Semi-Linear Diffusive Representations for Non-Linear Fractional Differential Systems

Jacques AUDOUNET[1], Denis MATIGNON[2⋆], and Gérard MONTSENY[3]

[1] MIP / CNRS, Université Paul Sabatier. 118, route de Narbonne. F-31068 Toulouse Cedex 4, France. email: audounet@mip.ups-tlse.fr.
[2] ENST, TSI Dept. & CNRS, URA 820. 46, rue Barrault. F-75634 Paris Cedex 13, France. email: matignon@tsi.enst.fr, URL: http://www.tsi.enst.fr/~matignon/.
[3] LAAS / CNRS. 7, avenue du Colonel Roche. F-31077 Toulouse Cedex 4, France. email: montseny@laas.fr, URL: http://www.laas.fr/gt-opd/

Abstract. The stability of non-linear fractional differential equations is studied. A sufficient stability condition on the non-linearity is given for the input-output stability, thanks to many different reformulations of the system using diffusive representations of dissipative pseudo-differential operators. The problem of asymptotic internal stability is analyzed by a more involved functional analytic method. Finally, a *fractional* version of the classical Hartman–Grobman theorem for hyperbolic dynamical systems of order 1 is conjectured and reformulated, based upon known necessary and sufficient stability conditions for linear fractional differential equations.

1 Statement of the problem

We are interested in the following problem involving a non-linear dynamics f and a state $x \in \mathbb{R}$ (for simplicity sake):

$$d^\alpha x(t) \;=\; f(x(t)) + u(t); \qquad x(0) = x_0\,, \tag{1}$$

where d^α is the Caputo regularized version of the so-called Riemann-Liouville fractional derivative, with $0 < \alpha < 1$; meaning $d^\alpha x = I^{1-\alpha}\dot{x} = Y_{1-\alpha} \star \dot{x}$, with causal kernel $Y_\beta(t) = \frac{1}{\star(\beta)} t_+^{\beta-1}$ for the fractional integral operator I^β of order β.

Problem (1) can be advantageously reformulated in the equivalent Abel–Volterra equation:

$$x(t) \;=\; x_0 + I^\alpha \left[f(x(t)) + u(t) \right]. \tag{2}$$

Then, using $Y_\alpha \star Y_{1-\alpha} = Y_1$ the Heaviside unit step, (2) can also be written as:

$$x(t) \;=\; I^\alpha \left[f(x(t)) + u(t) + x_0\, Y_{1-\alpha}(t) \right]; \tag{3}$$

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alternatively, with the help of the new variable $z(t) = x(t) - x_0$, and of the new function $\widetilde{f}(z) = f(z + x_0)$, (2) can also be written as:

$$z(t) \;=\; I^\alpha \left[\widetilde{f}(z(t)) + u(t)\right] . \qquad (4)$$

As far as stability is concerned, trying to use geometrical or standard analytical techniques (such as those used in the integer case, see [11]), i.e. trying to extend them to the *fractional* differential case is of little help, unfortunately; for the main reason that quadratic forms prove hard to fractionally differentiate, since the fractional derivative is intrinsically a *non-local* pseudo-differential operator.

On the contrary, using diffusive representations of pseudo-differential operators (see [8,6,2]) proves useful, in so far as the problem can first be reformulated into one (or many equivalent) way(s) that is classical, namely a first order in time diffusion equation, on an infinite-dimensional state-space endowed with an appropriate Hilbert structure. Quite standard energy methods (Lyapunov functionals, LaSalle invariance principle) can therefore be used.

The paper is organized as follows:

- in section 2, the problem is reformulated in equivalent ways with many advantages for the analysis; in particular stability properties are more easily examined in this context; a main comparison result is established;
- in section 3, the problem is examined with *null* initial condition $x_0 = 0$, it requires LaSalle invariance principle, and gives strong stability of the internal state;
- in section 4, the problem of the initial condition alone is addressed: it requires more specific analytical tools pertaining to the properties of the heat equation, the use of which will be sketched as closely as possible;
- finally in section 5 we will indicate some natural extensions of the results, either straightforward ($x \in \mathbb{C}$ or $x \in \mathbb{C}^n$, other diffusive pseudo-differential operators that are dissipative: $\mu > 0$), or that seem to be within reach but still need to be fully developped.

2 Diffusive formulations

In subsection 2.1, system (1) is transformed into a *diagonal* infinite-dimensional system with an extra variable $\xi > 0$, and a state $\psi(\xi, t)$. The *heat equation* formulation can be recovered as follows: first let $\xi = 4\pi^2\eta^2$ with $\eta \in \mathbb{R}$, then perform the inverse Fourier transform in the space of tempered distributions, a *heat* equation is then obtained with an extra space variable y and a state $\varphi(y, t)$ in subsection 2.2.

2.1 Diagonal diffusive formulations

Output form In the scalar case, problem (4) is *equivalent* to (see [10]):

$$\partial_t\psi(\xi,t) = -\xi\,\psi(\xi,t) + \widetilde{f}(z(t)) + u(t)\,; \quad \psi(.,0) \,=\, 0 \quad \xi > 0\,, \tag{5a}$$

$$z(t) = \int_0^\infty \mu_\alpha(\xi)\,\psi(\xi,t)\,d\xi\,; \tag{5b}$$

where μ_α stands for *the* diffusive representation of the fractional integral operator I^α, that is: $\mu_\alpha(\xi) = \frac{\sin\alpha\pi}{\pi}\,\xi^{-\alpha}$.
The energy associated to this equation is:

$$E_\alpha(t) \,=\, \frac{1}{2}\int_0^\infty \mu_\alpha(\xi)\,|\psi(\xi,t)|^2\,d\xi\,, \tag{6}$$

for which it is easily proved that the following equality holds:

$$\frac{dE_\alpha}{dt}(t) \,=\, -\int_0^\infty \xi\,\mu_\alpha(\xi)\,|\psi(\xi,t)|^2\,d\xi + z(t)\,\widetilde{f}(z(t)) + z(t)\,u(t)\,. \tag{7}$$

The functional spaces to be used are: $\mathcal{H}_\alpha = L^2_{\mu_\alpha}(\mathbb{R}^+)$, $\mathcal{V}_\alpha = L^2_{(1+\xi)\,\mu_\alpha}(\mathbb{R}^+)$ and ${\mathcal{V}_\alpha}' = L^2_{(1+\xi)^{-1}\,\mu_\alpha}(\mathbb{R}^+)$, and $\mathcal{V}_\alpha \hookrightarrow \mathcal{H}_\alpha \hookrightarrow {\mathcal{V}_\alpha}'$ with continuous and dense injections.

Balanced form Let us denote $\nu_\alpha(\xi) = \sqrt{\mu_\alpha(\xi)}$, which is meaningful thanks to $\mu > 0$ *only*; then by a straightforward change on ψ, and a slight abuse of notations, we get:

$$\partial_t\psi(\xi,t) = -\xi\,\psi(\xi,t) + \nu_\alpha(\xi)\,\left[\widetilde{f}(z(t)) + u(t)\right]\,; \quad \psi(.,0) \,=\, 0 \quad \xi > 0\,, \tag{8a}$$

$$z(t) = \int_0^\infty \nu_\alpha(\xi)\,\psi(\xi,t)\,d\xi\,. \tag{8b}$$

The energy associated to this equation is:

$$E(t) \,=\, \frac{1}{2}\int_0^\infty |\psi(\xi,t)|^2\,d\xi\,, \tag{9}$$

The functional spaces to be used are: $\mathcal{H} = L^2(\mathbb{R}^+)$, $\mathcal{V} = L^2_{(1+\xi)}(\mathbb{R}^+)$ and $\mathcal{V}' = L^2_{(1+\xi)^{-1}}(\mathbb{R}^+)$. They are *independent* of α.

2.2 Heat equation formulations

Now, tempered distributions will be used: $M_\alpha(y)$, with Fourier transform $m_\alpha(\eta) = 2\,\sin(\alpha\pi)\,|2\pi\,\eta|^{1-2\alpha}$ for the output form, and $N_\alpha(y)$, with Fourier transform $n_\alpha(\eta) = \sqrt{m_\alpha(\eta)}$ for the balanced form. It is clear that, for $\frac{1}{2} < \alpha < 1$ both $M_\alpha(y) \propto |y|^{-2(1-\alpha)}$ and $N_\alpha(y) \propto |y|^{-(\frac{3}{2}-\alpha)}$ are regular L^1_{loc} functions; for $\alpha = \frac{1}{2}$, they are proportional to the Dirac measure δ, and for $0 < \alpha < \frac{1}{2}$ they are distributions of order 1 involving only *finite parts*: hence, integral terms such as $\int_{\mathbb{R}} M_\alpha(y)\,\varphi(y,t)\,dy$ have to be understood in the sense of *duality* brackets $< M_\alpha,\,\varphi(t) >=< 1,\,\varphi(t) >_{{\mathcal{V}_\alpha}',\,\mathcal{V}_\alpha}$.

Output form System (5a)-(5b) is *equivalent* to:

$$\partial_t\varphi(y,t) = \partial_y^2\varphi(y,t) + \left[\widetilde{f}(z(t)) + u(t)\right]\,\delta(y)\,; \quad \varphi(.,0) \;=\; 0\,, \tag{10a}$$

$$z(t) = \int_{\mathbb{R}} M_\alpha(y)\,\varphi(y,t)\,dy \;= < M_\alpha,\,\varphi(t) > \;. \tag{10b}$$

Balanced form System (8a)-(8b) is *equivalent* to:

$$\partial_t\varphi(y,t) = \partial_y^2\varphi(y,t) + \left[\widetilde{f}(z(t)) + u(t)\right]\,N_\alpha(y)\,; \quad \varphi(.,0) \;=\; 0\,, \tag{11a}$$

$$z(t) = \int_{\mathbb{R}} N_\alpha(y)\,\varphi(y,t)\,dy \;= < N_\alpha,\,\varphi(t) > \;. \tag{11b}$$

The energy associated to this equation is:

$$E(t) \;=\; \frac{1}{2}\int_{\mathbb{R}} |\varphi(y,t)|^2\,dy\,, \tag{12}$$

for which it is easily proved that the following equality holds:

$$\frac{dE}{dt}(t) \;=\; -\int_{\mathbb{R}} |\partial_y\varphi(y,t)|^2\,dy + z(t)\,\widetilde{f}(z(t)) + z(t)\,u(t)\,. \tag{13}$$

The functional spaces to be used are: $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{V} = H^1(\mathbb{R})$ and $\mathcal{V}' = H^{-1}(\mathbb{R})$. They are *independent* of α.

Remark 1. Note that these equivalent reformulations are interesting results on their own, for the following reasons:

- the system is local in time,
- a natural energy functional E is provided on an energy space $\mathcal{H}$, which helps prove that the system is dissipative under some specific conditions on the non-linearity f,
- a classical $(\mathcal{V},\mathcal{H},\mathcal{V}')$ functional analytic framework is being used, in which regularity results can be more easily obtained,
- on the heat equation formulations, the (weak or strong) maximum principle can be used, especially for comparison results,
- numerical approximation of diagonal diffusive formulations is straightforward, using standard schemes of numerical analysis (see [10]).

These features can *not* be captured on the original system (1) nor on any of the Abel-Volterra forms (2)-(4).

2.3 A comparison result

On formulation (11a)-(11b) with an extra forced term denoted by $g(t)$, the following quadratic a priori estimate will be useful in the sequel:

$$\frac{1}{2}\,\partial_t\|\varphi\|^2 + \|\partial_y\varphi\|^2 \;=\; f(t, < N_\alpha,\,\varphi(t) >)\; < N_\alpha,\,\varphi(t) > + < g(t),\,\varphi(t) >$$

The following theorem is an extension to the case $\alpha \neq \frac{1}{2}$ of a result of [1].

Theorem 1. *Suppose $f(t,.)$ is strictly decreasing on $\mathbb{R}$, let us consider φ_1, φ_2 solutions of:*

$$\partial_t \varphi_j - \partial_y^2 \varphi_j = f(t, < N_\alpha, \varphi_j >) \otimes N_\alpha + g_j \tag{14}$$

such that $t \mapsto z_j(t) =< N_\alpha, \varphi_j(t) >=< n_\alpha, \widehat{\varphi_j}(t) >$ be of class $\mathcal{C}^1$ on $[0,T]$.

If $g_1 \geq g_2$ on $[0,\ T]$, then $\varphi_1 \geq \varphi_2$ and $z_1 \geq z_2$ on $[0,\ T]$.

Proof (Sketch of). Function $\Phi = \varphi_1 - \varphi_2$ is the solution of:

$$\partial_t \Phi - \partial_y^2 \Phi = [f(t, < N_\alpha, \varphi_1 >) - f(t, < N_\alpha, \varphi_2 >)] \otimes N_\alpha + g_1 - g_2\,; \qquad \Phi_0 = 0\,.$$

Multiplying this equation by Φ_- (where $\Phi = \Phi_+ - \Phi_-$ and $\Phi_+ \Phi_- = 0$) and integrating over $\mathbb{R}$ leads to:

$$\begin{aligned} &\frac{1}{2}\partial_t \|\Phi_-\|^2 + \|\partial_x \Phi_-\|^2 = \\ -\ &[f(t, < N_\alpha, \varphi_1 >) - f(t, < N_\alpha, \varphi_2 >)]\ < N_\alpha, \Phi_- > - < g_1 - g_2, \Phi_- > \end{aligned}$$

Then, thanks to f strictly decreasing, $-[f(t,z_1) - f(t,z_2)]\ (z_1 - z_2)_- \leq 0$, with $z_j =< N_\alpha, \varphi_j >$. Hence, together with $g_1 - g_2 \geq 0$, we get:

$$\frac{1}{2}\,\partial_t \|\Phi_-\|^2 + \|\partial_x \Phi_-\|^2 \ \leq\ 0 + 0$$

Then function $\|\Phi_-\|$ is positive decreasing, with initial value 0, thus null a.e. It follows that $\varphi_1 \geq \varphi_2$ a.e. and $z_1 \geq z_2$ on $[0,\ T]$. □

3 Analysis of the case $x_0 = 0$

First, we get a main theorem, the corollary of which is the stability of system (1) subject to specific conditions. Note that the proof needs to be performed on one of the four equivalent diffusive formulations only.

Theorem 2. *As soon as the input u has stopped, and provided f is strictly decreasing with $x\,f(x) < 0$, we get: $\|\psi(.,t)\|_{\mathcal{H}} \to 0$ and $x(t) \to 0$ as $t \to \infty$.*

Proof (Sketch of). The goal is to apply LaSalle invariance principle (see e.g. [3]); to this end, we proceed in six steps:

1. system (5a)-(5b) is dissipative: from (7), $\dot{E}_\alpha(t) \leq 0$ thanks to $x\,f(x) < 0$,
2. moreover $\dot{E}_\alpha = 0$ if and only if $\psi(.) = 0$ $\mu_\alpha-$ a.e.,
3. for any $\psi_0 \in \mathcal{H}_\alpha$, the trajectory $\{\psi(.,t)\}_{t\geq 0}$ is precompact (see [9]),
4. $\psi \to 0$ in $\mathcal{H}_\alpha$ strongly as $t \to \infty$,
5. $\psi \rightharpoonup 0$ in $\mathcal{V}_\alpha$ weakly,
6. hence, $x(t) =< 1, \psi(.,t) >_{\mathcal{V}_\alpha{}', \mathcal{V}_\alpha} \to 0$ as $t \to \infty$. □

Remark 2. In the case when f is *linear*, that is $f(x) = \lambda x$, the condition on f reads $\lambda < 0$; by a simple extension to the complex-valued case, one easily gets $\Re e(\lambda) < 0$, which happens to be sufficient but not necessary, since the *optimal* stability result for the linear case is: $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ (see [5,6]).

Remark 3. Function f can be *discontinuous* at 0, in which case the results are still valid, though care must be taken that f is a *multivalued* function with $f(0) \ni 0$, and the following *differential inclusion*:

$$\partial_t\varphi(y,t) - \partial_y^2\varphi(y,t) - \left[\widetilde{f}(z(t)) + u(t)\right] \otimes N_\alpha \ni 0\,; \quad \varphi(.,0) = 0\,, \tag{15a}$$

$$z(t) = < N_\alpha, \varphi(t) >\,, \tag{15b}$$

which is nothing but a diffusive representation for the non-linear fractional differential inclusion:

$$d^\alpha x(t) - f(x(t)) - u(t) \ni 0\,; \qquad x(0) = x_0\,. \tag{16}$$

Once the well-posedness nature of problem (15a)-(15b) has been established, the solution $x = x_0 + z$ of (16) is uniquely determined as an output.

4 Analysis of the case $x_0 \neq 0$

This seems to be a more difficult problem than the previous one, mostly because of the long-memory behaviour.

4.1 Formulation through an extra forced term

From reformulation (3), the *pseudo*-initial condition x_0 in (1) can be taken into account by an extra *input* v instead of u in *any* of the equivalent diffusive formulations of section 2 with f, and can therefore be interpreted as a forced term, namely: $v(t) = x_0\,\frac{1}{\star\,(1-\alpha)}t_+^{-\alpha} + u(t)$.

Unfortunately, we cannot expect to use the stability result above (theorem 2), for it is clear that the extra input will never stop: the everlasting behaviour of the extra input comes from the hereditary aspect of the problem.

4.2 Formulation by a change of function and variable

From reformulation (4), the *pseudo*-initial condition x_0 in (1) can be taken into account by a change of variable $z = x - x_0$ and a change of function $\widetilde{f}(z) = f(x_0 + z)$; we then use the heat equation formulation in balanced form (11a)-(11b). Suppose $u = 0$ from $t = 0$ on (the extension to $u = 0$ from $t = t_0$ will be addressed at the end of the section). Let $x_0 < 0$[1], then $\widetilde{f}(0) = f(x_0) > 0$.

[1] the case $x_0 < 0$ is treated similarly.

Lemma 1. *φ et z are increasing functions of t.*

Proof. With $g(t) = -N_\alpha \otimes \widetilde{f}(0)\,\mathbf{1}_{[0,T]}$, φ_g is a solution of:

$$\partial_t \varphi_g - \partial_y^2 \varphi_g = N_\alpha \otimes \widetilde{f}(z) + g(t); \qquad \varphi_{g0} = 0\,;$$

it is identically zero on $[0,\,T]$;[2] hence, thanks to the comparison result (theorem 1) with $g(t) \leq 0$, we get $\varphi_g(.,t) = \varphi(.,t-T) \leq \varphi(.,t)$, thus as $N_\alpha \geq 0$, $< N_\alpha, \varphi(.,t-T) > \leq < N_\alpha, \varphi(.,t) >$. □

Lemma 2. $\lim z \;\leq\; -x_0$.

Proof. Otherwise, by continuity, $\exists t_0$ such that $z(t_0) = -x_0 \Rightarrow \widetilde{f}(z(t_0)) = 0$, because φ is increasing; from what we deduce that:

- either $\varphi = cte$, that is an equilibrium state, implying φ is constant $\forall t > t_0$, hence $z(t)$ is constant.
- or $\varphi \neq cte$, in which case the concavity is of constant sign and negative, which contradicts φ increasing and $\varphi_0 = 0$. □

Lemma 3. *There exists a unique equilibrium state $\varphi_\infty = cte$, and $z_\infty = -x_0$.*

Proof. At the equilibrium, $\partial_y^2 \varphi_\infty = -N_\alpha \otimes \widetilde{f}(z_\infty)$. N_α being positive, the concavity of φ_∞ is of constant sign and negative, which is contradictory with φ increasing and $\varphi_0 = 0$, except if φ_∞ is constant. Thus, $\widetilde{f}(z_\infty) = 0$ necessarily and $z_\infty = -x_0$ for f is injective. □

Lemma 4. $z \to -x_0$.

Proof. $\lim z$ exists and $\leq -x_0$. If $z^* = \lim z < -x_0$, then $\widetilde{f}(z) \geq k > 0$, which implies that on any compact subset $[-Y,\,Y]$, $\partial_t \varphi > \partial_x^2 \varphi + N_\alpha \otimes k$. From which we can easily deduce that $\varphi \to +\infty$; more precisely:

$$\forall K,\, Y,\, \exists t_0, \qquad \varphi(y) \geq K \text{ for } y \in [-Y,\,Y],$$

hence $z = < N_\alpha, \varphi > \geq K \int_{-Y}^{Y} N_\alpha \, dy > z^*$ for K large enough, which is contradictory. □

Corollary 1. $x \to 0$ *as* $t \to \infty$.

Corollary 2. *It can be shown that the equilibrium state φ_∞ is asymptotically reached, in the following sense: $\varphi \to \varphi_\infty$ uniformly on any compact subset, that is in the weak-$*$ topology of $L^\infty(\mathbb{R})$.*

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[2] Here g has been computed in such a way as to delay the start of the diffusion process by T.

The previous analysis amounts to the *maximum principle* for the case $\alpha \neq \frac{1}{2}$. The formulation by a heat equation (namely, heat equation formulations) can *not* be overcome; it gives valuable information thanks to the evolution of an internal state of infinite dimension, from which the long-memory behaviour stems: this aspect is rather well controlled (from a functional analytic point of view) thanks to the specific properties of the heat equation. This is certainly one of the most *remarkable* applications of DR of PDOs; these techniques provide not only straightforward numerical schemes for the approximation, but also very sharp estimates for the analysis of the problem (and especially for *asymptotic* analysis).

Remark 4. It is noteworthy that $< N_\alpha, \varphi(.,t) >$ tends to $-x_0$ as $t \to \infty$, but $< N_\alpha, \varphi_\infty >$ is *not* properly defined, because N_α and φ_∞ do *not* belong to dual spaces (except in the case $\alpha = \frac{1}{2}$). To some extent, the maximum principle *forces* this limit to exist without degenerating, but $\varphi(t)$ *diverges* in the energy space $L^2(\mathbb{R})$ (weak-$*$ convergence in $L^\infty(\mathbb{R})$).

When $u = 0$ for $t > t_0$, the state φ is initialized by $\varphi_0(y) = \varphi(t_0, y)$ at time t_0, with null input but $x_0 = x(t_0) \neq 0$ in general. Then, as the autonomous dynamics generated by $\varphi_0 \neq 0$ is stable (diffusion), it is the stability/unstability generated by x_0 which will play the major role and enable to conclude in a similar way.

5 Further extensions

The conditions can easily be extended to the complex-valued case, namely $\Re e(x^* f(x)) < 0$, and also to the vector-valued case, as $\Re e(x^H f(x)) < 0$; and the monotonicity of f must be translated in an appropriate way.

Moreover, the whole set of results obtained thanks to diffusive formulations can be extended to any other diffusive pseudo-differential operator of *dissipative* nature, that is $\mu > 0$.

Finally, in order to extend the sufficient stability condition, a more accurate result can be *conjectured*, as a *fractional* version of the Hartman–Grobman theorem, namely:

Theorem 3 (Conjecture). *The local stability of the equilibrium $x^* = 0$ of the non-linear fractional differential system $d^\alpha x = f(x)$ is governed by the global stability of the linearized system near the equilibrium $d^\alpha x = \lambda x$, where $\lambda = f'(0) \in \mathbb{C}$, namely:*

- *$x^* = 0$ is locally asymptotically stable if $|\arg(\lambda)| > \alpha\frac{\pi}{2}$,*
- *$x^* = 0$ is not locally stable if $|\arg(\lambda)| < \alpha\frac{\pi}{2}$.*

Note that nothing can be said if $|\arg(\lambda)| = \alpha\frac{\pi}{2}$, in which case the linearized system is asymptotically oscillating.

The idea is to use a semi-linear diffusive reformulation of the system, and then an infinite-dimensional version of the Hartman-Grobman theorem; more precisely:

$$\partial_t \varphi = \partial_y^2 \varphi + \widetilde{f}(< N_\alpha, \varphi >) \otimes N_\alpha, \qquad \varphi_0 = 0, \tag{17a}$$

$$x(t) = x_0 + < N_\alpha, \varphi(t) >, \tag{17b}$$

is of the form $\partial_t \varphi = F(\varphi)$ with F linearizable in a weak sense (unbounded operators), $F = L + B$ with $L = \partial_y^2 + l$ the linear part and B a non-linear term of lower differential order; the solution and stability of (17a)-(17b) is known exactly when F reduces to L. Care must be taken that the equilibrium state φ_∞ does *not* belong to the energy space: specific methods from functional analysis and semi-linear diffusion PDEs must be investigated in order to tackle the problem properly.

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