and nonlinear n^{th} order FIVP of various orders. Finally, in Chapter 9 are the conclusions of this study.

CHAPTER 2

MATHEMATICAL BACKGROUNDS

2.1 Introduction

Fuzzy sets theory were introduced by Lotfi A. Zadeh [202-205] and considered as a generalization of crisp (classical) set theory. In crisp sets theory the membership of elements in relation to a set is assessed in binary terms - an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in relation to a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets [176] are an extension of classical set theory since, for a certain universe, a membership function may act as an indicator function, mapping all elements to either 1 or 0, as in the classical notion. A crisp set is normally defined as a collection of elements or objects $x \in X$ which can be, countable, or over countable. Each single element can be either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belongs to A" is true, whereas in the latter case the statement is false.

The idea of the fuzzy differential equation was to form a suitable setting for mathematical modeling of real world problem. FDEs take into account the information about the behavior of a dynamical system which is uncertain in order to obtain a more realistic and flexible model [54]. So, we have r (the fuzzy number) in the equation, whereas ODEs do not have the fuzzy number.

In this chapter we define the elements of fuzzy sets and FDEs. This is to provide the necessary backdrop for the form of this thesis.

Definition 2.1 [176]: If x is a collection of objects denoted generally by X, then a $\tilde{A}$ fuzzy set in X is expressed as a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

where $\mu_{\tilde{A}}(x): x \to [0,1]$ is a membership function (MF) of the fuzzy set $\tilde{A}$.

Also MF can be called a degree of compatibility or degree of truth such that fuzzy $\tilde{A}$ set is totally characterized by this membership function MF, and the range of membership function is a subset of the non-negative real numbers whose supremum is finite. The membership function $\mu_{\tilde{A}}(x)$ quantifies the grade of membership of the elements x to the fundamental set X. An element mapping to the value 0 means that the member is not included in the given set, 1 describes a fully included member. Values strictly between 0 and 1 characterize the fuzzy members.

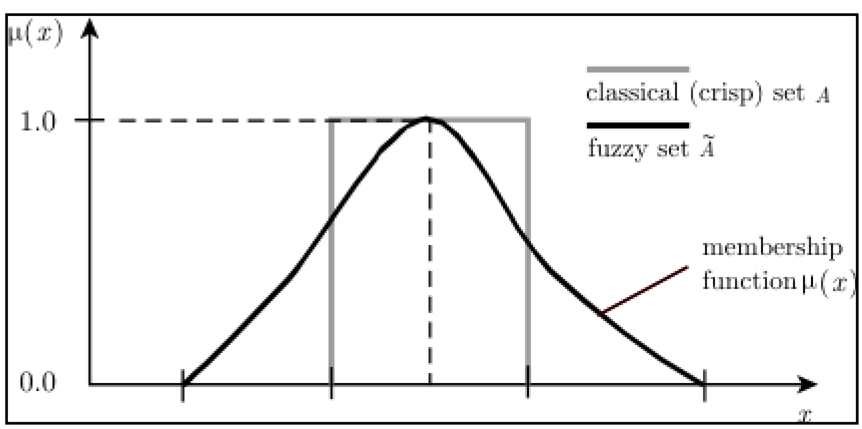

Figure 2.1: Fuzzy set $\tilde{A}$ with classical crisp set

Definition 2.2 (Support of a fuzzy set) [75]: The support of a fuzzy set $\tilde{A}$ within the universal set X is the set

$$Supp(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}$$

The support of a fuzzy set $\tilde{A}$ is the set $Supp(\tilde{A})$ that contains all the elements in X that have nonzero membership grades in $\tilde{A}$.

Definition 2.3 (Convex fuzzy set) [61]: Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $E(\mathbb{R}^n) = \tilde{E}$ denote the set of all nonempty fuzzy sets in $\mathbb{R}^n$.

A fuzzy set with membership function $\mu_{\tilde{E}}: \mathbb{R}^n \to [0,1]$ is called convex if

$$\mu(\theta x_1 + (1-\theta)x_2) \geq min\{\mu(x_1), \mu(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $\theta \in (0,1)$. According to [26] a fuzzy set with membership function $\mu: \mathbb{R}^n \to [0,1]$, is called a cone if $\mu(\theta x) = \mu(x)$, for all $x \in \mathbb{R}^n$ and $\theta > 0$. A convex fuzzy cone is a fuzzy cone, which is also a convex fuzzy set.

2.2 The Extension Principle

One of the most fundamental concepts of fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle.

Definition 2.4 [79, 147]: Let X be the Cartesian product of universes $X_1, X_2, \dots, X_n$ which is denoted by X and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ be n-fuzzy subsets in $X_1, X_2, \dots, X_n$, respectively, with Cartesian product $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2 \times \dots \times \tilde{A}_n$ and f is a mapping from X to a universe Y,$\left(y = f(x_1, x_2, \dots, x_n)\right)$. Then, the extension principle allows defining a fuzzy subset $\tilde{B} = f\left(\tilde{A}\right)$ in Y by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) : y = f(x_1, x_2, \dots, x_n), x_1, x_2, \dots, x_n \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x_1, x_2, \dots, x_n \in f^{-1}(y)} min\{\mu_{\tilde{A}_1}(x_1), \dots, \mu_{\tilde{A}_n}(x_n)\} & if\ y \in range\ of\ (f)\ (f^{-1}(y) \neq \emptyset) \\ 0 & if\ y \notin range\ of\ (f)\ (f^{-1}(y) = \emptyset) \end{cases}$$

and f^{-1} is the inverse image of f.

Remark (2.1):

For $n = 1$, the extension principle will be:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) : y = f(x), x \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} min\{\mu_{\tilde{A}}(x)\}, & if\ y \in range\ of\ (f) \\ 0 & if\ y \notin range\ of\ (f) \end{cases}$$

which is one of the definitions of a fuzzy function [14].

2.3 The *r*-Level Sets

The *r*-level sets can be used to prove some results that are satisfied in ordinary sets are also satisfied here in fuzzy sets.

Definition 2.5 [51]: The *r*-level (or *r*-cut) set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e.,

$$\tilde{A}_r = \{x \in X | \mu_{\tilde{A}} > r, r \in [0,1]\}$$

Remark (2.2) [51]:

One can also define the strong *r*-level sets as:

$$A_r^+ = \{x \in X | \mu_{\tilde{A}} > r, r \in [0,1]\}$$

It is easily checked that the following properties are satisfied for all r, $s \in [0, 1]$:

1. $(\tilde{A} \cup \tilde{B})_r = \tilde{A}_r \cup \tilde{B}_r$.
2. $(\tilde{A} \cap \tilde{B})_r = \tilde{A}_r \cap \tilde{B}_r$
3. If $\tilde{A} \subseteq \tilde{B}$, then $\tilde{A}_r \subseteq \tilde{B}_r$
4. If $r \leq s$, then $\tilde{A}_r \supseteq \tilde{A}_s$.
5. $\tilde{A} = \tilde{B}$ if and only if $\tilde{A}_r = \tilde{B}_r$, $\forall\ r \in [0, 1]$.
6. $\tilde{A}_r \cap \tilde{A}_s = \tilde{A}_s$ and $\tilde{A}_r \cup \tilde{A}_s = \tilde{A}_r$,whenever $r \leq s$.

If $\tilde{A}$ is a fuzzy set, $\{\tilde{A}_r\}$, $\forall\ r \in [0, 1]$ is a family of subsets of the universal set X, then:

$$\tilde{A} = \cup_{r\in[0,1]} r\, \tilde{A}_r$$

This means that all *r*-levels corresponding to any fuzzy set form a family of nested crisp sets, as visually depicted in Figure (2.2).

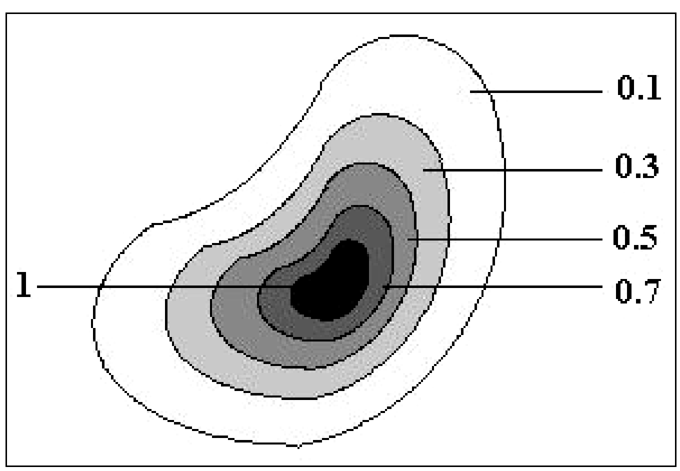

Figure 2.2: Nested *r*-level sets.

Proposition (2.1) [51]:

Let X and Y be two universal sets, and f: $X \times X \longrightarrow Y$ be an ordinary function, and $\tilde{A}$, $\tilde{B}$ be any two fuzzy subsets of X, then:

$$f(\tilde{A}, \tilde{B}) = \cup_{r \in [0,1]} r\, f(\tilde{A}_r, \tilde{B}_r)$$

2.4 Fuzzy Numbers

Fuzzy numbers [61, 113] are subsets of the real numbers set and represent vagueness values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. Also a fuzzy number can be expressed as a fuzzy set defining a fuzzy interval in the real number $\mathbb{R}$. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally a fuzzy interval is represented by two endpoints a_1 and a_3 peak point a_2 as $[a_1, a_2, a_3]$ Figure (2.3). The *r*-cut operation can be also applied to the fuzzy number. If we denote *r*-cut interval for fuzzy number $\tilde{A}$ as $[\tilde{A}]_r$, the obtained interval $[\tilde{A}]_r$, is defined as $[\tilde{A}]_r = [a_1{}^{(r)}, a_3{}^{(r)}]$, we can also know that it is an ordinary crisp interval (Figure 2.3).

Definition 2.6 [124, 175]: Let $\tilde{E}$ be the set of all upper semi-continuous normal convex fuzzy numbers with *r*-level bounded intervals such that:

$$[\mu]_r = \{t \in \mathbb{R}: \mu \geq r\}.$$

An arbitrary fuzzy number is represented by an ordered pair of membership functions $[\tilde{\mu}(t)]_r = [\underline{\mu}(t), \overline{\mu}(t)]_r$ for all $r \in [0,1]$ which is satisfying

1. $\mu(t)$ is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$.

2. $\mu(t)$ is convex fuzzy set, i.e. $\mu(\lambda t + (1-\lambda)s) \geq \min\{\mu(t), \mu(s)\} \quad \forall t, s \in \mathbb{R}, \lambda \in [0,1]$.

3. $\forall \mu \in \tilde{E}, \mu$ is an upper semi continuous on R; $\{x \in R; \mu(t) > 0\}$ is compact.

4. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over [0,1].

5. $\overline{\mu}(t)$ is a bounded left continuous non-increasing function over [0,1].

6. $\underline{\mu}(t) \leq \overline{\mu}(t)$, for all $r \in [0, 1]$, more details about the properties of fuzzy numbers mentioned in [35, 38, 176].

The *r*-level sets of any fuzzy number are much more effective as representation forms of fuzzy set than the above properties. Also, according to [202] fuzzy sets can be defined by the families of their *r*-level sets based on the resolution identity theorem.

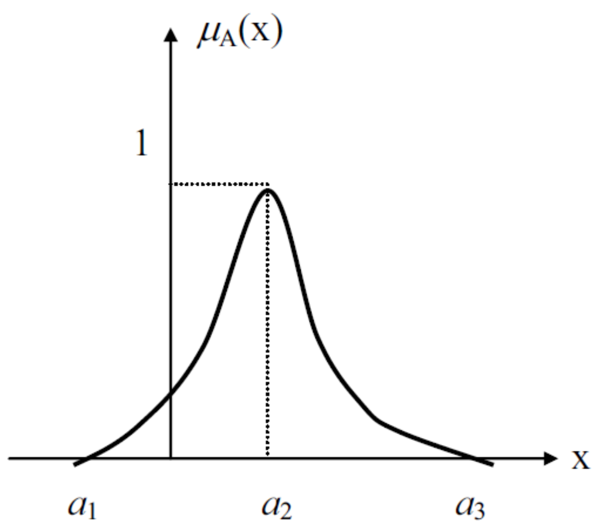

Figure 2.3: Fuzzy Number $\widetilde{A} = [a_1, a_2, a_3]$

Definition 2.7 (Trapezoidal Fuzzy Number):

A trapezoidal fuzzy number [62, 64] is defined by four real numbers $\alpha < \beta < \gamma < \delta$ .The base of the trapezoid is the interval $[\alpha, \delta]$ with vertices at $x = \beta$, $x = \gamma$. A trapezoidal fuzzy number will be denoted by $\mu = (\alpha, \beta, \gamma, \delta)$, the membership function is defined as the follows:

$$\mu\ (x;\alpha,\beta,\gamma,\delta)= \begin{cases} 0, & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \le x \le \beta \\ 1, & \text{if } \beta \le x \le \gamma \\ \dfrac{\delta-x}{\delta-\gamma}, & \text{if } \gamma \le x \le \delta \\ 0, & \text{if } x > \delta \end{cases} \tag{2.1}$$

Note that:
(1) $\mu > 0$ if $\alpha > 0$;
(2) $\mu > 0$ if $\beta > 0$;
(3) $\mu > 0$ if $\gamma > 0$; and
(4) $\mu > 0$ if $\delta > 0$.

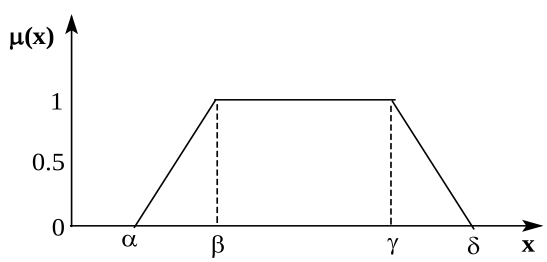

Figure 2.4: Trapezoidal Fuzzy Number

The r-level set of trapezoidal fuzzy number can be defined as follows:

$$\forall\, r \in [0,1], [\tilde{\mu}]_r = [(\beta-\alpha)r + \alpha, \delta - (\delta-\gamma)r]$$

Definition 2.8 (Triangular Fuzzy Number):

A fuzzy number μ, is called a triangular fuzzy number [124] if defined by three numbers $\alpha < \beta < \gamma$ where the graph of μ (x) is a triangle with the base on the interval [α, β] and vertex at x = β, and its membership function has the following form:

$$\mu\ (x;\alpha,\beta,\gamma) = \begin{cases} 0, & if\ \ x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & if\ \ \alpha \le x \le \beta \\ \dfrac{\gamma-x}{\gamma-\beta}, & if\ \ \beta \le x \le \gamma \\ 0, & if\ \ x > \gamma \end{cases} \tag{2.2}$$

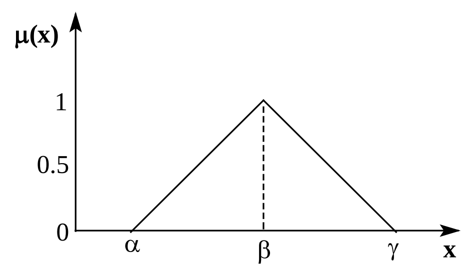

Figure 2.5: Triangular Fuzzy Number

and its *r*-level is

$[\tilde{\mu}]_r = [\alpha + r\ (\beta - \alpha),\ \gamma - r\ (\gamma - \beta)],\ r \in [0, 1]$

2.5 Fuzzy Function

Fuzzifying a crisp function of crisp variable is a function which produces images of crisp domain in a fuzzy set.

Definition 2.9 [72]: A mapping $\tilde{f}: T \to \tilde{E}$ (or $\tilde{P}(E)$) for some interval $T \subseteq \tilde{E}$ is called a fuzzy function or fuzzy process with non-fuzzy variable (crisp), and we denote *r*-level set by:

$$[\tilde{f}(t)]_r = \left[\underline{f}(t;r), \overline{f}(t;r)\right], t \in T, r \in [0,1]$$

where $\tilde{E}$ defined in definition (2.3). That is to say, the fuzzifying function is a mapping from a domain to a fuzzy set of range. Fuzzifying function and the fuzzy relation coincide with each other in the mathematical manner.

2.6 Fuzzy Differentiation

Definition 2.10[175]: Let $D_H\left(\left[\tilde{a}, \tilde{b}\right]_r\right)$ be the Hausdorff distance between two fuzzy set (or fuzzy numbers) $\tilde{a}, \tilde{b} \in \tilde{E}$ such that

$$D\left(\left[\tilde{a}, \tilde{b}\right]\right) = sup\left\{D_H\left(\left[\tilde{a}, \tilde{b}\right]_r\right) \middle| r \in [0,1]\right\}$$

and $(\tilde{E}\ , D)$ is a complete metric space [138].

Suppose $\tilde{E}$ is the set of all upper semi-continuous normal convex fuzzy numbers with bounded *r*-level sets. Since the *r*-cuts of fuzzy numbers are always closed and bounded [72], such that the intervals are $[\tilde{\mu}(t)]_r = \left[\underline{\mu}(t), \overline{\mu}(t)\right]_r, t \in \mathbb{R}, \forall\, r \in [0,1]$ .Let $[\tilde{a}(r)] = \left[\underline{a}(r), \overline{a}(r)\right], \left[\tilde{b}(r)\right] = \left[\underline{b}(r), \overline{b}(r)\right]$ be two fuzzy numbers in definitions (2.7-2.8), for $s \geq 0$. According to [121], we can define the addition and multiplication between two fuzzy numbers by *s* as follows

1- $\left(\underline{a+b}\right)(r) = \left(\underline{a}(r) + \underline{b}(r)\right)$

2- $\left(\overline{a+b}\right)(r) = \left(\overline{a}(r) + \overline{b}(r)\right)$

3- $\left(\underline{sa}\right)(r) = s.\underline{a}(r), (\overline{sa})(r) = s.\overline{a}(r)$.

Now let: $\tilde{E} \times \tilde{E} \to \mathbb{R} \cup \{0\}$, $D\left(\left[\tilde{a},\tilde{b}\right]_r\right) = sup_{\gamma\in[0,1]} Max\{|\underline{a}(r) - \underline{b}(r)|, |\overline{a}(r) - \overline{b}(r)|\}$ be the Hausdorff distance between fuzzy numbers [172] where the following properties are well-known:

i. $D\left(\left[\tilde{a}+\tilde{c},\tilde{b}+\tilde{c}\right]_r\right) = D\left(\left[\tilde{a},\tilde{b}\right]_r\right), \ \forall \tilde{a},\tilde{b},\tilde{c} \in \tilde{E}.$

ii. $D\left(\left[s.\tilde{a},s.\tilde{b}\right]_r\right) = |s| D\left(\left[\tilde{a},\tilde{b}\right]_r\right), \ \forall s \in \mathbb{R}, \forall \tilde{a},\tilde{b} \in \tilde{E}$.

iii. $D\left(\left[\tilde{a}+\tilde{b},\tilde{c}+\tilde{d}\right]_r\right) \le D\left(\left[\tilde{a},\tilde{b}\right]_r\right) + D\left(\left[\tilde{c},\tilde{d}\right]_r\right), \ \forall \tilde{a},\tilde{b},\tilde{c},\tilde{d} \in \tilde{E}$.

Definition 2.11 [177]: Consider $\tilde{x},\tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then z is called the H-difference (Hukuhara difference) of *x* and *y* and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.12[181]: If $\tilde{f}: I \to \tilde{E}$ and $y_0 \in$ I, where $I \in [t_0, T]$. We say that $\tilde{f}$ Hukuhara differentiable at y_0, if there exists an element $\left[\tilde{f'}\right]_r \in \tilde{E}$ such that for all *h* > 0 sufficiently small (near to 0), $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$and $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ exists with the limits are taken in the metric space $(\tilde{E}, D)$

$$\lim_{h\to 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}$$

The fuzzy set $\left[\tilde{f'}(y_0)\right]_r$ is called the Hukuhara derivative of $\left[\boldsymbol{\tilde{f'}(y)}\right]_{\boldsymbol{r}}$ at y_0.

These limits are taken in the space $(\tilde{E}, D)$ if t_0 or *T*, then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on *r*-level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r, \ \forall\, r \in [0,1]$. By consideration of definition of the metric *D* all the *r*-level sets $\left[\tilde{f}(0)\right]_r$ are Hukuhara differentiable at y_0, with Hukuhara derivatives $\left[\tilde{f'}(y_0)\right]_r$,when $\tilde{f}: I \to \tilde{E}$ is Hukuhara differentiable at y_0 with

Hukuhara derivative $[\tilde{f}'(y_0)]_r$ it leads to that $[\tilde{f}'(y)]_r$ is Hukuhara differentiable for all $r \in [0,1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r-levels $[\tilde{\boldsymbol{f}}'(\boldsymbol{y})]_{\boldsymbol{r}}$ are Hukuhara differentiable at t_0.

Theorem 2.1 [187]: Let $\tilde{f}: [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote

$$[\tilde{f}'(t)]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right].$$

Then the boundary functions $\underline{f}'(t;r), \overline{f}'(t;r)$ are both differentiable

$$[\tilde{f}'(t)]_r = \left[\left(\underline{f}(t;r)\right)', \left(\overline{f}(t;r)\right)'\right], \forall\, r \in [0,1]$$

Theorem 2.2[181]: Let $\tilde{f}: [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote $[\tilde{f}'(t)]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right]$. Then both of the boundary functions $\underline{f}'(t;r), \overline{f}'(t;r)$ are differentiable, we can write for nth order fuzzy derivative

$$[\tilde{f}^{(n)}(t)]_r = \left[\left(\underline{f}^{(n)}(t;r)\right)', \left(\overline{f}^{(n)}(t;r)\right)'\right], \forall\, r \in [0,1]\ .$$

According to Proposition (3.1) in [181] and the definition (3.1) in [175], we can define the fuzzy Hukuhara differentiability of n times as follows:

Definition 2.13: Define the mapping $\tilde{f}': I \to \tilde{E}$ and $y_0 \in$ I, where $I \in [t_0, T]$. We say that $\tilde{f}'$ is Hukuhara differentiable for $t \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exist $\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r), \tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}\ , D)$.

$$\lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0+h;r) \ominus \tilde{f}^{(n-1)}(y_0;r)}{h}$$

$$= \lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0;r) \ominus \tilde{f}^{(n-1)}(y_0-h;r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$.

2.7 Fuzzy Integration

Definition 2.14 [193]: Let the mapping $\tilde{f}: I \to \tilde{E}$ be a closed and bounded fuzzy valued function on the interval $I \in [t_0, T]$ such that $[\tilde{f}(t)]_r = \left[\underline{f}(t;r), \overline{f}(t;r)\right]$. Suppose $\underline{f}(t;r), \overline{f}(t;r)$ are Riemann integrable on the interval I for all $r \in [0,1]$. Let

$$\tilde{A}(t;r) = \left[\int_{t_0}^{T} \underline{f}(t;r)dt\, , \int_{t_0}^{T} \overline{f}(t;r)dt\right]$$

Then we say $[\tilde{f}(t)]_r$ is fuzzy Riemann- integrable on I denoted by $[\tilde{f}(t)]_r \in \tilde{E}_{RI}$ on the interval I for $s \in \tilde{A}(t;0)$ with $\int_{t_0}^{T} \tilde{f}(t;r)dt$ and is defined by

$$\mu_{\int_{t_0}^{T} \tilde{f}(t;r)dt}(\mathrm{t}) = sup_{r\in[0.1]} r 1_{\tilde{A}(t;r)}$$

It is clear that

$$\overline{f}(t;r_1) > \overline{f}(t;r_2) > \tilde{f}(t;r) > \underline{f}(t;r_1) > \underline{f}(t;r_2)$$

for $r_1 < r_2 < 1$ such that if we let

$$\int_{t_0}^{T} \underline{f}(t;r)dt = \underline{A}(t;r)\, , \int_{t_0}^{T} \overline{f}(t;r)dt = \overline{A}(t;r)$$

then

$$\int_{t_0}^{T} \tilde{f}(t;r)dt = \left[\underline{A}(t;r)\, , \overline{A}(t;r)\,\right] = \tilde{A}(t;r)$$

There is another definition of fuzzy integration

Definition 2.15 [187]: The fuzzy integral of fuzzy process, $\tilde{f}(t;r)$, $\int_a^b \tilde{f}(t;r)dt$ for $a, b \in T$ and $r \in [0,1]$ is defined by:

$$\int_a^b \tilde{f}(t;r)dt = \left[\int_a^b \underline{f}(t;r)dt\,, \int_a^b \overline{f}(t;r)dt\right]$$

2.8 Fuzzy Differential Equations

Many dynamical real life problems may be formulated as a mathematical model. Many of them can be formulated either as a system of ordinary or partial differential equations. Fuzzy differential equation is a useful and powerful tool to model dynamical system problems when information about its behavior is inadequate.

2.8.1 Defuzzification First Order FIVP:

Consider the first order FIVP involving ODE:

$$\begin{aligned} &\tilde{y}'(t) = \tilde{f}\big(t, \tilde{y}(t)\big), \ \ t \in [t_0, T] \\ &\tilde{y}(t_0) = \tilde{y}_0 \end{aligned} \qquad (2.3)$$

where $\tilde{y}(t)$ is a fuzzy function of the crisp variable t, $\tilde{f}$ is fuzzy function of the crisp variable t and the fuzzy variable $\tilde{y}$. Here $\tilde{y}'(t)$ is the fuzzy H-derivative of $\tilde{y}(t)$ and $\tilde{y}_0$ is the fuzzy number as in Section (2.4) that refers to the fuzzy initial condition $\tilde{y}(t_0)$ of Eq. (2.3).

We denote the fuzzy function y by $[\tilde{y}]_r = [\underline{y}, \overline{y}]_r$, for $t \in [t_0, T]$ and $r \in [0,1]$. It means that the r-level set of $\tilde{y}(t)$ can be defined as: