Diffusive Representation for Pseudo-Differentially Damped Non-Linear Systems

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Abstract. A large class of viscoelastic and elastoplastic systems, frequently encountered in physics, are based on causal pseudo-differential operators, which are hereditary: the whole past of the state is involved in the dynamic expression of the system evolution. This generally induces major technical difficulties.

We consider a specific class of pseudo-differential damping operators, associated to the so-called diffusive representation which enables to built augmented state-space realizations without heredity. Dissipativity property is expressed in a straightforward and precise way. Thanks to state-space realizations, standard analysis and approximation methods as well as control-theory concepts may therefore be used.

1 Introduction

Visco-elasticity and elasto-plasticity are difficult to take into account in control theory: modelling is most of time delicate and control of such systems is quite an open problem. In particular the presence of *discontinuous* functions generates non regular trajectories and therefore implies the possible existence of fast or "abnormal" behaviors in the solutions. However, accurate descriptions of such singular phenomena are of great importance in many concrete situations [6], [8], because they involve non negligible energy transfers and, due to non-linearities, have consequently significant effects on the system evolution at slow time-scale.

A large class of such systems, frequently encountered in physics, are based on causal *pseudo-differential* operators, sometimes with long-memory characteristics: classical examples are fractional derivatives or integrals and various combinations of them [9]. Pseudo-differential operators are hereditary: the whole past of the state is involved in the dynamic expression of the system evolution. This generally induces major technical difficulties. Furthermore, from the thermodynamical point of view, consistence of the model is a difficult question in most cases.

The pseudo-differential operators under consideration here are those which admit a so-called diffusive representation i.e. which can be simulated by using a non hereditary operator of diffusion type in a an augmented state space. Analysis and approximation as well as control of the related models are then performed in the context of this representation with classical tools of applied mathematics. The dissipativity of the models admits a quantitative evaluation by the way of internal (hidden) variables associated with the augmented state space.

The paper is organized as follows.

In section 2, we state the problem and the associated framework.

The definition of pseudo-differential damping is given in section 3.

The section 4 is devoted to a constructive approach of diffusive representations in the perspective of concrete analysis and numerical simulations.

In section 5, we state and prove the main result of the paper. This result enables to transform the initial hereditary problem into a Cauchy one with infinitesimal generator and energy functional.

In section 6, we apply this result to the problems introduced in section 2 and we exhibit the specific properties generated by the diffusive formulation, namely about asymptotic behaviors.

Through numerical simulations, we finally treat an example of pseudo-differentially damped second order system in section 7, in order to illustrate the efficiency of the approach from the point of view of approximations.

2 Framework

Let $\mathcal{E}$ a real separable Hilbert space with scalar product $(.|.)_{\mathcal{E}}$, V a potential, and:

$$\mathcal{H} : W^{2,\infty}_{loc}\left(\mathbf{R}^+_t;\mathcal{E}\right) \to L^{\infty}_{loc}\left(\mathbf{R}^+_t;\mathcal{E}\right) \tag{1}$$

a causal and continuous hereditary (the whole past $X_{[0,t]}$ of X at time t is involved in $\mathcal{H}(X)(t)$) non linear operator [2]. We consider the following autonomous functional dynamical equation:

$$X'' + \mathcal{H}\left(X\right) + \operatorname{grad} V(X) = 0, \tag{2}$$

with initial conditions: $X(0) = X_0$ such that $V(X_0) < +\infty,\ X'(0) = X'_0 \in \mathcal{E}$.

We define the (mechanical) energy of $(X,X')^T$ by:

$$E_m(t) = V(X(t)) + \frac{1}{2}\|X'(t)\|^2_{\mathcal{E}}. \tag{3}$$

If $X \in W^{2,\infty}_{loc}\left(\mathbf{R}^+_t;\mathcal{E}\right)$ is solution of (2), then we have:

$$\frac{dE_m(t)}{dt} = -\left(\mathcal{H}(X)(t)|X'(t)\right)_{\mathcal{E}}. \tag{4}$$

When $\mathcal{H} \equiv 0$, (2) is *conservative* and obviously, $E_m(t) = E_m(0)$.
If $\left(\mathcal{H}(X)(t)|X'(t)\right)_{\mathcal{E}} \geq 0 \quad t-$a.e., then (2) is *dissipative* (on the trajectory $(X,X')^T$).

Definition 1. The "position-force" relation defined by $\mathcal{H}(X)$ is said *thermodynamically consistent* if there exists a Hilbert space $\mathcal{F}$ and:
$\exists\,\Psi : W^{2,\infty}_{loc}\left(\mathbf{R}^+_t;\mathcal{E}\right) \to L^\infty_{loc}(\mathbf{R}^+_t;\mathcal{F})$ causal and continuous,
$\exists\, Q_t \geq 0$ a pseudo − potential on $\mathcal{F}$,
$\exists\, P \geq 0$ a non − negative potential on $\mathcal{F}$,
such that, for any $x \in W^{2,\infty}_{loc}\left(\mathbf{R}^+_t;\mathcal{E}\right)$:

$$\left(\mathcal{H}(x)(t)|x'(t)\right)_{\mathcal{E}} = Q_t\left(\Psi(x)(t)\right) + \frac{d}{dt}P\left(\Psi(x)(t)\right) \quad t-\text{a.e.} \tag{5}$$

In the decomposition (5) of the *mechanical power* $\left(\mathcal{H}(x)(t)|x'(t)\right)_{\mathcal{E}}$, the first term is the (positive) *dissipation rate* and the second term is the derivative of the *free-energy function* $P(\Psi(x)(t))$.

Let

$$E(t) := E_m(t) + P(\Psi(X)(t)) \tag{6}$$

denote the *energy* of system (2), we easily deduce:

Proposition 1. *If $\mathcal{H}(X)$ is thermodynamically consistent, then system (2) is dissipative: for any (X,X') solution of (2),*

$$\frac{dE(t)}{dt} = -Q_t\left(\Psi(X)(t)\right) \leq 0 \;\; t-a.e. \tag{7}$$

Remark 1. Controls may be considered, under the general form: $X''+\mathcal{H}(X)+\operatorname{grad}V(X) = u(t,X,X')$. Classical viscous damping defined by BX', B positive, may also be added without difficulty.

Examples.

1. Viscous damping: $\mathcal{H}(X)(t) = BX'(t),\; B \geq 0,\; \mathcal{F} = \mathcal{E},\; \Psi(X) = X',\; Q_t(\varphi) = (B\varphi|\varphi)_{\mathcal{E}},\; P(\varphi) = 0.$

2. Coulomb dry friction [1]:

$$\mathcal{E} = \mathbf{R},\ \mathcal{H}(X)(t) = \begin{cases} k\,\mathrm{sign}\,(X'(t))\, if X'(t) \neq 0,\ k > 0 \\ -V'\,(X(t))\, if X'(t) = 0, \end{cases}$$

$$\mathcal{F} = \mathbf{R},\ \Psi(X) = X',\ Q_t(\varphi) = k|\varphi|,\ P(\varphi) = 0.$$

Note that, with function sign understood in the multivalued sense:

$$\mathrm{sign}\,(0) = [-1, 1],$$

system (2) may be rewritten: $X'' + k\,\mathrm{sign}\,(X') + V'(X) \ni 0\ \ t$ - a.e.

3. Hysteresis damping [19].

3 Pseudo-differential damping

3.1 Pseudo-differential operators

The operators $\mathcal{H}$ under consideration in this paper involve pseudo-differential components [18]. They are causal and considered in a way parallel to the classical one using Laplace transform instead of the Fourier one.

The analogy will not be emphasized here, essentially because these operators belong to a subclass which is more conveniently directly described by a class of symbols.

For simplicity, we restrict the statement to scalar systems ($\mathcal{E} = \mathbf{R}$); extension to the vector framework requires further technical adaptations (in particular in infinite-dimensional cases, such as PDEs).

We denote by $\mathcal{S}'_+(\mathbf{R})$ the space of *causal* tempered distributions on $\mathbf{R}$ [16] and by $\mathcal{L}$ the Laplace transform defined by: $(\mathcal{L}u)(p) = \int_0^{+\infty} e^{-p\sigma} u(\sigma)\, d\sigma$.

A complex valued function on $\mathbf{R}^+ \times \mathbf{C}$ will be defined as a symbol and, when this expression makes sense, to such a symbol H we associate the causal operator:

$$\begin{array}{ll} H(\sigma, \partial_\sigma) : \mathcal{S}'_+(\mathbf{R}) & \to \mathcal{S}'_+(\mathbf{R}) \\ \qquad\qquad x & \mapsto z = H(\sigma, \partial_\sigma)\, x = \mathcal{L}^{-1}\left[H(\sigma, .)\, \mathcal{L}x\right]. \end{array} \tag{8}$$

In the case of Volterra (singular) operators: When $H(\sigma, .) = \mathcal{L}h(\sigma, .)$, the following is immediate:

Proposition 2. *Let H a symbol such that For any $\sigma > 0$, $H(\sigma, .) = \mathcal{L}h(\sigma, .)$. Then $H(\sigma, \partial_\sigma)$ is the Volterra operator:*

$$\left(H(\sigma, \partial_\sigma)\, x\right)(\sigma) = \int_0^\sigma h(\sigma, \sigma - \tau) x(\tau)\, d\tau = \int_0^\sigma \mathbf{h}(\sigma, \tau) x(\tau)\, d\tau. \tag{9}$$

Rigorously speaking, the symbol of a *causal* Volterra operator is not unique: it is only defined up to an algebraic quotient. Indeed, it is easy to see that any $\widetilde{h}$ such that $\widetilde{h}(\sigma,\tau)=h(\sigma,\tau)$ on $0<\tau<\sigma$ defines the *same* operator, but the associated symbol $\widetilde{H}(\sigma,.)=\mathcal{L}h(\sigma,.)$ may obviously be different.

Note that in the convolutive case, $h(\sigma,\tau)=h(\tau)$ and $H(p)$ reduces to the classical transfer function. Note also that various regularity properties with respect to the σ-variable may be considered, in accordance to the specific needs of the problem in which such operators are involved.

3.2 Pseudo-differential damping

We study the two following types of damping which are of particular interest in concrete situations:

- linear pseudo-differential *visco-elasticity* [6] defined by:

$$\mathcal{H}(X)=H(t,\partial_t)X', \tag{10}$$

- pseudo-differential *elasto-plasticity* defined by [17]:

$$s:=S(X)(t):=\int_0^t |X'|\,d\tau, \tag{11}$$

$$\mathcal{H}(X)=\left[H(s,\partial_s)\left(X\circ S(X)^{-1}\right)'\right]\circ S(X). \tag{12}$$

The model (11), (12), both non-linear and hereditary and introduced by P.-A. Bliman and M. Sorine [1], defines S as an intrinsic clock such that relatively to the intrinsic time $s=S(X(t))$, the definition

$$X_S:=X\circ S(X)^{-1} \tag{13}$$

gives the linear law:

$$\mathcal{H}_S(X_S)=H(s,\partial_s)X_S'. \tag{14}$$

Such (endochrone) phenomena are frequently encountered in hysteresis theories [19].

The main difficulties in the analysis of models of that type lie in their heredity: the expression of $\mathcal{H}(X(t))$ involves the whole past $(X_{0\leq\tau\leq t})$ of X. We will show in the sequel how to use diffusive representation:

- to state simple sufficient conditions on $H(\sigma,\partial_\sigma)$ ($\sigma=t,s$ resp.) for thermodynamical consistency,
- to build non hereditary augmented state equations including an auxiliary state-variable $\varphi(\sigma,\xi)$ associated to the free energy of $\mathcal{H}$.

4 Diffusive representation

In the following, we essentially consider the *convolutive* case, for simplicity. Most of results remain available in the general case which sometimes requires specific technical developments and will be presented in a further paper (see also [11]).

We present here a simplified introduction to diffusive representations for pseudo-differential operators of diffusive type. These causal operators are in fact defined so as to admit a representation, using an augmented state space, by a diffusive system.

As in the previous section, σ denotes a *time*-variable (t or s, with $\frac{ds}{dt} = |X'|$).

4.1 The algebra Δ' of convolutive diffusive symbols

We first introduce the concept of diffusive symbol, on which are based the diffusive state-space realizations.

Let $H(\sigma, \partial_\sigma)$ an operator with symbol $H(\sigma, p)$. This operator is of diffusive type when there exists $\overline{\mu}(\sigma, \xi)$ such that ??:

$$H(\sigma, p) = \int_0^{+\infty} \frac{\overline{\mu}(\sigma, \xi)}{p + \xi}\, d\xi,\ p = i\omega,\ \omega \in \mathbf{R},\ \sigma > 0; \tag{15}$$

The solution $\bar{\mu}$ of (15), when it exists, is unique and called the diffusive symbol of $H(\sigma, \partial_\sigma)$.

Theorem 1. *$\overline{\mu}$ is solution of (15) if and only if the impulse response of $H(\sigma, \partial_\sigma)$, denoted by h, is given by:*

$$h(\sigma, .) = \mathcal{L}\, \overline{\mu}(\sigma, .). \tag{16}$$

Proof. (formal) From Laplace transform inversion formula and Fubini theorem, for any $\sigma > 0$ and some $a > 0$:

$$h(\sigma, \tau) = \tfrac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} e^{p\tau} H(\sigma, p)\, dp = \tfrac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} e^{p\tau} \int_0^{+\infty} \tfrac{\overline{\mu}(\sigma,\xi)}{p+\xi}\, d\xi\, dp =$$
$$= \int_0^{+\infty} \left(\tfrac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \tfrac{e^{p\tau}}{p+\xi}\, dp \right) \overline{\mu}(\sigma, \xi)\, d\xi = \int_0^{+\infty} e^{-\xi\tau}\, \overline{\mu}(\sigma, \xi)\, d\xi = (\mathcal{L}\overline{\mu})\,(\sigma, \tau).$$

As a consequence of the analyticity of $h(\sigma, .)$, we have the so-called "pseudo-local property":

Corollary 1. *If $H(\sigma, \partial_\sigma)$ is of diffusive type, then:*

$$\operatorname{sing\,supp}\ (H(\sigma, \partial_\sigma)x) \subset \operatorname{sing\,supp} x \tag{17}$$

for any x.

Examples.

1. We consider the particular case of fractional integrators and derivators which are interesting due to their simplicity and popularity. The diffusive symbol of $H(t,\partial_t)=\partial_t^{-\alpha(t)}$, $\Re e\,(\alpha(t))>0$, is expressed by[1]:
$$\overline{\mu}(t,\xi)=\frac{\sin(\pi\alpha(t))}{\pi}\,\mathrm{fp}\,\xi^{-\alpha(t)},\ \xi>0. \tag{18}$$
2. The diffusive symbol of $H(\partial_t)=\partial_t^{-1}\ln(\partial_t)$ is given by[2]:
$$\overline{\mu}(\xi)=\mathrm{fp}\frac{1}{\xi}-\gamma\,\delta(\xi); \tag{19}$$
the associated impulse response is $h(t)=-\ln(t)-\gamma$.
3. The diffusive symbol of $H(\partial_t)=e^{a\partial_t}\,\mathrm{Ei}\,(a\partial_t)$ is given by[3]:
$$\overline{\mu}(\xi)=e^{-a\xi}; \tag{20}$$
the associated impulse response is $h(t)=\frac{1}{t+a}$.
4. Any stable rational transfer function with real poles is the symbol of a diffusive operator. Many other examples can be found in [10].

Obviously, thanks to linearity, the space of diffusive symbols is isomorphic to a subspace of pseudo-differential operators. Let us now consider two convolutive operators $H(\partial_\sigma)$, $K(\partial_\sigma)$, with respective diffusive symbols $\overline{\mu}$, $\overline{\nu}$. We have the following results [10]:

Theorem 2. *The product operator $H(\partial_\sigma)\circ K(\partial_\sigma)$ is also diffusive. Its diffusive symbol is defined by an internal product denoted by $\overline{\mu}\#\overline{\nu}$. When $\overline{\mu},\overline{\nu}$ are regular, this product is expressed:*
$$\overline{\mu}\#\overline{\nu}=-\overline{\mu}\,(\overline{\nu}*\mathrm{pv}\frac{1}{\xi})-\overline{\nu}\,(\overline{\mu}*\mathrm{pv}\frac{1}{\xi}). \tag{21}$$

Theorem 3. *Equipped with product $\#$, the space Δ' of convolutive diffusive symbols is a commutative algebra of causal tempered distributions on $\mathbf{R}_\xi$, with Fréchet topology.*

This induces an isomorphic algebra of convolutive pseudo-differential operators. Thanks to closedness of Δ' and continuity of $\#$, both algebraic and analytical developments[4] can therefore be performed in Δ' which is the general mathematical framework fitted to diffusive representation. We do not describe here the topology of Δ', we only give here-after a simply sufficient condition for $\overline{\mu}\in\Delta'$ [10], [11]:

[1] $\mathrm{fp}\,f$ and $\mathrm{pv}\,f$ respectively denote the "finite part" and "principal value" distributions associated to non locally integrable functions f [16]. They may be viewed as the derivative of sufficiently high order in the sense of **distributions,** of some locally integrable functions. For example, $\mathrm{pv}\frac{1}{x}$ is the (causal) derivative of $\ln(|x|)$ and $\mathrm{fp}\frac{1}{x}$ is the derivative of the causal function $\ln(x)$.

[2] γ denotes the Euler constant.

[3] $\mathrm{Ei}(a)=\int_a^{+\infty}\frac{e^{-u}}{u}\,du$.

[4] Namely numerical analysis.

Proposition 3. *If* $\frac{\overline{\mu}}{1+\xi} \in L^1(\mathbf{R}^+)$, *then* $\overline{\mu} \in \Delta'$.

4.2 Diffusive realizations of $H(\partial_\sigma)$

Standard realizations We consider the following input-output equation (in a suitable Hilbert state-space):

$$\begin{cases} \partial_\sigma \psi(\sigma,\xi) + \xi\, \psi(\sigma,\xi) = x(\sigma), \quad \psi(0,\xi) = 0,\ \sigma > 0 \\ y(\sigma) = \displaystyle\int_0^{+\infty} \overline{\mu}(\sigma,\xi)\, \psi(\sigma,\xi)\, d\xi. \end{cases} \tag{22}$$

If $\overline{\mu}$ is the diffusive symbol of $H(\sigma,\partial_\sigma)$, then we have:

Theorem 4. *The input-output correspondence* $x \mapsto y$ *defined by (22) satisfies:* $y = H(\sigma,\partial_\sigma)\, x$.

Proof. From (22), and Fubini theorem:

$$y = \int_0^{+\infty} \overline{\mu}(\sigma,\xi) \int_0^{\sigma} e^{-\xi\tau}\, x(\sigma-\tau)\, d\tau\; d\xi =$$
$$= \int_0^{\sigma} \left(\int_0^{+\infty} \overline{\mu}(\sigma,\xi)\, e^{-\xi\tau} d\xi \right) x\,(\sigma-\tau)\, d\tau = \int_0^{\sigma} (\mathcal{L}\overline{\mu})\,(\sigma,\tau)\, x\,(\sigma-\tau)\, d\tau =$$
$$\int_0^{\sigma} h(\sigma,\sigma-\tau)\, x(\tau)\, d\tau = (H(\sigma,\partial_\sigma)x)\,(\sigma).$$

Definition 2. The input-output state equation (22) is called the standard diffusive realization of $H(\sigma,\partial_\sigma)$.

Various other state-space realizations may be built (see [10]); in particular, by using Fourier transform with respect to η, with $\xi = 4\pi^2\eta^2$:

$$\begin{cases} \partial_\sigma \Phi(\sigma,\lambda) - \partial_\lambda^2 \Phi(\sigma,\lambda) = x(\sigma)\, \delta(\lambda)\,, \quad \Phi(0,\lambda) = 0,\ \lambda \in \mathbf{R} \\ y(\sigma) = \displaystyle\int_{-\infty}^{+\infty} \overline{M}(\sigma,\lambda)\, \Phi(\sigma,\lambda)\, d\lambda. \end{cases} \tag{23}$$

Remark 2. This last formulation, which gives to diffusive pseudo-differential operators a physical meaning, is at the origin of the term "diffusive representation".

The following result will be fundamental in the sequel:

Lemma 1. *The pseudo-differential operator* $\partial_\sigma^{-1} H(\partial_\sigma)$ *has diffusive symbol* $\delta \# \overline{\mu}$.

Proof. It is sufficient to prove that δ is the diffusive symbol of ∂_σ^{-1}. From the well-known property $\xi\, \delta(\xi) = 0$:

$$\begin{aligned} \partial_\sigma \int_0^{+\infty} \delta\, \psi\, d\xi = \int_0^{+\infty} \delta\, \partial_\sigma \psi\, d\xi = \int_0^{+\infty} \delta\, (-\xi\psi + x)\, d\xi = \\ = -\int_0^{+\infty} \xi\, \delta\, \psi\, d\xi + x \int_0^{+\infty} \delta\; d\xi = x. \end{aligned} \tag{24}$$

Extended diffusive realizations Extended realizations enable to take into account more general pseudo-differential operators. We consider here the following, which is well-adapted to visco-elastic and elasto-plastic phenomena. It consists in *derivating* the output, which obviously leads to the state-space realization of $x \mapsto z = \partial_\sigma H(\sigma, \partial_\sigma) x$:

$$\begin{cases} \partial_\sigma \psi + \xi\,\psi = x, \quad \psi(0,\xi) = 0, \ \sigma > 0 \\ z = \partial_\sigma \displaystyle\int_0^{+\infty} \overline{\mu}\,\psi\,d\xi = \int_0^{+\infty} \overline{\mu}\,(-\xi\,\psi + x)\,d\xi. \end{cases} \tag{25}$$

Note that this last formulation is of the abstract form:

$$\begin{cases} \frac{dX}{dt} = AX + Bx, \ X_0 = 0 \\ z = C(X + Dx). \end{cases} \tag{26}$$

From lemma 1 and according the previously introduced notions and notations, the following result is obvious:

Theorem 5. *The correspondence $x \mapsto y = H(\partial_\sigma)x$ realized by (22), is also realized by:*

$$\begin{cases} \partial_\sigma \psi + \xi\,\psi = x, \quad \psi(0,\xi) = 0, \ \sigma > 0 \\ y = \displaystyle\int_0^{+\infty} \delta \# \overline{\mu}\,(-\xi\,\psi + x)\,d\xi. \end{cases} \tag{27}$$

Finite-dimensional approximate diffusive realizations They are obtained from discretization of the ξ-variable in (25), following standard methods of partial differential equations and numerical analysis. We only give some indications, more details will be found in the referenced papers.

Given a finite mesh $\chi_K = \{\xi_k\}_{1 \leq k \leq K} \subset \mathbf{R}^+$, and $\Omega_K = \{\Lambda_k(\xi)\}$ a suitable set of interpolating functions, a finite-dimensional approximation of ψ defined by (22) is obtained by:

$$\widetilde{\psi}(\sigma,\xi) = \sum_{k=1}^{K} \psi(\sigma,\xi_k)\,\Lambda_k(\xi), \tag{28}$$

and an approximation of y (defined by (22)) is then deduced:

$$\begin{aligned} \widetilde{y}(\sigma) = \int_0^{+\infty} \overline{\mu}\ \widetilde{\psi}\,d\xi &= \sum_{k=1}^{K} \psi(\sigma,\xi_k) \int_0^{+\infty} \overline{\mu}(\sigma,\xi)\,\xi\,\Lambda_k(\xi)\,d\xi \\ &= \sum_{k=1}^{K} \overline{\mu}_k(\sigma)\,\psi_k(\sigma). \end{aligned} \tag{29}$$

Under simple and natural hypothesis on χ_K and Ω_K and according to fitted topologies, we may state:

Proposition 4. *The finite-dimensional approximate realization of* $x \mapsto y = H(\sigma, \partial_\sigma)x$ *:*

$$\begin{cases} \frac{d\psi_k}{dt} = -\xi_k \, \psi_k + x, \; \psi_k(0) = 0 \\ \widetilde{y} = \sum_{k=1}^{K} \overline{\mu}_k \, \psi_k \end{cases} \tag{30}$$

is convergent when $K \to +\infty$*:*

$$\widetilde{y} - H(\sigma, \partial_\sigma)x \to 0. \tag{31}$$

Corollary 2. *The finite-dimensional approximate realization of* $x \mapsto z = \partial_\sigma H(\sigma, \partial_\sigma)x$ *:*

$$\begin{cases} \frac{d\psi_k}{dt} = -\xi_k \, \psi_k + x, \; \psi_k(0) = 0 \\ \widetilde{z} = \sum_{k=1}^{K} \overline{\mu}_k \, (-\xi_k \, \psi_k + x) = \sum_{k=1}^{K} \mu_k \; \psi_k + \mu_0 \, x \end{cases} \tag{32}$$

is convergent:

$$\widetilde{z} - \partial_\sigma H(\sigma, \partial_\sigma)x \to 0. \tag{33}$$

From a different point of view, thanks to topological density of the space of measures in Δ', *optimal* K-dimensional diffusive realizations of the form (32) may easily be obtained by solving (15). Solutions are built in the *pseudo-inversion* sense[5], with $\overline{\mu} \in \mathcal{M}_K \subset \Delta'$, the K-dimensional space of Dirac measures with support χ_K. This requires Hilbertian formulations and is not presented here (see [10]). An example of optimal approximate diffusive realization is given in section 7.

5 Main result

In order to built dynamical models for pseudo-differential visco-elasticity and elasto-plasticity, we prove the following result on which will be based the thermodynamical consistency of $\mathcal{H}$. It gives a sufficient (and probably necessary) condition to get positiveness of operator $H(\partial_\sigma)$.

Theorem 6. *If the diffusive symbol* $\overline{\mu}$ *of* $\partial_\sigma^{-1} H(\partial_\sigma)$ *is such that:*

$$\exists \, \mu, \nu \in L^1_{loc}(\mathbf{R}^+) \cap \Delta', \; \mu, \nu \geq 0, \; \overline{\mu} = \delta \# \mu + \nu, \tag{34}$$

then we have the following balanced diffusive realization of $z = H(\partial_\sigma)x$*:*

$$\begin{cases} \partial_\sigma \varphi + \xi \, \varphi = \left(\sqrt{\mu} + \sqrt{\xi \, \nu}\right) \, x, \;\; \varphi(0, \xi) = 0, \; \sigma > 0 \\ z = \int_0^{+\infty} \left[\left(\sqrt{\mu} - \sqrt{\xi \, \nu}\right) \varphi + \nu \, x \, \right] \, d\xi. \end{cases} \tag{35}$$

Furthermore we have the estimate for any $\sigma > 0$*:*

$$x(\sigma) \, H(\partial_\sigma)x(\sigma) = \frac{d}{d\sigma} \frac{1}{2} \, \|\varphi\|^2_{L^2(\mathbf{R}^+)} + \int_0^{+\infty} \left(\sqrt{\xi} \, \varphi - x(\sigma)\sqrt{\nu} \right)^2 d\xi. \tag{36}$$

[5] Orthogonal projection.

Proof. 1. By change of function $\psi = \frac{\varphi}{\sqrt{\mu}+\sqrt{\xi\,\nu}}$ and theorem 5,

$$z = \int_0^{+\infty} ((\mu - \xi\nu)\psi + \nu\, x)\, d\xi = \int_0^{+\infty} \mu\,\psi\, d\xi + \int_0^{+\infty} \nu\,(-\xi\psi + x)\, d\xi =$$
$$= \int_0^{+\infty} \delta\#\mu\,(-\xi\psi + x)\, d\xi + \int_0^{+\infty} \nu\,(-\xi\psi + x)\, d\xi =$$
$$= \int_0^{+\infty} \overline{\mu}\,(-\xi\psi + x)\, d\xi = \partial_\sigma \int_0^{+\infty} \overline{\mu}\,\psi\, d\xi = \partial_\sigma \partial_\sigma^{-1} H(\partial_\sigma)x = H(\partial_\sigma)x.$$

2. Furthermore,

$$xH(\partial_\sigma)x = x\int_0^{+\infty} \left[\left(\sqrt{\mu} - \sqrt{\xi\,\nu}\right)\varphi + x\,\nu\right]\, d\xi =$$
$$= \int_0^{+\infty} \left[x\sqrt{\mu}\,\varphi - x\sqrt{\xi\nu}\,\varphi + x^2\,\nu\right]\, d\xi =$$
$$= \int_0^{+\infty} \left[-\xi\,\varphi^2 + x\,\left(\sqrt{\mu} + \sqrt{\xi\nu}\right)\,\varphi + \left(\xi\,\varphi^2 - 2\sqrt{\xi\nu}\,\varphi\, x + x^2\,\nu\right)\right]\, d\xi =$$
$$= \int_0^{+\infty} \varphi\left[-\xi\,\varphi + x\,\left(\sqrt{\mu} + \sqrt{\xi\nu}\right)\right]\, d\xi + \int_0^{+\infty} \left(\sqrt{\xi}\,\varphi - x\,\sqrt{\nu}\right)^2 d\xi =$$
$$= \partial_\sigma\left(\frac{1}{2}\int_0^{+\infty} \varphi^2\, d\xi\right) + \int_0^{+\infty} \left(\sqrt{\xi}\,\varphi - x\,\sqrt{\nu}\right)^2 d\xi.$$

Remark 3. 1. Property (36) is in fact much more precise than *positiveness* $(\int_0^\Sigma xH(\partial_\sigma)x\, d\sigma \geq 0)$ which, in the context of diffusive representation, appears as a simple corollary.

2. Besides positiveness, relation (36) suggests the natural Hilbert energy state-space for (35): $\mathcal{F} = L^2(\mathbf{R}_\xi^+)$.

As a consequence, we have in the particular case of *fractional* operators:

Corollary 3. *For* $H(\partial_\sigma) = k_1\,\partial_\sigma^{\alpha_1} + k_2\,\partial_\sigma^{-\alpha_2},\ k_1, k_2 \geq 0,\ 0 < \alpha_1, \alpha_2 < 1$, *properties (34), (35), (36) are verified, with:*

$$\begin{aligned} &\mu(\xi) = k_2 \tfrac{\sin(\pi\alpha_2)}{\pi}\tfrac{1}{\xi^{\alpha_2}},\ \nu(\xi) = k_1\tfrac{\sin(\pi(1-\alpha_1))}{\pi}\tfrac{1}{\xi^{1-\alpha_1}},\\ &\overline{\mu}(\xi) = k_1\tfrac{\sin(\pi(1-\alpha_1))}{\pi}\tfrac{1}{\xi^{1-\alpha_1}} + k_2\tfrac{\sin(\pi(1+\alpha_2))}{\pi}\,\mathrm{fp}\tfrac{1}{\xi^{1+\alpha_2}}. \end{aligned} \tag{37}$$

Proof. Obvious from (18) and lemma 1.

6 Application to pseudo-differentially damped systems.

6.1 Thermodynamical consistence of $\mathcal{H}$

The following results are then deduced from theorem 6, by ordinary computations:

Theorem 7. *If $H(\partial_t)$ satisfies hypothesis of theorem 6, then $\mathcal{H}(X) = H(\partial_t)X'$ is thermodynamically consistent, by taking:*

$$\mathcal{F} = L^2(\mathbf{R}_\xi^+),$$

$$\begin{aligned} \varphi_X(t) = \varphi(t,.), \varphi \text{ defined by (35), with } x(t) := X'(t), \\ Q_t(\varphi) = \int_0^{+\infty} \left(\sqrt{\xi}\,\varphi - \sqrt{\nu(\xi)}\,X'(t)\right)^2 d\xi, \\ P(\varphi) = \tfrac{1}{2}\,\|\varphi\|^2_{L^2(\mathbf{R}^+)}\,. \end{aligned} \tag{38}$$

Proof. Obvious from theorem 6 and definition 1 with $\mathcal{E} = \mathbf{R}$.

Theorem 8. *If $H(\partial_s)$ satisfies hypothesis of theorem 6, then $\mathcal{H}_S(X_S) = H(s,\partial_s)X'_S$ (see section 3.2) is thermodynamically consistent, by taking:*

$$\begin{aligned} s = \int_0^t |X'|\,d\tau, \\ \mathcal{F} = L^2(\mathbf{R}_\xi^+), \\ \varphi_X(t) = \varphi(s(t),.),\ \varphi \text{ defined by (35) with } x(s) := \tfrac{d}{ds}X(t(s)), \\ Q_t(\varphi) = \tfrac{1}{(X'(t))^2}\int_0^{+\infty} \left(|X'(t)|\sqrt{\xi}\,\varphi - \sqrt{\nu(\xi)}\,X'(t)\right)^2 d\xi, \\ P(\varphi) = \tfrac{1}{2}\,\|\varphi\|^2_{L^2(\mathbf{R}^+)}\,. \end{aligned} \tag{39}$$

Proof. Similar to theorem 7, with :

$$Q_{t(s)}(\varphi) = \int_0^{+\infty} \left(\sqrt{\xi}\,\varphi - \sqrt{\nu(\xi)}\,\tfrac{dX}{ds}\right)^2 d\xi =$$

$$= \int_0^{+\infty} \left(\sqrt{\xi}\,\varphi - \sqrt{\nu(\xi)}\,X'\,\tfrac{dt}{ds}\right)^2 d\xi = \int_0^{+\infty} \left(\sqrt{\xi}\,\varphi - \sqrt{\nu(\xi)}\,\tfrac{X'(t(s))}{|X'(t(s))|}\right)^2 d\xi.$$

6.2 Time-local state-space realizations of (2)

By coupling the diffusive realization of $\mathcal{H}$ and the main state equation, we obtain suitable global models for pseudo-differentially damped systems, with existence of an infinitesimal generator (time-local system):

Corollary 4. ***(concrete state-space realizations)*** *Denoting:*

$$M(\xi) := \sqrt{\mu(\xi)} - \sqrt{\xi\,\nu(\xi)} \text{ and } M^\dagger(\xi) := \sqrt{\mu(\xi)} + \sqrt{\xi\,\nu(\xi)}, \tag{40}$$

non hereditary global state-space realizations of (2) (Cauchy problems) are then explicitly built:

visco-elastic model :

$$\begin{cases} \partial_t^2 X + \displaystyle\int_0^{+\infty} \left[M^\dagger\varphi + \partial_t X \otimes \nu\right] d\xi + V'(X) = 0 \\ \partial_t\varphi + \xi\,\varphi - \partial_t X \otimes M\ = 0, \end{cases} \tag{41}$$

elasto-plastic model:

$$\begin{cases} \partial_t^2 X + \displaystyle\int_0^{+\infty} \left[M^\dagger \varphi + \mathrm{sign}\,(\partial_t X) \otimes \nu \right] d\xi + V'(X) \ni 0 \\ \partial_t \varphi + \xi\, \varphi\, |\partial_t X| - \partial_t X \otimes M \;= 0, \end{cases} \tag{42}$$

with initial condition (X_0, X_0', φ_0) *and energy functional:*

$$E(t) = V(X(t)) + \frac{1}{2}\,(\partial_t X(t))^2 + \frac{1}{2}\,\|\varphi(t,.)\|^2_{L^2(\mathbf{R}^+)}\,, \tag{43}$$

such that:

$$\frac{dE(t)}{dt} = -Q_t\,(\varphi(t,.)) \le 0 \quad \forall t > 0. \tag{44}$$

Proof. Obvious from:

$$\frac{dX}{ds} = \frac{dX}{dt}\frac{dt}{ds} = \frac{X'}{|X'|} \in \mathrm{sign}(X') \tag{45}$$

and:

$$\partial_t \varphi = \partial_s \varphi\, \frac{ds}{dt} = \partial_s \varphi\, |X'|. \tag{46}$$

Remark 4. Dry friction dissipation is obtained by (42) with $M = M^\dagger = 0$, $\nu = \delta$.

Under weak hypothesis, *existence* and *uniqueness* of the solution of (41), (42) in a fitted Hilbert state-space $\mathcal{G}$ can therefore be proved from classical energy-based methods of partial differential equations (Galerkin method for example). Note that in the *non Lipschitz* case (42), existence of *a priori* energy estimates proves to be decisive in order to suppress mathematical ambiguousness inherent to such systems [13].

Furthermore, finite-dimensional *convergent* approximations of (41), (42) can efficiently be elaborated from energy error estimates. This enables to build finite-dimensional differential approximated models with arbitrary precision, of the form: $\frac{dz}{dt} = F(z)$, $z(t) \in \mathbf{R}^N$.

Finally, thanks to the existence of an infinitesimal generator for (41) and (42) (induced by (35)), classical tools of control theory may be employed. Note that the damping function, defined by the abstract form $\mathcal{H}(X)$ and equivalently by the concrete state-space realization (35), may also be considered as a pseudo-differential (closed-loop) control, constructed for example by minimization of a cost functional $\mathcal{J}(\overline{\mu})$. Indeed, pseudo-differential diffusive controls have proved to be of particular interest for *robustness* purposes in linear control problems (see [3], [4], [5], [12]). From a slightly different point of view, such methodologies have also successfully been used in pseudo-differential passive control of linear infinite-dimensional systems in [14], [15], [3].

6.3 Analysis of asymptotic behaviors

From (44), specific techniques like *LaSalle invariance principle* [7] then enable to find asymptotic equilibrium states. In (42), they systematically depend on the initial condition:

$$\begin{aligned}&\forall\,(X_0,X_0',\varphi_0)\in\mathcal{G},\ \exists\,!\,(X_\infty,0,\varphi_\infty)\in\mathcal{G},\ \text{such that :}\\ &E(t)\downarrow E_\infty=V(X_\infty)+\tfrac{1}{2}\left\|\varphi_\infty\right\|^2,\\ &X(t)\to X_\infty,\ \partial_t X(t)\to 0,\ \varphi(t,.)\longrightarrow\varphi_\infty\ \text{strongly in}\ L^2(\mathbf{R}_\xi^+),\end{aligned}\tag{47}$$

with the following characteristic equation for equilibrium:

$$\begin{aligned}&\int_0^{+\infty}\left[M^\dagger(\xi)\,\varphi_\infty(\xi)+\alpha_\infty\,\nu(\xi)\right]\,d\xi=-V'(X_\infty),\\ &\alpha_\infty\in\operatorname{sign}(0)=[-1,1].\end{aligned}\tag{48}$$

Note that this last expression *explicitly* involves the diffusive realization of $\mathcal{H}(X)$, through its characteristic parameters M and ν. Excepted in very simple cases (dry friction), such an explicit characterization is not accessible from initial formulation (2).

7 An example of numerical simulation

7.1 Problem statement

In order to highlight the efficiency of diffusive representation from the point of view of numerical simulations, we consider the second order oscillator with viscoelastic damping $\mathcal{H}(X):=\lambda\,\partial_t^{-\alpha}X',\ 0<\alpha<1,\ \lambda>0$:

$$\partial_t^2 X=-\lambda\,\partial_t^{1-\alpha}X-f(X).\tag{49}$$

From corollary 4 and (18), model (49) is equivalently transformed into:

$$\left\{\begin{aligned}&\partial_t^2 X=-\lambda\int_0^{+\infty}\tfrac{\sin(\alpha\pi)}{\pi\,\xi^\alpha}\,\psi\,d\xi-f(X)=0\\ &\partial_t\psi=-\xi\,\psi+\partial_t X\,.\end{aligned}\right.\tag{50}$$

A K-dimensional optimal diffusive approximation of $\partial_t^{-\alpha}$ has been performed (see section 4.2), with the following parameters[6]:

$$\begin{aligned}&\alpha=0.75\\ &K=25\\ &\xi_1=0.001\\ &\xi_{25}=50\,000\\ &\tfrac{\xi_{k+1}}{\xi_k}=\ 2.093102\,,\end{aligned}$$

[6] $K=25$ has been chosen for high precision. Smaller values of K are generally sufficient in physical situations.

$(\mu_k)_{1\leq k\leq 25} = (0.43463$, -0.32584, 0.16799, 0.02225, 0.07070, 0.06959, 0.09374, 0.10293, 0.13561, 0.14745, 0.19909, 0.20862, 0.29512, 0.29128, 0.44231, 0.39891, 0.67268, 0.52850, 1.04661, 0.64882, 1.69943, 0.74395, 1.87083, 3.89598, 3.38926).

Under such conditions, with:

$$A = 1 \leq k \leq K\mathrm{diag}(-\xi_k),\ B = (1,1,...,1)^T,\ C = (\mu_k)_{1\leq k\leq 25},\ X_1 := X,$$

model (50) can be rewritten under the form:

$$\begin{cases} \frac{dX_1}{dt} = X_2 \\ \frac{dX_2}{dt} = -f(X_1) - \lambda\, C\, \psi \\ \frac{d\psi}{dt} = A\,\psi\ + B\, X_2. \end{cases} \tag{51}$$

System (51) has been simulated by classical Runge-Kutta method, with $\lambda = 2$, in the *linear:* $f(X) = X$, and *non-linear:* $f(X) = \sin X$ cases.

7.2 Numerical results

The frequency response and the pole-zero map[7] of the approximation of $\partial_t^{0.75}$ are given in figures 1, 2. Note that on 6 decades, phase is *constant* (67.5°) and magnitude decreases at rate of $0.75 \times 20\,\mathrm{dB/dec}$; these properties are characteristic of fractional integrators.

Evolution of the linear system is shown in figures 3, 4, 5. Long memory viscoelastic behavior is clearly visible: after a few oscillations generated by the elastic component of $\mathcal{H}(X)$, $X(t)$ slowly decreases to 0, involving both the viscous and elastic component of the pseudo-differential damping.

In figures 6, 7, 8, non-linearity significantly affects the evolution: due to the elastic component of $\mathcal{H}(X)$, small overshoots appear at the beginning, while the viscoelastic counterpart considerably slackens the system. This is the consequence of the particular choice of initial conditions, near an unstable equilibrium point ($\sin(X_0) \simeq 0$).

More detailed simulations (namely in presence of elastoplastic damping) will be presented in a further paper devoted to numerical approximation.

[7] Only the domain $[-0.1, 0] + i[-0.002, 0.002]$ is visible in the figure.

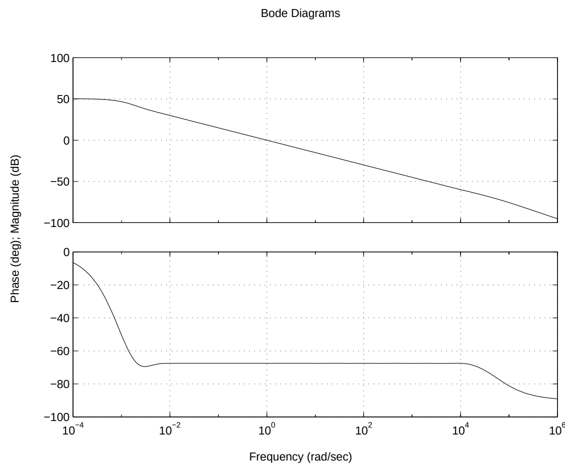

Fig. 1. Frequency response of the approximate $\partial_t^{0.75}$

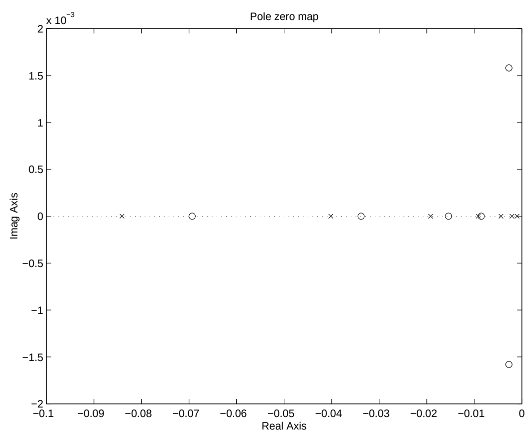

Fig. 2. Pole-zero (partial) map of the approximate $\partial_t^{0.75}$

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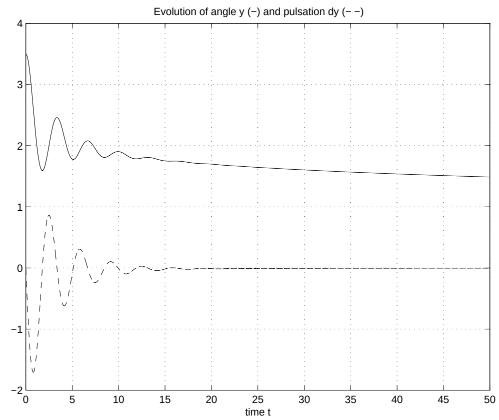

Fig. 3. Linear model $\partial_t^2 X + \lambda \partial_t^{1+\alpha} X + X = 0$

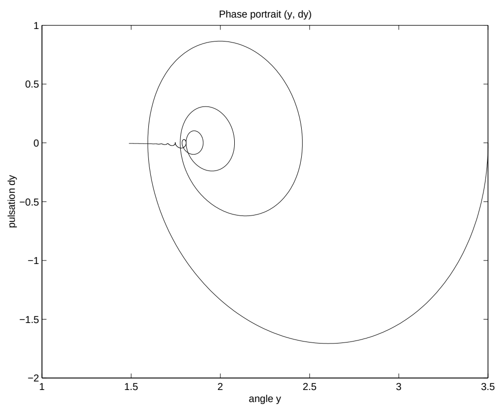

Fig. 4. Linear model $\partial_t^2 X + \lambda \partial_t^{1+\alpha} X + X = 0$

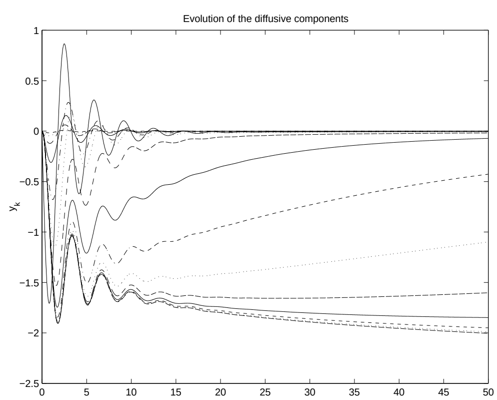

Fig. 5. Linear model $\partial_t^2 X + \lambda \partial_t^{1+\alpha} X + X = 0$

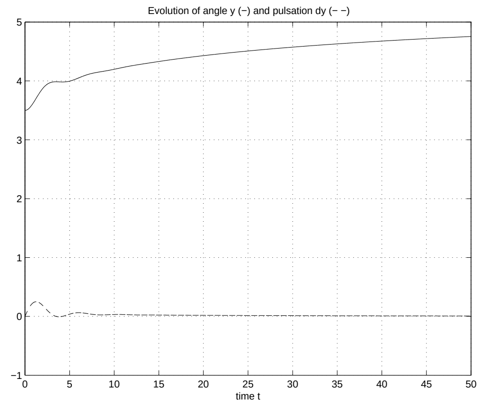

Fig. 6. Non-linear model $\partial_t^2 X + \lambda \partial_t^{1+\alpha} X + \sin(X) = 0$

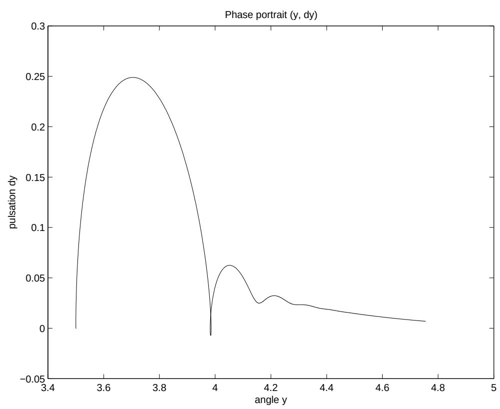

Fig. 7. Non-linear model $\partial_t^2 X + \lambda \partial_t^{1+\alpha} X + \sin(X) = 0$

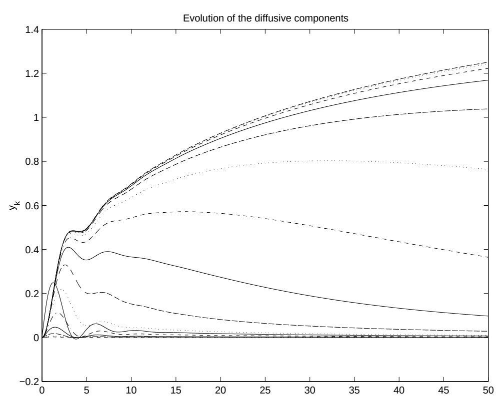

Fig. 8. Non-linear model $\partial_t^2 X + \lambda \partial_t^{1+\alpha} X + \sin(X) = 0$

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