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Part III

Disturbance energies and stability criteria in reacting flows

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Chapter 5

Introduction

This chapter presents the topic of disturbance energies in flows and the advanced post-processing tool developed during this PhD to check the closure of balances of disturbance energy conservation equations.
Section 5.1 gives a review of previous studies related to this topic. The attention is particularly focused on the hypothesis done by each authors to derive conservation equations for disturbance energies.
Section 5.2 presents the post-processing tool used during this thesis to compute the balances of such quantities. Details of the implementation are provided.
Section 5.3 gives simple validations of the tool on a 2D reacting configuration checking the closure of the balances of mass, axial momentum and total energy conservation equations.

5.1 Previous studies

Combustion stability has received sustained attention in both the academic and industrial communities over the last fifty years in particular. During this time, the literature on this problem has grown enormously, and now spans numerous applications including rockets [45, 35], afterburners [12], gas turbines [39, 107] and industrial burners [118]. The sustained research on this problem is primarily because manufacturers still rely heavily on in situ testing and tuning of the complete, operating device to avoid instability. This continued reliance on testing has several causes, including incompleteness in our fundamental understanding of the problem, as argued recently by Nicoud and Poinsot[93].

The following sections present previous studies concerning this particular issue.

- First a simple derivation leading to an extended Rayleigh stability criterion is discussed. It points out the main hypothesis that are necessary for this criterion to be relevant.
- Further developments of Morfey [85] and Bloxsidge et al. [12] on acoustic energy are then discussed.

- Finally, the notion of ”disturbance energy” in non-reacting and reacting flows is introduced thanks to the works of Myers [88] and Chu [23].

5.1.1 Classical acoustic energy

The Rayleigh stability criterion is the most common argument for explaining combustion stability. Whilst Rayleigh himself only first stated this criterion in prose form [120], it is often written as

$$\int_{\Omega} \overline{p'\omega'_T} d\mathbf{x} > 0, \quad (5.1)$$

where p', $\omega_T{}'$ and Ω are the static pressure and heat release rate disturbances at a point in space and the combustor volume respectively. $(\bar{\ })$ denotes the time average. This criterion states that the combustor is unstable when the relative phase of the pressure and heat release disturbances over the combustor volume are such that the integral is positive.

The following paragraphs presents a simple derivation of the acoustic energy in a reacting medium [1] which (with the right assumptions) leads to an extended Rayleigh criterion. The starting point of this derivation is the Navier-Stockes equations expressed here in tensor form and using total derivatives. The first two used assumptions are:

- zero volume forces
- zero volume heat sources

The required equations are mass and momentum conservation equations:

$$\frac{D\rho}{Dt} + \rho \vec{\nabla}.\vec{u} = 0 \quad (5.2)$$

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \vec{\nabla} \tau \quad (5.3)$$

Since the flow is not adiabatic, the energy equation is also required :

$$\rho c_p \frac{DT}{Dt} = \dot{\omega}'_T + \frac{Dp}{Dt} + \tau : \vec{\nabla}(\vec{u}) - \left(\rho \sum_{k=1}^{N} C_{p,k} Y_k \vec{V}_k \right) .\vec{\nabla} T \quad (5.4)$$

where $\vec{V}_k$ is the diffusion velocity vector of species k. By dividing equation 5.4 by $\rho c_p T$ and using the equation of state $p = \rho r T$, one obtains a conservation equation for $ln(p)$:

$$\frac{1}{\gamma} \frac{Dln(p)}{Dt} + \vec{\nabla}.\vec{u} = \frac{1}{\rho c_p T} \left[\dot{\omega}_T + \tau : \vec{\nabla}(\vec{u}) - \rho \sum_{k=1}^{N} C_{p,k} Y_k \vec{V}_k \right] + \frac{1}{r} \frac{Dr}{Dt} \quad (5.5)$$

[1] The major part of this derivation can be found in [107]

Furthermore, if the following hypothesis are made:

- low-speed mean flow,
- identical molecular weights for all species,
- negligible viscous terms,
- all convective derivatives are negligible compared to time derivatives,

the following simplification can be made:

- γ is constant and the mean pressure p_0 is also constant so that $\gamma p_0 = \rho_0 c_0^2$ is constant.

Equation 5.5 then simplifies to :

$$\frac{1}{\gamma}\frac{\partial ln(p)}{\partial t} + \vec{\nabla}.\vec{u} = \frac{1}{\rho c_p T}\dot{\omega}_T \tag{5.6}$$

Equation 5.3 simplifies to :

$$\rho\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}p \tag{5.7}$$

Linearizing equations 5.6 and 5.7, one derives the following set of equations for the fluctuations of pressure p_1 and velocity $\vec{u_1}$

$$\frac{\partial \vec{u}_1}{\partial t} = -\frac{1}{\rho_0}\vec{\nabla}p_1 \tag{5.8}$$

$$\frac{1}{\gamma p_0}\frac{\partial p_1}{\partial t} + \vec{\nabla}.\vec{u}_1 = \frac{\gamma - 1}{\gamma p_0}\dot{\omega}_{T1} \tag{5.9}$$

Taking the scalar product of equation 5.8 with $\vec{u_1}$ and multiplying equation 5.9 by p_1 and adding both results, leads to :

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u_1^2 + \frac{1}{2\rho_0 c_0^2}p_1^2\right) + \vec{\nabla}.p_1\vec{u_1} = \frac{\gamma - 1}{\gamma p_0}p_1\dot{\omega}_{T1} \tag{5.10}$$

Equation 5.10 stands for the local evolution of the acoustic energy. To assess the increase of the acoustic energy in a configuration, equation 5.10 must be integrated over the total volume and over one period of oscillation which gives :

$$E_1(T_{n+1}) - E_1(T_n) = \int_V \frac{\gamma - 1}{\gamma p_0}\overline{p_1\dot{\omega}_{T1}}dv - \int_S \overline{p_1\vec{u_1}}.\vec{n}ds \tag{5.11}$$

where $E(T_n) = \frac{1}{2}\rho_0 u_1^2(t = T_n) + \frac{1}{2\rho_0 c_0^2}p_1^2(t = T_n)$ accounts for the acoustic energy at time $t = T_n$. The stability criterion emerging from equation 5.11 states that the configuration will become unstable (i.e the acoustic energy will grow from one cycle of excitation to the other) if :

$$\int_V \frac{\gamma - 1}{\gamma p_0}\overline{p_1\dot{\omega}_{T1}}dv > \int_S \overline{p_1\vec{u_1}}.\vec{n}ds \tag{5.12}$$

the term $\int_S \overline{p_1 \vec{u_1}}.\vec{n}ds$ therefore represent the 'acoustic losses' due to acoustic fluxes across the boundaries.

Table 5.1 sums up the hypothesis used to derive an extended Rayleigh stability criterion. The first remark is that many hypothesis are needed, therefore restricting the analysis to special cases. To sum up, species must have the same thermal capacities and the mean flow velocity must remain small in order to conserve a small mean Mach number. Finally, the fluctuations of all variables must remain small which is a strong hypothesis especially in reacting flows where the flame moves from one place to the other. This move creates temperature fluctuations which have the same amplitude as the mean temperature.

For all these reasons many authors have tried to define other quantities that would have the same properties as the acoustic energy well defined in non reacting flows but that would not rely on the same hypothesis. In the following section, emphasis is made on the works of Morfey [85] and Bloxsidge et al. [12].

5.1.2 Further development of the concept of acoustic energy

The generalised concept of acoustic energy

Morfey start his analysis with some statements about the definition of acoustic energy. Some parts of the following are directly taken from [85]. In an ideal acoustic medium at rest, the sound power crossing any closed surface S is defined as :

$$W = \int_S I ds \tag{5.13}$$

where $I = \overline{p'\vec{v'}.\vec{n}}$ is the acoustic intensity.
"The usefulness of sound power derives from its continuity property : if S_1 and S_2 are two surfaces enclosing the same source of sound, the same value of W is found for both surfaces, provided that the intervening region is in a steady state. This property is so useful that the question arises whether in a more general situation, such as a moving medium, a quantity can be found analogous to sound power in the classical situation."
Morfey proposes the following approach:
A vector $\vec{I}$ may be sought which is based on first-order estimates of the sound field, and reduces to the classical expression in that $\vec{\nabla}.\vec{I}$ vanishes (to second order in fluctuating quantities) over a wider range of conditions than $\vec{\nabla}.(p'\vec{v'})$. The wider the range of conditions, the more general is the continuity property and the more useful the definition.

Separing the velocity fluctuation into rotational (induced by vorticity w') and irrotational (induced by acoustic u') parts ($v' = u' + w'$), Morfey defines a conservation equation for a generalized acoustic energy ($E^\star$) in a rotational mean flow, in presence of entropy gradients combined with mean pressure gradients or flow. Heat conduction and viscous dissipation is also taken into account.

$$\frac{\partial E^\star}{\partial t} + \vec{\nabla}.\vec{N}^\star = P_0 + P_1 \tag{5.14}$$

	Description	**Comments**
H1	zero volume forces	Usual hypothesis, the flow is not forced using external forced
H2	zero volume heat sources	Usual hypothesis, the flow is not heated using an external source of heat.
H3	low-speed mean flow : $M_0 << 1$	In practice the derivation can be done for small mean Mach number configuration only.
H4	$C_{p,i} = \bar{c_p}$, $C_{v,i} = \bar{c_v}$ for all species	The derivation is enclosed in the analysis of mixture with low differences between species thermodynamic quantities. This excludes H_2/O_2 or H_2/Air flames.
H5	negligible viscous terms : $\tau << 1$	Often true for low Reynolds configurations. Especially, in reacting flows, the variation of the pressure due to the combustion process is much bigger than the variation due to the turbulence
H6	$\vec{u}.\vec{\nabla}(f) << \frac{\partial f}{\partial t}$	Strongly linked with H3
H7	Low fluctuation level : $u_1 << c_0$, $p_1 << p_0$, etc...	One of the most important hypothesis. If this condition is not satisfied the linearization process cannot be accomplished. A priori not always true, especially for temperature fluctuations in reacting configurations.

Table 5.1 - *Necessary hypothesis to derive an extended Rayleigh stability criterion.*

$E^\star$ represents a generalized acoustic energy density and $N^\star$ is the generalized acoustic energy flux, given by :

$$E^\star = \frac{1}{2\rho c^2}p'^2 + \frac{1}{2}\rho u'^2 + (\vec{V}.\vec{u'})\frac{p'}{c^2} \tag{5.15}$$

and

$$\vec{N^\star} = p'\vec{u'} + \frac{p'^2}{\rho c^2}\vec{V} + \frac{p'}{c}(\vec{V}.\vec{u'})\vec{V} + \rho(\vec{V}.\vec{u'})\vec{V} \tag{5.16}$$

where $\vec{V}$ is the mean velocity. P_0 and P_1[2] are respectively the production rate in uniform fluid at rest and the additional production term when sound travels through a rotational medium, or through a medium in which mean entropy gradients are combined with mean pressure gradients or flow.

The acoustic energy $E^\star$ defined by Morfey differs from the the classical acoustic energy in that it contains an additional term : $(\vec{V}.\vec{u'})\frac{p'}{c^2}$. This term appears as one considers the propagation of sound in a moving medium and accounts for the acoustic energy contained in the interaction between the fluctuations of velocity and pressure and the mean field. The acoustic energy flux also differs from the classical one in that it contains three terms that are related to the interaction between the mean moving flow and the fluctuations of pressure and velocity. The sound power crossing two enclosed surface is in general different of an amount equal to the volume integral of the production term in the intervening space.

This analysis states that part of the sound production can be related to fluctuations of the entropy field. Though, the sources terms linked to entropy fluctuations are not developed and therefore no direct link with the Rayleigh criterion can be made.

Morfey[85] considers a viscous, heat conducting fluid, and first splits the disturbance velocity field into irrotational and solenoidal components which are defined as the acoustic and unsteady vortical motions respectively. He then applies the definitions of acoustic energy density and flux proposed by Cantrell[18] for inviscid, non heat conducting flows to his acoustic field. Any resulting entropy disturbances in Morfey's analysis are then shifted into the source term. However, the definition of all irrotational velocity disturbances as 'acoustic' is problematic in combusting flows because they usually feature significant irrotational velocity disturbances due to heat addition. These are entropy disturbances and therefore convective rather than acoustic [24].

On the production rate terms

Bloxsidge et al. [12] address this particular point theoretically by deriving a 1D conservation equation for acoustic energy in a reacting configuration. In their case, drag forces in the momentum equation have been neglected. Heat capacities for the species are equal and constant. Conservation equations of velocity, entropy and mass are linearized around the mean flow field denoted $(\bar{\ })$. All fluctuating

[2] The reader is referred to [85] for the exact definitions of these terms

quantities are denoted $()'$. The following production term for the acoustic energy defined by Morfey[85] is found:

$$P = \frac{\gamma - 1}{\bar{\rho}\bar{c}^2}\omega'_T(p' + \bar{\rho}\bar{u}u')$$
$$-\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\left[(\gamma - 1 - \gamma\bar{M}^2)\left(\frac{p'^2}{\gamma^2\bar{p}^2} + \frac{\bar{u}p'u'}{\gamma\bar{\rho}\bar{c}^2}\right) + (1-\bar{M}^2)\frac{u'^2}{\bar{c}^2} + \frac{\bar{M}\rho'}{\bar{\rho}}\left(\frac{u'}{\bar{c}} + \frac{\bar{M}p'}{\gamma\bar{p}}\right)\right]$$
$$-\frac{\bar{M}^2}{(1-\bar{M}^2}(\bar{u}p' + \bar{\rho}\bar{c}^2u')\frac{s'}{c_p}\frac{dA}{dx} \quad (5.17)$$

The main interest of this derivation lies in the fact that it generalizes the Rayleigh stability criterion. Since it also introduces explicitly the mean and fluctuating heat release $\bar{\omega_T}$ and ω'_T, it also enables a simpler physical interpretation of their results. It appears at first sight that P contains the Rayleigh source term in the form $\frac{\gamma-1}{\bar{\rho}\bar{c}^2}\omega'_T p'$. But P also contains many other terms that are interesting to consider. Eq.(6.1) can be reordered using the linearized state equation $\frac{\rho'}{\bar{\rho}} = \frac{p'}{\bar{p}} - \frac{T'}{\bar{T}}$:

$$P = \frac{\gamma - 1}{\bar{\rho}\bar{c}^2}\omega'_T(p' + \bar{\rho}\bar{u}u')$$
$$-\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\frac{1}{\rho c^2}\left[(2\gamma-1)\frac{p'^2}{2\rho\bar{c}^2} + (2\gamma - 1 - \gamma M^2)\frac{\bar{u}p'u'}{\bar{c}^2} + 2\frac{\rho u'^2}{2}\right]$$
$$+\bar{M}\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\frac{T'}{\bar{T}}\left(\frac{u'}{\bar{c}} + \frac{\bar{M}p'}{\gamma\bar{p}}\right)$$
$$-\frac{\bar{M}^2}{(1-\bar{M}^2}(\bar{u}p' + \bar{\rho}\bar{c}^2u')\frac{s'}{c_p}\frac{dA}{dx} \quad (5.18)$$

Table 5.2 gives an attempt of physical interpretation for each term composing P. Compared to the classical acoustic energy production term, new terms appear in this derivation that come from the interaction between the fluctuations of velocity and the heat release fluctuation, the interaction between the acoustic energy itself and the mean heat release, the interactions between the fluctuations of velocity and pressure and the fluctuations of temperature coupled with the mean heat release and finally the coupling between entropy fluctuations and the axial gradient of section area.

This study reveals that many physical phenomena can be responsible for the fluctuation of the acoustic energy. Yet, the main drawback of this study is the use of Morfey extended acoustic energy to a reacting case. In contrary to the non-reacting case, the separation between rotational and irrotational components of the velocity field cannot be done. The combustion phenomenon creates disturbance of velocity that are irrotationnal but not of an acoustic nature. Furthermore, this analysis states that the disturbances of entropy should be seen as a production term for the acoustic. This production term is not developed here and its interpretation may be somewhat hard to find.

5.1.3 Development of the ”disturbance energy” concept

Because of these conceptual issues, an other path has been followed by other authors, following the idea that the acoustic energy is not the only disturbance energy that can be linked to fluctuations in the flow,

	Mathematical form	Interpretation
P1	$\frac{\gamma-1}{\gamma\bar{p}}\omega'_T p'$	Classical Rayleigh source term.
P2	$(\gamma-1)\bar{M}\omega'_T\frac{u'}{\bar{c}}$	Interaction between heat release fluctuations and velocity fluctuations. Note that this term vanishes if the mean mach number $\bar{M}$ is zero.
P3	$-\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\frac{1}{\rho c^2}\left[(2\gamma-1)\frac{p'^2}{2\rho\bar{c}^2}+(2\gamma-1-\gamma M^2)\frac{\bar{u}p'u'}{\bar{c}^2}+2\frac{\rho u'^2}{2}\right]$	Interaction between the different component of the instantaneous extended acoustic energy and the mean heat release field. the ratio for the three components are : $2\gamma-1$ for $\frac{p'^2}{2\rho\bar{c}^2}$, $(2\gamma-1-\gamma M^2)$ for $\frac{\bar{u}p'u'}{\bar{c}^2}$ and 2 for $\frac{\rho u'^2}{2}$. Note that this term vanishes if the mean Mach number is zero because no mean heat release field can be considered.
P4	$+\bar{M}\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\frac{T'}{\bar{T}}\left(\frac{u'}{\bar{c}}+\frac{\bar{M}p'}{\gamma\bar{p}}\right)$	Interactions between fluctuations of temperature and fluctuations of velocity and pressure. Note that this term vanishes if the mean Mach number is zero.
P5	$-\frac{\bar{M}^2}{(1-\bar{M}^2)}(\bar{u}p'+\bar{\rho}\bar{c}^2u')\frac{s'}{c_p}\frac{dA}{dx}$	Interactions between fluctuations of entropy and the axial gradient of cross section. This term is somewhat particular to the 1-D context. Note that this term vanishes if the mean Mach number is zero.

Table 5.2 - *Physical interpretation of the different components of the production term P*

especially in a reacting case. They consider the energy linked with entropy fluctuations as being part of an extended ”disturbance energy” involving acoustic, turbulence and entropy fluctuating energies. In this section, we will focus on the concept of disturbance energy developed by Chu [23] and by Myers [88].

Chu starts from the Navier-stockes equations for any newtonian fluid in movement. Adding a conservation equation for mass and internal energy, he states that whatever a disturbance in that fluid should be, it must be a disturbance of one of the following six variables : $p', \rho', T', u_i' (i = 1, 3)$. The aim of his study is to construct a definite positive quantity representing the energy contained in the disturbances in the flow. Linearizing the five corresponding equations around the mean steady flow, he finally describes that the evolution of all disturbances in a fluid can be represented by the following equations.

$$\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla}.\vec{u'} = 0 \tag{5.19}$$

$$\rho_0 \frac{\partial \vec{u'}}{\partial t} = -\vec{\nabla} p' + \vec{\nabla}(\tau') \tag{5.20}$$

$$\rho_0 c_v \frac{\partial T'}{\partial t} + p_0 \vec{\nabla}.\vec{u'} = \vec{\nabla}.(\lambda \vec{\nabla} T') \tag{5.21}$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0} \tag{5.22}$$

Using the set of equations 5.19-5.22, he constructs a conservation equation for a ”disturbance energy” (E) defined as :

$$E = \frac{1}{2}\rho_0 \vec{u'}.\vec{u'} + \frac{1}{2}\rho_0 c_0^2 \left(\frac{p'}{\gamma p_0}\right)^2 + \frac{1}{2}\frac{\gamma - 1}{\gamma} p_0 \left(\frac{s'}{r}\right)^2 \tag{5.23}$$

where the fluctuation of entropy is part of the disturbance energy. Chu shows that this quantity is a definite positive quantity.

The fact that the fluctuations of entropy should appear in the definition of the disturbance energy is made clear by taking a simple example. If the fluid is originally at rest with no gradient of pressure but with a gradient of entropy, this gradient through heat diffusion will introduce a gradient of mass density that will introduce a movement of the fluid. We see that in this particular case, the energy contained in the $\frac{1}{2}\frac{\gamma-1}{\gamma} p_0 \left(\frac{s'}{r}\right)^2$ has been transfered to the $\frac{1}{2}\rho_0 \vec{u'}.\vec{u'}$ term. This example shows a conversion of potential entropy disturbance energy into kinematic disturbance energy.

For Chu, the disturbance energy should be ”a positive definite quantity which in the absence of heat transfer at the boundaries and of work done by them must be a monotone non-increasing function of time.” The quantity E defined by equation 5.23 has the properties required by this definition. The main hypothesis of his analysis are

- the mean flow is an uniform medium at rest : $\vec{u}_0 = 0$
- all species have the same thermodynamic properties

- thermodynamic quantities are constant over space and time
- mean variables are constant over space (for example : $\vec{\nabla} T_0 = \vec{0}$)
- the perturbations are small compared to the mean flow variables.

Chu shows that under the limitations of the previous hypothesis, the conservation equation for E can be written as follows

$$\frac{\partial E}{\partial t} = \frac{T'}{T_0}\omega'_T - \frac{\lambda_0}{T_0}(\nabla T')^2 - \Phi' \tag{5.24}$$

where Φ' is the fluctuation of the dissipation function defined as

$$\Phi = \frac{1}{2}\tau_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Equation 5.24 states that the source term for the disturbance energy is linked with the heat release fluctuation in a different way as for the Rayleigh criterion. This term is proportional to the correlation between temperature and heat release fluctuations. A criterion can be derived from this equation. In a reacting case, if T_n is the period of oscillation of the fluctuation of the disturbance energy, the energy contained in disturbances will grow only if :

$$\frac{1}{T_0}\int_{T_n}\int_V T'\omega'_T dvdt > \int_{T_n}\int_V [\frac{\lambda_0}{T_0}(\nabla T')^2 + \Phi']dvdt \tag{5.25}$$

The main output of this study is that if a disturbance (defined as depending of velocity, pressure and entropy fluctuations) grows in a combustor, the source term for the energy transported by it is not the classical Rayleigh term which only account for the source term of mecanical disturbance energy (i.e the parts depending of u' and p') but is the correlation between local fluctuations of temperature and heat release.

Even if this analysis is of first interest because it involves the entropy disturbance energy in a more general type of disturbance energy, the definition of the disturbance energy still relies on the linearization hypothesis and on the hypothesis that the mean Mach number is zero.

Exact non linear disturbance energy equation in non-reacting flows

Myers [88] addresses the issue of the linearization process. He defines a non-linear conservation equation for a disturbance energy in flow. This analysis is valid for non-reacting cases and the following paragraphs focus on the derivation of this conservation equation.

In contrary to what has been considered so far, Myers does not linearizes quantities using the decomposition in mean part and fluctuating part (small enough to linearize the equations) but considers that any quantity can be written as follows[3]:

$$q(x,t) = \bar{q}(x) + \sum_{n=1}^{\inf} \delta^n q_n(x,t) \tag{5.26}$$

The goal of this analysis is to give a nonlinear conservation equation for a quantity which account for the energy of disturbances in any non-reacting flow. The conditions that this equation must meet are that the zeroth order and first order of the above decomposition of the disturbance energy must be zero. If linearized to second order, this equation equation should only let appear first order terms for the primitive variables. These properties are used to avoid any conceptual problem having to consider two different orders for the fluctuations of the same quantity. After quite heavy algebra that may be found in [88], Myers derives the following exact nonlinear conservation equation for the energy of the disturbances in a non-reacting flow:

$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{W} = -D \tag{5.27}$$

where

$$\begin{aligned}
E &= \rho[H - \overline{H} - \overline{T}(s - \overline{s})] - \overline{\vec{m}}.(\vec{u} - \overline{\vec{u}}) - (p - \overline{p}) && (5.28)\\
\vec{W} &= (\vec{m} - \overline{\vec{m}}).[H - \overline{H} - \overline{T}(s - \overline{s})] + \overline{\vec{m}}(T - \overline{T})(s - \overline{s})\\
&\quad -(\vec{m} - \overline{\vec{m}}).\left(\frac{\tau}{\rho} - \frac{\overline{\tau}}{\overline{\rho}}\right) + (T - \overline{T})\left(\frac{\vec{q}}{T} - \frac{\overline{\vec{q}}}{\overline{\overline{T}}}\right) && (5.29)\\
D &= (\vec{m} - \overline{\vec{m}}).[\vec{\zeta} \times \vec{u} - \overline{\vec{\zeta}} \times \overline{\vec{u}} + (s - \overline{s})\vec{\nabla}\overline{T}] - (s - \overline{s})\overline{\vec{m}}.\vec{\nabla}(T - \overline{T})\\
&\quad + \left(\frac{\tau}{\rho} - \frac{\overline{\tau}}{\overline{\rho}}\right).\nabla(m - \overline{m}) - (\vec{m} - \overline{\vec{m}})\left(\frac{\tau}{\rho^2}.\vec{\nabla}\rho - \frac{\overline{\tau}}{\overline{\rho}^2}.\vec{\nabla}\overline{\rho}\right)\\
&\quad (T - \overline{T})\left(\frac{\Phi}{T} - \frac{\overline{\Phi}}{\overline{\overline{T}}}\right) - \left(\frac{\vec{q}}{T} - \frac{\overline{\vec{q}}}{\overline{\overline{T}}}\right).\vec{\nabla}(T - \overline{T})\\
&\quad +(T - \overline{T})\left(\frac{\vec{q}.\vec{\nabla}T}{T^2} - \frac{\overline{\vec{q}}.\vec{\nabla}\overline{T}}{\overline{T}^2}\right) && (5.30)
\end{aligned}$$

A simple treatment can be applied to equation 5.27 to enable the comparison with previous studies. One can apply the decomposition given by equation 5.26 to it. In this case, one finds the following equation for the second order of the disturbance energy decomposition[4] :

$$\frac{\partial E_2}{\partial t} + \vec{\nabla}.\vec{W}_2 = -D_2 \tag{5.31}$$

[3] q' verifies : $\sum_{n=1}^{\inf} \delta^n q_n = q'$

[4] It can be shown that E_0, E_1, $\vec{W}_0$, $\vec{W}_1$, D_0, and D_1 all vanish with this decomposition

where

$$E_2 = \frac{p_1^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u_1^2 + \rho_1 \vec{u}_0.\vec{u}_1 + \frac{\rho_0 T_0 s_1^2}{2C_{p0}} \quad (5.32)$$

$$\vec{W}_2 = \vec{m}_1.\left(\frac{p_1}{\rho_0} + \vec{u}_0.\vec{u}_1\right) - \vec{m}_1\left(\frac{\tau}{\rho}\right)_1 + \vec{m}_0 T_1 s_1 + T_1\left(\frac{\vec{q}}{T}\right)_1 \quad (5.33)$$

$$\begin{aligned} D_2 = & -\rho_0 \vec{u}_0 \vec{\zeta}_1 \times \vec{u}_1 - \rho_1 \vec{u}_1 \vec{\zeta}_0 \times \vec{u}_0 + \vec{m}_1 s_1 \vec{\nabla} T_0 - s_1 \vec{m}_0.\vec{\nabla} T_1 \\ & + \left(\frac{\tau}{\rho}\right)_1 .\nabla m_1 - \vec{m}_1\left(\frac{\tau}{\rho^2}.\vec{\nabla}\rho\right)_1 \\ & - T_1\left(\frac{\Phi}{T}\right)_1 - \left(\frac{\vec{q}}{T}\right)_1 .\vec{\nabla} T_1 + T_1\left(\frac{\vec{q}.\vec{\nabla} T}{T^2}\right)_1 \end{aligned} \quad (5.34)$$

If furthermore one considers that the first order of fluctuation is the only one which is not zero for all primitive variables (i.e $q_1 = q'$ for all variables), that the reference flow is at rest (i.e $u_0 = 0$ and $\tau_0 = 0$), that the gas is thermodynamically perfect (i.e $c_p = c_v$ =constant), one finally can write the following equation for the disturbance energy in a non-reacting flow :

$$\frac{\partial E_2}{\partial t} + \vec{\nabla}.\vec{W}_2 = -D_2 \quad (5.35)$$

where

$$E_2 = \frac{p'^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u'^2 + \frac{1}{2}\frac{\gamma-1}{\gamma} p_0 \left(\frac{s'}{r}\right)^2 \quad (5.36)$$

$$\vec{W}_2 = \vec{u'} p' - \vec{u'}\tau' - \frac{\lambda_0}{2T_0}\vec{\nabla}(T'^2) + \frac{\lambda_0 T'^2}{T_0^2}\vec{\nabla} T_0 \quad (5.37)$$

$$\begin{aligned} D_2 = & \rho_0 s' \vec{u'} \vec{\nabla} T_0 + \underbrace{\tau'.\nabla u'}_{\Phi'} + \frac{\lambda_0}{T_0}(\vec{\nabla} T')^2 + \lambda_0 \frac{T'^2 (\nabla T_0)^2}{T_0^3} \\ & - \underbrace{T'\left(\frac{\Phi'}{T_0}\right)}_{=0 \ to \ second \ order} \end{aligned} \quad (5.38)$$

Equations 5.35-5.38 can be compared to equation 5.24 given by Chu for the temporal evolution of the disturbance energy in a reacting flow because they refer to the same quantity E. The first thing to notice (apart from the fact that Chu has a source term related to heat release fluctuation because he considers a reacting case) is that the equation 5.24 is not a transport equation because it contains no transport term for the energy E. The transport term 5.37 has four terms. Actually, the first three are also found by Chu [5], but only when the mean flow is not zero. The first term is the classical acoustic flux, the second one is usually small and corresponds to the transport of disturbance energy thanks to the viscous fluxes. The third one is the thermal flux of disturbance energy and the last one is the transport of disturbance energy by the mean gradient of temperature.

[5] The last one is not present in the analysis from Chu because he considers that the mean gradient of temperature is zero

Concerning the source terms, both find two equivalent terms : dissipation of the disturbance energy thanks to viscous and thermal effects. In Myers analysis, two more terms appear. Equation 5.38 shows a dissipation term linked with the mean gradient of temperature and a term coming from the correlation of the velocity and entropy fluctuations in presence of a gradient of mean temperature. Note that if the mean gradient of temperature is neglected, one finds the equations given by Pierce for the transport of disturbance energy [100].

Myers[88] allows entropy disturbances to remain in both the energy density and flux terms. Myers' equation is consistent with those developed earlier by Chu[23] and Pierce[100] for zero mean flow. However, because the energy density and flux terms contain entropy disturbances, the resulting energies are not 'acoustic' and are properly called 'disturbance energies' [23, 88].

Remarks on the definition of disturbance energies and stability criteria.

In combusting flows, any equation stating disturbance energy conservation must start from equations of motion that at least include non-zero mean flow quantities and entropy variation. To ignore either the mean flow or entropy variation causes conceptual problems. It appears that only the energies defined by Morfey[85] and Myers[88] do this. Viscous dissipation and heat conduction, whilst included in these works, are not essential and are usually small.

Nicoud and Poinsot [93] rederive the fluctuating energy equation of Chu[23] and argues that the Rayleigh criterion is an incomplete description of the significant sources of 'disturbance' energy in combustion. In the limit of small disturbance amplitude, a source term proportional to $T_1\omega_{T1}$ is found where ω_{T1} and T_1 are the first terms in the heat release and static temperature asymptotic expansions. This term is analogous but significantly different to the 'Rayleigh term' in equation 5.1. Entropy disturbances through the flame are also argued to be a significant source of disturbance energy. Bloxsidge[12] and Dowling[38] also show that terms other than the Rayleigh term existed for their differently defined acoustic energy equation, but both argue that these terms are small in practice.

This difference of opinion on such a fundamental and practically important problem needs resolving. This can be achieved by first deriving a general equation for disturbance energy, as done in section 6.3 and then study numerically the magnitudes of all identified source terms. The basic equation should not be linear as it is often the case when dealing with acoustics. Indeed the temperature, entropy and velocity disturbances in particular can be large within flames and nonlinear effects are already known to be significant in the acoustic energy analysis of solid rocket combustion [45, 35].

The work of Myers[88] also justifies the inclusion of entropy disturbances in the energy density and flux terms. Large entropy or pressure disturbances can be accompanied by convectively and sonically travelling disturbances in the pressure and entropy fields respectively [24]. Thus, a useful disturbance energy should incorporate pressure, velocity and entropy disturbances for nonlinear studies.

In order to know what are the main source terms of this energy, one needs to investigate the balance closure of conservation equations for the different disturbance energies that can be defined in reacting flows[6]. Here LES is chosen to reach this goal. This choice comes from the following remark that resolved structures should account for the major part of the disturbance energy in the flow.

An advanced post-processing tool is required to get all fields needed to close the budget of such an energy. Thanks to this kind of tool, one would expect to gain insight into the understanding of this issue. Section 5.2 presents in more details the implementation of this tool in the LES code ”AVBP” which is used at CERFACS and section 5.3 presents simple validation test cases for this tool.

[6]Note that this energy can be ”purely acoustic”, ”acoustic and turbulent”, or also contain ”entropy disturbances”

5.2 An advanced post-processing tool for LES : POSTTIT

5.2.1 Motivations

Two main motivations support the development of an advanced post-processing tool for LES:

- **The balance closure of conservation equation in LES codes.**
- **The visualization of complex variables not directly available in the code.**

Balance of conservation equations in AVBP LES code

It is important in LES to know the conservation of equations that are used to advance the different variables in the code. An example of such equation is the total energy equation that is advanced by the LES code ”AVBP” :

$$\frac{\partial \rho E}{\partial t} = -\vec{\nabla}.(\rho E\vec{u}) + \dot{\omega_T} - \vec{\nabla}.\vec{q} - \vec{u} : \vec{\nabla}\sigma \tag{5.39}$$

This equation is local and stands for the variation of the total energy in a fluid element. To obtain the balance of the total energy in a given configuration, one has to integrate this equation over the entire volume. It becomes :

$$\int_v \frac{\partial \rho E}{\partial t} dv = -\int_v \vec{\nabla}.(\rho E\vec{u}) dv + \int_v \dot{\omega_T} dv - \int_v \vec{\nabla}.\vec{q} dv - \int_v \vec{u} : \vec{\nabla}\sigma dv \tag{5.40}$$

$$\int_v \frac{\partial \rho E}{\partial t} dv = \int_s \rho E\vec{u}.\vec{n} ds + \int_v \dot{\omega_T} dv - \int_s \vec{q}.\vec{n} ds - \int_v \vec{u} : \vec{\nabla}\sigma dv \tag{5.41}$$

Equations 5.40 and 5.41 are equivalent. Whatever is the method chosen to obtain the total variation of the total energy inside the configuration, one either has to be able to compute the operator $\vec{\nabla}.\vec{f}$ and/or to be able to get the fluxes crossing the boundaries.

Usual post-processing visualization tools already exist that are able to do such operations. But there is no certitude that the algorithm for these operators are the same in the LES code and in the post-processing tool. Moreover, numerical terms are often added during the advancement of equations to ensure the stability of the code. These terms should also appear in the balance of conservation equations since they are global sink (or source) numerical terms for the variables of interest.

In order to better understand the motivations for an advanced post-processing tool in LES, it is now of interest to give the conservation equations that are advanced in the LES code AVBP, to describe the advancement of these equations and have a few comments about the numerical terms that influence them. The conservation equations in the LES code AVBP have been described on page 35 but are recalled here for the reader comprehension. The transport equations to advance species, velocity and total energy are:

$$\frac{\partial \rho Y_k}{\partial t} + \vec{\nabla}.(\rho(\vec{u} + \vec{V^c})Y_k) = \vec{\nabla}.(\rho D_k \vec{\nabla} Y_k) + \dot{\omega_k} \tag{5.42}$$

$$\frac{\rho \partial \vec{u}}{\partial t} + \rho(\vec{u} \times \vec{\nabla})\vec{u} = -\vec{\nabla} p + \vec{\nabla} \tau \tag{5.43}$$

$$\rho \frac{\partial E}{\partial t} + \rho \vec{u}.\vec{\nabla} E = \dot{\omega_T} - \vec{\nabla}.\vec{q} - \vec{u} : \vec{\nabla} \sigma \tag{5.44}$$

Adding all species equations, the following set of local equations appears which represent the quantities of interest for the conservativity of the LES code AVBP:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}.(\rho \vec{u}) \tag{5.45}$$

$$\frac{\partial \rho \vec{u}_j}{\partial t} = -\vec{\nabla}.(\rho \vec{u}_j \vec{u}) - \vec{\nabla}(p)_j + \vec{\nabla}(\tau)_j \tag{5.46}$$

$$\frac{\partial \rho E}{\partial t} = -\vec{\nabla}.(\rho E \vec{u}) + \dot{\omega_T} - \vec{\nabla}.\vec{q} - \vec{u} : \vec{\nabla} \sigma \tag{5.47}$$

These equations stand for the conservation of mass,momentum and total energy.
In the code, all variables are updated explicitely from one time step to the other. The way this update is done only relyies on the number of Runge-Kutta time steps chosen for each iteration.
Both TTGC spatial scheme [29] which is third order and Lax-Wendroff spatial scheme which is second order are available in the AVBP code. In this work, only the Lax-Wendroff scheme has been used because of its simplicity in terms of advancement which directly impacts on the post-processing tool. In this case, the time scheme is a simple corrected one-step Runge-Kutta scheme. Doing so, the update of E from iteration n to iteration $n+1$ can be split into three stages :

- First, all local physical terms of the equation are computed which gives the predicted value for the variable at iteration $n+1$

  $$E(n+1)_{pre1} = E(n) + A(n) \times dt(n) \tag{5.48}$$

  where $A(n)$ is the update function at iteration n. To be consistent with all equations presented previously, A is therefore the sum of all local source terms in Eqs.(5.45)-(5.47).

- Then, the correction terms due to the artificial viscosity and Lax-Wendroff scheme are applied.

  $$E(n+1)_{pre2} = E(n+1)_{pre1} + \mathbf{LW}(n) \times dt(n) + \mathbf{AVI}(n) \times dt(n) \tag{5.49}$$

  where $LW(n)$ and $AVI(n)$ are respectively the contributions of the scheme and of the artificial viscosity to the time advancement of variable E. These contributions are different in nature.

  – The artificial viscosity correction is a numerical term which is applied to ensure that the dispersion and dissipation properties of the scheme do not impact on the stability of the computation (see Section 1.5).

- The contribution of the Lax-Wendroff scheme is quite different. The goal of this term is to ensure that the space derivative of variable E is second order in space. This term therefore has some importance when sharp gradients must be resolved as it is the case in flames (see Section 1.6).

- Finally, boundary corrections are applied to E_{pre2} field on the nodes at the boundaries. This term should not be seen as a numerical correction since the prescription of boundaries is part of the mathematical problem involving the whole geometry. Therefore, the separation of this term from the rest of the residual $A(n)$ is more a numerical artifact that comes from the way the time advancement is done in the code. Yet, in the present case, this term adds to the previous ones to give the final value of E for all nodes at iteration $n+1$:

$$E(n+1) = E(n+1)_{pre2} + \mathbf{BND}(n) \times dt(n) \tag{5.50}$$

Summarizing these steps, in AVBP, when using the Lax-Wendroff scheme, the local advancement of a variable E from iteration n to $n+1$ is given by the following equation :

$$\frac{E(n+1) - E(n)}{dt(n)} = A(n) + \mathbf{LW}(n) + \mathbf{AVI}(n) + \mathbf{BND}(n) \tag{5.51}$$

Note that $\frac{E(n+1)-E(n)}{dt(n)}$ also represents a discrete time derivative of variable E at iteration n.

To evaluate the conservation of the advanced equations in the code, Eq.(5.51) must be integrated over the whole volume of the mesh. Taking the volume integral of Eq.(5.51) gives:

$$\int_V \left(\frac{E(n+1) - E(n)}{dt(n)} \right) dv = \int_V \left(A(n) + \mathbf{LW}(n) + \mathbf{AVI}(n) + \mathbf{BND}(n) \right) dv \tag{5.52}$$

Applying this general scheme to Eqs.(5.45)-(5.47) gives the integral equations using results at iterations n and $n+1$ that should conserve in the code :

$$\int_V \left(\frac{\rho(n+1)-\rho(n)}{dt(n)}\right) dv = \int_V \left(\vec{\nabla}.(\rho\vec{u}(n))\right) dv + \int_V \left(\mathbf{LW}_\rho(n) + \mathbf{AVI}_\rho(n) + \mathbf{BND}_\rho(n)\right) dv \quad (5.53)$$

$$\int_V \left(\frac{\rho\vec{u}(n+1)-\rho\vec{u}(n)}{dt(n)}\right) dv = \int_V \left(\rho(\vec{u}\times\vec{\nabla})\vec{u}(n)\right) dv + \int_V \left(\vec{\nabla}\tau(n) - \vec{\nabla}p(n)\right) dv + \int_V \left(\mathbf{LW}_{\rho\mathbf{u}}(n) + \mathbf{AVI}_{\rho\mathbf{u}}(n) + \mathbf{BND}_{\rho\mathbf{u}}(n)\right) dv \quad (5.54)$$

$$\int_V \left(\frac{\rho E(n+1)-\rho E(n)}{dt(n)}\right) dv = \int_V \left(\vec{\nabla}.(\rho H\vec{u}(n) + \vec{q}(n))\right) dv + \int_V \left(\dot{\omega_T}(n) - \vec{u} : \vec{\nabla}(\tau)(n)\right) dv + \int_V \left(\mathbf{LW}_{\rho\mathbf{E}}(n) + \mathbf{AVI}_{\rho\mathbf{E}}(n) + \mathbf{BND}_{\rho\mathbf{E}}(n)\right) dv \quad (5.55)$$

For these equations, the time derivative term is calculated as $\int_V \frac{E(n+1)-E(n)}{dt(n)} dv$ and therefore only requires to know the values of the variables at the beginning and at the end of the iteration.

Remarks on correction terms

- **Boundary correction**

  Boundary conditions are imposed in AVBP via characteristic and non-characteristic conditions at the nodes.
  During one iteration of AVBP, the predicted values for the fields on boundaries are given by the numerical scheme. Corrected values correspond to the new fields on boundaries taking into account the characteristic waves entering the domain :

$$E(n+1) = E(n+1)_{pre2} + \sum_i L_i \times dt(n) \quad (5.56)$$

  where $\sum_i L_i$ are all characteristic waves crossing the boundaries. They impose weakly the target values at the patch nodes.
  Of course, $E(n+1)_{pre2}$ and $E(n+1)$ are different because computing Euler/Navier-Stockes equations for a given flow is not enough to account for the target values at the boundaries. The equivalent contribution of this correction on the advancement of a characteristic quantity is : $\int_V \mathbf{BND}(n) \times dt(n)dv$.
  As this term only applies on the boundary nodes, its global importance greatly depends on the total volume of these nodes compared to the inner volume. In practice, the importance of this term

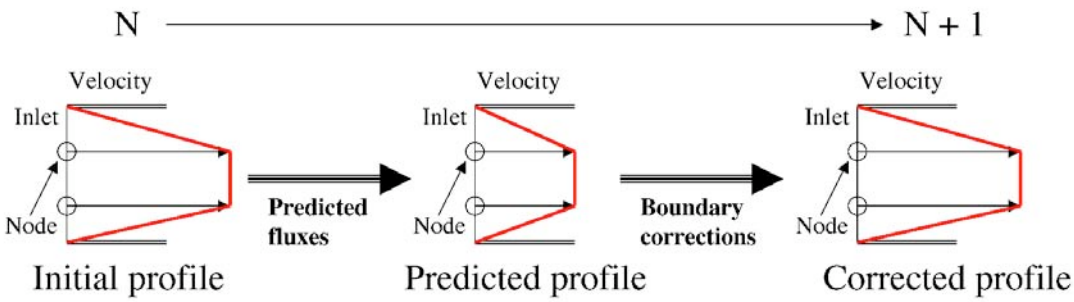


Figure 5.1 - *Schematic representation of the influence of boundary conditions*

therefore increases in case of poor discretization of the boundaries. Fig.(5.1) shows how boundary conditions apply to impose the velocity profile on an inlet.

- **Lax-Wendroff correction**

  The evaluation of the R-H-S terms of the conservation equations is done using a central difference spatial scheme. But, as described on page 60, the Lax-Wendroff spatial scheme requires the addition of a correction to the results given by this scheme. Doing so, it adds a global numerical correction denoted **LW** to the global balance of conservation equations.

- **Artificial viscosity correction**

  The artificial viscosity correction, denoted here **AVI**, is used to dissipate numerical spatial high frequency waves. This term should remain small as long as the mesh discretization is good enough to resolve the gradients in the flow. In practice, attention should be paid to this term in reacting configurations, especially when no thickening flame model is used as it is the case in this part.

Visualization of complex variables

Post-processing and analyze techniques of reacting/turbulent/two-phase flows now require the use of complex treatments of the output fields of the simulation. Many examples can be found in the literature of complex criteria useful for the analyzis of such flows.

- The Q criterion of Hunt et al.[58] and further developed by Jeong and Hussain [60] which detects the presence of coherent structures in the flow is one of them. It requires the use of the gradient operator applied to the velocity field.

  For a three-dimensional smooth velocity field $v(x,t)$, available Galilean invariant vortex criteria uses the velocity gradient decomposition

$$\vec{\nabla}.\vec{u} = S + \Omega \tag{5.57}$$

where $S = \frac{1}{2}\left[\vec{\nabla}.\vec{u} + \left(\vec{\nabla}.\vec{u}\right)^T\right]$ is the rate-of-strain tensor, and $\Omega = \frac{1}{2}\left[\vec{\nabla}.\vec{u} - \left(\vec{\nabla}.\vec{u}\right)^T\right]$ is the vorticity tensor. In historical order, the first three-dimensional vortex criterion using 5.57 is the Q criterion of Hunt et al.[58] which defines a vortex as a spatial region where

$$Q = \frac{1}{2}\left[|\Omega|^2 - |S|^2\right] > 0 \tag{5.58}$$

- The criterion $M(\vec{x}, t)$ defined by Pope [115] is used to assess the quality of an LES simulation.

  The evaluation of $M(\vec{x}, t)$ requires the determination (locally in space and time) of the turbulent kinetic energy of the resolved motions $K(\vec{x}, t) \equiv \frac{1}{2}(W - \langle W\rangle).(W - \langle W\rangle)$, and that of the residual motions (that can be evaluated using the formula given by Sagaut [123])

$$kr(\vec{x}, t) = \left(\frac{\nu_{sgs}}{C_m\Delta}\right)^2.$$

  where ν_{sgs} is the sub-grid scale turbulent viscosity, Δ a characteristic mesh length and C_m a constant provided by DNS of homogeneous isotropic turbulence. This constant only depends on the shape of the spectrum of the kinetic turbulent energy.
  This criterion is defined as :

$$M(\vec{x}, t) = \frac{k_r(\vec{x}, t)}{K(\vec{x}, t) + k_r(\vec{x}, t)} \tag{5.59}$$

  Thus the value of M is between 0 and 1: $M = 0$ corresponds to DNS and $M = 1$ to RANS. Smaller values of M correspond to the resolution of more of the turbulent motions. (Although this definition of M is conceptually simple, in LES, the approximation of means, denoted here by angled brackets$\langle\rangle$, is non-trivial.)

As the complexity of analysis in flows will increase, the need for tools which can give access to such complex variables will increase in order to avoid discrepancies between the LES code operators and the post-processing tool operators.

5.2.2 Objectives

Regarding to the previously presented motivations, three main objectives are prescribed to the POSTTIT tool.

- **Access to all variables :** The user should be able to visualize any variable that can be computed from the inner variables of the LES code. This includes :
  - **Numerical variables :** The user should be able to have access to all numerical or boundary corrections added to the variables advanced by the LES code (boundary corrections, wall law corrections, numerical viscosity, numerical time derivative corrections, etc...).
  - **Physical variables :** The user should be able to compute any complex physical variable that can derive from the resolved variables. An example of a physical quantity that is usually not directly available from the LES code is the classical local Rayleigh source term (i.e. $\frac{\gamma-1}{\gamma}p'\dot{\omega}'$). If the pressure, heat release, calorific ratio are computed in the LES code, by first reading a mean field for the case of interest, the user must be able to construct and store this term in a solution file.
- **Volume integrals and fluxes :** The second objective is directly linked with the goal of closing budgets in LES. After having calculated all terms one needs for closing a conservation equation balance, the user must be able to integrate the volume terms over the whole volume of the configuration and evaluate the flux terms when needed.
- **Exact spatial operators :** An other objective, common to the two previous ones is that the operators used to compute variables must be the exact ones used in the LES code. This should be done to ensure consistency. Gradient ($\vec{\nabla}()$) and divergence ($\vec{\nabla}.\vec{()}$) operators should therefore use the routines of the LES code.

5.2.3 Algorithmic organization

Table 5.3 presents the algorithmic organization of the POSTTIT tool and its interactions with the LES code AVBP. On the left is the evolution of an LES iteration from the declaration of variables to the writing of an output solution. In the middle, one can find the specific POSTTIT modules that are filled by the user to investigate the variables that are not directly computed by the LES code. On the right, the inputs and outputs of the tool are described. The **!** sign precedes the names of files that must be filled to run the POSTTIT tool.
To reach the objectives described in section 5.2.2, the specific POSTTIT modules are directly integrated in the LES code during the compilation. All these files have the ".inc" extension. In addition to these files, an ascii file "posttit.choice" has to be filled to prescribe the major features of the tool. The user therefore needs to fill seven files that are specific to the POSTTIT tool :

- **! posttit.choices:** with this file the user specifies if he needs to read a mean solution file to compute his own variables. In this case, the name of the mean solution file is provided. Then the name pattern of the solutions is given. The first iteration number, the last one and the iteration step number are read. The user then states the number of fluxes he wants to calculate. Finally, a flag is given which tells if fields are to be saved in the solution and how many fields are saved.

- **! perso_declare.inc:** It prescribes all needed scalars and vectors for the POSTTIT tool. Included in "commonbl_slave.H" file.
- **! perso_allocate.inc:** It allocates all POSTTIT fields. Included in "slave_array.F" file.
- **! perso_define.inc:** Maybe the most important file of the POSTTIT tool. In this file, the user calculates the different variables he needs by using the already existing variables of the LES code. Operators (grad,div,etc...) can be applied to the LES fields to compute the complex variables of interest. Included in "slave_norm_flux.F" file.
- **perso_intv.inc:** Only used when using POSTTIT in balance equation mode. With this file, the user computes the desired volume integrals of any variables. Included in "slave_norm_flux.F" file.
- **perso_flux.inc:** Only used when using POSTTIT in balance equation mode. With this file, the user computes the desired surface fluxes of any vector. Note that when this file is used, the fluxes of the vector are calculated across all patches and not only across inlet and outlet patches. Included in "slave_norm_flux.F" file.
- **perso_save.inc:** Only used when using POSTTIT in visualization mode. With this file, the user specifies the variables he wants to save as additional fields. Included in "slave_norm_flux.F" file.

There are two kinds of output files in the POSTTIT tool :

- **avbp_post :** This file contains all volume integrals and fluxes of the variables that have been computed by the tool. It is mostly useful when the user is interested in checking a conservation equation balance closure.
- **posttit.sol :** This file contains the fields of the variables that the user has computed with the POSTTIT tool and wants to visualize.

5.2.4 Remarks on the practical use of the POSTTIT tool

The POSTTIT tool uses the snapshots obtained from the LES computation.
Therefore, before running the code, the user should first have a reflection on the main frequencies that will be present in his results. This reflection is made easier if the simulation concerns acoustically unstable or harmonically forced configurations as it is the case here.
The user should consider to record at least 10 snapshots per period for the highest identified frequency. Note that enough periods are often also required in order to have access to the long-term fluctuations.
This may lead to approximately 40 to 100 stored solutions. In his reflection, the user therefore also has to consider the size of the memory that will be needed to store these solutions, especially for 3D turbulent reacting cases.

In this work, 40 to 1600 snapshots have been used, depending on the investigated physical phenomena.

Once the snapshots are recorded, POSTTIT can be applied to each of them. The LES tool AVBP is called by POSTTIT for each solution during one iteration and the outputs are recorded as previously described. The local time derivative for each node is obtained by subtracting the value before the iteration to the one obtained at the end and dividing the result by the local time step (for a one step Runge-Kutta scheme).

Since it uses the AVBP code, POSTTIT benefits of all its characteristics. This post-processing tool is therefore massively parallel and uses the exact gradient and divergence operators that are in the code.

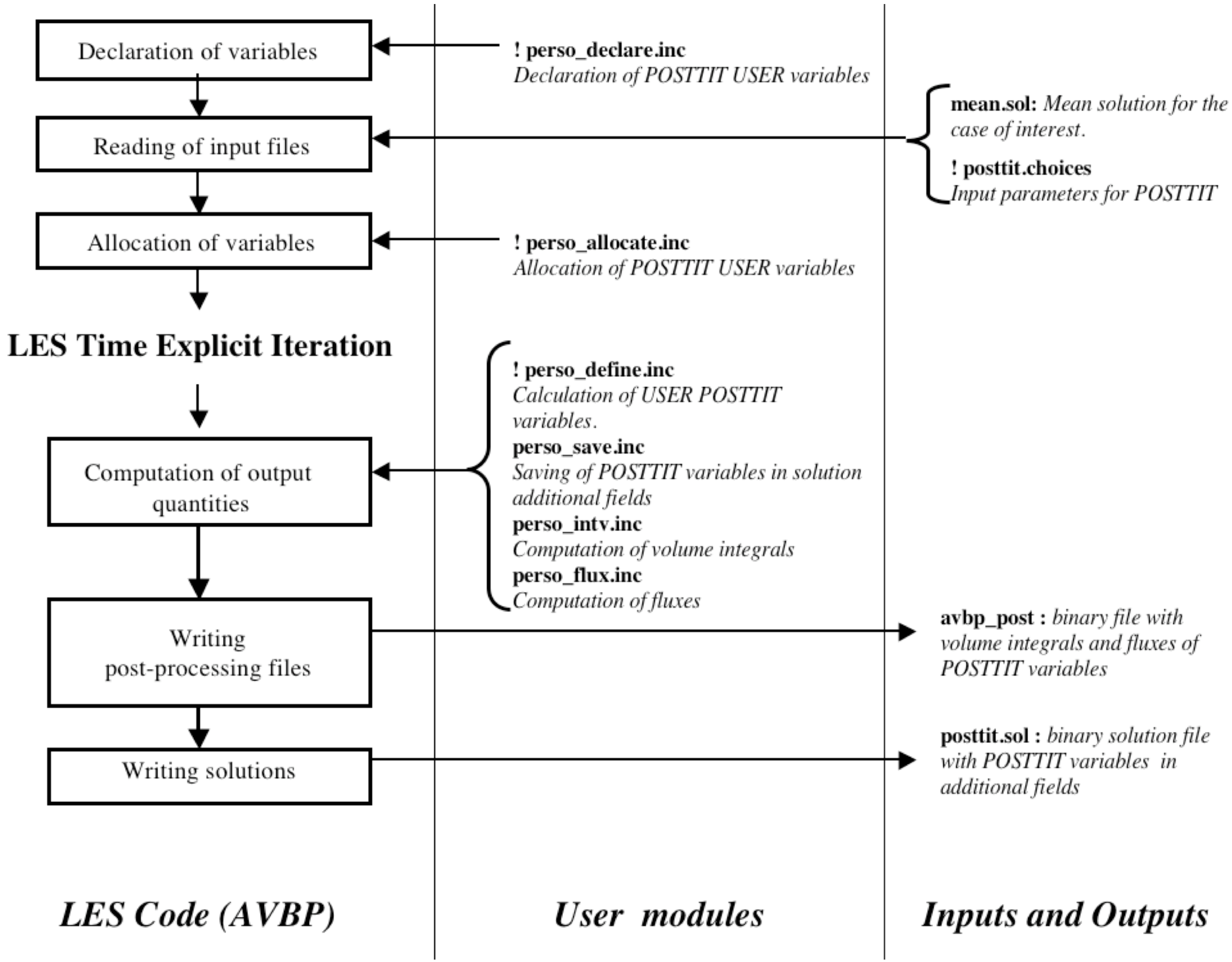


Table 5.3 - *Simplified algorithmic organization of the POSTTIT tool and interactions with the LES code AVBP. The* **!** *sign precedes the names of files that must be filled to run the POSTTIT tool.*

5.3 Examples of conservation equation balance closure

This section presents three applications of the POSTTIT tool in a 2-D unstable reacting case. This tool is used here to test the closure of the equations advanced by the LES code AVBP :

- mass conservation (Eq.(5.53))
- momentum conservation (Eq.(5.54) axial component)
- Total energy conservation (Eq.(5.55))

5.3.1 Description of the case/mesh

The test case studied here is a 2-D configuration. It reproduces the experimental test rig of Le Helley [52]. It contains approximately 26000 triangle elements. This burner corresponds to a laminar flame. The unstructured mesh used for this study takes into account for a large part of the feeding line for acoustic reasons. During the computation, the configuration is made unstable by increasing instantaneously the acoustic reflection coefficient at the outlet. All primitive variables are perturbed and start to oscillate (see Fig.(5.2)).

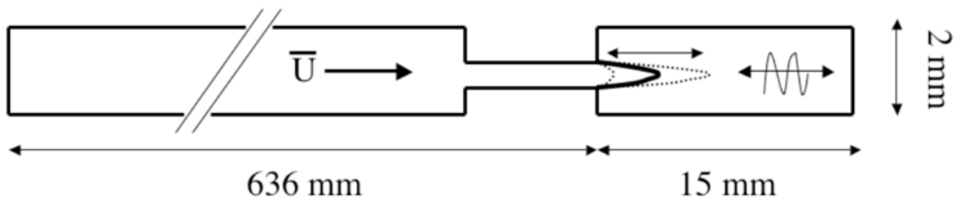


Figure 5.2 - *Schematic representation of the studied case.*

5.3.2 Mass balance closure

The mass balance closure is checked in this configuration. The integral shape of the mass conservation equation is :

$$\frac{\partial m}{\partial t} = \int_S \rho \vec{u}.\vec{n} ds + \int_V (\mathbf{LW}_\rho(n) + \mathbf{AVI}_\rho(n) + \mathbf{BND}_\rho(n))\, dv \tag{5.60}$$

$\frac{\partial m}{\partial t}$ is therefore identified to $\int_V \left(\frac{\rho(n+1)-\rho(n)}{dt(n)}\right) dv$ where m is the total mass enclosed in the configuration.

Eq.(5.60) states that the time derivative of the total mass inside the domain is equal to the sum of the mass fluxes entering and leaving the domain plus numerical and boundary corrections. These corrections are added to the sum of terms shown on Fig.(5.3). Note that in this case they are almost zero. Fig.(5.3) a) shows a perfect agreement between the time derivative of mass and the sum of all mass fluxes. Fig.(5.3) b) shows the inlet mass flux (positive) and the outlet mass flux (negative). The fluctuation of the mass as the instability is growing is mainly due to the fluctuation of the inlet mass flux. The outlet mass flux first reacts to the change of the reflection coefficient by decreasing. It means that during a small amount of time, the equilibrium between entering and leaving mass is broken and some mass leaves the domain.

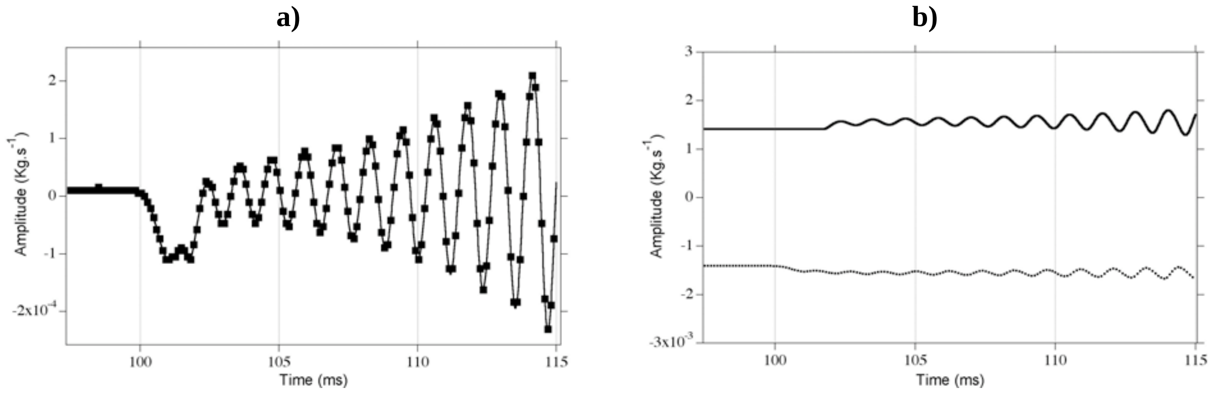


Figure 5.3 - **a)** *Total mass balance closure : ▪ time derivative of total mass —— sum of all terms.* **b)** *Main terms of the balance : —— inlet mass flux ······ outlet mass flux*

5.3.3 Axial momentum balance closure

The axial momentum balance closure is checked in this configuration. The integrated axial momentum conservation equation is :

$$\int_V \frac{\partial \rho u_x}{\partial t} dv = \int_S \rho u_x \vec{u}.\vec{n} ds - \int_V \left(\vec{\nabla}.\vec{x}\right) p dv + \int_V \left(\vec{\nabla}.\vec{\vec{\tau}}\right).\vec{x} dv + \int_V \left(\mathbf{LW}_{\rho \mathbf{u_x}}(n) + \mathbf{AVI}_{\rho \mathbf{u_x}}(n) + \mathbf{BND}_{\rho \mathbf{u_x}}(n)\right) dv \quad (5.61)$$

$\int_V \frac{\partial \rho u_x}{\partial t} dv$ is therefore identified to $\int_V \left(\frac{\rho u_x(n+1) - \rho u_x(n)}{dt(n)}\right) dv$. Numerical and boundary corrections are added to the sum of terms on Fig.(5.4) a).

Fig.(5.4) a) shows a perfect agreement between the time derivative of the axial momentum and the sum of R-H-S terms in Eq.(5.61). Fig.(5.4) b) shows the major terms of the axial momentum balance. The major terms in this case are the axial gradient of pressure and the viscous dissipation due to wall friction. Both terms equilibrate before the excitation leading to a zero variation of the axial momentum. As the reflection coefficient is increased at the outlet, the axial pressure gradient begins to oscillate and the variation of the time derivative term is mainly due to it. The wall friction oscillations are much lower and affect the global balance by making the time derivative of axial momentum oscillate around its the equilibrium position.

Fig.(5.5) shows the inlet and outlet fluxes of axial momentum. They have almost no influence on the total balance because they are three order of magnitude lower than the major terms. Though, it can be noticed that the outlet flux of axial momentum more strongly reacts to the excitation.

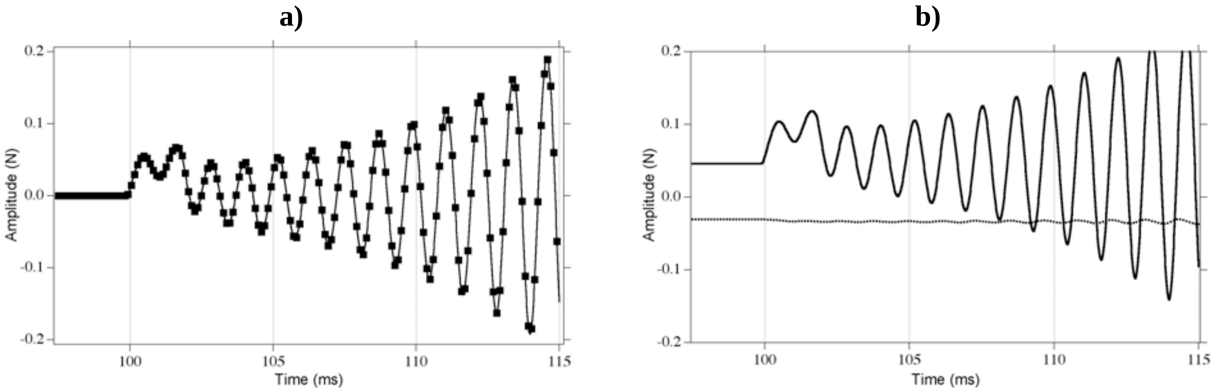


Figure 5.4 - **a)** *axial momentum balance closure : ■ time derivative of axial momentum —— sum of all terms.* **b)** *Main terms of the balance : —— volume integral of the axial gradient of pressure ········ viscous dissipation of axial momentum due to wall friction*

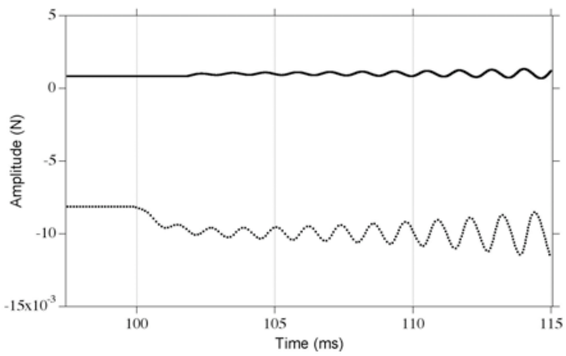


Figure 5.5 - *Main terms of the balance : —— inlet axial momentum flux ········ outlet axial momentum flux*

5.3.4 Total energy balance closure

The total energy balance closure is checked in this configuration. The integrated total energy conservation equation is :

$$\int_V \frac{\partial \rho E}{\partial t} dv = \int_S (\rho \vec{u} E).\vec{n} ds + \int_V \dot{\omega_T} dv + \int_S \vec{q}.\vec{n} ds - \int_V \vec{u} : \vec{\nabla} (\sigma) \, dv \tag{5.62}$$

$\int_V \frac{\partial \rho E}{\partial t} dv$ is therefore identified to $\int_V \left(\frac{\rho E(n+1) - \rho E(n)}{dt(n)} \right) dv$. Numerical and boundary corrections are added to the sum of R-H-S terms of Eq.(5.62) on Fig.(5.6) a).

Fig.(5.6) a) shows a good agreement between the time derivative of the total energy and the sum of all R-H-S terms of Eq.(5.62). Fig.(5.6) b) shows the major terms of the balance. The variation of the time derivative of the total energy is due to two main sources terms :

- the inlet flux of total energy.
- the source heat release term.

and one sink term which is the outlet flux of total energy.
As the instability starts, both heat release source term and outlet flux absolute amplitudes increase and oscillate which results in a global oscillation of the time derivative of the total energy.

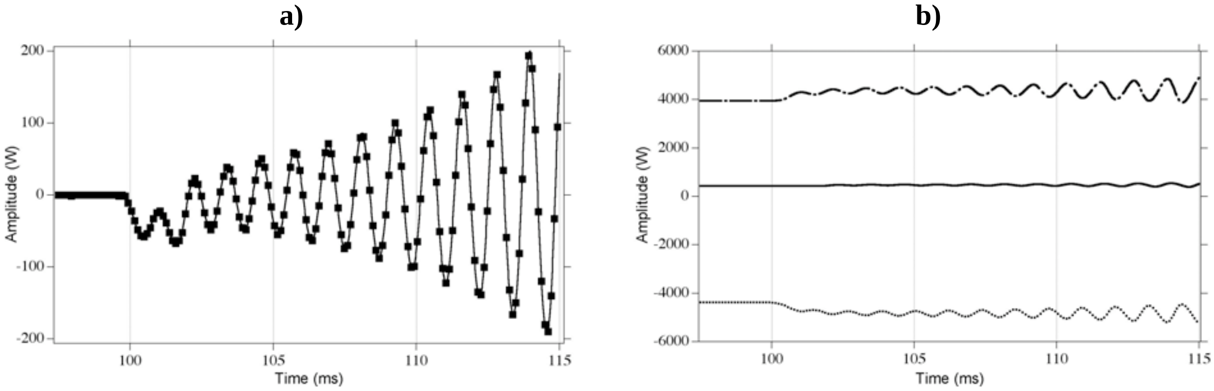


Figure 5.6 - **a)** *Total energy balance closure :* ■ *time derivative of total energy* —— *sum of all terms*
b) *Main terms of the balance :* —— *inlet total energy flux* ········ *outlet energy flux*

Chapter 6

Disturbance energies in flow

This chapter focuses on the derivation of disturbance energy conservation equations. Three conservation equations and disturbance energies are considered:

- In section 6.1 a nonlinear conservation equation for quadratic pressure and velocity fluctuations is proposed. When linearized, this equation shrinks into a linear conservation equation for the disturbance energy defined by $e_1 = \frac{1}{2\rho_0 c_0} p_1^2 + \frac{\rho_0}{2} \vec{u}_1.\vec{u}_1$. For a laminar, isentropic and small mean Mach flow, this disturbance energy is the acoustic disturbance energy. In all other cases, this quantity is composed of fluctuations due to acoustic, turbulence and entropy disturbances[1].
- In section 6.2 a nonlinear conservation equation for quadratic entropy fluctuations is proposed. When linearized, this equation shrinks into a linear conservation equation for the entropy disturbance energy defined by $e_s = \frac{\rho_0 T_0}{2C_{p0}} s_1^2$.
- In section 6.3 a nonlinear conservation equation is proposed for a disturbance energy (E_d) that a priori contains acoustic, turbulence and entropy fluctuations.
- Section 6.4 focuses on the choice of the baseline flow.
- Section 6.5 provides three tables summarizing the equations that are derived here. A comparison with other similar disturbance energies conservation equations is also provided.

6.1 Pressure-Velocity (PV) disturbance energy Eq.(1)$_{\text{[Eq.(6.10)]}}$

6.1.1 Nonlinear conservation equation for pressure fluctuations

For this derivation, when needed, all variables will be written as being the sum of their mean part (i.e. $\overline{\rho}$) and their fluctuating part (i.e. ρ'). All the missing details of the algebra leading to Eq.(6.11) may be

[1] due for example in reacting cases to the dilatation due to the flame.

found in Appendix A.

This derivation uses a conservation equation for pressure (Eq.(6.1)) which writes

$$\frac{\partial p}{\partial t} = -\gamma p \vec{\nabla}.\vec{u} - \vec{u}.\vec{\nabla} p + (\gamma - 1)\left[\dot{\omega_T} - \vec{\nabla}.\left(\vec{q}\right) - \sum_{k=1}^{N} h_{sk}\rho \frac{DY_k}{Dt} + \tau : \vec{\nabla}\left(\vec{u}\right)\right] + \frac{\gamma p}{r}\frac{Dr}{Dt}, \qquad (6.1)$$

and a mean conservation equation (Eq.(6.2)) for this quantity :

$$cons_p - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}} - \vec{\bar{u}}.\vec{\nabla}\bar{p} + (\bar{\gamma} - 1)\left[\overline{\dot{\omega_T}} - \vec{\nabla}.\vec{\bar{q}} - \sum_{k=1}^{N}\overline{h_{sk}}\bar{\rho}\frac{D\overline{Y_k}}{Dt} + \bar{\tau} : \vec{\nabla}\left(\vec{\bar{u}}\right)\right] + \frac{\bar{\gamma}\bar{p}}{\bar{r}}\frac{D\bar{r}}{Dt} = 0. \qquad (6.2)$$

where $cons_p$ contains all mean cross-correlations of fluctuations (ie.$\overline{\gamma' p'}$, $\overline{\gamma' \vec{u}'}$,etc...). Note that the identity $\overline{\frac{\partial p}{\partial t}}$ is assumed. It either means that the final and initial values of the pressure are the same, or that the solutions are statistically stationary.

Substracting Eq.(6.2) to Eq.(6.1), multiplying by p', and dividing by γp gives :

$$\begin{aligned} \frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} = -\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right] - \frac{p'}{\gamma p}\left[\vec{u}.\vec{\nabla}p - \vec{\bar{u}}.\vec{\nabla}\bar{p}\right] \\ +\frac{p'}{\gamma p}\left[(\gamma - 1)\dot{\omega_T} - (\bar{\gamma} - 1)\overline{\dot{\omega_T}}\right] - \frac{p'}{\gamma p}\left[(\gamma - 1)\vec{\nabla}.\vec{q} - (\bar{\gamma} - 1)\vec{\nabla}.\vec{\bar{q}}\right] \\ -\frac{p'}{\gamma p}\left[(\gamma - 1)\sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} - (\bar{\gamma} - 1)\sum_{k=1}^{N}\overline{h_{sk}}\bar{\rho}\frac{D\overline{Y_k}}{Dt}\right] \\ +\frac{p'}{\gamma p}\left[(\gamma - 1)\left(\tau : \vec{\nabla}\left(\vec{u}\right)\right) - (\bar{\gamma} - 1)\left(\bar{\tau} : \vec{\nabla}\left(\vec{\bar{u}}\right)\right)\right] \\ +\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt} - \frac{\bar{\gamma}\bar{p}}{\bar{r}}\frac{D\bar{r}}{Dt}\right] - cons_p\frac{p'}{\gamma p} \end{aligned} \qquad (6.3)$$

Eq.(6.3) is a nonlinear equation for the quadratic fluctuations of pressure. As it will be presented in the next subsection, the linearization of Eq.(6.3) gives a conservation equation for the disturbance energy contained in pressure fluctuations.

Linearization

All further developments assume that all fluctuations are very small compared to their mean value. To avoid any confusion between the fluctuating part and the linearization of the variable to first order, the latter will be underscored 1. The order of the mean flow is therefore 0. Table 6.1 summarizes the transformations used for the linearization process:

Type of Derivation	Variable	Mean Flow	Fluctuation
Non Linear	β	$\overline{\beta}$	β'
Linear	β	β_0	β_1

Table 6.1 - *Notations for mean and fluctuating parts of variable β.*

Linearizing equation 6.3 and retaining only second order terms gives:

$$\begin{aligned} \frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} &= -p_1\vec{\nabla}.\vec{u_1} - \frac{p_1^2}{p_0}\vec{\nabla}.\vec{u_0} - \frac{p_1\gamma_1}{\gamma_0}\vec{\nabla}.\vec{u_0} \\ &-\frac{\vec{u_0}}{2\gamma_0 p_0}.\vec{\nabla}p_1^2 - \frac{p_1}{\gamma_0 p_0}\vec{u_1}.\vec{\nabla}p_0 + \frac{\gamma_0 - 1}{\gamma_0 p_0}p_1\dot{\omega_{T1}} + \frac{\gamma_1 p_1}{\gamma_0 p_0}\dot{\omega_{T0}} \\ &-\frac{\gamma_0 - 1}{\gamma_0 p_0}p_1\vec{\nabla}.\vec{q_1} - \frac{\gamma_1 p_1}{\gamma_0 p_0}\vec{\nabla}.\vec{q_0} \\ -\frac{p_1\gamma_1}{\gamma_0 p_0}\sum_{k=1}^{N}h_{sk0}\rho_0\frac{DY_{0k}}{Dt} &- \frac{p_1(\gamma_0-1)}{\gamma_0 p_0}\sum_{k=1}^{N}\left[h_{sk1}\rho_0\frac{DY_{0k}}{Dt} + h_{sk0}\rho_1\frac{DY_{0k}}{Dt} + h_{sk0}\rho_0\frac{DY_{k1}}{Dt}\right] \\ &+\frac{p_1\gamma_1}{\gamma_0 p_0}\tau_0 : \vec{\nabla}(\vec{u_0}) + \frac{p_1(\gamma_0-1)}{\gamma_0 p_0}\left[\tau_1 : \vec{\nabla}(\vec{u_0}) + \tau_0 : \vec{\nabla}(\vec{u_1})\right] \\ &+\frac{p_1}{r_0}\frac{Dr_1}{Dt} + \frac{Dr_0}{Dt}\left[\frac{\gamma_1 p_1}{\gamma_0 p_0} + \frac{p_1^2}{r_0 p_0} - \frac{p_1 r_1}{r_0^2}\right] - cons_p\frac{p_1}{\gamma_0 p_0} \end{aligned} \tag{6.4}$$

6.1.2 Nonlinear conservation equation for velocity fluctuations

The same reasoning is applied to the velocity to get an exact nonlinear conservation equation for the disturbance energy contained in quadratic velocity fluctuations. The mean conservation equation for velocity is :

$$co\vec{n}s_v - \overline{\rho}\left(\overline{\vec{u}}\times\vec{\nabla}\right)\overline{\vec{u}} - \vec{\nabla}\overline{p} + \vec{\nabla}\overline{\tau} = 0 \tag{6.5}$$

where $co\vec{n}s_v$ contains all mean cross-correlations of fluctuations (ie.$\overline{\vec{u'}.\vec{u'}}$, etc...).
Therefore, a conservation equation for the fluctuating velocity is:

$$\begin{aligned} \rho\frac{\partial\vec{u'}}{\partial t} &= -\rho\left(\vec{u}\times\vec{\nabla}\right)\vec{u} - \vec{\nabla}p + \vec{\nabla}\tau \\ &+\overline{\rho}\left(\overline{\vec{u}}\times\vec{\nabla}\right)\overline{\vec{u}} + \vec{\nabla}\overline{p} - \vec{\nabla}\overline{\tau} - co\vec{n}s_v, \end{aligned} \tag{6.6}$$

and gives:

$$\rho\frac{\partial\vec{u'}}{\partial t} = -\rho\left(\vec{u}\times\vec{\nabla}\right)\vec{u'} - \vec{\nabla}p' + \vec{\nabla}\tau' + \left[\left[\overline{\rho}\overline{\vec{u}} - \rho\vec{u}\right]\times\vec{\nabla}\right]\overline{\vec{u}} - co\vec{n}s_v \tag{6.7}$$

taking the scalar product of Eq.(6.7) by $\vec{u'}$ gives a nonlinear conservation equation for u'^2:

$$\frac{\rho}{2}\frac{\partial u'^2}{\partial t} = -\rho\vec{u}.\vec{\nabla}\left(\frac{u'^2}{2}\right) - \vec{u'}.\vec{\nabla}p' + \vec{u'} : \vec{\nabla}\tau' + \vec{u'}.\left[\left[\overline{\rho}\overline{\vec{u}} - \rho\vec{u}\right] \times \vec{\nabla}\right]\overline{\vec{u}} - \vec{u'}.co\vec{n}s_v \tag{6.8}$$

Linearization

Linearizing Eq.(6.8) to second order gives:

$$\begin{aligned}\frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} &= -\rho_0\vec{u_0}.\vec{\nabla}\left(\frac{u_1^2}{2}\right) - \vec{u_1}.\vec{\nabla}p_1 + \vec{u_1} : \vec{\nabla}\tau_1 \\ &-\rho_1\vec{u_1}.\vec{\nabla}\left(\frac{u_0^2}{2}\right) - \rho_0 u_1^2\vec{\nabla}.\vec{u_0} - \vec{u_1}.co\vec{n}s_v\end{aligned} \tag{6.9}$$

6.1.3 Nonlinear conservation equation for coupled pressure and velocity fluctuations

Combining Eq.(6.3) and Eq.(6.8), one obtains the following exact equation in p'^2 and u'^2 :

$$\begin{aligned}\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} + \frac{\rho}{2}\frac{\partial u'^2}{\partial t} &= -\rho\vec{u}.\vec{\nabla}\left(\frac{u'^2}{2}\right) - \vec{u'}.\vec{\nabla}p' + \vec{u'} : \vec{\nabla}\tau' \\ &+\vec{u'}.\left[\left[\overline{\rho}\overline{\vec{u}} - \rho\vec{u}\right] \times \vec{\nabla}\right]\overline{\vec{u}} - \vec{u'}.co\vec{n}s_v \\ &-\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \overline{\gamma}\overline{p}\vec{\nabla}.\overline{\vec{u}}\right] - \frac{p'}{\gamma p}\left[\vec{u}.\vec{\nabla}p - \overline{\vec{u}}.\vec{\nabla}\overline{p}\right] \\ +\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right] &- \frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\overline{\vec{q}}\right] \\ &-\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} - (\overline{\gamma}-1)\sum_{k=1}^{N}\overline{h_{sk}}\overline{\rho}\frac{D\overline{Y_k}}{Dt}\right] \\ &+\frac{p'}{\gamma p}\left[(\gamma-1)\left(\tau : \vec{\nabla}\vec{u}\right) - (\overline{\gamma}-1)\left(\overline{\tau} : \vec{\nabla}\overline{\vec{u}}\right)\right] \\ &+\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt} - \frac{\overline{\gamma}\overline{p}}{\overline{r}}\frac{D\overline{r}}{Dt}\right] - cons_p\frac{p'}{\gamma p}\end{aligned} \tag{6.10}$$

This equation will be called Eq.(1)[Eq.(6.10)]

Eq.(**1**)[Eq.(6.10)] is constructed here supposing negligible numerical corrections. Yet, the code also introduces dissipation and dispersion that must be taken into account for the balance analysis of this equation. These terms ($\mathbf{LW_{E_1}}$, $\mathbf{BND_{E_1}}$ and $\mathbf{AVI_{E_1}}$) may be found in Annex D.

6.1.4 Linear conservation equation for disturbance energy in pressure and velocity fluctuations

Combining Eqs.(6.4) and Eq.(6.9) gives the following linear equation:

$$\begin{aligned}
\frac{1}{2\gamma_0 p_0}\frac{D_0 p_1^2}{Dt} + \frac{\rho_0}{2}\frac{D_0 u_1^2}{Dt} + \vec{\nabla}.\left(p_1 \vec{u_1}\right) = \\
\frac{\gamma_0 - 1}{\gamma_0 p_0} p_1 \left[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1} - \sum_{k=1}^{N}\left(h_{sk1}\rho_0\frac{D_0 Y_{0k}}{Dt} + h_{sk0}\rho_1\frac{D_0 Y_{0k}}{Dt} + h_{sk0}\rho_0\frac{D_0 Y_{k1}}{Dt}\right)\right] \\
+\frac{\gamma_0 - 1}{\gamma_0 p_0} p_1 \left[\tau_1 : \vec{\nabla}\vec{u_0} + \tau_0 : \vec{\nabla}\vec{u_1}\right] \\
+\frac{p_1\gamma_1}{\gamma_0 p_0}\left[-p_0\vec{\nabla}.\vec{u_0} + \dot{\omega_{T0}} - \vec{\nabla}.\vec{q_0} - \sum_{k=1}^{N}\left(h_{sk0}\rho_0\frac{D_0 Y_{0k}}{Dt} + \tau_0 : \vec{\nabla}\vec{u_0}\right)\right] \\
-\frac{p_1^2}{p_0}\vec{\nabla}.\vec{u_0} - \frac{p_1}{\gamma_0 p_0}\vec{u_1}.\vec{\nabla}p_0 - \rho_1\vec{u_1}.\vec{\nabla}\left(\frac{u_0^2}{2}\right) - \rho_0 u_1^2\vec{\nabla}.\vec{u_0} \\
+\frac{p_1}{r_0}\frac{D_0 r_1}{Dt} + \frac{D_0 r_0}{Dt}\left[\frac{\gamma_1 p_1}{\gamma_0 p_0} + \frac{p_1^2}{r_0 p_0} - \frac{p_1 r_1}{r_0^2}\right] + \vec{u_1} : \vec{\nabla}\tau_1 - \vec{u_1}.\vec{cons_v} - cons_p\frac{p_1}{\gamma_0 p_0}
\end{aligned} \tag{6.11}$$

This equation will be called Linear Eq.(1)

$\frac{1}{\gamma_0 p_0}\frac{p_1^2}{2} + \rho_0\frac{u_1^2}{2}$ **will be called Energy 1**

$\frac{D_0}{Dt}$ denotes the total derivative with a spatial component relying only on the mean velocity (i.e $\frac{D_0 f}{Dt} = \frac{\partial f}{\partial t} + \vec{u_0}.\vec{\nabla}f$). Eq.(6.11) is a conservation equation for the quantity $\frac{1}{2\gamma_0 p_0}p_1^2 + \frac{\rho_0}{2}u_1^2$ which represents the acoustic energy in a laminar, isentropic and small mean Mach flow. In other cases, this quantity a priori contains a part of the acoustic, turbulent and entropic disturbance energies.

6.1.5 Discussion

Eq.(6.11) shows that applying the linear hypothesis is not enough to obtain the acoustic energy conservation equation. The following paragraph addresses the necessary hypothesis to obtain such an equation. **If all spatial dependencies of mean fields and all mean cross-correlations are neglected**, Eq.(6.11) shrinks to:

$$\begin{aligned}
\frac{D_0 e_1^2}{Dt} + \vec{\nabla}.\left(p_1\vec{u_1}\right) = \frac{\gamma_0 - 1}{\gamma_0 p_0}p_1\left[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1} - \sum_{k=1}^{N} h_{sk0}\rho_0\frac{D_0 Y_{k1}}{Dt} + \tau_0 : \vec{\nabla}\left(\vec{u_1}\right)\right] \\
+\frac{p_1\gamma_1}{\gamma_0 p_0}\dot{\omega_{T0}} + \frac{p_1}{r_0}\frac{D_0 r_1}{Dt} + \vec{u_1} : \vec{\nabla}\left(\tau_1\right)
\end{aligned} \tag{6.12}$$

with $e_1 = \frac{1}{2\gamma_0 p_0} p_1^2 + \frac{\rho_0}{2} u_1^2$.
Assuming an homogeneous mixing and a negligible viscosity gives a conservation equation for the acoustic energy including the transport of acoustic energy by the mean flow:

$$\frac{\partial e_1}{\partial t} + \vec{u_0}.\vec{\nabla} e_1 + \vec{\nabla}.\left(p_1 \vec{u_1}\right) = \frac{\gamma_0 - 1}{\gamma_0 p_0} p_1 \left[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1}\right] \tag{6.13}$$

This equation will be called Simplified linear Eq.(1)

Summarizing the necessary assumptions that this acoustic energy conservation equation implies:

- Linear hypothesis (i.e $\beta_1 << \beta_0$ (except for $u_1 << c_0$))
- Negligible mean cross-correlations (i.e $cons_p = co\vec{n}s_u = 0$)
- Negligible mean and fluctuating viscosity
- Homogeneous mixing
- Negligible mean gradients (i.e $\vec{\nabla}\beta_0 << 1$)

These assumptions will be tested in chapter 7 through the closure of Eq.(6.11).

6.2 Entropy disturbance energy Eq.(2)[Eq.(6.18)]

The goal of this section is to derive a nonlinear equation for quadratic entropy fluctuations (Eq.(6.18)) and a linear conservation equation for the disturbance energy contained in entropy fluctuations (Eq.(6.19)). All the missing details of the algebra leading to Eqs.(6.18) and(6.19) may be found in Appendix B.

6.2.1 Nonlinear conservation equation for entropy fluctuations

The conservation equation for entropy in a multicomponent, reacting medium writes :

$$\frac{\partial s}{\partial t} + \vec{u}.\vec{\nabla} s = \frac{1}{\rho T}\left[\dot{\omega}_T - \vec{\nabla}.\vec{q} + \phi\right] - \frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] \tag{6.14}$$

Averaging all terms of Eq.(6.14), one may write a mean conservation equation for entropy :

$$\begin{aligned} &-\overline{\vec{u}}.\vec{\nabla}\overline{s} + \frac{1}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right] \\ &-\frac{1}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] + cons_s = 0 \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} cons_s = &-\overline{\vec{u'}.\vec{\nabla}s'} + \overline{\frac{1}{\rho T}\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right]} - \frac{1}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right] \\ &-\overline{\frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right]} + \frac{1}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] \end{aligned} \tag{6.16}$$

Note that the identity $\overline{\frac{\partial s}{\partial t}}$ is assumed. It either means that the final and initial values of the entropy are the same, or that the flow is statistically stationary.
Taking the difference between Eq.(6.14) and Eq.(6.15) gives:

$$\begin{aligned} \frac{\partial s'}{\partial t} = &-\vec{u}.\vec{\nabla}s + \frac{1}{\rho T}\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right] \\ -\frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] &+ \overline{\vec{u}}.\vec{\nabla}\overline{s} - \frac{1}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right] \\ &+\frac{1}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] - cons_s \end{aligned} \tag{6.17}$$

Multiplying the previous equation by $\frac{ps'}{rc_p}$ gives a nonlinear conservation equation for s'^2 :

$$\begin{aligned}\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} &= -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right]\\ &+\frac{s'}{c_p}\left[\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right] - \frac{\rho T}{\overline{\overline{\rho T}}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right]\right]\\ &-\frac{s'}{c_p}\left[\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] - \frac{\rho T}{\overline{\overline{\rho T}}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]\right]\\ &-\frac{ps'}{rc_p}cons_s\end{aligned} \tag{6.18}$$

This equation will be called Eq.(2)[Eq.(6.18)]

Eq.(**2**)[Eq.(6.18)] is constructed here supposing negligible numerical corrections. Yet, the code also introduces dissipation and dispersion that must be taken into account for the balance analysis of this equation. These terms ($\mathbf{LW_{E_2}}$, $\mathbf{BND_{E_2}}$ and $\mathbf{AVI_{E_2}}$) may be found in Annex D.

Linearization

Linearizing Eq.(6.18) to second order gives :

$$\begin{aligned}\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} &= -\frac{p_0 s_1}{r_0 c_{p0}}\left(\vec{u_1}.\vec{\nabla}s_0 + \vec{u_0}.\vec{\nabla}s_1\right)\\ &+\frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} + \phi_1 - \vec{\nabla}.\vec{q_1}\right]\\ &+\frac{s_1}{c_{p0}}\left[\left(-\frac{\rho_1}{\rho_0} - \frac{T_1}{T_0}\right)\left(\dot{\omega}_{T0} + \phi_0 - \vec{\nabla}.\vec{q_0}\right)\right]\\ &-\frac{\rho_0 s_1}{c_{p0}}\left[-\frac{T_1}{T_0}\sum_k g_{sk0}\frac{DY_{k0}}{Dt} + \sum_k g_{sk1}\frac{DY_{k0}}{Dt} + \sum_k g_{sk0}\frac{\partial Y_{k1}}{\partial t}\right]\end{aligned} \tag{6.19}$$

This equation will be called Linear Eq.(2)

$\frac{p_0}{r_0c_{p0}}\frac{s_1^2}{2}$ **will be called Energy 2**

6.2.2 Links between acoustic and entropy disturbance energies

This equation, giving the evolution of the disturbance energy contained in quadratic fluctuations of entropy cannot be directly compared to Eq.(6.11). s_1 is rewritten in function of $T_1,\ p_1$ and Y_{k1} to enable this comparison. Using gibbs equation for entropy in a multispecies system to determine s_1 gives:

$$Tds = dh - \frac{1}{\rho}dp - \sum_k g_k dY_k \tag{6.20}$$

$$Ts_1 = c_p T_1 - \frac{1}{\rho}p_1 - \sum_k g_k Y_{k1} \tag{6.21}$$

and therefore :

$$\frac{s_1}{c_{p0}} = \frac{c_p T_1}{c_{p0} T} - \frac{r}{p c_{p0}} p_1 - \sum_k \frac{g_k}{c_{p0} T} Y_{k1} \tag{6.22}$$

Keeping only first order terms in s_1 gives:

$$\frac{s_1}{c_{p0}} = \frac{T_1}{T_0} - \frac{r_0}{p_0 c_{p0}} p_1 - \sum_k \frac{g_{k0}}{c_{p0} T_0} Y_{k1} \tag{6.23}$$

substituing Eq.(6.23) into Eq.(6.19) gives:

$$\begin{aligned}
\frac{p_0}{2 r_0 c_{p0}} \frac{\partial s_1^2}{\partial t} &= -\frac{p_0 s_1}{r_0 c_{p0}} \left(\vec{u_1}.\vec{\nabla}\left(s_0\right) + \vec{u_0}.\vec{\nabla}\left(s_1\right) \right) \\
&+ \left[\frac{T_1}{T_0} - \frac{r_0}{c_{p0}} \frac{p_1}{p_0} - \sum_k \frac{g_{k0}}{c_{p0} T_0} Y_{k1} \right] \left[\dot{\omega}_{T1} + \phi_1 - \vec{\nabla}.\left(\vec{q_1}\right) \right] \\
&+ \left[\frac{T_1}{T_0} - \frac{r_0}{c_{p0}} \frac{p_1}{p_0} - \sum_k \frac{g_{k0}}{c_{p0} T_0} Y_{k1} \right] \left[\left(-\frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} \right) \left(\dot{\omega}_{T0} + \phi_0 - \vec{\nabla}.\left(\vec{q_0}\right) \right) \right] \\
&- \rho_0 \left[\frac{T_1}{T_0} - \frac{r_0}{c_{p0}} \frac{p_1}{p_0} - \sum_k \frac{g_{k0}}{c_{p0} T_0} Y_{k1} \right] \\
&\left[-\frac{T_1}{T_0} \sum_k g_{sk0} \frac{DY_{k0}}{Dt} + \sum_k g_{sk1} \frac{DY_{k0}}{Dt} + \sum_k g_{sk0} \frac{\partial Y_{k1}}{\partial t} \right]
\end{aligned} \tag{6.24}$$

By doing the same assumptions as for the simplification of the linear disturbance energy conservation Eq.(6.11) and using the fact that under these assumptions $\frac{\gamma_0 - 1}{\gamma_0} = \frac{r_0}{C_{p0}}$ and $\frac{\rho_1}{\rho_0} = -\frac{T_1}{T_0} + \frac{p_1}{p_0}$, Eq.(6.24) shrinks to :

$$\begin{aligned}
\frac{p_0}{2 r_0 c_{p0}} \frac{\partial s_1^2}{\partial t} + \frac{p_0}{2 r_0 c_{p0}} \vec{u_0}.\vec{\nabla} s_1^2 &= \left[\frac{T_1}{T_0} - \frac{\gamma_0 - 1}{\gamma_0} \frac{p_1}{p_0} \right] \left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1} \right] \\
&- \left[\frac{T_1}{T_0} - \frac{\gamma_0 - 1}{\gamma_0} \frac{p_1}{p_0} \right] \frac{p_1}{p_0} \dot{\omega}_{T0}
\end{aligned} \tag{6.25}$$

This equation will be called Simplified linear Eq.(2)

At this stage two important remarks should be made concerning source terms related to $\dot{\omega}_{T1}$ and $\dot{\omega}_{T0}$. The classical Rayleigh term (i.e. $\frac{\gamma_0-1}{\gamma_0}\frac{p_1}{p_0}\dot{\omega}_{T1}$) appears as a sink term for the entropy disturbance energy, whereas its source term is $\frac{\dot{\omega}_{T1}T_1}{T_0}$. It can therefore be suspected that in the linear framework, the rayleigh term is a transfer term which takes energy from the entropy disturbance energy to increase pressure and velocity disturbance energy, showing thereby a direct link between the two types of energies.
In contrary to what is observed for the disturbance energy defined in Eq (6.11), terms related to $\dot{\omega}_{T0}$ do not vanish in the linearization. Unless the hypothesis $M = 0$ is made, the source term $\frac{\gamma_0-1}{\gamma_0}\left(\frac{p_1}{p_0}\right)^2\dot{\omega}_{T0}$ implies the creation of entropy disturbance energy by the quadratic fluctuations of pressure correlated to the mean heat release.

In the two previous sections 6.1 and 6.2, linear conservation equations of disturbance energies are derived. Yet, a linearization of the flow variables is not always possible, especially in reacting flows. In this case, the flame front moves inside the combustor and the local temperature fluctuation level can be as high as the mean local temperature. In this case, it is expected that non-linear phenomena will have an important contribution to the level of disturbance energy in the flow. Following this remark, in this section, one derives a nonlinear conservation equation for a disturbance energy in reacting flows.

6.3 Nonlinear disturbance energy Eq.(3)[Eq.(6.37)]

In a mathematical form, this equation should write :

$$\frac{DE_d}{Dt} + \vec{\nabla}.\vec{W} = D \tag{6.26}$$

where E_d is a nonlinear disturbance energy, $\vec{W}$ the associated flux across boundaries and D the source term.

The quantity E_d should therefore meet several properties to be useful when applied to flow instabilities studies. When needed[2] E_d will be expanded as $f(\mathbf{x},t) = f_0(\mathbf{x}) + \sum \epsilon^i f_i(\mathbf{x},t)$.

- P_0**:** E_d should be zero when there are no fluctuations, that is $E_{d0} = 0$,
- P_1**:** E_d should be quadratic in the primitive variables fluctuations, that is $E_{d1} = 0$
- P_2**:** the leading order term of E_d should only depend on the first order terms of the primitive variable fluctuations (i.e ρ_1, etc...)
- P_3**:** E_{d2} should be definite positive so that it increases with the amplitude of the fluctuations.

[2]One therefore has the following relation between this decomposition and the decomposition in mean and fluctuating part : $\sum \epsilon^i f_i(\mathbf{x},t) = f'$

Notation	Expression	Notation	Expression
$\vec{q}$	$-\lambda\vec{\nabla}T + \rho\sum_{k=1}^{N} h_{sk}Y_k\vec{V_k}$	$\vec{q_k}$	$\rho Y_k\vec{V_k}$
$\vec{\zeta}$	$(\vec{\nabla}\otimes\vec{u})\otimes\vec{u}$	$\vec{\xi}$	$\vec{\nabla}\otimes\vec{u}$
ϕ	$\sum_i\sum_j \tau_{ij}\frac{\partial u_j}{\partial x_i}$	ψ_j	$\sum_i \frac{1}{\rho}\frac{\partial\tau_{ij}}{\partial x_i}$
Q	$\frac{\dot{\omega_T}-\vec{\nabla}.\vec{q}+\phi}{T}$	$\dot{\omega_T}$	$\sum_{k=1}^{N}\Delta h_{f,k}^0\dot{\omega_k}$
h_s	$h-\sum_{k=1}^{N}\Delta h_{f,k}^0 Y_k$	H	$h_s+\frac{\vec{u}.\vec{u}}{2}$
g_{sk}	$h_{sk}-Ts_k$	g_k	$g_{sk}+\Delta h_{f,k}$

Table 6.2 - *Notations*

6.3.1 Derivation of a disturbance energy conservation equation

In contrary to what is done for the derivation of Eqs.(6.10) and (6.18), here the baseline flow is the steady state flow. This implies that no mean cross-correlations terms are considered anymore. This choice is made here to ease the presentation of this derivation but also has important consequences on the level of fluctuations in the flow. A discussion on these consequences may be found in section 6.5 and the influence of the choice of the baseline flow on balance closure is addressed in Chapter 7. Note that details of the derivation with an unsteady averaged baseline flow may be found in appendix C. To simplify the comprehension, a table 6.2 recalling the major terms involved in this derivation is provided.

The derivation of this disturbance energy equation requires Crocco's formulations for the entropy and momentum transport equations. The set of Eqs.(6.27-6.30) describe any reactive laminar flow:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}.\vec{m}=0 \tag{6.27}$$

$$\frac{\partial\vec{u}}{\partial t}+\vec{\zeta}+\vec{\nabla}H-T\vec{\nabla}s=\vec{\psi}+\vec{\psi^\star} \tag{6.28}$$

$$\frac{\partial\rho Y_k}{\partial t}+\vec{\nabla}.(\vec{m}Y_k+\vec{q_k})=\dot{\omega}_k,\ for\ k\ =\ 1,2,...,n \tag{6.29}$$

$$\frac{\partial\rho s}{\partial t}+\vec{\nabla}.\left(\vec{m}s\right)=Q+Q^\star \tag{6.30}$$

where $Q^\star\ =\ -\sum_k g_{sk}\left[\dot{\omega}_k-\vec{\nabla}.\vec{q_k}\right]/T$ and $\vec{\psi^\star}\ =\ \sum_{k=1}^{N}g_{sk}\vec{\nabla}Y_k$, with g_{sk} being the sensible free enthalpy of species k, $\dot{\omega}_k$ the volumic mass rate of consumption for species k and $\vec{q_k}$ the species flux defined as : $\vec{q_k}=\rho Y_k\vec{V_k}$. All other variables correspond to the analysis of Myers [88], except that in this reacting case derivation: $Q=\frac{\dot{\omega_T}-\vec{\nabla}.\vec{q}+\Phi}{T}$.
The corresponding equations for the steady mean fields are:

$$\vec{\nabla}.\overline{\vec{m}}=0 \tag{6.31}$$

$$\bar{\vec{\zeta}}+\vec{\nabla}\overline{H}-\bar{T}\vec{\nabla}\bar{s}-\overline{\vec{\psi^\star}}-\overline{\vec{\psi}}=0 \tag{6.32}$$

$$\vec{\nabla}.\overline{\vec{m}Y_k}+\vec{\nabla}.\overline{\vec{q_k}}=\overline{\dot{\omega_k}} \tag{6.33}$$

$$\vec{\nabla}.\left(\overline{\vec{m}}\bar{s}\right)-\bar{Q}-\bar{Q}^\star=0 \tag{6.34}$$

The derivation of a disturbance energy will also be using the specific stagnation enthalpy transport instantaneous and mean equations:

$$\frac{\partial}{\partial t}\left(\rho H - p\right) + \vec{\nabla}.\left(\vec{m}H\right) - \vec{m}.\vec{\psi} - TQ = 0 \qquad (6.35)$$

$$\vec{\nabla}.\left(\overline{\vec{m}H}\right) - \overline{\vec{m}}.\overline{\vec{\psi}} - \bar{T}\bar{Q} = 0 \qquad (6.36)$$

The sum of $\left(\overline{H} - \bar{T}\overline{s} - \sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\right)$ times Eq.(6.27), $\bar{T}$ times Eq.(6.30), $\overline{g_{sk}}$ times Eq.(6.29) and the scalar product of $\overline{\vec{m}}$ with Eq.(6.28) are substracted from Eq.(6.35). Doing so and after quite heavy algebra given in Appendix C, it gives:

$$\begin{aligned}
&\underbrace{\frac{\partial}{\partial t}\left[\rho\left(H - \overline{H}\right) - \rho\bar{T}\left(s - \overline{s}\right) - \overline{\vec{m}}.(\vec{u} - \overline{\vec{u}}) - (p - \overline{p}) - \sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k - \overline{Y_k})\right]}_{E_d} \\
&+\vec{\nabla}.\underbrace{\left[\left(\vec{m} - \overline{\vec{m}}\right)\left[\left(H - \overline{H}\right) - \bar{T}(s - \overline{s})\right] - \overline{\vec{m}}(T - \bar{T})\left(s - \overline{s}\right)\right]}_{W} \\
&\underbrace{+(\vec{m} - \overline{\vec{m}})(\vec{\zeta} - \overline{\vec{\zeta}})}_{D_\zeta}\underbrace{+(s - \overline{s})(\vec{m} - \overline{\vec{m}})\vec{\nabla}\bar{T} - \overline{\vec{m}}(s - \overline{s})\vec{\nabla}(T - \bar{T})}_{D_s} \\
&\underbrace{-(T - \bar{T})(Q - \bar{Q})}_{D_Q}\underbrace{-(\vec{m} - \overline{\vec{m}}).(\vec{\psi} - \overline{\vec{\psi}})}_{D_{\vec{\psi}}} \\
&\underbrace{-(T - \bar{T})(Q^\star - \overline{Q^\star})}_{D_{Q^\star}}\underbrace{-(\vec{m} - \overline{\vec{m}}).(\vec{\psi^\star} - \overline{\vec{\psi^\star}})}_{D_{\vec{\psi}^\star}} \\
&\underbrace{-\sum_{k=1}^{n} g'_{sk}\Omega'_k - \sum_{k=1}^{n} g'_{sk}Y_k\vec{\nabla}.\vec{m'} - \sum_{k=1}^{n}\overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m'}}_{D_{Yk}} = 0
\end{aligned} \qquad (6.37)$$

where $\Omega'_k = [\dot{\omega}'_k - \vec{\nabla}.\vec{q}'_k - \vec{\nabla}.(\vec{m}Y_k)']$.

This equation will be called Eq.(3)[Eq.(6.37)]

E_d will be called Energy 3

6.3.2 Linearization

It is obvious from Eq.(6.37) that E_d satisfies property **P0**. Disturbances of the form $()' = () - (^-) = \sum_{i=1}^{\infty}\epsilon^i()_i$ are then substituted into the exact Eq.(6.37), and only the lowest order terms in ϵ are retained.

In keeping with the other studies of disturbance energy in isentropic and homentropic flows [85, 18, 100], the remaining terms are of second order in the disturbances, meaning that E_d also satisfies **P1**. Retention of all second-order terms in the exact flux vector and source terms results in a rather complex disturbance energy equation, where much of the complexity is contained in viscous stress, dissipation, and heat conduction terms that can be argued to be negligible in most combusting flows. Ignoring such terms as well as the vorticity terms results in the following linearized disturbance energy equation,

$$\frac{\partial E_{d2}}{\partial t} + \vec{\nabla}.\vec{W}_2 = D_2, \tag{6.38}$$

This equation will be called Linear Eq.(3)

where the disturbance energy density E_{d2}, flux vector $\vec{W}_2$, and source D_2 terms are

$$\begin{aligned} E_{d2} &= \frac{p_1^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 \vec{u}_1.\vec{u}_1 + \rho_1 \vec{u}_0.\vec{u}_1 + \frac{\rho_0 T_0 s_1^2}{2c_{p0}} + E_{Y2}, & (6.39)\\ \vec{W}_2 &= (p_1 + \rho_0 \vec{u}_0.\vec{u}_1)\left(\vec{u}_1 + \frac{\rho_1}{\rho_0}\vec{u}_0\right) + \vec{m}_0 T_1 s_1, & (6.40) \end{aligned}$$

and

$$\begin{aligned} D_2 &= s_1 \vec{m}_1.\vec{\nabla} T_0 + s_1 \vec{m}_0.\vec{\nabla} T_1 \\ &+ \frac{T_1}{T_0}(\omega_{T1} - \vec{\nabla}.\vec{q}_1) + T_1(\omega_{T0} - \vec{\nabla}.\vec{q}_0)\left(\frac{1}{T} - \frac{1}{T_0}\right) \\ &+ T_1 Q_1^* + \vec{m}_1.\vec{\psi}_1^* \\ &+ \sum_{k=1}^{n} g_{sk1}\Omega_{k1} + \sum_{k=1}^{n}(g_{sk1}Y_{k1} + g_{sk0}Y_{k1})\,\vec{\nabla}.\vec{m}_1. \end{aligned} \tag{6.41}$$

The E_{Y2} term in Eq.(6.39) is the contribution from the $\rho g_{sk0}(Y_k - Y_{k0})$ terms in Eq.(6.37) and is equal to

$$\begin{aligned} E_{Y2} &= \frac{p_0}{2}\sum_{k=1}^{n}\left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right)\frac{g_{sk} - e_{sk}}{C_v T} + \frac{W}{W_k}\left(1 + \frac{1}{Y_k}\right)\right] Y_{k1}^2 \\ &+ p_0 \sum_{k=1}^{n}\sum_{j\neq k}\left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right)\frac{g_{sj} - e_{sj}}{C_v T} + \frac{W}{W_j}\right] Y_{k1} Y_{j1} \\ &+ \frac{\rho}{C_v}\sum_{k=1}^{n}(g_{sk} - e_{sk})\, s_1 Y_{k1} + \sum_{k=1}^{n}\left[(\gamma - 1)(g_{sk} - e_{sk}) + \frac{RT}{W_k}\right]\rho_1 Y_{k1}. \end{aligned} \tag{6.42}$$

From Eqs.(6.39)-(6.42), the proposed disturbance energy also satisfies property **P2** since its leading order term only depends on first-order quantities. Note also that Eq.(6.38) simplifies to other, existing acoustic energy conservation equations under the condition of homentropic flow and homogeneous mixture. The energy density E_{d2} and flux $\vec{W}_2$ terms then become those defined by Cantrell[18] for acoustic propagation in a non-stationary medium. Under the zero Mach number flow assumption and calorific perfection, Eq.(6.38) reduces to the form given in [93].

6.4 The choice of the baseline flow

A mendatory property of the reference field considered in Eq.(**1**)[Eq.(6.10)], (**2**) and (**3**) is that it should be easily obtained from unsteady computations. Two possibilities are investigated in this work.

- When available, the reference field can be the steady state solution of the flow. This is only possible if the flow is laminar. By definition, no such field can be defined in turbulent cases. In practice, the fluctuating instantaneous solutions are therefore compared to the stable one.
  Fig.(6.1) presents the implications of this choice on heat release fields in a 1D flame case. Here the flame moves periodically from left to right (for example under the influence of an incoming acoustic wave). Fig.(6.1) shows the reference, the instantaneous and the fluctuating fields of heat release.
  Since the flame has moved compared to its steady state position, fluctuations are not zero. The maximum amplitude is found on the maximum of the instantaneous field, and the minimum amplitude is found at the location of the maximum reference field. Note that if the reference and instantaneous curves are symmetric in fresh and burnt gases, the integral of the fluctuation on the line is zero.

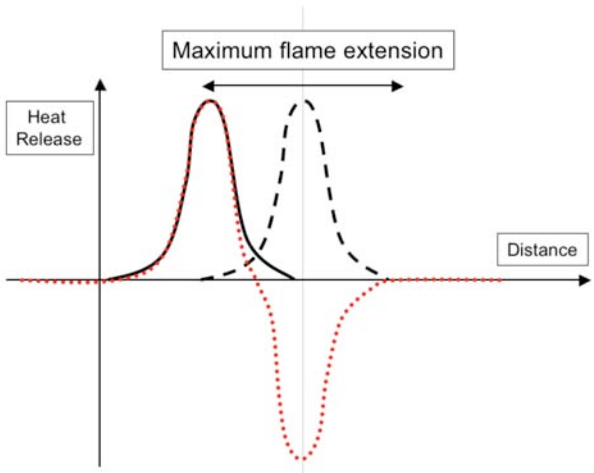


Figure 6.1 - *Fields of heat release in a 1D flame case(***Steady state reference field***):* —— *Instataneous* - - - *Reference* ········ *Fluctuation*

- Another possible choice for the reference field is the averaged mean field. Fluctuations will then be obtained by comparing instantaneous fields to this reference. Fig.(6.2) takes the same example of a 1D forced flame. Since the flame moves following the incoming acoustic wave, the averaged field obtained after a few periods looks very different from the steady state. The averaged field is not straight because the flame spends more time during a cycle at the locations of the maximum flame extension.
  Since in the two cases the instantaneous fields are the same, fluctuating fields look very differ-

ent. Two zones of maximum and minimum fluctuation can also be found but their positions have changed and their amplitudes are smaller than in the previous case.

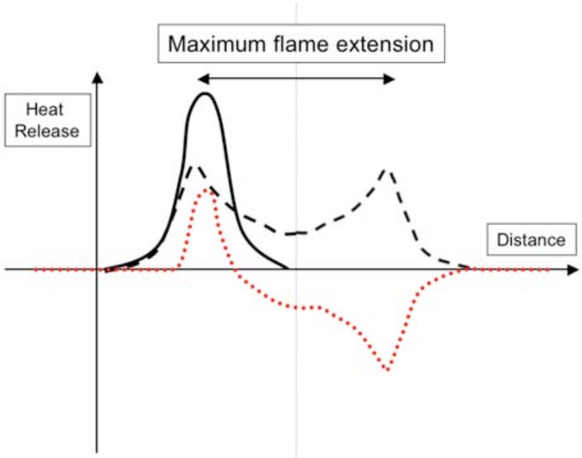


Figure 6.2 - *Fields of heat release in a 1D flame case(***Averaged reference field***):*
—— *Instataneous* - - - *Reference* ········ *Fluctuation*

This comparison shows that this choice is critical since it changes the amplitude of local fluctuations of heat release. Note that this phenomenon has nothing to do with turbulence. In practice, when dealing with turbulent cases, only the second choice is possible and no steady reference field can be defined. Using the averaged reference field on a laminar case is done in this work and should therefore be seen as a step towards the analysis of disturbance energies in turbulent cases using LES.

6.5 Summary of disturbance energies conservation equations

In order to help the reader in his comprehension, and to introduce simple notations that will used latter in the text, Tables (6.3), (6.4) and (6.5) provide a clear presentation of Eqs.(6.10), (6.18) and (6.37). These tables present for the three equations, the energy, the fluxes and the source terms that they involve.
A comparison with other conservation equations found in the Literature is also provided.

Type of equation	Energy	∇.(Fluxes)	Source terms
Nonlinear Eq.(**1**)[Eq.(6.10)]	$\int_T \frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} + \frac{\rho}{2}\frac{\partial u'^2}{\partial t}dt$	$\vec{u'}.\vec{\nabla}p' +$ $\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right]$	$-\rho\vec{u}.\vec{\nabla}\left(\frac{u'^2}{2}\right) + \vec{u'} : \vec{\nabla}\tau' + \vec{u'}.\left[\left[\bar{\rho}\vec{\bar{u}} - \rho\vec{u}\right] \times \vec{\nabla}\right]\vec{\bar{u}} -$ $\frac{p'}{\gamma p}\left[\vec{u}.\vec{\nabla}p - \vec{\bar{u}}.\vec{\nabla}\bar{p}\right] + \frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\bar{\gamma}-1)\,\overline{\dot{\omega_T}}\right] -$ $\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\bar{\gamma}-1)\,\vec{\nabla}.\vec{\bar{q}}\right] -$ $\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^N h_{sk}\rho\frac{DY_k}{Dt} - (\bar{\gamma}-1)\sum_{k=1}^N \overline{h_{sk}}\bar{\rho}\frac{D\overline{Y_k}}{Dt}\right] +$ $\frac{p'}{\gamma p}\left[(\gamma-1)\left(\tau : \vec{\nabla}\vec{u}\right) - (\bar{\gamma}-1)\left(\bar{\tau} : \vec{\nabla}\vec{\bar{u}}\right)\right] +$ $\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt} - \frac{\bar{\gamma}\bar{p}}{\bar{r}}\frac{D\bar{r}}{Dt}\right]$
Linear Eq.(**1**)[Eq.(6.11)] Hyp. : Linearization	$\frac{p_1^2}{2\gamma_0 p_0} + \frac{\rho_0 u_1^2}{2}$	$\vec{\nabla}.\left(p_1\vec{u_1}\right)$	$\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1}\right] -$ $\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\sum_{k=1}^N\left(h_{sk1}\rho_0\frac{D_0 Y_{0k}}{Dt} + h_{sk0}\rho_1\frac{D_0 Y_{0k}}{Dt} + h_{sk0}\rho_0\frac{D_0 Y_{k1}}{Dt}\right)\right] +$ $\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\tau_1 : \vec{\nabla}\vec{u_0} + \tau_0 : \vec{\nabla}\vec{u_1}\right] +$ $\frac{p_1\gamma_1}{\gamma_0 p_0}\left[-p_0\vec{\nabla}.\vec{u_0} + \dot{\omega_{T0}} - \vec{\nabla}.\vec{q_0}\right] -$ $\frac{p_1\gamma_1}{\gamma_0 p_0}\left[\sum_{k=1}^N\left(h_{sk0}\rho_0\frac{D_0 Y_{0k}}{Dt} + \tau_0 : \vec{\nabla}\vec{u_0}\right)\right] - \frac{p_1^2}{p_0}\vec{\nabla}.\vec{u_0} -$ $\frac{p_1}{\gamma_0 p_0}\vec{u_1}.\vec{\nabla}p_0 - \rho_1\vec{u_1}.\vec{\nabla}\left(\frac{u_0^2}{2}\right) - \rho_0 u_1^2\vec{\nabla}.\vec{u_0} + \frac{p_1}{r_0}\frac{D_0 r_1}{Dt} +$ $\frac{D_0 r_0}{Dt}\left[\frac{\gamma_1 p_1}{\gamma_0 p_0} + \frac{p_1^2}{r_0 p_0} - \frac{p_1 r_1}{r_0^2}\right] + \vec{u_1} :$ $\vec{\nabla}\tau_1 - \vec{u_0}.\vec{\nabla}\left[\frac{p_1^2}{2\gamma_0 p_0} + \frac{\rho_0 u_1^2}{2}\right]$
Simplified Linear Eq.(**1**)[Eq.(6.13)] Hyps. : Summarized on page 179	$\frac{p_1^2}{2\gamma_0 p_0} + \frac{\rho_0 u_1^2}{2}$	$\vec{\nabla}.\left(p_1\vec{u_1}\right)$	$\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1}\right] - \vec{u_0}.\vec{\nabla}\left[\frac{p_1^2}{2\gamma_0 p_0} + \frac{\rho_0 u_1^2}{2}\right]$
Acoustic equation **(reacting flow)**	$\frac{p_1^2}{2\gamma_0 p_0} + \frac{\rho_0 u_1^2}{2}$	$\vec{\nabla}.\left(p_1\vec{u_1}\right)$	$\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1}\right]$
Bloxsidge linear equation [12] **(reacting flow)**	$\frac{1}{2\rho c^2}p'^2 + \frac{1}{2}\rho u'^2 +$ $(\vec{V}.\vec{u'})\frac{p'}{c^2}$	$\vec{\nabla}.p'\vec{u'} + \vec{\nabla}.\frac{p'^2}{\rho c^2}\vec{V} +$ $\vec{\nabla}.\frac{p'}{c}(\vec{V}.\vec{u'})\vec{V} +$ $\vec{\nabla}.\rho(\vec{V}.\vec{u'})\vec{V}$	$\frac{\gamma-1}{\bar{\rho}\bar{c}^2}\omega_T'(p' + \bar{\rho}\bar{u}u') -$ $\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\left[(\gamma - 1 - \gamma\bar{M}^2)\left(\frac{p'^2}{\gamma^2\bar{p}^2} + \frac{\bar{u}p'u'}{\gamma\bar{\rho}\bar{c}^2}\right)\right] -$ $\frac{(\gamma-1)\bar{\omega_T}}{1-\bar{M}^2}\left[(1-\bar{M}^2)\frac{u'^2}{\bar{c}^2} + \frac{\bar{M}\rho'}{\bar{\rho}}\left(\frac{u'}{\bar{c}} + \frac{\bar{M}p'}{\gamma\bar{p}}\right)\right] - \frac{\bar{M}^2}{(1-\bar{M}^2)}(\bar{u}p' +$ $\bar{\rho}\bar{c}^2 u')\frac{s'}{c_p}\frac{dA}{dx}$

Table 6.3 - *Conservation equations related to acoustic disturbance energy*

Type of equation	Energy	∇.(Fluxes)	Source terms
Nonlinear Eq.(**2**)[Eq.(6.18)]	$\int_T \frac{p}{2rc_p}\frac{\partial s'^2}{\partial t}dt$	NO	$-\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right] +$ $\frac{s'}{c_p}\left[\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right] - \frac{\rho T}{\overline{\rho T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right]\right] -$ $\frac{s'}{c_p}\left[\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] - \frac{\rho T}{\overline{\rho T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]\right]$
Linear Eq.(**2**)[Eq.(6.19)] Hyp. : Linearization	$\frac{p_0}{2r_0 c_{p0}}s_1^2$	NO	$-\frac{p_0 s_1}{r_0 c_{p0}}\left(\vec{u_1}.\vec{\nabla}s_0 + \vec{u_0}.\vec{\nabla}s_1\right) +$ $\frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} + \phi_1 - \vec{\nabla}.\vec{q_1}\right] +$ $\frac{s_1}{c_{p0}}\left[\left(-\frac{\rho_1}{\rho_0} - \frac{T_1}{T_0}\right)\left(\dot{\omega}_{T0} + \phi_0 - \vec{\nabla}.\vec{q_0}\right)\right] -$ $\frac{\rho_0 s_1}{c_{p0}}\left[-\frac{T_1}{T_0}\sum_k g_{sk0}\frac{DY_{k0}}{Dt} + \sum_k g_{sk1}\frac{DY_{k0}}{Dt}\right] -$ $\frac{\rho_0 s_1}{c_{p0}}\left[\sum_k g_{sk0}\frac{\partial Y_{k1}}{\partial t}\right]$
Simplified linear Eq.(**2**)[Eq.(6.25)] Hyps. : summarized on page 183	$\frac{p_0}{2r_0 c_{p0}}s_1^2$	NO	$-\frac{p_0}{2r_0 c_{p0}}\vec{u_0}.\vec{\nabla}s_1^2 +$ $\left[\frac{T_1}{T_0} - \frac{\gamma_0 - 1}{\gamma_0}\frac{p_1}{p_0}\right]\left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1}\right] -$ $\left[\frac{T_1}{T_0} - \frac{\gamma_0 - 1}{\gamma_0}\frac{p_1}{p_0}\right]\frac{p_1}{p_0}\dot{\omega}_{T0}$
Chu linear equation [23] **(reacting flow)**	$\frac{1}{2}\rho_0\vec{u_1}.\vec{u_1} +$ $\frac{1}{2}\rho_0 c_0^2\left(\frac{p_1}{\gamma_0 p_0}\right)^2 + \frac{p_0}{2r_0 c_{p0}}s_1^2$	NO	$\frac{T_1}{T_0}\omega_{T1} - \frac{\lambda_0}{T_0}(\nabla T_1)^2 - \Phi_1$

Table 6.4 - *Conservation equations related to entropy disturbance energy*

Type of equation	Energy	Flux	Source terms
Nonlinear Eq.**(3)**[Eq.(6.37)]	$\rho\left(H-\overline{H}\right)-\rho\bar{T}\left(s-\overline{s}\right)-\overline{\vec{m}}.(\vec{u}-\overline{\vec{u}})-(p-\overline{p})-\sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k-\overline{Y_k})$	$\vec{\nabla}.\left(\vec{m}-\overline{\vec{m}}\right)\left(H-\overline{H}\right)-\vec{\nabla}.\bar{T}(s-\overline{s})-\vec{\nabla}.\overline{\vec{m}}(T-\bar{T})\left(s-\overline{s}\right)$	$-(\vec{m}-\overline{\vec{m}})(\vec{\zeta}-\overline{\vec{\zeta}})-(s-\overline{s})(\vec{m}-\overline{\vec{m}})\vec{\nabla}\bar{T}+\overline{\vec{m}}(s-\overline{s})\vec{\nabla}(T-\bar{T})+(T-\bar{T})(Q-\bar{Q})+(\vec{m}-\overline{\vec{m}}).(\vec{\psi}-\overline{\vec{\psi}})+(T-\bar{T})(Q^{\star}-\overline{Q^{\star}})+(\vec{m}-\overline{\vec{m}}).(\vec{\psi^{\star}}-\overline{\vec{\psi^{\star}}})+\sum_{k=1}^{n}g'_{sk}\Omega'_k+\sum_{k=1}^{n}g'_{sk}Y_k\vec{\nabla}.\vec{m'}+\sum_{k=1}^{n}\overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m'}$
Linear Eq.**(3)**[Eq.(6.38)] Hyp. : Linearization	$\frac{p_1^2}{2\rho_0 c_0^2}+\frac{1}{2}\rho_0\vec{u}_1.\vec{u}_1+\rho_1\vec{u}_0.\vec{u}_1+\frac{\rho_0 T_0 s_1^2}{2c_{p0}}+E_{Y2}$	$(p_1+\rho_0\vec{u}_0.\vec{u}_1)\left(\vec{u}_1+\frac{\rho_1}{\rho_0}\vec{u}_0\right)+\vec{m}_0 T_1 s_1$	$s_1\vec{m}_1.\vec{\nabla}T_0+s_1\vec{m}_0.\vec{\nabla}T_1+\frac{T_1}{T_0}(\omega_{T1}-\vec{\nabla}.\vec{q}_1)+T_1(\omega_{T0}-\vec{\nabla}.\vec{q}_0)\left(\frac{1}{T}-\frac{1}{T_0}\right)+T_1Q_1^*+\vec{m}_1.\vec{\psi}_1^*+\sum_{k=1}^{n}g_{sk1}\Omega_{k1}+\sum_{k=1}^{n}\left(g_{sk1}Y_{k1}+g_{sk0}Y_{k1}\right)\vec{\nabla}.\vec{m}_1$
Myers nonlinear equation [88] **(non reacting flow)**	$\rho[H-\overline{H}-\overline{T}(s-\overline{s})]-\overline{\vec{m}}.(\vec{u}-\overline{\vec{u}})-(p-\overline{p})$	$(\vec{m}-\overline{\vec{m}}).[H-\overline{H}-\overline{T}(s-\overline{s})]+\overline{\vec{m}}(T-\overline{T})(s-\overline{s})-(\vec{m}-\overline{\vec{m}}).\left(\frac{\tau}{\rho}-\frac{\overline{\tau}}{\overline{\rho}}\right)+(T-\overline{T})\left(\frac{\vec{q}}{T}-\frac{\overline{\vec{q}}}{\overline{T}}\right)$	$-(\vec{m}-\overline{\vec{m}}).[\vec{\zeta}\times\vec{u}-\overline{\vec{\zeta}}\times\overline{\vec{u}}+(s-\overline{s})\vec{\nabla}\overline{T}]+(s-\overline{s})\overline{\vec{m}}.\vec{\nabla}(T-\overline{T})-\left(\frac{\tau}{\rho}-\frac{\overline{\tau}}{\overline{\rho}}\right).\nabla(m-\overline{m})+(\vec{m}-\overline{\vec{m}})\left(\frac{\tau}{\rho^2}.\vec{\nabla}\rho-\frac{\overline{\tau}}{\overline{\rho}^2}.\vec{\nabla}\overline{\rho}\right)+(T-\overline{T})\left(\frac{\Phi}{T}-\frac{\overline{\Phi}}{\overline{T}}\right)+\left(\frac{\vec{q}}{T}-\frac{\overline{\vec{q}}}{\overline{T}}\right).\vec{\nabla}(T-\overline{T})-(T-\overline{T})\left(\frac{\vec{q}.\vec{\nabla}T}{T^2}-\frac{\overline{\vec{q}}.\vec{\nabla}\overline{T}}{\overline{T}^2}\right)$
Myers linear equation [88] **(non reacting flow)**	$\frac{p_1^2}{2\rho_0 c_0^2}+\frac{1}{2}\rho_0 u_1^2+\rho_1\vec{u}_0.\vec{u}_1+\frac{\rho_0 T_0 s_1^2}{2C_{p0}}$	$\vec{m}_1.\left(\frac{p_1}{\rho_0}+\vec{u}_0.\vec{u}_1\right)-\vec{m}_1\left(\frac{\tau}{\rho}\right)_1+\vec{m}_0 T_1 s_1+T_1\left(\frac{\vec{q}}{T}\right)_1$	$\rho_0\vec{u}_0\vec{\zeta}_1\times\vec{u}_1+\rho_1\vec{u}_1\vec{\zeta}_0\times\vec{u}_0-\vec{m}_1 s_1\vec{\nabla}T_0+s_1\vec{m}_0.\vec{\nabla}T_1-\left(\frac{\tau}{\rho}\right)_1.\nabla m_1+\vec{m}_1\left(\frac{\tau}{\rho^2}.\vec{\nabla}\rho\right)_1+T_1\left(\frac{\Phi}{T}\right)_1+\left(\frac{\vec{q}}{T}\right)_1.\vec{\nabla}T_1-T_1\left(\frac{\vec{q}.\vec{\nabla}T}{T^2}\right)_1$

Table 6.5 - *Conservation equations related to total disturbance energy*

Chapter 7

Results

This chapter presents the results concerning balance closure of disturbance energy equations defined in chapter 6. Notations for the different terms involved in each balance may be found in Annex E.

- **General organisation**

  Two different configurations are studied and Fig.(7.1) presents a schematic representations of these configurations.

  For each configuration, different test cases are computed :

  - Configuration **A** (Section 7.1) a 1-D laminar flame. Two cases are studied :
    - ∗ Case **A1** : the inlet is forced with an entering acoustic wave at 287Hz and an amplitude corresponding to $0.1m.s^{-1}$ for the acoustic inlet velocity. The baseline flow is the initial steady solution.
    - ∗ Case **A2** : the inlet is forced with an entering acoustic wave at 57Hz and an amplitude corresponding to $0.1m.s^{-1}$ for the acoustic inlet velocity. The baseline flow is the initial steady solution.
  - Configuration **B** (Section 7.2) A 2-D laminar flame. Three cases are studied:
    - ∗ Case **B1** : the inlet is forced with an entering acoustic wave at 600Hz and an amplitude corresponding to $0.05m.s^{-1}$ for the acoustic velocity. The baseline flow is the initial steady solution.
    - ∗ Case **B2** : the outlet acoustic reflection coefficient is increased so that the flame becomes unstable at 856Hz. After 44ms, the reflection coefficient is released and the system relaxes to its steady state. The baseline flow is the initial steady solution.
    - ∗ Case **B2bis** : It is the same case as **B2**, but the post-processing of solutions differ. The baseline flow is the mean flow defined using all the snapshots gathered during the computation.

  Three different equations balance closure are tested.

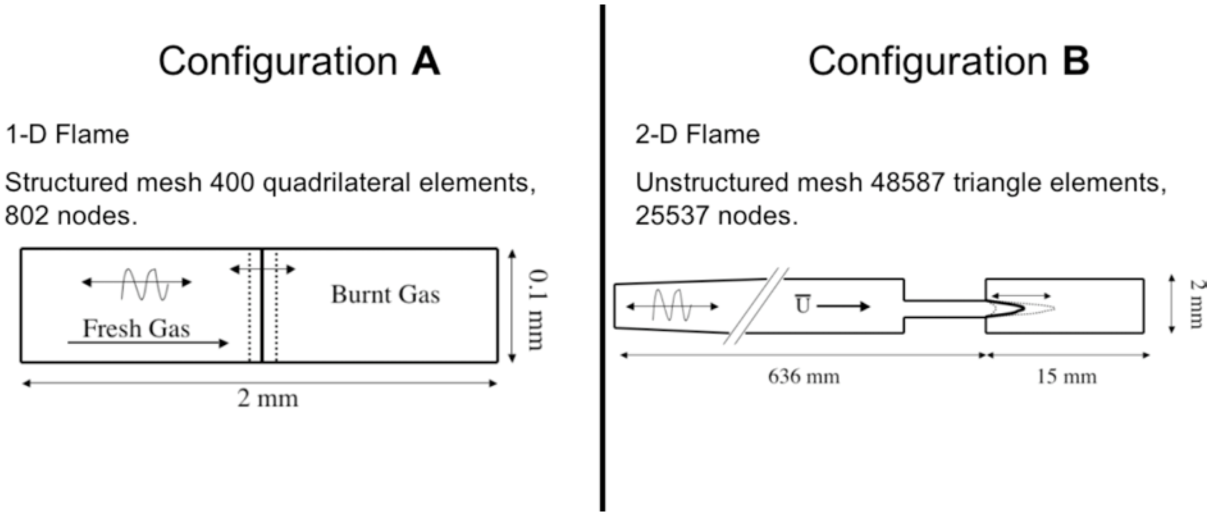


Figure 7.1 - *Scheme of the configurations with mesh informations.*

- Eq.(**1**)[Eq.(6.10)] : a nonlinear conservation equation in p'^2 and u'^2 defined by Eq.(6.10)
- Eq.(**2**)[Eq.(6.18)] : a nonlinear conservation equation in s'^2 defined by Eq.(6.18)
- Eq.(**3**)[Eq.(6.37)] : a nonlinear conservation equation for a general disturbance energy defined by Eq.(6.37)

Tables 6.3, 6.4 and 6.5 summarize these equations. **Note that "minimum" and "minimum linear" equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**
When needed, one uses simplified notations for the different R-H-S terms of Eqs.(**1**), (**2**) and (**3**). These notations may be found in appendix E on page 313

- **Presentation**

  Table 7 sums up the investigated balance closures and gives for each test done (marked with a $\times$) the page where this test can be found.

  For each case, the results will be presented as shown on Fig.(7.2) for Eqs.(**1**),(**2**) and (**3**).

  - First, the nonlinear equation is closed. This step mainly checks the accuracy of the POSTTIT post-processing tool applied to complex equations closures.
  - Then, the balance closure of the minimum equation is checked. This step answers the following question : what are the terms responsible for the evolution of the chosen disturbance energy? The arbitrary level of 10% of the absolute maximum amplitude of the time derivative term is the limit under which terms are not kept. This step will be used in chapter 8 to derive a nonlinear stability criterion in flow for the disturbance energy defined by Eq.(**3**)[Eq.(6.37)].
  - Finally, the balance closure of the minimum linear equation is presented. This step will be used in chapter 8 to derive stability criteria in flow for disturbance energies defined by linearized Eqs.(**1**),(**2**).

Case / Equation	**A1** (sec 7.1.1)	**A2** (sec 7.1.2)	
Eq **1**	$\times$ (p 199)	$\times$ (p 204)	
Eq **2**	$\times$ (p 199)	$\times$ (p 205)	
Eq **3**	$\times$ (p 201)	$\times$ (p 205)	
Case / Equation	**B1** (sec 7.2.1)	**B2** (sec 7.2.2)	**B2bis** (sec 7.2.3)
Eq **1**	$\times$ (p 210)	$\times$ (p 223)	$\times$ (p 240)
Eq **2**	$\times$ (p 216)	$\times$ (p 229)	$\times$ (p 247)
Eq **3**	$\times$ (p 220)	$\times$ (p 235)	$\times$ (p 253)

Table 7.1 - *Cases studied and corresponding balance equations tests*

P_0 is the mean linear power of the flame and only depends here on the configuration used. For case **A**, $P_0 = 5W.m^{-1}$. For case **B**, $P_0 = 3950W.m^{-1}$ Nonlinear Eq.(**3**)$_{[Eq.(6.37)]}$ already describes the evolution of a disturbance energy in contrary to Eq.(**1**)$_{[Eq.(6.10)]}$ and Eq.(**2**)$_{[Eq.(6.18)]}$. The minimum Eq.(**3**)$_{[Eq.(6.37)]}$ therefore gives a nonlinear stability criterion in flow which is one of the goals of this study. For this reason, no balance closure for the linearization of the minimum Eq.(**3**)$_{[Eq.(6.37)]}$ will be presented here.

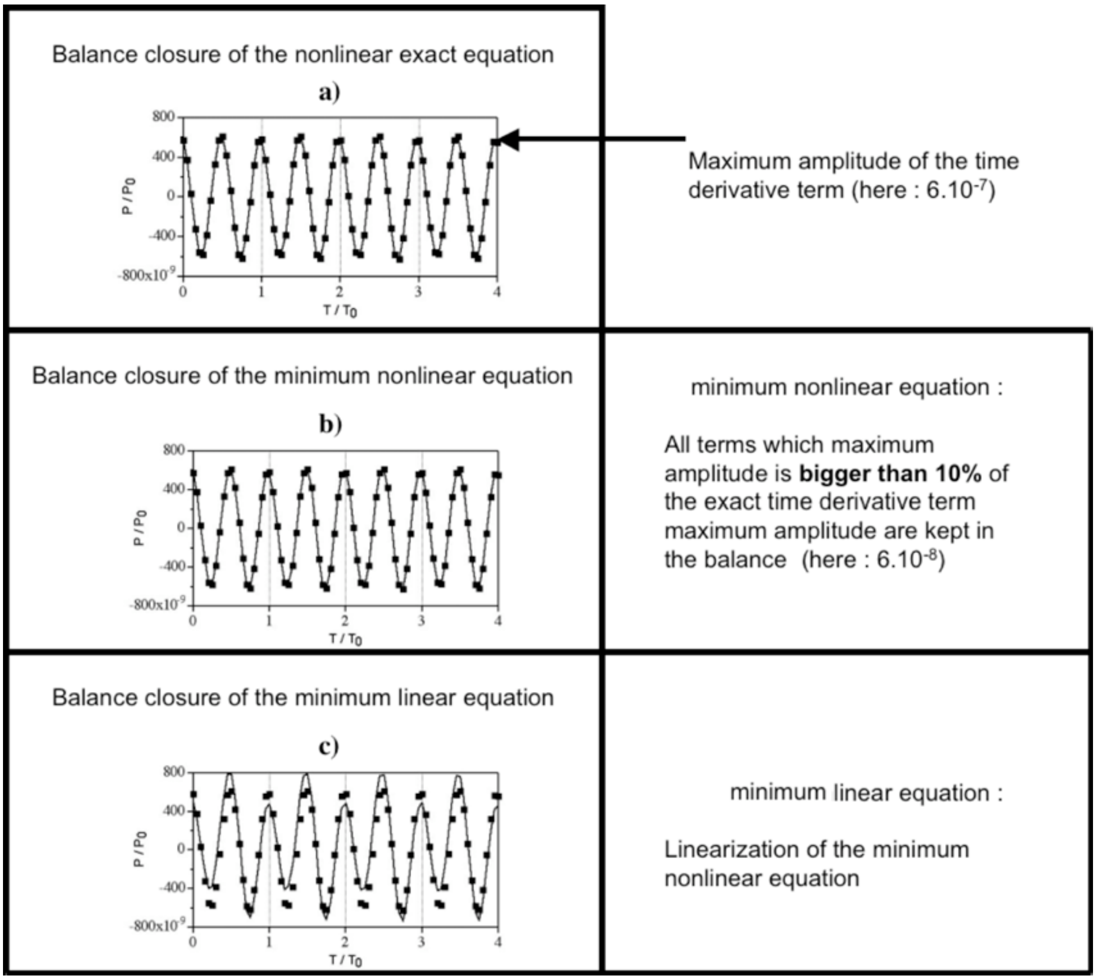


Figure 7.2 - *Presentation of the results*

7.1 Configuration A (1-D Reacting Case)

7.1.1 Case A1 (Forced case : F_0=287Hz)

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** . **Note that "minimum" and "minimum linear" equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**

Equation 1[Eq.(6.10)]

Fig.(7.3) **a)** shows a perfect agreement between the sum $\mathbf{deudt} + \mathbf{depdt}$ and the sum of the volume integrals of all R-H-S terms of Eq.(**1**)[Eq.(6.10)]. As already mentioned, this step mainly checks the ability of the post-processing tool (POSTTIT) to close the balance of Eq.(**1**)[Eq.(6.10)]. The maximum power reached never exceeds $4.0\ 10^{-4}\%$ of the mean power of the flame. This means that the pressure and velocity fluctuations only contain a very small power compared to the chemical power released by the flame.
The major terms are $\mathbf{Ac_1}$, $\mathbf{Ac_2}$. These terms are respectively:

- $\mathbf{Ac_1} : -\frac{p'}{\gamma p}\left[\gamma p \vec{\nabla}.\vec{u} - \overline{\gamma p}\vec{\nabla}.\vec{\overline{u}}\right]$
- $\mathbf{Ac_2} : -\vec{u'}.\vec{\nabla}p'$

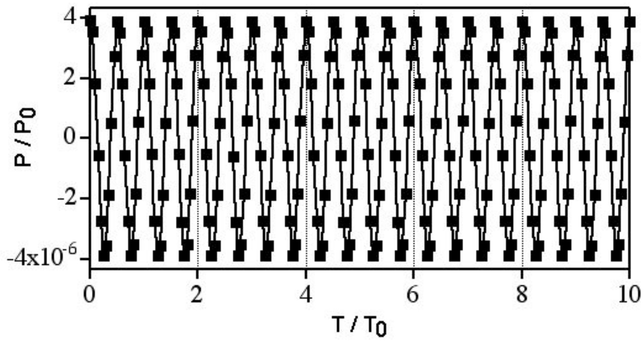
Fig.(7.3) **b)** shows the closure of the minimum Eq.(**1**). The fluctuations of heat release and heat flux (terms $\mathbf{Ac_3}$, $\mathbf{Ac_4}$ (not shown)) are very small when compared to the major terms. In this case, the flame does not directly interacts with the acoustic forcing. The numerical terms are small and do not influence the sum of R-H-S terms of Eq.(**1**).

Fig.(7.3) **c)** shows the closure of the minimum linear Eq.(**1**) (not shown). The linearizing process does not influences the closure, and the minimum linearized Eq.(**1**) shrinks to the classical non reacting linear acoustic energy conservation equation. It is as if in this case, the acoustic disturbance energy introduced at the inlet by the forcing was just propagating through the system without interacting with the flame.

Equation 2[Eq.(6.18)]

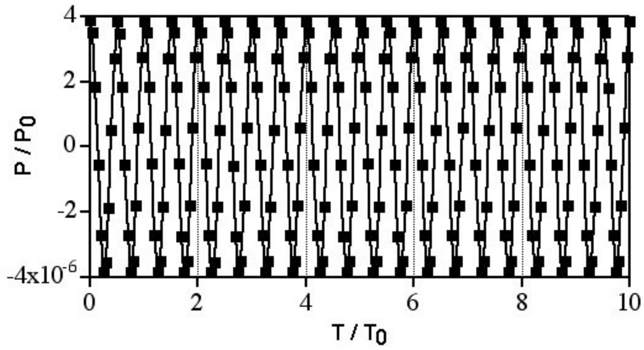
Figs.(7.4) **a)** and **b)** respectively show the balance of the exact and minimum Eq.(**2**). The only term which amplitude is bigger than 10% of the maximum absolute amplitude of the time derivative term is $\mathbf{En_1} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \vec{\overline{u}}.\vec{\nabla}\overline{s}\right]$. In this case, $\mathbf{En_2}$ and $\mathbf{En_3}$ standing for the influence of heat release and heat flux fluctuations are almost negligible. The flame therefore has a negligible direct influence of the flame on the time derivative term $\mathbf{de_s dt}$.

a)



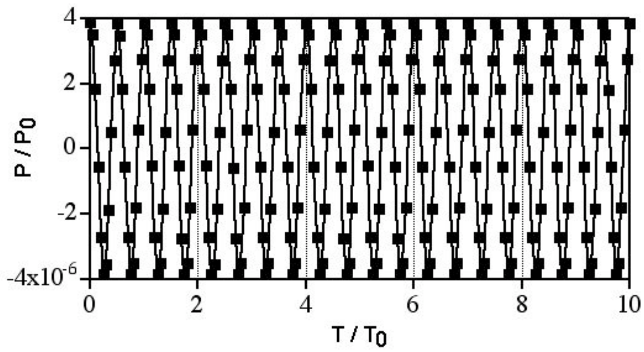
Maximum amplitude of time derivative : 4.10^{-6}

b)



$$\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} + \frac{\rho}{2}\frac{\partial u'^2}{\partial t} = -\vec{u'}.\vec{\nabla}p' - \frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right]$$

c)



$$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} + \frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} + \vec{\nabla}.(p_1\vec{u_1}) = 0$$

Figure 7.3 - *Case* **A1** *: Eq.***(1)**

a) Exact nonlinear ▪ $\mathbf{dedt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dedt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{dedt}_l$ —— sum of terms

As shown in Fig.(7.4) **c)**, linearizing the minimum Eq.(**2**) has no influence on the closure. Therefore, keeping only the major terms of the balance leads to the following conservation equation for the entropy disturbance energy:

$$\frac{\partial e_s}{\partial t} = -\frac{p_0 s_1}{r_0 c_{p0}} \vec{u_1}.\vec{\nabla} s_0 \tag{7.1}$$

where $e_s = \frac{p_0}{2r_0 c_{p0}} s_1^2$. It is interesting to note that Eq.(7.1) is the linearized conservation equation of the entropy disturbance energy in a non-reacting flow but with mean gradient of entropy. This equation states that in this case, the entropy disturbance energy travels through the configuration without creating significant fluctuations of heat release and heat flux.
Having a closer look to the only remaining source term, and using the fact that under the present assumptions $s_1 = C_{p0}T_1$ and $s_0 = C_{p0}T_0$, it can be rewritten as $-\rho_0 C_{p0} T_1 \vec{u_1}.\vec{\nabla}\left(\frac{T_0^2}{2}\right)$. In this case, the creation or dissipation of the entropy disturbance energy therefore depends on the correlation of the temperature fluctuations with the velocity fluctuations. When both are in phase, the entropy disturbance energy is dissipated. If they are in opposition of phase, the entropy disturbance energy is created. Note that in the present case which is pulsated, the entropy disturbance energy neither grow or decay over a period of forcing.

Equation 3[Eq.(6.37)]

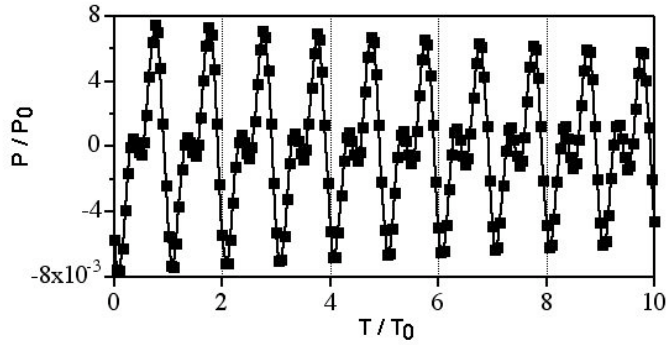
Fig.(7.5) **a)** shows the balance of the exact Eq.(**3**)[Eq.(6.37)].
The maximum amplitude of the time derivative of the disturbance energy represents almost 0.7% of the mean power of the flame. Fig.(7.5) **a)** shows many similarities with Fig.(7.4) **c)**. The disturbance energy is supposed to contain all disturbance energies in the flow. In this case, it mostly contains entropy disturbance energy as Fig.(7.5) **a)** and Fig.(7.4) **c)** show very close time derivatives for both disturbance energies.

The major term in this case is $\mathbf{D_s}$ and Fig.(7.5) **b)** shows the balance of the minimum Eq.(**3**).
The term related to the fluxes of disturbance energy through the boundaries is almost negligible, and is only responsible for 0.2% of the fluctuation of the time derivative of the disturbance energy (not shown). Considering that $\mathbf{D_s}$ is the only significant term and that $s'\overline{\vec{m}}.\vec{\nabla}T'$ is negligible (as shown in Fig.(7.6)), a simplified conservation equation for the disturbance energy can be written :

$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T} \tag{7.2}$$

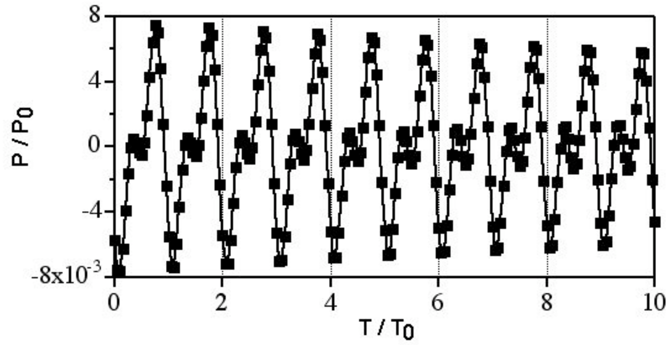
Eq.(7.2) states that in this case (a 1-D flame forced at 287Hz with a velocity amplitude equal to $0.1 m.s^{-1}$) the growth or decay of the disturbance energy is linked to fluctuations of entropy and momentum in presence of a gradient of mean temperature. This term is indirectly related to the flame since the flame is the cause of the gradient of temperature in the flow.

a)



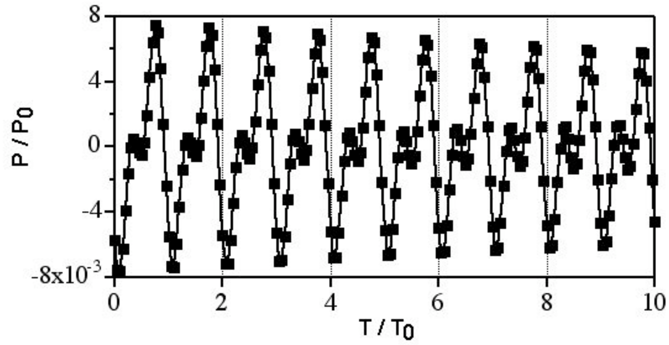
Maximum amplitude of time derivative : 8.10^{-3}

b)



$$\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right]$$

c)

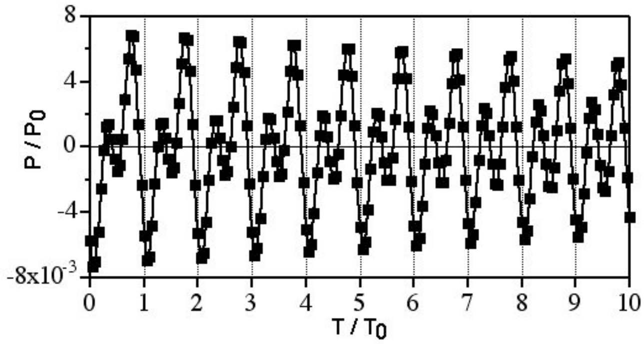


$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} = -\frac{p_0s_1}{r_0c_{p0}}\vec{u_1}.\vec{\nabla}s_0$$

Figure 7.4 - *Case* **A1** *: Eq.***(2)**

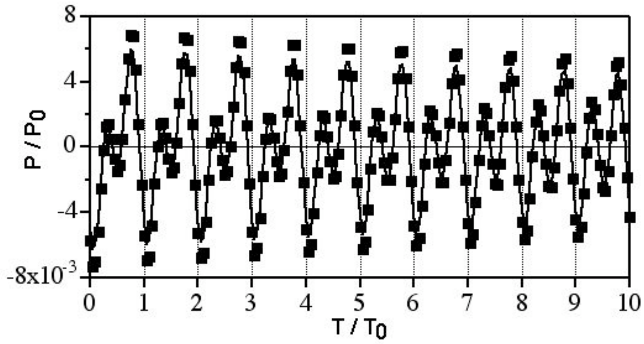
a) Exact nonlinear ■ $\mathbf{de_s dt}$ —— sum of terms
b) Minimum nonlinear ■ $\mathbf{de_s dt}$ —— sum of terms
c) Minimum linear ■ $\mathbf{de_s dt}_l$ —— sum of terms

a)



Maximum amplitude of time derivative : 7.10^{-3}

b)



$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T}$$

Figure 7.5 - *Case* **A1** *: Eq.***(3)**

a) Exact nonlinear ■ $\mathbf{dE_d dt}$ —— sum of terms

b) Minimum nonlinear ■ $\mathbf{dE_d dt}$ —— sum of terms

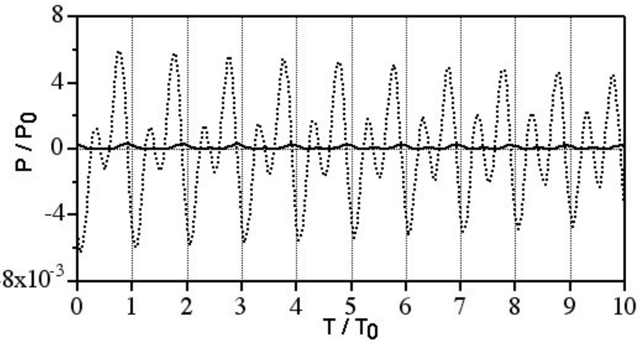


Figure 7.6 - *Case* **A2** *: Eq.***(3)** —— $s'\overline{\vec{m}}.\vec{\nabla}T'$ ········ $-s'\vec{m}'\vec{\nabla}\bar{T}$

7.1.2 Case A2 (Forced case : F_0=57Hz)

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** . **Note that "minimum" and "minimum linear" equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**

Equation 1$_{\text{[Eq.(6.10)]}}$

Fig.(7.7) **a)** shows the balance of the nonlinear conservation equation for pressure and velocity fluctuations. Fig.(7.7) **a)** shows a perfect agreement between the time derivative term $\mathbf{dedt}$ and the sum of the volume integrals of the R-H-S terms of Eq.(**1**)$_{\text{[Eq.(6.10)]}}$ providing that the influence of the numerical corrections is taken into account (see Fig.(7.8)). The time derivative term is perfectly quadratic showing two oscillations per period of forcing.
Fig.(7.7) **b)** shows the balance of the minimum Eq.(**1**). The major physical terms that have to be kept are :

- $\mathbf{Ac_1}$: $-\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \overline{\gamma p}\vec{\nabla}.\overline{\vec{u}}\right]$
- $\mathbf{Ac_2}$: $-\vec{u'}.\vec{\nabla}p'$
- $\mathbf{Ac_3}$: $\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right]$
- $\mathbf{Ac_4}$: $-\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\overline{\vec{q}}\right]$

Fig.(7.8) shows the numerical correction linked with the 2nd spatial scheme used during the computation. The maximum absolute amplitude of this term almost represents 50% of the maximum amplitude of the time derivative term. In this case, this correction term is obviously necessary. Note that it is not in case for **A1** for which the first step of the scheme already gives a second order discretization since no numerical correction appear in Fig.(7.3). This is due to the behavior of the flame which is different between the two cases. In **A1**, the amplitude of the move of the flame front is much smaller than in the present case **A2**. This acts on the fluxes that have to be resolved by the code. If their value becomes important because the flame goes from one place to the other, the errors of discretization will also tend to increase which is the case here.

As shown in Fig.(7.7) **c)**, the linearization only has a small impact on the closure of the balance of the minimum Eq.(**1**). Compared to the previous case (see sections 7.1.1), the terms $\frac{\gamma_0-1}{\gamma_0 p_0}p_1(\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1})$ and $-\frac{p_1\gamma_1}{\gamma_0 p_0}(\dot{\omega_{T0}} - \vec{\nabla}.\vec{q_0})$ are much bigger when compared to $-\vec{u}_1.\vec{\nabla}p_1$. Therefore, direct interaction terms with the flame (involving fluctuations of heat release and heat flux) are present in the linear minimum Eq.(**1**). These terms have two major components :

- $\frac{\gamma_0 - 1}{\gamma_0 p_0} p_1 (\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1})$
  This term is composed of the fluctuations of heat release and heat flux coupled with the fluctuations of pressure. This term is an extension of the Rayleigh term with fluctuations of heat flux.

- $-\frac{p_1 \gamma_1}{\gamma_0 p_0} (\dot{\omega_{T0}} - \vec{\nabla}.\vec{q_0})$
  This term depends on the correlation of heat specific ratio fluctuations with pressure fluctuations. This term appears in multi-species cases.[1]

Note that the presence of terms based on γ_1 in the minimum linear Eq.(**1**) shows that even if γ_1 is generally considered as small enough to be neglected, the product $\gamma_1 \dot{\omega_{T0}}$ may not be because of the high local values that the mean heat release can take.

Equation 2[Eq.(6.18)]

Fig.(7.9) **a)** shows the balance of the nonlinear conservation equation for the s'^2 quantity. The derivative term $\mathbf{de_s dt}$ is not perfectly quadratic and has two frequency components at 57Hz (frequency of forcing) and 114Hz (quadratic component).

Fig.(7.9) **b)** shows the closure of the minimum Eq.(**2**).
Compared to case **A1** (see sections 7.1.1), the relative importance of the terms $\mathbf{En_2}$ and $\mathbf{En_3}$ compared to $\mathbf{En_1}$ has increased.
Fig.(7.10) compares the term $\mathbf{En_2}$ for cases **A1** and **A2**. The signals are normalized by the maximum absolute value of $\mathbf{de_s dt}$ in the two cases. The relative importance of this term is multiplied by 5 in case **A2**. This result is due to the cinematic effect of the velocity fluctuation on the flame front. When the period increases, this gives more time for the flame to move and therefore increases its covered area. This increase in area directly impacts the volume integral of heat release and heat flux fluctuations terms leading to the observed difference in Fig.(7.10).

Fig.(7.9) **c)** shows the closure of the minimum linear Eq.(**2**).
The linearizing process has no influence on the closing of the balance. In this case, the linearization of minimum Eq.(**2**) leads to :

$$\frac{\partial e_s}{\partial t} = -\frac{p_0 s_1}{r_0 c_{p0}} \vec{u_1}.\vec{\nabla} s_0 + \frac{s_1}{c_{p0}} \left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1} \right] \tag{7.3}$$

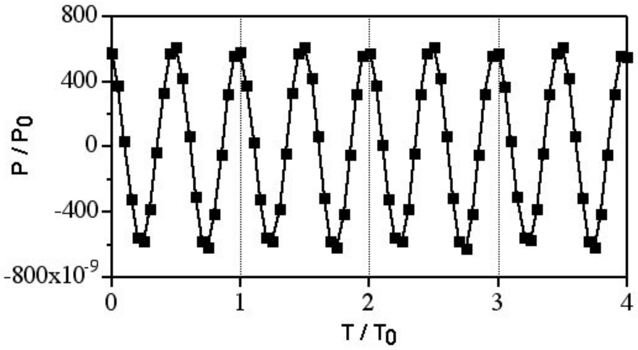
where $e_s = \frac{p_0}{2 r_0 C_{p0}} s_1^2$.

Equation 3[Eq.(6.37)]

Fig.(7.11) **a)** shows the balance of the nonlinear disturbance energy. The maximum amplitude of the time derivative term represents 3.4% of the mean power of the flame.

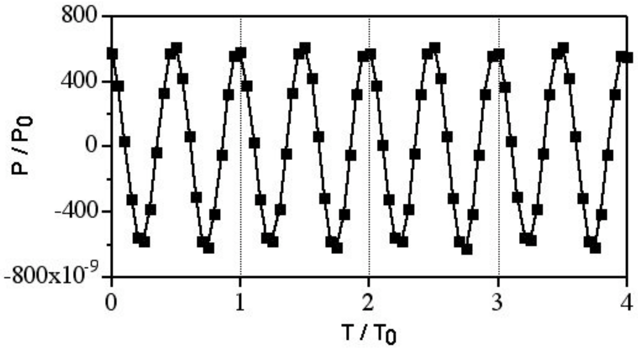
[1]This term is therefore present in any reacting case.

a)



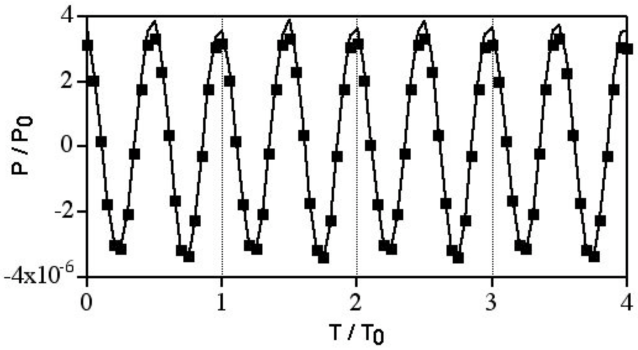
Maximum amplitude of time derivative : 6.10^{-7}

b)



$$\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} + \frac{\rho}{2}\frac{\partial u'^2}{\partial t} = -\vec{u'}.\vec{\nabla}p' - \frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \overline{\gamma p}\vec{\nabla}.\vec{\overline{u}}\right] + \frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right] - \frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\vec{\overline{q}}\right] + LW_{E_1}$$

c)



$$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} + \frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} = -\vec{u_1}.\vec{\nabla}p_1 + \frac{\gamma_0-1}{\gamma_0 p_0}p_1[\dot{\omega_{T1}} - \vec{\nabla}.\vec{q_1}] + \frac{p_1\gamma_1}{\gamma_0 p_0}[-p_0\vec{\nabla}.\vec{u_0} + \dot{\omega_{T0}} - \vec{\nabla}.\vec{q_0}] + LW_{E_1}$$

Figure 7.7 - *Case* **A2** *: Eq.***(1)**

a) Exact nonlinear ■ $\mathbf{dedt}$ —— sum of terms
b) Minimum nonlinear ■ $\mathbf{dedt}$ —— sum of terms
c) Minimum linear ■ $\mathbf{dedt}_l$ —— sum of terms

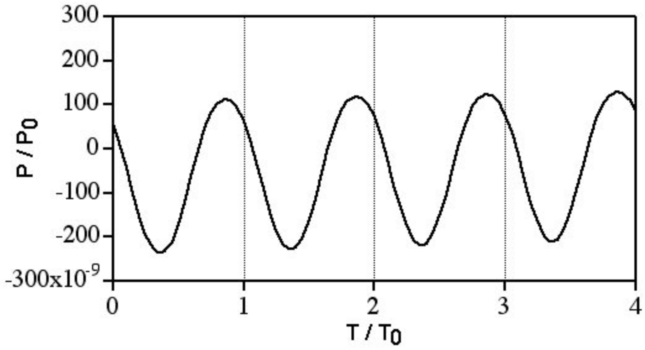


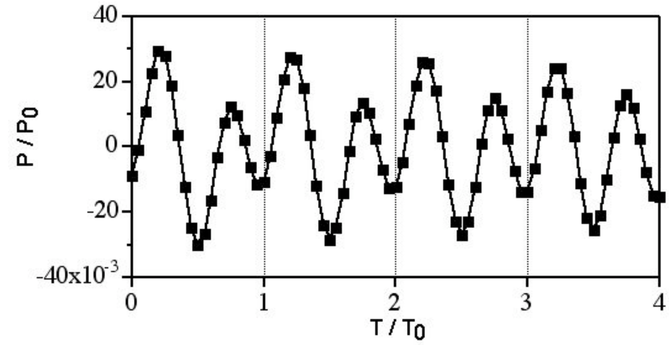
Figure 7.8 - *Case* **A2** *: Eq.***(1)** *Numerical correction* LW_{E_1}

Fig.(7.11) **b)** shows the closure obtained for minimum Eq.(**3**). Considering that $\mathbf{D_s}$, $\mathbf{D_Q}$ are the only significant terms, the minimum conservation equation for the disturbance energy writes in this case :

$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}}s'\vec{\nabla}T' + T'Q' \quad (7.4)$$

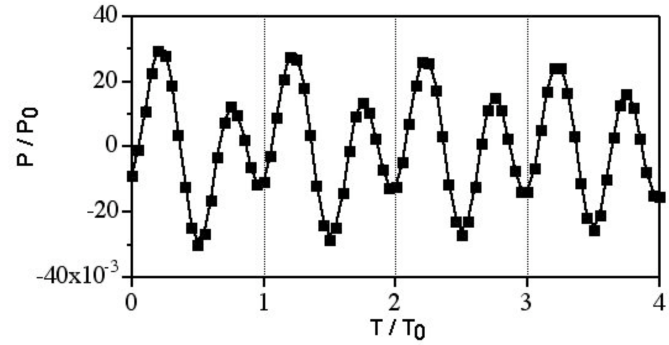
The relative amplitude of the term $T'Q'$ is increased compared to the case **A1** (not shown). This term represents the influence of the fluctuations of heat release and heat flux on the nonlinear disturbance energy. This increase is therefore consistent with what is observed for Eq.(**1**)[Eq.(6.10)] and (**2**).
Figs.(7.11) **a)** and Fig.(7.9) **a)** can also be compared. It shows that in this case, the nonlinear disturbance energy is almost entirely composed of entropy disturbance energy, disturbance energy in pressure and velocity fluctuations being negligible.

a)



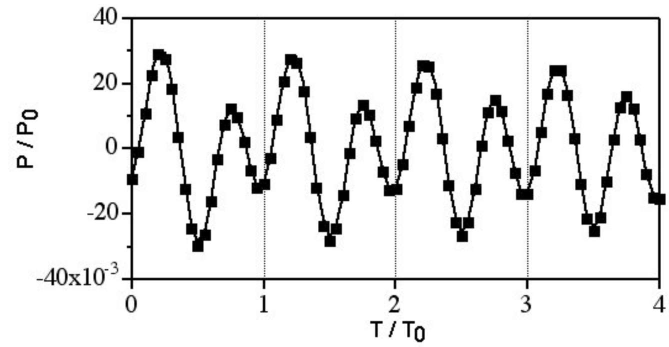
Maximum amplitude of time derivative : 3.10^{-2}

b)



$$\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right] + \frac{ps'}{rc_p}\left[\frac{1}{\rho T}\left[\dot{\omega}_T - \vec{\nabla}.\vec{q}\right] - \frac{1}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} - \vec{\nabla}.\overline{\vec{q}}\right]\right]$$

c)



$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} = -\frac{p_0 s_1}{r_0 c_{p0}}\vec{u_1}.\vec{\nabla}s_0 + \frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1}\right]$$

Figure 7.9 - *Case* **A2** *: Eq.***(2)**

a) Exact nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{de_s dt}_l$ —— sum of terms

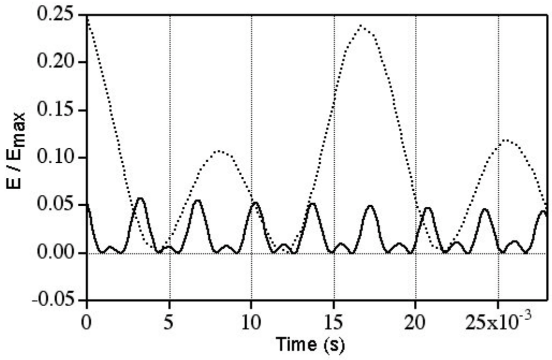
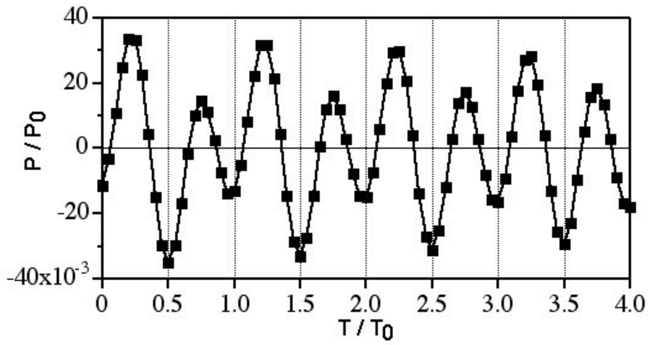


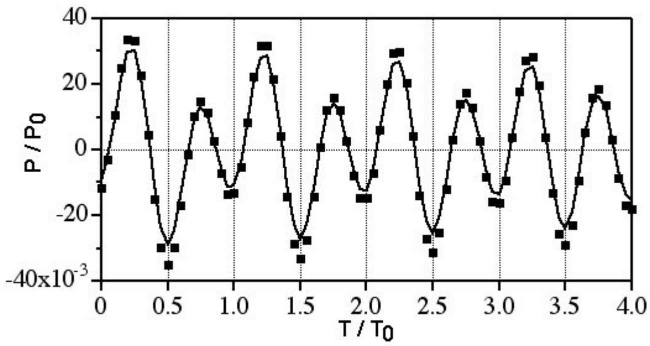
Figure 7.10 - *Normalized amplitudes of* $\mathbf{En_2}$ **A1**(——), **A2**(········)
Amplitudes are normalized by the maximum absolute amplitude of $\mathbf{dedt}$ *:* $\mathbf{E_{max}}$. (0.03 *for* **A1** *and* 0.008 *for* **A2**).
Time is shifted to compare both cases.

a)



Maximum amplitude of time derivative : $3, 4.10^{-2}$

b)



$$\frac{\partial E_d}{\partial t} = -s'\vec{m'}\vec{\nabla}\bar{T} + \vec{\overline{m}}s'\vec{\nabla}T' + T'Q'$$

Figure 7.11 - *Case* **A2** *: Eq.***(3)**

a) Exact nonlinear ▪ $\mathbf{dE_ddt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dE_ddt}$ —— sum of terms

7.2 Configuration B (2-D Reacting Case)

7.2.1 Case B1 (Forced Case : F_0=600 Hz)

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** . **Note that ”minimum” and ”minimum linear” equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**

Equation 1[Eq.(6.10)]

Figs.(7.16, 7.17) present the balance of the nonlinear conservation equation for quadratic pressure and velocity fluctuations. The evolution of the balance is split in two main phases. First the flame is forced at 600Hz during almost 14 periods of oscillation and then the forcing is stopped and the system relaxes to its initial state.
In the following, the analysis of the balance of Eq.(**1**)[Eq.(6.10)] is first done for the forced phase and then the relaxation phase is studied.

- Forced phase

  Fig.(7.16) **a)** shows the balance of the exact nonlinear conservation equation for quadratic pressure and velocity fluctuations during the forced phase.
  The maximum amplitude of the time derivative term is small compared to the mean power of the flame (3.10^{-4}%). The first striking particularity of the time derivative term $\mathbf{dedt}$ compared to the previous configuration **A** is that no particular frequency component appears. This term is not quadratic. Fig.(7.12) shows the frequency spectrum of $\mathbf{dedt}$. The predominant component is the quadratic component at 1200Hz, but a strong component is also present at 1450 Hz (55% of the amplitude of the quadratic component). Fig.(7.12) also shows that many harmonics of these two frequencies are present in the signal with for example a non negligible component at 250Hz which can be observed in Fig.(7.16) **a)** with a repeated pattern every two periods of forcing.

  Fig.(7.13) shows the two components of the time derivative term $\mathbf{dedt}$. During the first one and a half period, both deudt and depdt are in phase and strictly positive. After this first phase, both are almost quadratic, in opposition of phase and their absolute maximum amplitude grows rapidly and then shows patterns at approximately 250Hz.

  Fig.(7.16) **b)** shows the closure of the balance of the minimum Eq.(**1**). The major terms involved in this equation (having an amplitude greater than 10 % of the maximum amplitude of the time derivative term) are :

  – $\mathbf{Ac_1}$: $-\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right]$
  – $\mathbf{Ac_2}$: $-\vec{u'}.\vec{\nabla}p'$

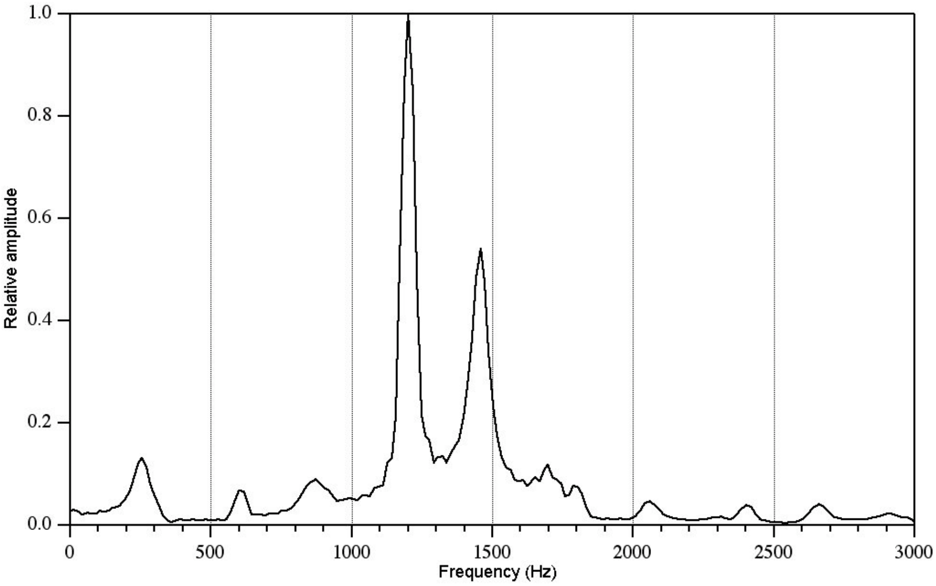


Figure 7.12 - *Frequency spectrum of the time derivative term* **dedt**. *All amplitudes are scaled by the quadratic component amplitude (1200Hz)*

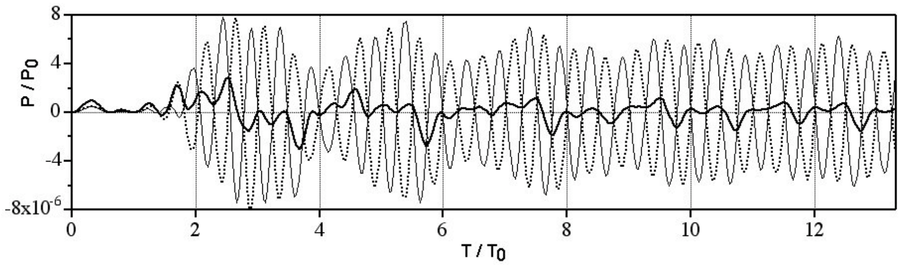


Figure 7.13 - *Case* **B1**, *Eq.***(1)** *:* **Forced phase** *Composition of the time derivative term* —— **dedt** —— **deudt** ······· **depdt**

- $\mathbf{Ac_3}$: $\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T}-(\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right]$
- $\mathbf{Ac_4}$: $-\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q}-(\overline{\gamma}-1)\,\vec{\nabla}.\overline{\vec{q}}\right]$
- $\mathbf{Ac_5}$: $\vec{u'}:\vec{\nabla}\tau'$
- $\mathbf{Ac_6}$: $-\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N}h_{sk}\rho\vec{u}.\vec{\nabla}Y_k-(\overline{\gamma}-1)\sum_{k=1}^{N}\overline{h_{sk}}\overline{\rho}\overline{\vec{u}}.\vec{\nabla}\overline{Y_k}\right]$
- $\mathbf{Ac_7}$: $\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{\partial r}{\partial t}\right]$
- $\mathbf{Ac_8}$: $\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\vec{u}.\vec{\nabla}r-\frac{\bar{\gamma}\bar{p}}{\bar{r}}\vec{u}.\vec{\nabla}\overline{r}\right]$
- $\mathbf{LW_{E_1}}$

Compared to the previous forced configuration **A1**, the direct influence of the flame through the terms $\mathbf{Ac_3}$ and $\mathbf{Ac_4}$ is important. Furthermore, contrasting with cases **A1** and **A2**, here the viscous term $\mathbf{Ac_5}$ is not negligible anymore.

All terms except $\mathbf{Ac_1}$ and $\mathbf{Ac_2}$ (that shows a very small peak in Fig.(7.14)) are exactly zero during almost one and a half period after the beginning of the forcing. This delay approximately corresponds to the time needed by the acoustic perturbation to go from the inlet to the flame. After this first phase, $\mathbf{Ac_3}$ which represents the combustion source term is strictly positive. Note that the viscous term ($\mathbf{Ac_5}$) although not shown here is a strictly dissipative term.

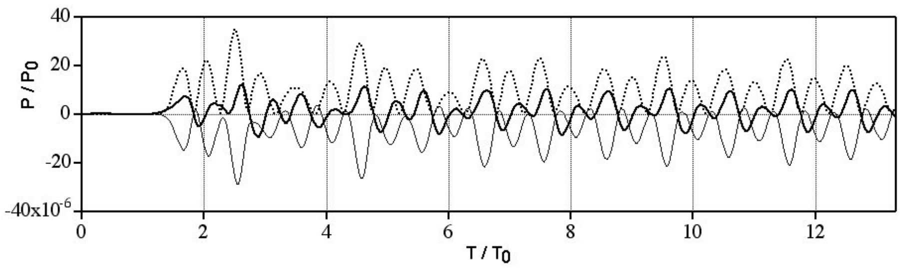


Figure 7.14 - *Case* **B1**, *Eq.*(**1**) *:* **Forced phase** *Representative terms*
—— $\mathbf{Ac_2}$ —— $\mathbf{Ac_6}$ ········ $\mathbf{Ac_3}$

The minimum Eq.(**1**) closure requires the Lax-Wendroff numerical correction term. Its maximum amplitude is two times bigger than the maximum amplitude of the time derivative term. In comparison, the numerical term introduced by the boundaries remains almost negligible (not shown).

Fig.(7.16) **c)** shows that the linearization of the minimum Eq.(**1**) lowers the quality of the closure. Periodically, the sum of the R-H-S terms of the minimum linear Eq.(**1**) overestimate the time derivative $\mathbf{dedt}_l$. Note that the closure is almost perfect during the first one and a half period which corresponds to the time needed for the excitation to reach the flame. It can therefore be assumed that when the flame moves, it introduces nonlinearities that are visible in Fig.(7.16) **c)**.

- Relaxation

Fig.(7.17) **a)** shows a perfect closure of Eq.(**1**)[Eq.(6.10)] provided that the numerical correction term is taken into account. During the first two periods, the derivative term keeps its complex frequency structure. Then it oscillates at approximately 1700Hz and its maximum amplitude decreases. After two periods, the time derivative term related to pressure fluctuations almost vanishes so that the total time derivative term follows the term $\mathbf{deudt}$ which oscillates harmonically at 1700Hz (see Fig.(7.15)). This shows that the relaxation process appends at a different frequency from the frequency of forcing. Since all terms are quadratic, it means that all fluctuations have a frequency around 850Hz as the system goes back to its steady state. In the following case **B2** which is self-excitated, it will be shown that this frequency is an eigen-frequency of the system. Note that (not shown here) the numerical correction term decreases less rapidly than the other terms and therefore becomes more and more significant as the steady state is reached.

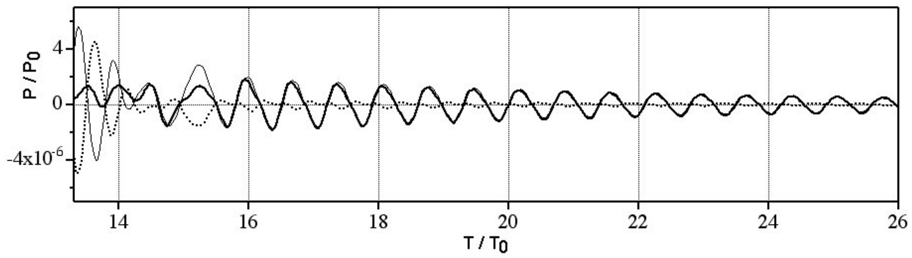
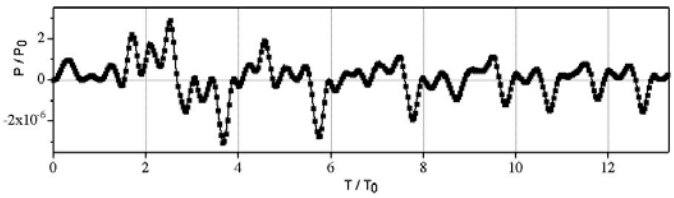


Figure 7.15 - *Case* **B1**, *Eq.***(1)** *:* **Relaxation** *Composition of the time derivative term*
—— **dedt** —— **deudt** ········ **depdt**

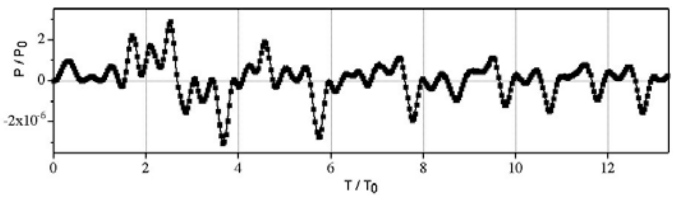
Fig.(7.17) **c)** shows that the minimum linear Eq.(**1**) better closes as the system returns to a steady state. Overshoots almost disappear 10 periods after the excitation is stopped. This behavior let think that it is indeed nonlinearities created by the excitation that explain the differences between Fig.(7.17) **b)** and Fig.(7.17) **c)**.

a)



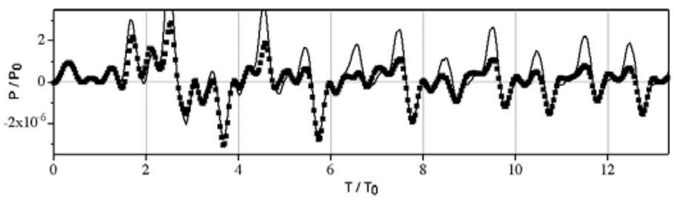
Maximum amplitude of time derivative : 3.10^{-6}
ALL PHASES

b)



$$\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} + \frac{\rho}{2}\frac{\partial u'^2}{\partial t} = -\vec{u'}.\vec{\nabla}p' - \frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right] + \frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\bar{\gamma}-1)\,\overline{\dot{\omega_T}}\right] - \frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\bar{\gamma}-1)\,\vec{\nabla}.\vec{\bar{q}}\right] + \vec{u'} : \vec{\nabla}\tau' - \frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N} h_{sk}\rho\vec{u}.\vec{\nabla}Y_k\right] + \frac{p'}{\gamma p}\left[(\bar{\gamma}-1)\sum_{k=1}^{N} \overline{h_{sk}}\bar{\rho}\vec{\bar{u}}.\vec{\nabla}\overline{Y_k}\right] + \frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\vec{u}.\vec{\nabla}r - \frac{\bar{\gamma}\bar{p}}{\bar{r}}\vec{\bar{u}}.\vec{\nabla}\bar{r}\right] + \frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{\partial r}{\partial t}\right] + \mathbf{LW_{E_1}}$$

c)



$$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} + \frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} = -\vec{u_1}.\vec{\nabla}p_1 - p_1\vec{\nabla}.\vec{u}_1 - \frac{\gamma_1}{\gamma_0}p_1\vec{\nabla}.\vec{u}_0 + \frac{\gamma_0-1}{\gamma_0 p_0}p_1\dot{\omega_{T1}} + \frac{\gamma_1 p_1}{\gamma_0 p_0}\dot{\omega_{T0}} - \frac{\gamma_0-1}{\gamma_0 p_0}p_1\vec{q}_1 - \frac{\gamma_1 p_1}{\gamma_0 p_0}\vec{q}_0 + \vec{u}_1 : \vec{\nabla}\tau_1 - \frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk0}\rho_0\vec{u}_0.\vec{\nabla}Y_{k1}\right] - \frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk0}\rho_0\vec{u_1}.\vec{\nabla}Y_{k0}\right] - \frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk0}\rho_1\vec{u_0}.\vec{\nabla}Y_{k0}\right] - \frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk1}\rho_0\vec{u_0}.\vec{\nabla}Y_{k0}\right] + \frac{p_1}{r_0}\vec{u}_1.\vec{\nabla}r_0 + \frac{p_1}{r_0}\vec{u}_0.\vec{\nabla}r_1 + \frac{p_1}{r_0}\frac{\partial r_1}{\partial t} + \mathbf{LW_{E_1}}$$

Figure 7.16 - *Case* **B1**, *Eq.***(1)** *:* **Forced phase**

a)	Exact nonlinear	▪ $\mathbf{dedt}$ ——	sum of terms
b)	Minimum nonlinear	▪ $\mathbf{dedt}$ ——	sum of terms
c)	Minimum linear	▪ $\mathbf{dedt}_l$ ——	sum of terms

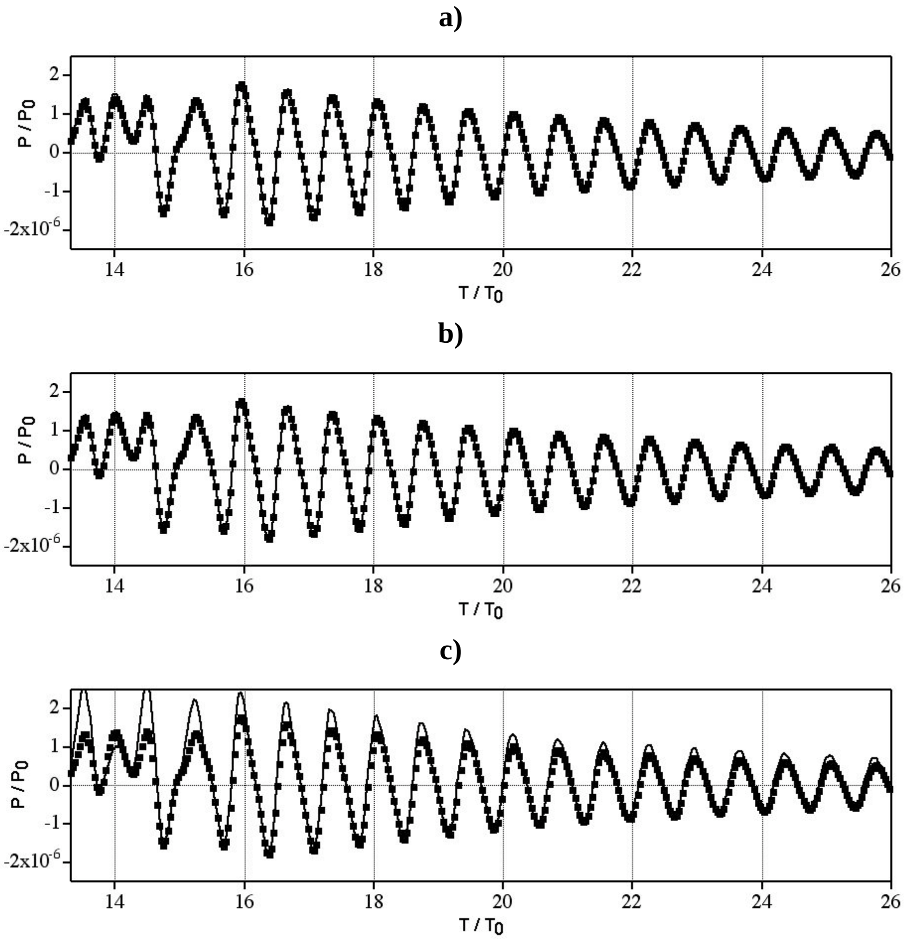


Figure 7.17 - *Case* **B1**, *Eq.***(1)** *:* **Relaxation**

a) Exact nonlinear ▪ **dedt** —— sum of terms

b) Minimum nonlinear ▪ **dedt** —— sum of terms

c) Minimum linear ▪ $\mathbf{dedt}_l$ —— sum of terms

Equation 2[Eq.(6.18)]

Figs.(7.20) and (7.21) show the balance closure of the conservation equation of the quadratic entropy fluctuations. As for the previous energy, the analysis is split in two parts.

- Forced phase

  Fig.(7.20) **a)** shows the balance of the exact nonlinear conservation equation for the entropy fluctuations during the phase of excitation. The time derivative term is zero during a bit more than one period which is the time needed by the incoming acoustic wave to reach the flame. After that, bursts of entropy disturbance energy occur with a frequency of approximately 250Hz ($\approx 2.5T_0$). The amplitude of the bursts decay rapidly and the time derivative term oscillates at almost 1200Hz after 10 periods of forcing. The maximum amplitude reached by this term represents 2% of the mean power of the flame which is much more important than for the disturbance energy contained in quadratic pressure and velocity fluctuations.

  Fig.(7.20) **b)** shows the closure of the minimum Eq.(**2**). The terms that are kept in this minimum equation are :

  – $\mathbf{En_1}$: $-\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right]$

  – $\mathbf{En_2}$: $\frac{s'}{c_p}\left[\dot{\omega}_T - \frac{\rho T}{\overline{\overline{\rho T}}}\overline{\dot{\omega}_T}\right]$

  – $\mathbf{En_3}$: $\frac{s'}{c_p}\left[-\vec{\nabla}.\vec{q} + \frac{\rho T}{\overline{\overline{\rho T}}}\vec{\nabla}.\overline{\vec{q}}\right]$

  – $\mathbf{En_4}$: $-\frac{s'}{c_p}\left[\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] - \frac{\rho T}{\overline{\overline{\rho T}}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]\right]$

  The term $\mathbf{En_1}$ accounts for the transport of the disturbance energy. As shown in Fig.(7.18), $\mathbf{En_2}$ is a source term for the disturbance energy and its maximum amplitude represents 15% of the mean power of the flame. $\mathbf{En_3}$ and $\mathbf{En_4}$ compensate this term so that the derivative term amplitude never exceeds 2% of the mean power of the flame. Note that in this case, $\mathbf{LW_{E_2}}$, $\mathbf{AVI_{E_2}}$ and $\mathbf{BND_{E_2}}$ corrections are negligible (Not shown).

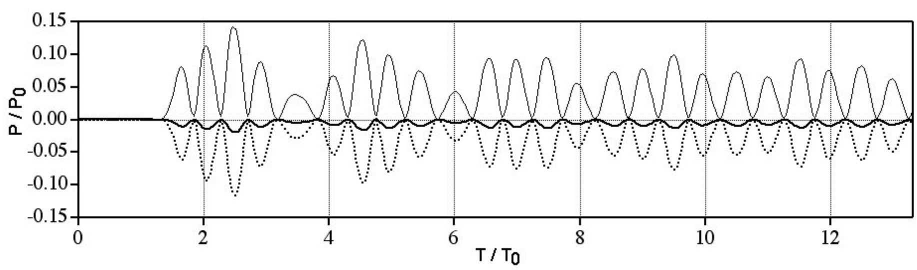
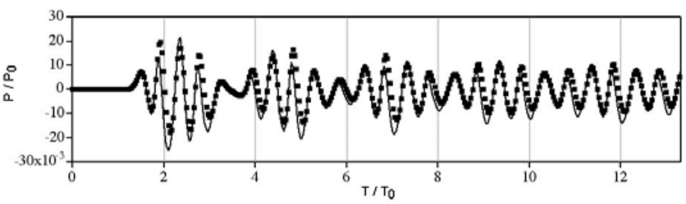


Figure 7.18 - *Case* **B1**, *Eq.***(2)** *:* **Forced phase** *(Representative terms)*
—— $\mathbf{En_2}$ —— $\mathbf{En_3}$ ········ $\mathbf{En_4}$

Fig.(7.20) **c)** shows that the minimum linear Eq.(**2**) does not close at all after one and a half period of forcing (i.e when the velocity fluctuation reaches the flame). This implies that nonlinearities are important in minimum Eq.(**2**).
In this kind of linearization, it often assumed that only terms related to the fluctuation of heat release should be kept. Doing so, one throws the term

$$-\frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2}+\frac{T_1}{{T_0}^2}\right]\left[\dot{\omega}_{T0}-\vec{\nabla}.\vec{q_0}\right].$$

Fig.(7.19) shows that the closure of the balance of the minimum linear Eq.(**2**) is much better in this case. This implies that nonlinearities compensate the term $-\frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2}+\frac{T_1}{{T_0}^2}\right]\left[\dot{\omega}_{T0}-\vec{\nabla}.\vec{q_0}\right]$ so that keeping only the term related to $\left[\dot{\omega}_{T1}-\vec{\nabla}.\vec{q_1}\right]$ gives the right answer. This shows that one should be very cautious when linearizing such disturbance energy equations.



$$\frac{p_0}{2r_0 c_{p0}}\frac{\partial s_1^2}{\partial t}=$$
$$-\frac{p_0 s_1}{r_0 c_{p0}}\vec{u_1}.\vec{\nabla}s_0+\frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1}-\vec{\nabla}.\vec{q_1}\right]$$

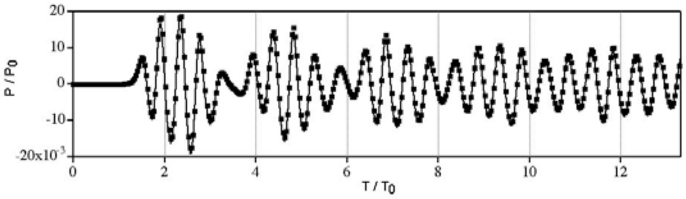
Figure 7.19 - *Case* **B1**, *Eq.***(2)** *:* **Forced phase**
Approximated minimum linear ■ $\mathbf{de_s dt}$ —— sum of terms

- Relaxation

Fig.(7.21) **a)** shows the balance closure of the exact nonlinear conservation equation for the quadratic entropy fluctuations during the relaxation phase. The excitation is stopped, and during approximately one and a half period, $\mathbf{de_s dt}$ is not influenced. Then its maximum amplitude decays rapidly and this term shows two components, one at twice the forcing frequency (1200Hz) and one around 1700Hz. The first component almost vanish after 24 period of forcing and $\mathbf{de_s dt}$ oscillates at 1700Hz. Since this term involves the square of the entropy fluctuation, it means that the entropy is fluctuating at 850Hz as observed for velocity fluctuations in the previous closure of Eq.(**1**)[Eq.(6.10)].
After two periods of forcing, all terms fluctuate at 1700Hz and their maximum amplitude decays rapidly (Not shown).

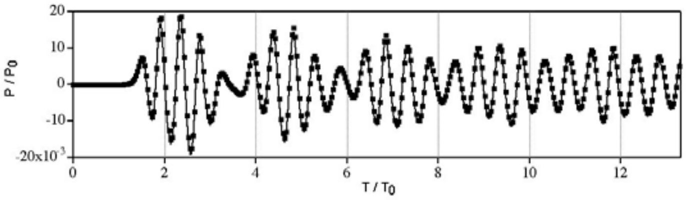
Fig.(7.21) **c)** shows that one period after the forcing is stopped, both terms start to decrease so that the discrepancy between the two also decreases. Yet, the error only becomes acceptable at least 10 periods after that (i.e when the flame does not move much anymore).

a)



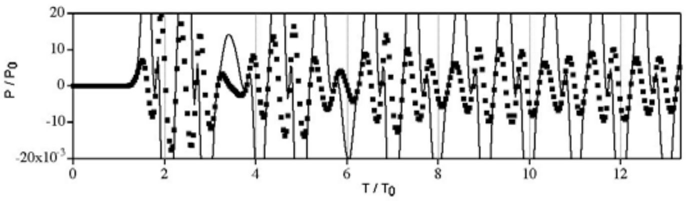
Maximum amplitude of time derivative : 2.10^{-2}
ALL PHASES

b)



$$\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right] +$$
$$\frac{ps'}{rc_p}\left[\frac{1}{\rho T}\dot{\omega}_T - \frac{1}{\overline{\overline{\rho T}}}\overline{\dot{\omega}_T}\right] +$$
$$\frac{ps'}{rc_p}\left[-\frac{1}{\rho T}\vec{\nabla}.\vec{q} + \frac{\rho T}{\overline{\overline{\rho T}}}\vec{\nabla}.\overline{\vec{q}}\right] -$$
$$\frac{ps'}{rc_p}\frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] +$$
$$\frac{ps'}{rc_p}\frac{1}{\overline{\overline{\rho T}}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]$$

c)



$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} =$$
$$-\frac{p_0 s_1}{r_0 c_{p0}}\vec{u_1}.\vec{\nabla}s_0 + \frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1}\right] -$$
$$\frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2} + \frac{T_1}{{T_0}^2}\right]\left[\dot{\omega}_{T0} - \vec{\nabla}.\vec{q_0}\right]$$

Figure 7.20 - *Case* **B1**, *Eq.***(2)** : **Forced phase**

a)	Exact nonlinear	▪ $\mathbf{de_s dt}$	—— sum of terms
b)	Minimum nonlinear	▪ $\mathbf{de_s dt}$	—— sum of terms
c)	Minimum linear	▪ $\mathbf{de_s dt}_l$	—— sum of terms

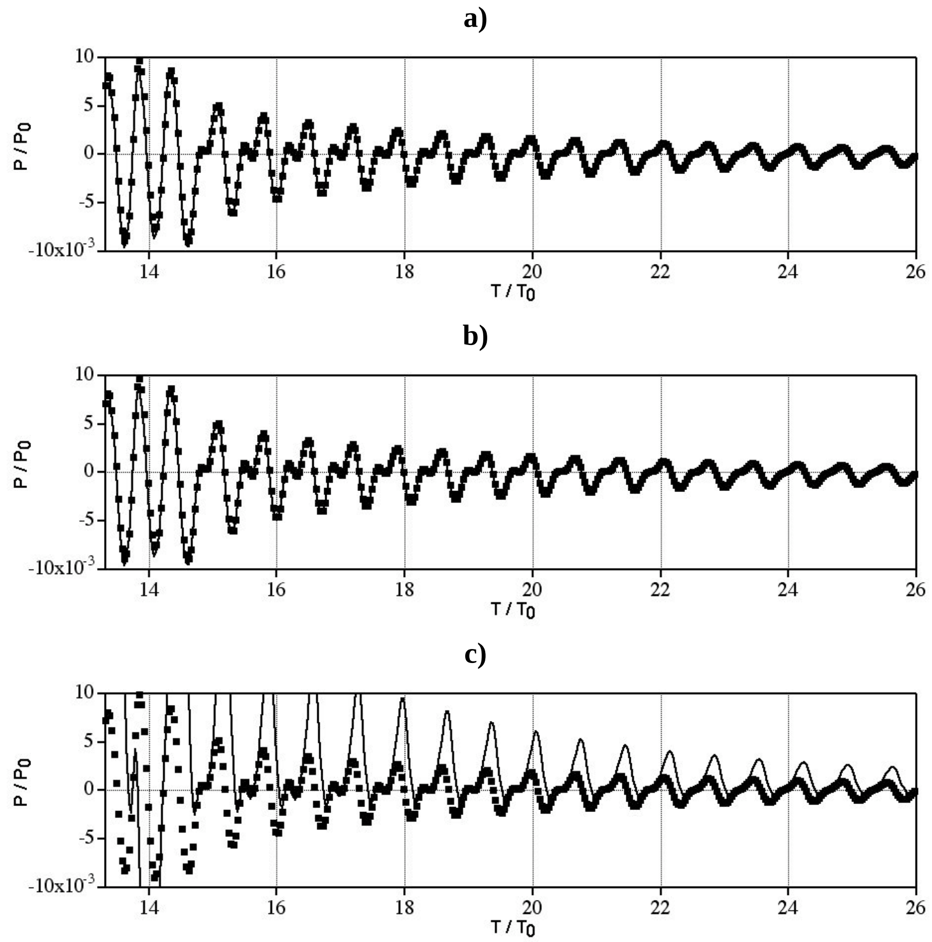


Figure 7.21 - *Case* **B1**, *Eq.***(2)** *:* **Relaxation**

a)	Exact nonlinear	▪ $\mathbf{de_s dt}$ —— sum of terms
b)	Minimum nonlinear	▪ $\mathbf{de_s dt}$ —— sum of terms
c)	Minimum linear	▪ $\mathbf{de_s dt}_l$ —— sum of terms

Equation 3[Eq.(6.37)]

Figs.(7.23) and (7.24) show the balance closure of the conservation equation of the nonlinear disturbance energy. As for the previous equations, the analysis is split in two parts.

- Forced phase

  Fig.(7.23) **a)** shows the balance closure of the exact nonlinear Eq.(**3**)[Eq.(6.37)] during the phase of excitation. As for the previous equations, the time derivative term is almost zero for approximately one and a half period of forcing. It then starts to oscillate at the quadratic frequency but its maximum amplitude also shows bursts at 250Hz as observed for the balance of Eq.(**2**)[Eq.(6.18)]. The amplitude of the bursts decreases, and after ten periods of forcing, the time derivative of the disturbance energy is almost quadratic. The maximum amplitude reached by the derivative term corresponds to 4% of the mean power of the flame.
  Fig.(7.40) **b)** shows the balance closure of the minimum nonlinear Eq.(**3**)[Eq.(6.37)]. The main terms are $\mathbf{D_s}$, $\mathbf{D_Q}$ and $\mathbf{D_Y}$. Fig.(7.22) shows that $\mathbf{D_s}$ is the only source term of the balance. $\mathbf{D_Q}$ and $\mathbf{D_Y}$ tend to compensate it but $\mathbf{D_Q}$ is the dissipative term having the most important amplitude. Although involving the fluctuation of heat release, $\mathbf{D_Q}$ is a sink term for Eq.(**3**)[Eq.(6.37)]. All these terms have a quadratic behavior but also show the bursts at 250Hz present in the time derivative term.

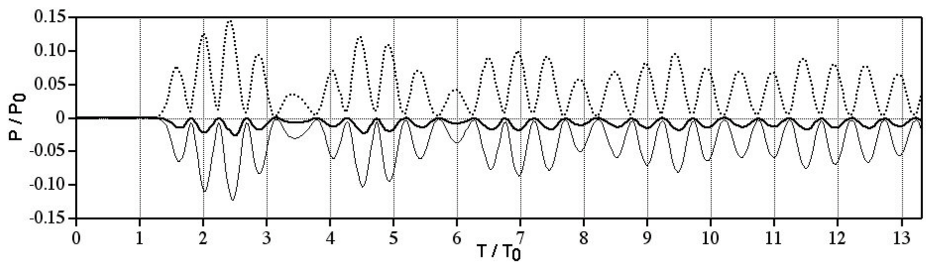
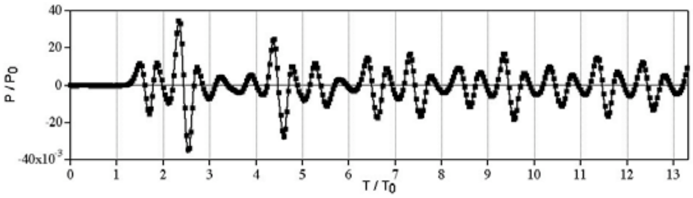


Figure 7.22 - *Case* **B1**, *Eq.*(**3**) *:* **Forced phase** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

- Relaxation

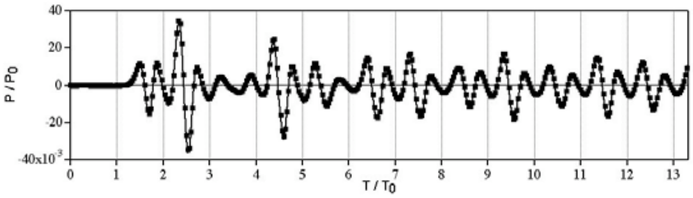
  Figs.(7.24) **a)** and **b)** respectively show the balance closure of the exact and minimum nonlinear Eq.(**3**)[Eq.(6.37)] during the relaxation. Approximately two periods after the forcing has been stopped, the maximum amplitude of the derivative term starts to decay. It shows two major frequency components, the first one is quadratic and decays rapidly as the forcing is stopped and the other one corresponds to approximately 1700Hz. After 22 periods, the time derivative oscillates almost harmonically at 1700Hz.
  All R-H-S terms oscillate at 1700Hz two periods after the forcing is stopped and their amplitude decreases rapidly. This delay corresponds to the time needed by an upstream propagating acoustic information to go from the outlet to the inlet.

a)



Maximum amplitude of time derivative : 4.10^{-2}
ALL PHASES

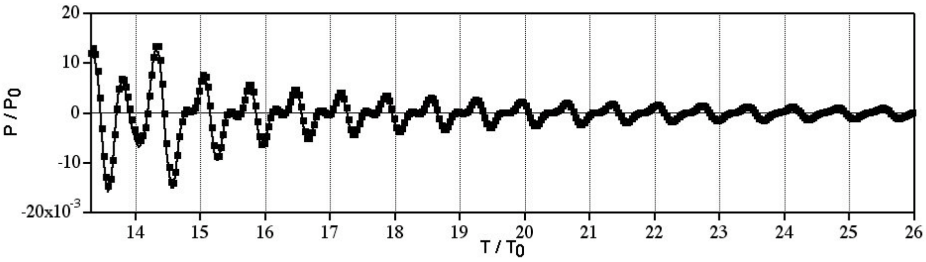
b)



$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}}s'\vec{\nabla}T' + T'Q' + \mathbf{D_Y}$$

Figure 7.23 - *Case* **B1**, *Eq.***(3)** *:* **Forced phase**
a) Exact nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

a)



b)

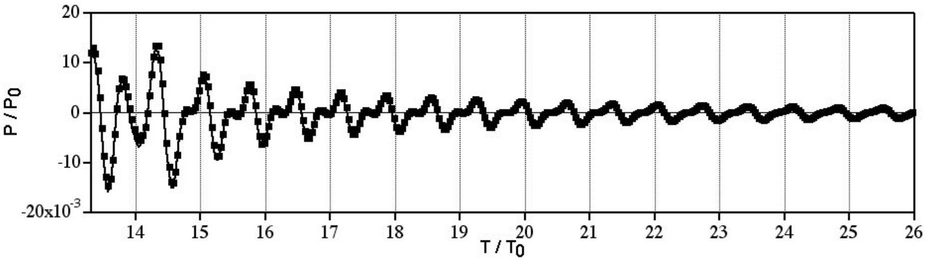


Figure 7.24 - *Case* **B1**, *Eq.***(3)** *:* **Relaxation**

a) Exact nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

b) Minimum nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

7.2.2 Case B2 Unstable Case: F_0=856 Hz Reference : Steady state flow

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** . **Note that ”minimum” and ”minimum linear” equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**

Equation 1$_{\text{[Eq.(6.10)]}}$

Figs.(7.28), (7.29), (7.30) present the balance of the conservation equation for quadratic pressure and velocity fluctuations. The evolution of the balance is split in three steps. First the configuration is made unstable by increasing the outlet acoustic reflection coefficient and the system becomes unstable. Then a limit cycle is reached and the maximum amplitude of the time derivative of the disturbance energy is constant. Finally, after 44ms (38 periods of instability), the reflection coefficient is decreased to its initial value and the system goes back to a steady state.
In the following, the analysis of the balance of Eq.(**1**)$_{\text{[Eq.(6.10)]}}$ is first done for the onset of the instability, then the limit cycle is studied and finally the relaxation phase is analyzed.

- onset of the instability

  Fig.(7.28) **a)** shows the balance of the exact nonlinear conservation equation for quadratic pressure and velocity fluctuations during the onset of the instability. There is a perfect agreement between $\mathbf{dedt}$ and the sum of the volume integrals of the R-H-S terms of Eq.(**1**)$_{\text{[Eq.(6.10)]}}$ as the instability grows. Yet, this imposes to take into account the numerical correction term ($\mathbf{LW_{E_1}}$) in Fig.(7.25). The time derivative term starts to oscillate harmonically at 856Hz as soon as the outlet reflection coefficient is increased. The maximum amplitude of the time derivative term increases exponentially from zero to 0.001% of the mean power of the flame in almost 14 periods of oscillation. During all the onset of the instability, deudt and depdt are in phase and their maximum amplitude grows exponentially (not shown).

  Fig.(7.28) **b)** shows the balance of the minimum Eq.(**1**).
  After a delay of half a period, all major terms start to oscillate at 856Hz with a growing maximum amplitude. These terms are:

  – $\mathbf{Ac_1}$: $-\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right]$
  – $\mathbf{Ac_2}$: $-\vec{u'}.\vec{\nabla}p'$
  – $\mathbf{Ac_3}$: $\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right]$
  – $\mathbf{Ac_4}$: $-\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\vec{\overline{q}}\right]$
  – $\mathbf{Ac_5}$: $\vec{u'} : \vec{\nabla}\tau'$
  – $\mathbf{LW_{E_1}}$

As expected from the results on case **A2**, $\mathbf{Ac_3}$ is the strongest source term with an amplitude growing to 4.10^{-3}% of the mean power of the flame (not shown). The main difference between this case and the case **A** is the presence of the term $\mathbf{Ac_5}$ in the minimum nonlinear Eq.(**1**). This comes from the wall friction which dissipates a part of the energy contained in quadratic velocity fluctuations. In this case, Fig.(7.25) shows that the numerical $\mathbf{LW_{E_1}}$ correction is not negligible and behaves as a sink term for the instability with a maximum absolute amplitude of 8.10^{-4}% of the mean power of the flame. Also note $\mathbf{BND_{E_1}}$ in Fig.(7.25) remains small enough to be neglected

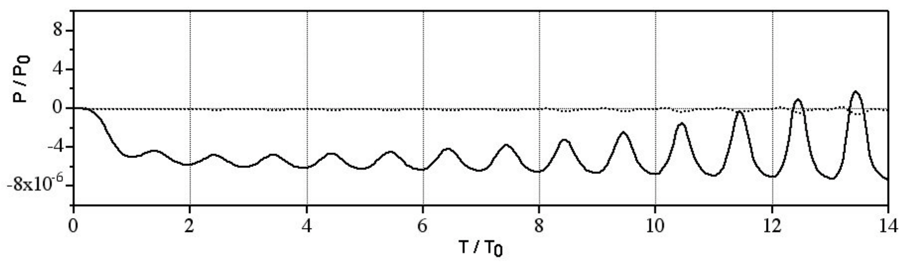


Figure 7.25 - *Case* **B2**, *Eq.***(1)** *:* **Onset of the instability** —— $\mathbf{LW_{E_1}}$ ········ $\mathbf{BND_{E_1}}$

The linearization of the minimum Eq.(**1**) influences the closure of the balance as shown in Fig.(7.28) **c)**. After one period of instability, the time derivative $\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} + \frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t}$ is under-predicted.
Direct interaction terms (involving fluctuations of heat release and heat flux) may be found in the minimum linear Eq.(**1**). Note that as already mentioned in case **A2**, terms related to the fluctuation of γ must be kept since they are not negligible.

- Limit cycle

  Fig.(7.29) **a)** shows the balance of the exact nonlinear conservation equation for quadratic pressure and velocity fluctuations during the limit cycle. The time derivative term maximum amplitude grows until 20 periods of forcing and then reaches a limit cycle. It then oscillates with two major frequency components, one at 856Hz and one at 1700Hz, so that it shows two peaks per period. The maximum absolute amplitude reached corresponds to 0.002% of the mean power of the flame. Fig.(7.26) shows that the two components of the time derivative term are in phase with a small predominance of the time derivative linked to pressure fluctuations.

  As for the onset of the instability, the major terms are $\mathbf{Ac_1}$, $\mathbf{Ac_2}$, $\mathbf{Ac_3}$, $\mathbf{Ac_4}$, $\mathbf{Ac_5}$ and the $\mathbf{LW_{E_1}}$ correction. The agreement in Fig.(7.29) **b)** shows that all other terms are indeed negligible.

  Fig.(7.29) **c)** shows the closure of the linear minimum Eq.(**1**). Obviously, nonlinearities are necessary for the closure of minimum Eq.(**1**) since here the maximum absolute amplitude of the time derivative term is under predicted. The sum of R-H-S terms goes not far beyond 1.10^{-3}% of the mean power of the flame though $\mathbf{dedt}$ reaches 2.10^{-3}%.

- Relaxation

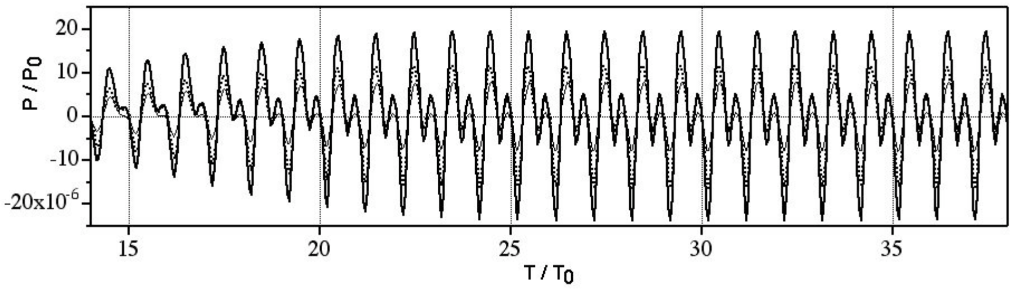


Figure 7.26 - *Case* **B2**, *Eq.***(1)** *:* **Limit cycle** *Composition of the time derivative term*
—— **dedt** —— **deudt** ········ **depdt**

Fig.(7.30) **a)** shows the balance of the exact nonlinear conservation equation for quadratic pressure and velocity fluctuations during the relaxation. As the acoustic outlet reflection coefficient is decreased, the time derivative term decreases and the 856Hz component vanishes. After one period, it oscillates at 1700Hz.
Fig.(7.27) shows that the two components of the time derivative have a small shift in phase during the relaxation but their amplitude remains of the same order in contrary to what is observed during the relaxation phase in case **B1**(see Fig.(7.15)).

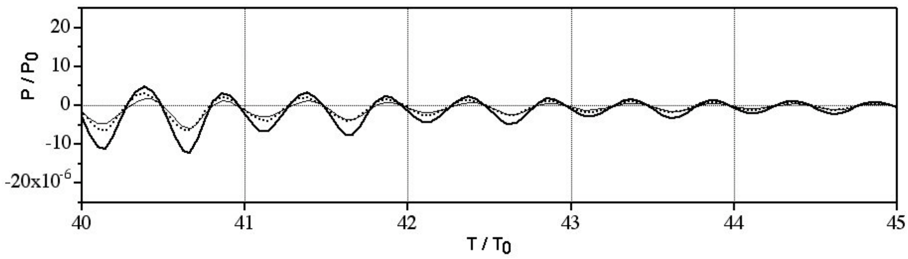
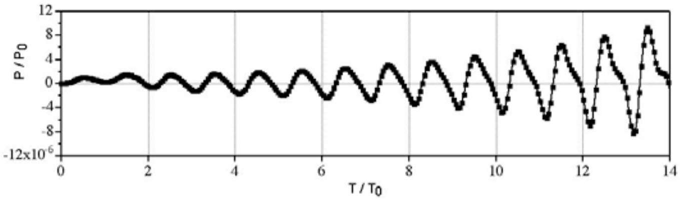


Figure 7.27 - *Case* **B2**, *Eq.***(1)** *:* **Relaxation** *Composition of the time derivative term*
—— **dedt** —— **deudt** ········ **depdt**

Fig.(7.30) **b)** shows the balance of the minimum nonlinear Eq.(**1**)
After one period of instability, all major terms tend to oscillate at 1700Hz. The absolute maximum amplitude of all terms decreases to zero. As the system goes back to a steady state, $\mathbf{Ac_3}$ and $\mathbf{Ac_1}$ are the most significant terms (not shown).

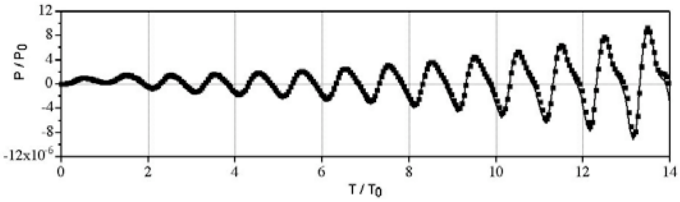
Fig.(7.30) **c)** show that nonlinearities in the minimum nonlinear Eq.(**1**) decrease as the system goes back to a steady state and almost disappear after 42 periods.

a)



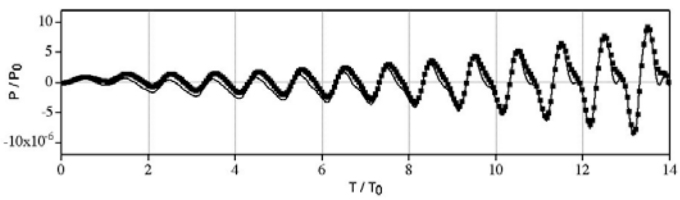
Maximum amplitude of time derivative : 2.10^{-5}
ALL PHASES

b)



$$\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t}+\frac{\rho}{2}\frac{\partial u'^2}{\partial t}= -\vec{u'}.\vec{\nabla} p' - \frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \overline{\gamma}\overline{p}\vec{\nabla}.\overline{\vec{u}}\right] + \frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right] - \frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\overline{\vec{q}}\right] + \vec{u'} : \vec{\nabla}\tau' + \mathbf{LW_{E_1}}$$

c)



$$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t}+\frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t}= -\vec{u_1}.\vec{\nabla} p_1 - p_1\vec{\nabla}.\vec{u}_1 - \frac{\gamma_1}{\gamma_0}p_1\vec{\nabla}.\vec{u}_0 + \frac{\gamma_0-1}{\gamma_0 p_0}p_1\dot{\omega_{T1}} + \frac{\gamma_1 p_1}{\gamma_0 p_0}\dot{\omega_{T0}} - \frac{\gamma_0-1}{\gamma_0 p_0}p_1\vec{q}_1 - \frac{\gamma_1 p_1}{\gamma_0 p_0}\vec{q}_0 + \vec{u}_1 : \vec{\nabla}\tau_1 + \mathbf{LW_{E_1}}$$

Figure 7.28 - *Case* **B2**, *Eq.***(1)** *:* **Onset of the instability**

a)	Exact nonlinear	▪ $\mathbf{dedt}$	—— sum of terms
b)	Minimum nonlinear	▪ $\mathbf{dedt}$	—— sum of terms
c)	Minimum linear	▪ $\mathbf{dedt}_l$	—— sum of terms

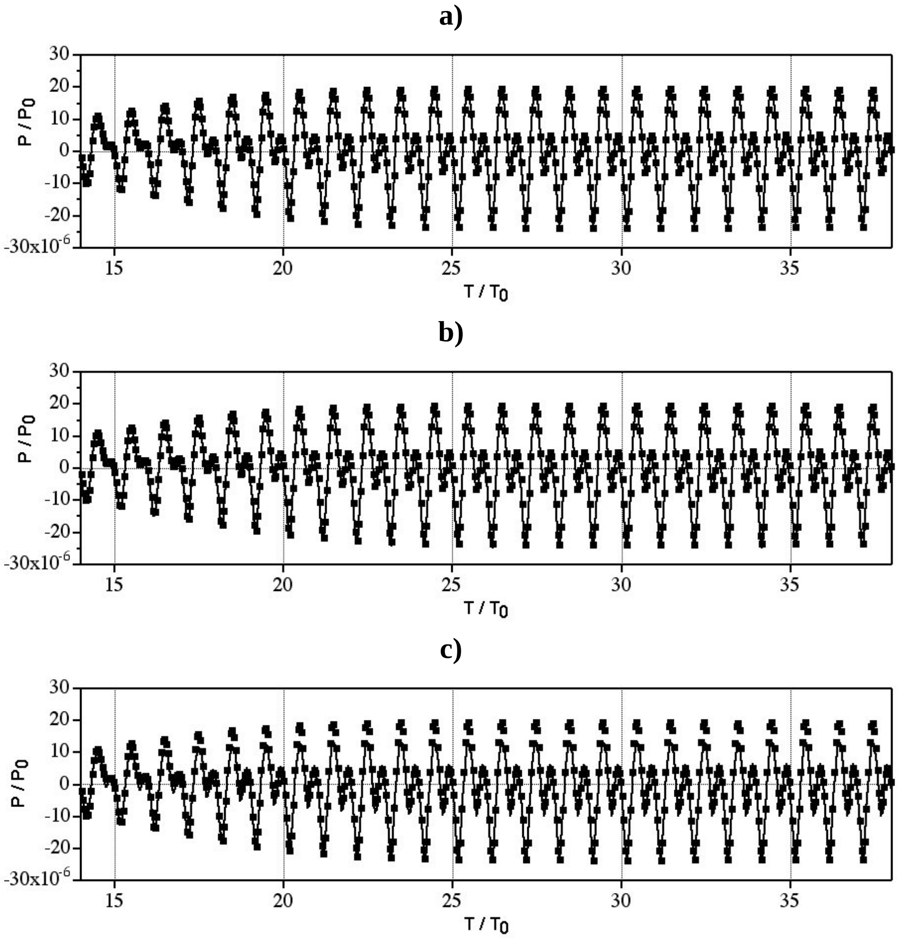


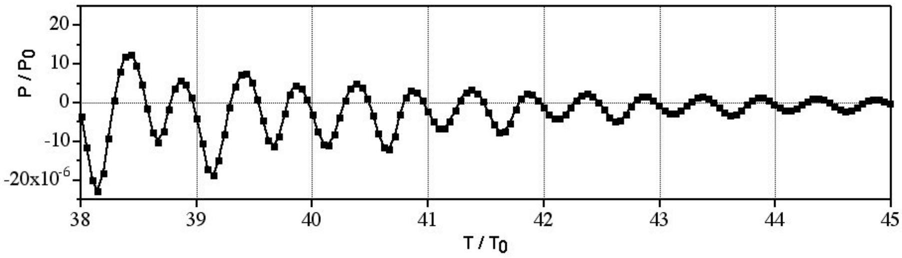
Figure 7.29 - *Case* **B2**, *Eq.***(1)** *:* **Limit cycle**

a) Exact nonlinear ■ **dedt** —— sum of terms

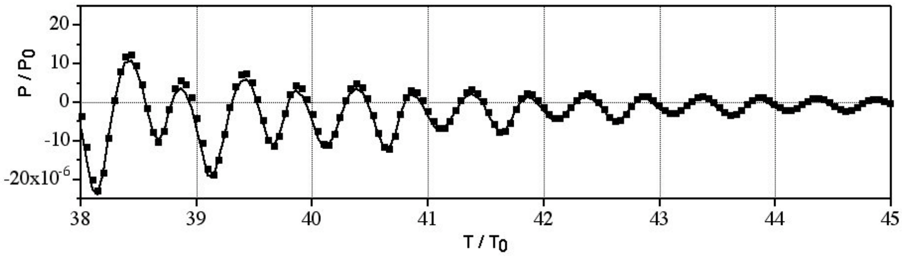
b) Minimum nonlinear ■ **dedt** —— sum of terms

c) Minimum linear ■ $\mathbf{dedt}_l$ —— sum of terms

a)



b)



c)

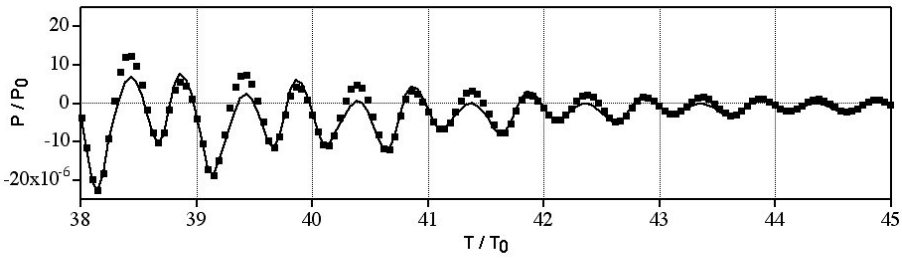


Figure 7.30 - *Case* **B2**, *Eq.***(1)** *:* **Relaxation**

a)	Exact nonlinear	▪ $\mathbf{dedt}$	—— sum of terms
b)	Minimum nonlinear	▪ $\mathbf{dedt}$	—— sum of terms
c)	Minimum linear	▪ $\mathbf{dedt}_l$	—— sum of terms

Equation 2[Eq.(6.18)]

Figs.(7.34), (7.35) and (7.36) present the balance of the nonlinear conservation equation for the disturbance energy in entropy fluctuations. As for Eq.(**1**)[Eq.(6.10)], the evolution of the balance is split in three steps.

- Onset of the instability

  Fig.(7.34) **a)** shows the balance of the exact nonlinear conservation equation for quadratic entropy fluctuations during the onset of the instability. Fig.(7.31) **a)** shows the positive envelop of the time derivative term $\mathbf{de_s dt}$. The time derivative term maximum absolute amplitude grows exponentially during the first 11 periods of the instability and Fig.(7.31) **b)** gives the exponential coefficient (calculated using regression on curve in Fig.(7.31) **a)**) along the onset of the instability. Note that this coefficient is not constant during the onset of the instability but increases during the first 11 periods from 120 to 160 and then decreases to zero as the limit cycle is reached. Finally note that the time derivative term oscillates harmonically at 856Hz during 11 periods and then a component at 1700Hz appears.

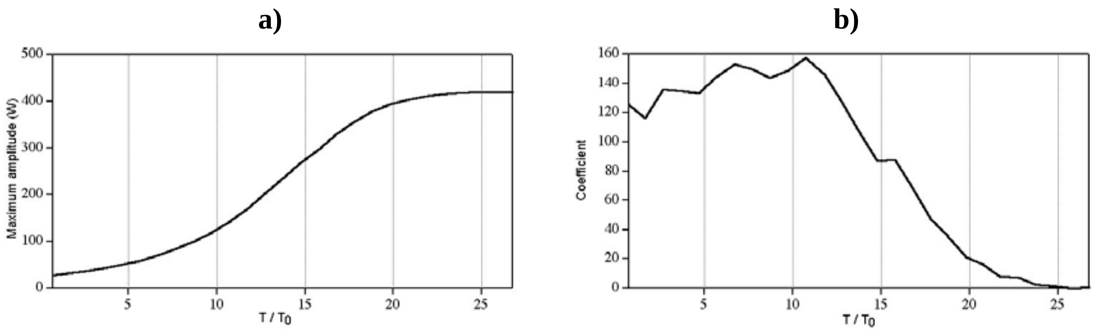


Figure 7.31 -
a) —— *Envelop of the maximum amplitude of the time derivative term during the onset of the instability*
b) —— *Exponential growth coefficient.*

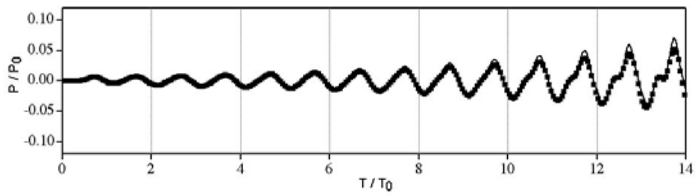
Fig.(7.34) **b)** shows the closure of the minimum nonlinear Eq.(**2**).
The major terms are :

- **En$_\mathbf{1}$** : $-\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right]$
- **En$_\mathbf{2}$** : $\frac{s'}{c_p}\left[\dot{\omega}_T - \frac{\rho T}{\overline{\rho T}}\overline{\dot{\omega}_T}\right]$
- **En$_\mathbf{3}$** : $\frac{s'}{c_p}\left[-\vec{\nabla}.\vec{q} + \frac{\rho T}{\overline{\rho T}}\vec{\nabla}.\overline{\vec{q}}\right]$
- **En$_\mathbf{4}$** : $-\frac{s'}{c_p}\left[\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] - \frac{\rho T}{\overline{\rho T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]\right]$

Linearizing minimum Eq.(**2**) results in major errors as soon as the instability starts as shown in Fig.(7.34) **c)**. The time derivative of the entropy disturbance energy is largely overpredicted. As for the case **B1**, neglecting the term $-\frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2}+\frac{T_1}{{T_0}^2}\right]\left[\dot{\omega}_{T0}-\vec{\nabla}.\vec{q_0}\right]$ leads paradoxically to the right answer as shown in Fig.(7.32). This means that nonlinearities are important in

$$+\frac{s'}{c_p}\left[\dot{\omega}_T-\frac{\rho T}{\overline{\rho}\overline{T}}\overline{\dot{\omega}_T}\right]+\frac{s'}{c_p}\left[-\vec{\nabla}.\vec{q}+\frac{\rho T}{\overline{\rho}\overline{T}}\vec{\nabla}.\overline{\vec{q}}\right]$$

.



$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t}=$$
$$-\frac{p_0 s_1}{r_0 c_{p0}}\vec{u_1}.\vec{\nabla}s_0+\frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1}-\vec{\nabla}.\vec{q_1}\right]-$$
$$\frac{s_1}{c_{p0}}\sum_k g_{sk1}\left[\dot{\omega}_{k0}-\vec{\nabla}.\vec{q_{k0}}\right]-$$
$$\frac{s_1}{c_{p0}}\sum_k g_{sk0}\left[\dot{\omega}_{k1}-\vec{\nabla}.\vec{q_{k1}}\right]+$$
$$\frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2}+\frac{T_1}{{T_0}^2}\right]\sum_k g_{sk0}\left[\dot{\omega}_{k0}-\vec{\nabla}.\vec{q_{k0}}\right]$$

Figure 7.32 - *Case* **B2**, *Eq.*(**2**) *:* **Onset of the instability**
Approximated minimum linear ▪ $\mathbf{de_s dt}$ —— sum of terms

- Limit cycle

  Figs.(7.35) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**2**) during the limit cycle. The time derivative term maximum amplitude is almost constant after 25 periods of oscillation as also shown by Fig.(7.31) **a)** and represents 10% of the mean power released by the flame. The signal has two major frequency components, one at 856Hz and one at the quadratic frequency (1700Hz).
  During this step, $\mathbf{En_2}$ and $\mathbf{En_3}$ are the two most important terms, but they tend to compensate each other so that $\mathbf{En_1}$ and $\mathbf{En_4}$ have a non-negligible influence on the balance (not shown).

  As for the onset of the instability, Fig.(7.35) **c)** shows the impossibility of deriving a minimum linear Eq.(**2**) in case **B2**

- Relaxation

  Fig.(7.36) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**2**) during the relaxation. Approximatly one period after the outlet acoustic reflection coefficient is lowered, the 856Hz frequency component vanishes in the $\mathbf{de_s dt}$ signal. The time derivative term maximum absolute amplitude decays exponentially and this term oscillates at 1700Hz as the instability disappears.
  All major terms absolute amplitudes decay progressively and $\mathbf{En_2}$ and $\mathbf{En_3}$ are the two major

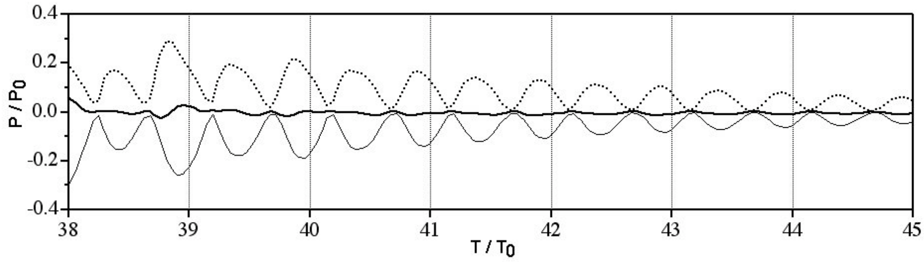
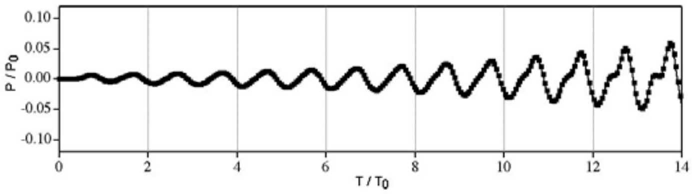


Figure 7.33 - *Case* **B2**, *Eq.***(2)** *:* **Relaxation** *(Representative terms)*
—— $\mathbf{En_1}$ —— $\mathbf{En_3}$ ········ $\mathbf{En_2}$

terms of the balance and tend to compensate (see Fig.(7.33)). Also note that the term $\mathbf{En_1}$ is almost zero about one period after the relaxation has started.

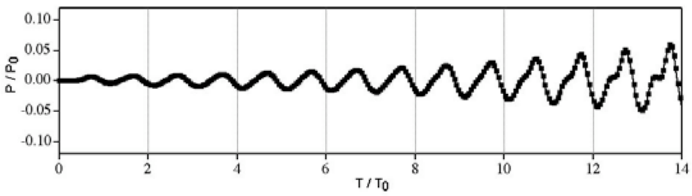
The relaxation phase is very much different for the time derivative of the entropy disturbance energy and for the the minimum linear Eq.(**1**). Even 7 periods after that the acoustic reflection coefficient has been reset to its initial value, nonlinearities are still important.

a)



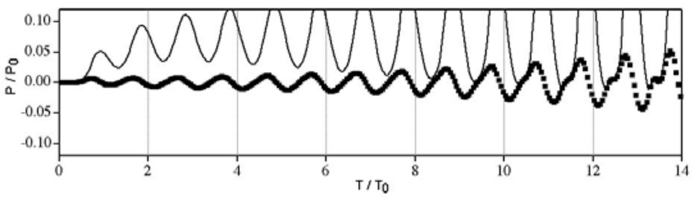
Maximum amplitude of time derivative : 1.10^{-1}
ALL PHASES

b)



$$\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right] + \frac{s'}{c_p}\left[\dot{\omega}_T - \frac{\rho T}{\overline{\rho}\overline{T}}\overline{\dot{\omega}_T}\right] + \frac{s'}{c_p}\left[-\vec{\nabla}.\vec{q} + \frac{\rho T}{\overline{\rho}\overline{T}}\vec{\nabla}.\overline{\vec{q}}\right] - \frac{ps'}{rc_p}\frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] + \frac{ps'}{rc_p}\frac{1}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]$$

c)

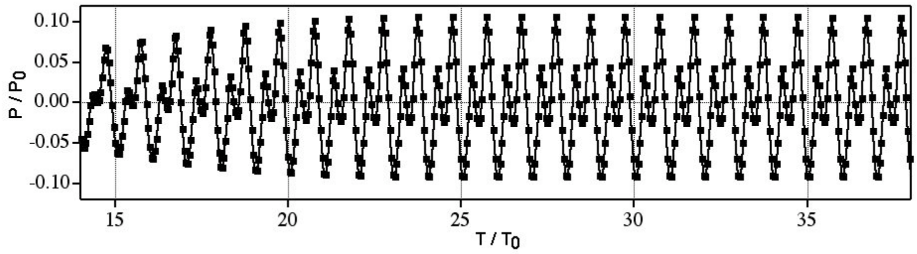


$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} = -\frac{p_0 s_1}{r_0 c_{p0}}\vec{u_1}.\vec{\nabla}s_0 + \frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1}\right] - \frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2} + \frac{T_1}{{T_0}^2}\right]\left[\dot{\omega}_{T0} - \vec{\nabla}.\vec{q_0}\right] - \frac{s_1}{c_{p0}}\sum_k g_{sk1}\left[\dot{\omega}_{k0} - \vec{\nabla}.\vec{q_{k0}}\right] - \frac{s_1}{c_{p0}}\sum_k g_{sk0}\left[\dot{\omega}_{k1} - \vec{\nabla}.\vec{q_{k1}}\right] + \frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2} + \frac{T_1}{{T_0}^2}\right]\sum_k g_{sk0}\left[\dot{\omega}_{k0} - \vec{\nabla}.\vec{q_{k0}}\right]$$

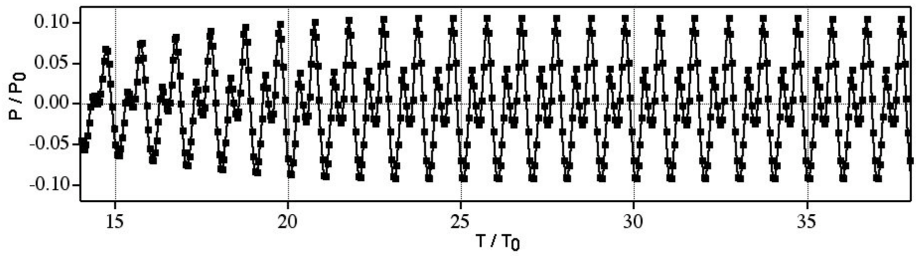
Figure 7.34 - *Case* **B2**, *Eq.***(2)** *:* **Onset of the instability**

a) Exact nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{de_s dt}_l$ —— sum of terms

a)



b)



c)

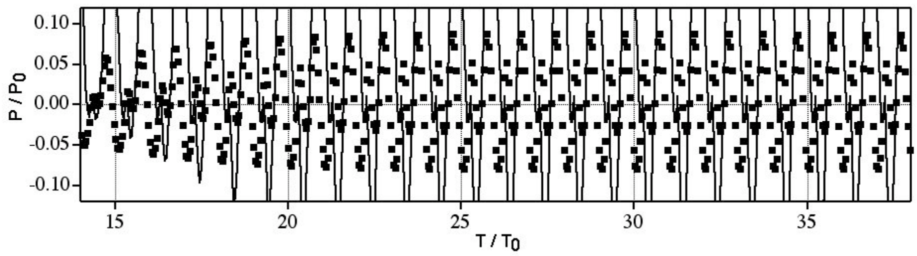
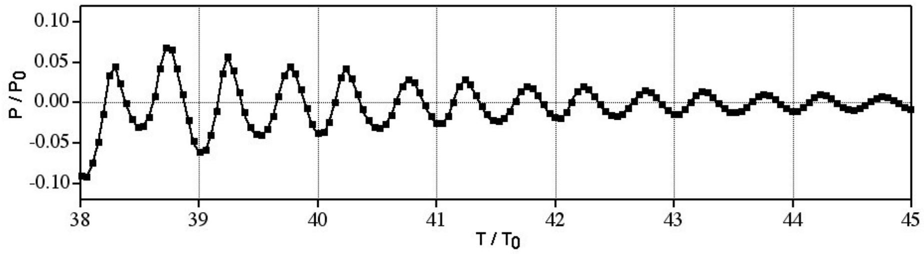


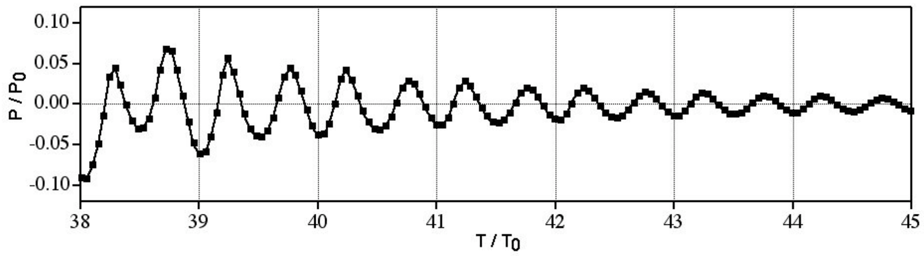
Figure 7.35 - *Case* **B2**, *Eq.***(2)** *:* **Limit cycle**

a)	Exact nonlinear	■ $\mathbf{de_s dt}$	—— sum of terms
b)	Minimum nonlinear	■ $\mathbf{de_s dt}$	—— sum of terms
c)	Minimum linear	■ $\mathbf{de_s dt}_l$	—— sum of terms

a)



b)



c)

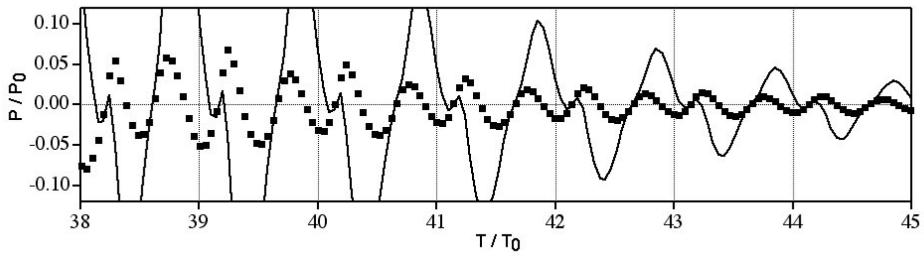


Figure 7.36 - *Case* **B2**, *Eq.***(2)** *:* **Relaxation**

a) Exact nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{de_s dt}_l$ —— sum of terms

Equation 3[Eq.(6.37)]

Figs.(7.40), (7.41) and (7.42) present the balance of the nonlinear conservation equation of the nonlinear disturbance energy. As for Eqs.(**1**) and (**2**), the evolution of the balance is split in three steps.

- Onset of the instability

  Fig.(7.40) **a)** shows the balance of the exact nonlinear Eq.(**3**)[Eq.(6.37)] during the onset of the instability. This figure shows a perfect agreement between the time derivative term $\mathbf{dE_d dt}$ and the sum of the volume integrals of the R-H-S terms of Eq.(**3**)[Eq.(6.37)] as the instability grows. This validates the ability of the POSTTIT tool to compute all R-H-S terms of Eq.(**3**)[Eq.(6.37)] precisely enough to ensure its closure. The maximum amplitude of the time derivative term grows from zero to 10% of the mean flame power in 10 periods of instability. This term oscillates harmonically during the first 6 periods at 856Hz. After this, a component at 1700Hz appears.

  Figs.(7.40) **b)** shows the balance of the minimum nonlinear Eq.(**3**)[Eq.(6.37)]. The major terms are:

  - $\mathbf{D_s}$: $-s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}}s'\vec{\nabla}T'$
  - $\mathbf{D_Q}$: $T'Q'$
  - $\mathbf{D_Y}$: $\vec{m}'.\vec{\psi}^{\star\prime} + T'Q^{\star\prime} - \sum_{k=1}^{n} g'_{sk}\Omega'_k + \sum_{k=1}^{n} g'_{sk}\overline{Y_k}\vec{\nabla}.\vec{m}' + \sum_{k=1}^{n} \overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m}'$

  All major terms oscillate harmonically during the first 6 periods at 856Hz. After that, these terms show nonlinearities which grow with the instability (see Fig.(7.37)). In contrary to what is observed for Eqs.(**1**) and (**2**), $\mathbf{D_Q}$ which includes fluctuations of heat release is a sink term for the nonlinear disturbance energy. the phase between $\mathbf{D_s}$ and $\mathbf{D_Q}$ (shown in Fig.(7.37)) increases from zero at the beginning of the instability to almost a quarter of the instability period at the end of the growth phase. This shift of phase may be one of the reasons for the growth of the nonlinear disturbance energy maximum amplitude.

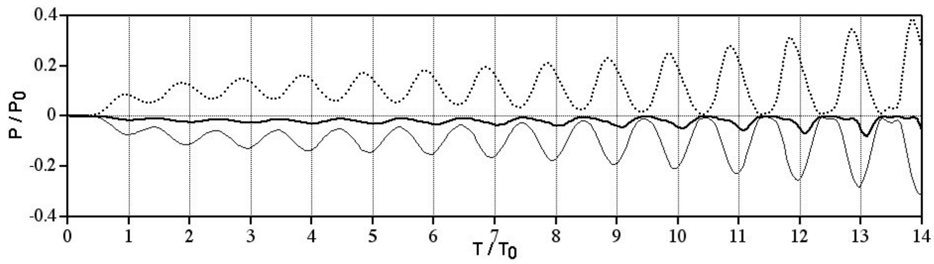


Figure 7.37 - *Case* **B2**, *Eq.*(**3**) *:* **Onset of the instability** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

- Limit cycle

  Figs.(7.41) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**3**)[Eq.(6.37)] during the limit cycle. After 20 periods of the instability, the limit cycle is reached.

The time derivative term oscillates with two major frequency components, one at 856Hz, and one at 1700Hz. The maximum absolute amplitude reached by $\mathbf{dE_d dt}$ represents 30% of the mean power of the flame.
During the limit cycle, the phase between $\mathbf{D_s}$ and $\mathbf{D_Q}$ keeps constant and equal to a quarter of the instability period (see Fig.(7.38)).

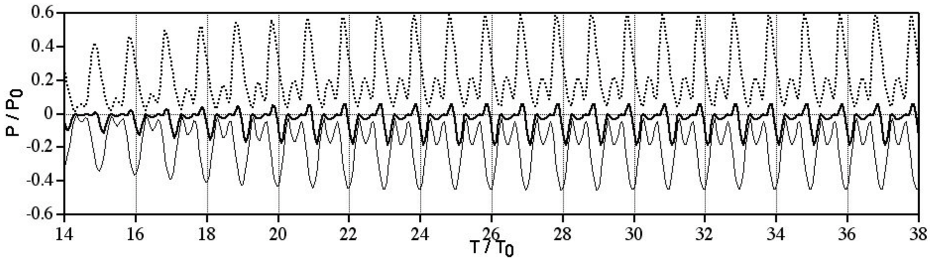


Figure 7.38 - *Case* **B2**, *Eq.***(3)** *:* **Limit cycle** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

- Relaxation

Fig.(7.42) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**3**)[Eq.(6.37)] during the relaxation. The time derivative maximum absolute amplitude decays rapidly, and in almost three periods of the instability is lower than 5% of the mean flame power. Yet, this term decreases in amplitude but does not oscillates harmonically even for small amplitudes.
The major terms are almost harmonic after 40 periods and oscillate at approximately 1700Hz (two maximum per instability period) (see Fig.(7.39)).

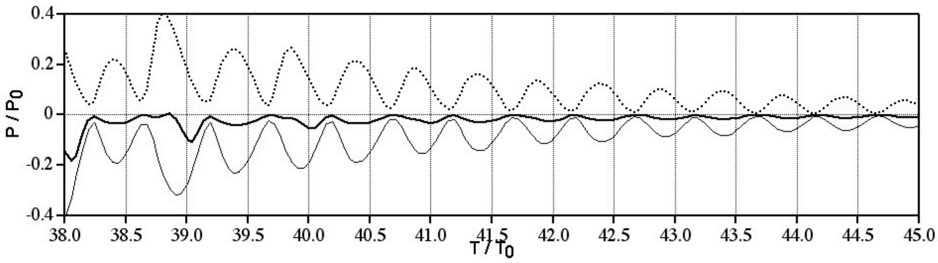
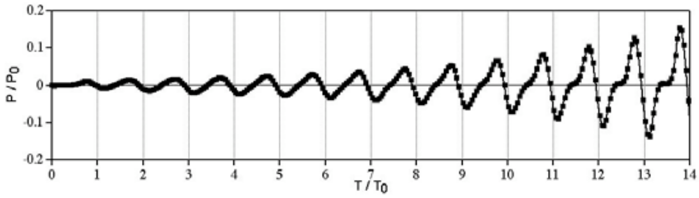


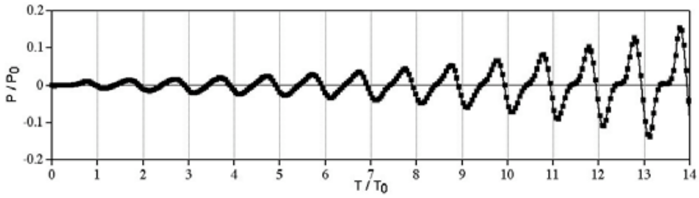
Figure 7.39 - *Case* **B2**, *Eq.***(3)** *:* **Relaxation** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

a)



Maximum amplitude of time derivative : $3, 2.10^{-1}$
ALL PHASES

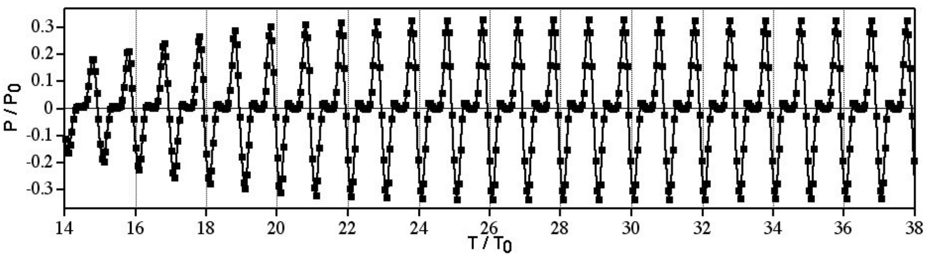
b)



$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}}s'\vec{\nabla}T' + T'Q' + \vec{m}'.\vec{\psi^{\star}}' + T'Q^{\star'} - \sum_{k=1}^{n} g'_{sk}\Omega'_k + \sum_{k=1}^{n} g'_{sk}\overline{Y_k}\vec{\nabla}.\vec{m}' + \sum_{k=1}^{n} \overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m}'$$

Figure 7.40 - *Case* **B2**, *Eq.***(3)** *:* **Onset of the instability**
a) Exact nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

a)



b)

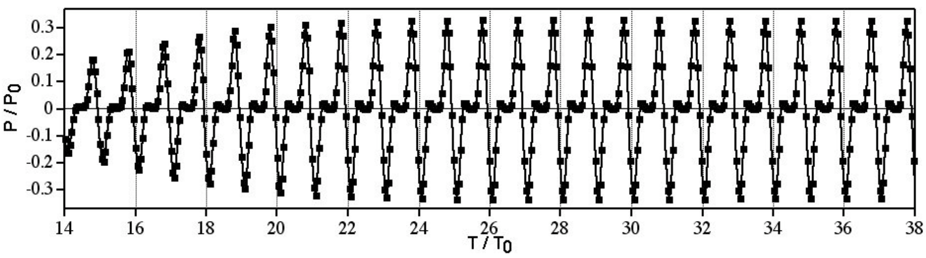
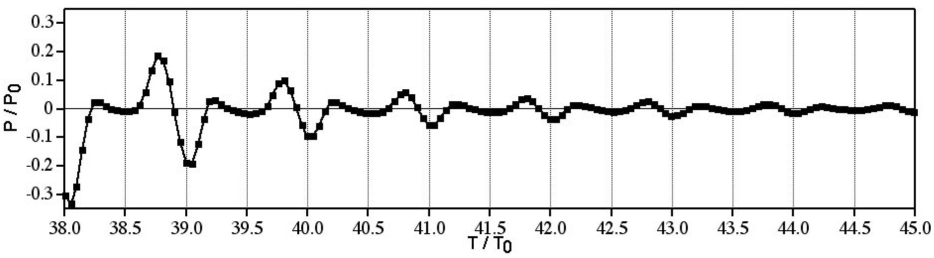


Figure 7.41 - *Case* **B2**, *Eq.***(3)** *:* **Limit cycle**
a) Exact nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

a)



b)

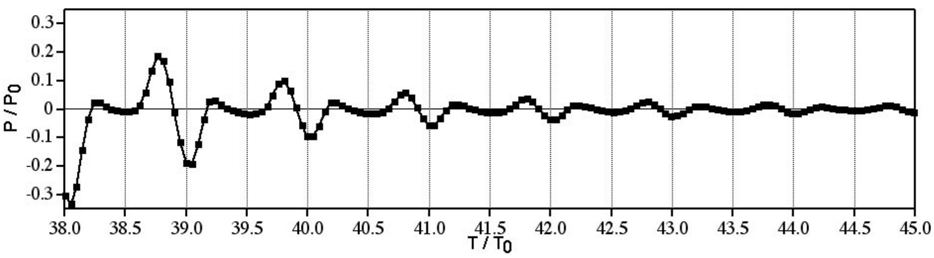


Figure 7.42 - *Case* **B2**, *Eq.***(3)** *:* **Relaxation**

a) Exact nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

b) Minimum nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

7.2.3 Case B2bis Unstable Case : F_0=856 Hz Reference : Mean perturbed flow

In this case, the mean reference field is obtained from averaging 1200 snapshots obtained during the computation. It therefore involves the influence of the instability. This impacts significantly the values of the mean field as shown in Fig.(7.43). As one can observe, even if the mean temperature field is only slightly changed in shape and intensity, the mean heat release is completely different. Indeed, the reference heat release field for case **B1** and **B2** is the steady one and therefore shows a maximum on the axial line equal to 3.10^{10} $W.m^{-3}$ and coherent with instantaneous values of heat release.
The reference field used in case **B2bis** is the averaging of instantaneous fields of heat release. It therefore takes into account the fluctuation of the flame position which influences the position of its maximum. Even if the maximum instantaneous values may be found on the axial line, the amplitude of the flame front move makes that the maximum will not be found here after averaging. Two maximum are found, one on each side of the axial line and their value is equal to 6.10^9 $W.m^{-3}$

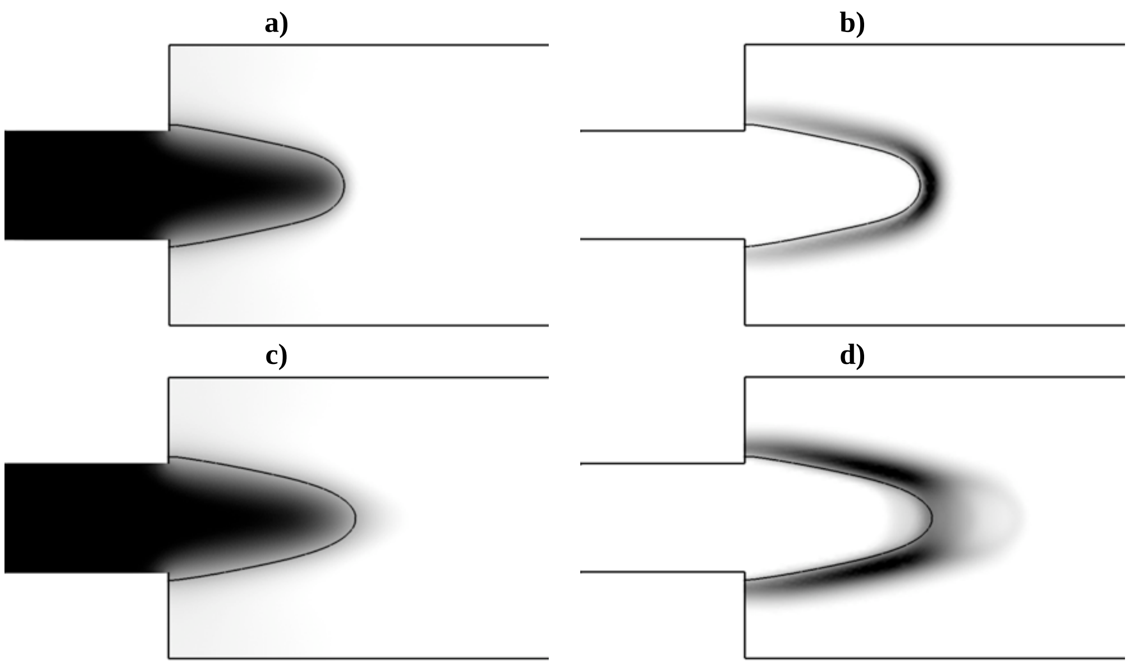


Figure 7.43 - *Reference fields :* —— *Isoline* $1500K$
Case **B1,B2 a)** *Temperature* **b)** *Heat release*
Case **B2bis c)** *Temperature* **d)** *Heat release*

As already discussed in section 6.5, choosing the mean field as the reference field to obtain the fluctuations therefore has many consequences that are pointed out in this section. When dealing with turbulent cases, no steady solution is available. The procedure used here is then the only one which can be applied. The case **B2bis** can therefore be seen as a first step towards the study of disturbance energies in reacting turbulent cases.

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** . **Note that "minimum" and "minimum linear" equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**

Equation 1[Eq.(6.10)]

Figs.(7.48), (7.49), (7.50) present the balance closure of the conservation equation for quadratic pressure and velocity fluctuations. As for case **B2**, the analysis of the balance of Eq.(**1**)[Eq.(6.10)] is first done for the onset of the instability, then the limit cycle is studied and finally the relaxation phase is analyzed. This section emphasizes the differences between cases **B2** and **B2bis**.

- onset of the instability

  Fig.(7.48) **a)** shows the balance of the exact nonlinear conservation equation for quadratic pressure and velocity fluctuations during the onset of the instability. Compared to **B2**, the time derivative term $\mathbf{dedt}$ does not oscillate harmonically but shows a strong component at 1700Hz as soon as the instability starts. The maximum amplitude reached at the end of the growing phase is only 7.10^{-4}% of the mean power of the flame (10^{-3}% in case **B2**).

  Fig.(7.48) **b)** shows the balance of the minimum Eq.(**1**).
  In contrary to case **B2**, all major terms oscillate as soon as the instability starts and no delay before this oscillation is observed. These terms are:

  - $\mathbf{Ac_1}: -\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\overline{u}}\right]$
  - $\mathbf{Ac_2}: -\vec{u'}.\vec{\nabla}p'$
  - $\mathbf{Ac_3}: \frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right]$
  - $\mathbf{Ac_4}: -\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\vec{\overline{q}}\right]$
  - $\mathbf{Ac_5}: \vec{u'} : \vec{\nabla}\tau'$
  - $\mathbf{Ac_6}: -\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N} h_{sk}\rho\vec{u}.\vec{\nabla}Y_k - (\overline{\gamma}-1)\sum_{k=1}^{N}\overline{h_{sk}}\overline{\rho}\vec{\overline{u}}.\vec{\nabla}\overline{Y_k}\right]$
  - $\mathbf{Ac_7}: \frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{\partial r}{\partial t}\right]$
  - $-\vec{u'}.co\vec{n}s_v$
  - $\mathbf{LW_{E_1}}$

  Compared to case **B2**, three terms are added in minimum Eq.(**1**). The first two terms $\mathbf{Ac_6}$ and $\mathbf{Ac_7}$ come from the fluctuation of the species mass fractions and thermodynamic quantities such as r due to the chemical reaction. Compared to **B2**, the term $\mathbf{Ac_7}$ does not have a much bigger amplitude in **B2bis** (see Fig.(7.44) **a)**). The main reason for the fact that this term should be kept here is that major terms such as $\mathbf{Ac_3}$ (shown in Fig.(7.44 **b)**) have a smaller amplitude in this case.

Actually, choosing the averaged mean flow as the baseline flow minimizes the fluctuation level in the flow. $\mathbf{Ac_6}$ and $\mathbf{Ac_7}$ are not influenced much by this phenomenon since they are constructed using the fluctuation of pressure which (as the temperature in Fig.(7.43) **a)** and **c)**) is almost the same in the two cases.

The last new term in minimum Eq.(**1**) is specific to this case. It is due to the choice of reference field. Since it should be zero if the reference field was steady, it measures the influence of the mean correlations on the level of fluctuations in the flow. This term is shown in Fig.(7.45). After half a period this term is a strictly positive source term. It then oscillates at 856Hz. Note that after 11 periods, it starts to periodically dissipate fluctuations.

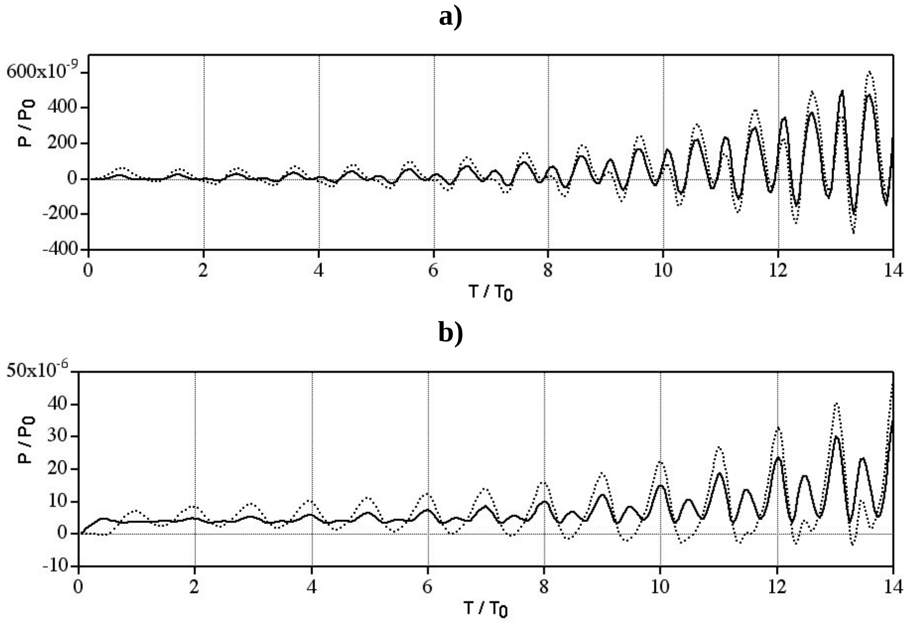


Figure 7.44 - *Case* **B2bis**, *Eq.*(**1**) *:* **Onset of the instability**
a) : $\mathbf{Ac_7}$ —— **B2bis** ········ **B2**
b) : $\mathbf{Ac_3}$ —— **B2bis** ········ **B2**

The linearization of the minimum Eq.(**1**) does not influences the closure of the balance as shown in Fig.(7.48) **c)** which is also a difference with case **B2** (see Fig.(7.28)). Linear terms coming from $\mathbf{Ac_6}$ and $\mathbf{Ac_7}$ are found in the minimum linear Eq.(**1**). Once more, terms related to the fluctuation of γ must be kept since they are not negligible.

- Limit cycle

Figs.(7.49) **a)** and **b)** respectively show the balance closures of the exact and minimum Eq.(**1**). The maximum amplitude of the time derivative term is 25% smaller than in **B2**. The remarks concerning the main terms of minimum Eq.(**1**) and their differences with case **B2** still apply during this phase.

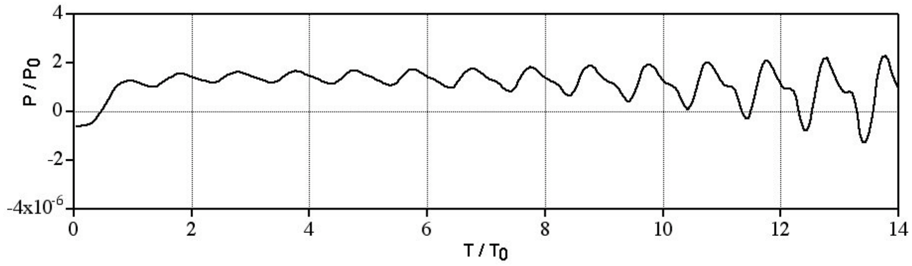


Figure 7.45 - *Case* **B2bis**, *Eq.***(1)** *:* **Onset of the instability** —— *mean-correlations term*

Fig.(7.49) **c)** shows that in **B2bis**, no underestimation of the time derivative of $\frac{1}{2\gamma_0 p_0} p_1^2 + \frac{\rho_0}{2} u_1^2$ is observed provided that the three new terms (once linearized) are taken into account.
Fig.(7.46) shows the mean-correlations term during the limit cycle. Even if its absolute minimum amplitude is bigger than its maximum one, its integral over one period is almost zero. In contrary to what is seen during the onset of the instability, this term is therefore not a source term any more.

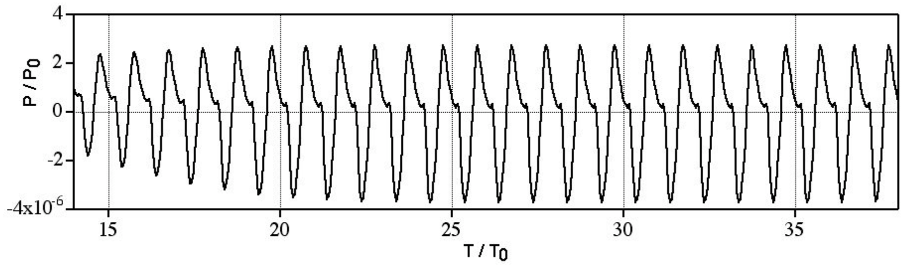


Figure 7.46 - *Case* **B2bis**, *Eq.***(1)** *:* **Limit cycle** —— *mean-correlations term*

- Relaxation

Figs.(7.50) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**1**) during the relaxation.

Fig.(7.50) **c)** shows that the closure of the minimum linear Eq.(**1**) is almost perfect during the relaxation phase.
Fig.(7.47) shows the mean-correlations term. As soon as the acoustic coefficient is decreased, its absolute minimum amplitude sinks to get lower than $4.10^{-4}\%$ of the mean power of the flame. The integral of this term over one period is therefore negative and this term dissipates the disturbance energy defined by minimum linear Eq.(**1**)

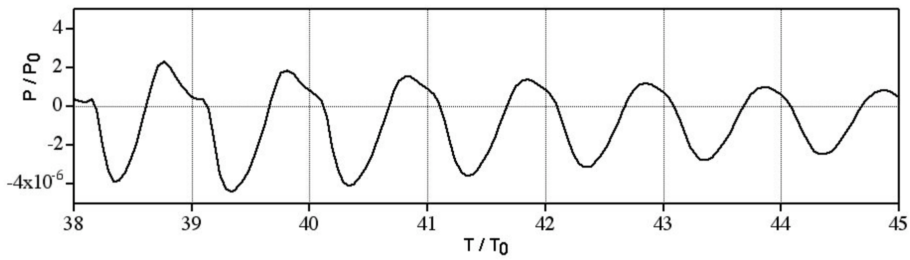
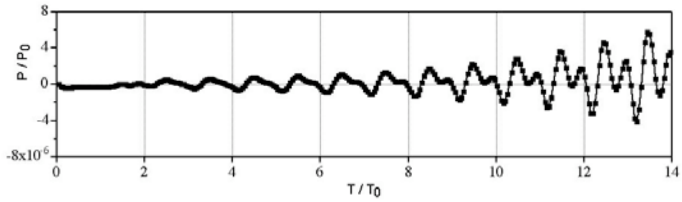


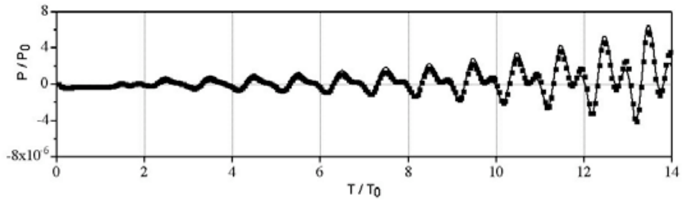
Figure 7.47 - *Case* **B2bis**, *Eq.***(1)** *:* **Relaxation** —— *mean-correlations term*

a)



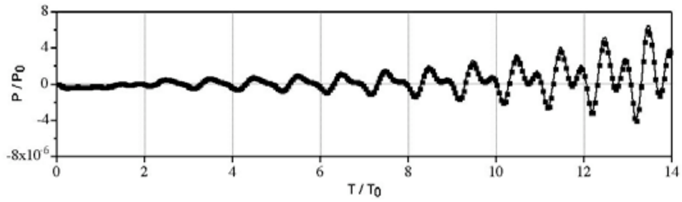
Maximum amplitude of time derivative : $1,5.10^{-5}$
ALL PHASES

b)



$$\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} + \frac{\rho}{2}\frac{\partial u'^2}{\partial t} =$$
$$-\vec{u'}.\vec{\nabla}p' - \frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \bar{\gamma}\bar{p}\vec{\nabla}.\vec{\bar{u}}\right] +$$
$$\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\bar{\gamma}-1)\,\overline{\dot{\omega_T}}\right] -$$
$$\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\bar{\gamma}-1)\,\vec{\nabla}.\vec{\bar{q}}\right] + \vec{u'} :$$
$$\vec{\nabla}\tau' - \frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N} h_{sk}\rho\vec{u}.\vec{\nabla}Y_k\right] +$$
$$\frac{p'}{\gamma p}\left[(\bar{\gamma}-1)\sum_{k=1}^{N} \overline{h_{sk}}\bar{\rho}\vec{\bar{u}}.\vec{\nabla}\overline{Y_k}\right] + \frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{\partial r}{\partial t}\right] -$$
$$\vec{u'}.\vec{cons}_v + \mathbf{LW_{E_1}}$$

c)



$$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} + \frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} =$$
$$-\vec{u_1}.\vec{\nabla}p_1 - p_1\vec{\nabla}.\vec{u}_1 - \frac{\gamma_1}{\gamma_0}p_1\vec{\nabla}.\vec{u}_0 + \frac{\gamma_0-1}{\gamma_0 p_0}p_1\dot{\omega_{T1}} +$$
$$\frac{\gamma_1 p_1}{\gamma_0 p_0}\dot{\omega_{T0}} - \frac{\gamma_0-1}{\gamma_0 p_0}p_1\vec{q}_1 - \frac{\gamma_1 p_1}{\gamma_0 p_0}\vec{q}_0 + \vec{u}_1 :$$
$$\vec{\nabla}\tau_1 - \frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk0}\rho_0\vec{u}_0.\vec{\nabla}Y_{k1}\right] -$$
$$\frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk0}\rho_0\vec{u_1}.\vec{\nabla}Y_{k0}\right] -$$
$$\frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk0}\rho_1\vec{u_0}.\vec{\nabla}Y_{k0}\right] -$$
$$\frac{p_1}{\gamma_0 p_0}\left[(\gamma_0-1)\sum_{k=1}^{N} h_{sk1}\rho_0\vec{u_0}.\vec{\nabla}Y_{k0}\right] +$$
$$\frac{p_1}{r_0}\frac{\partial r_1}{\partial t} - \vec{u'}.\vec{cons}_v + \mathbf{LW_{E_1}}$$

Figure 7.48 - *Case* **B2bis**, *Eq.***(1)** *:* **Onset of the instability**

a) Exact nonlinear ▪ $\mathbf{dedt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dedt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{dedt}_l$ —— sum of terms

a)

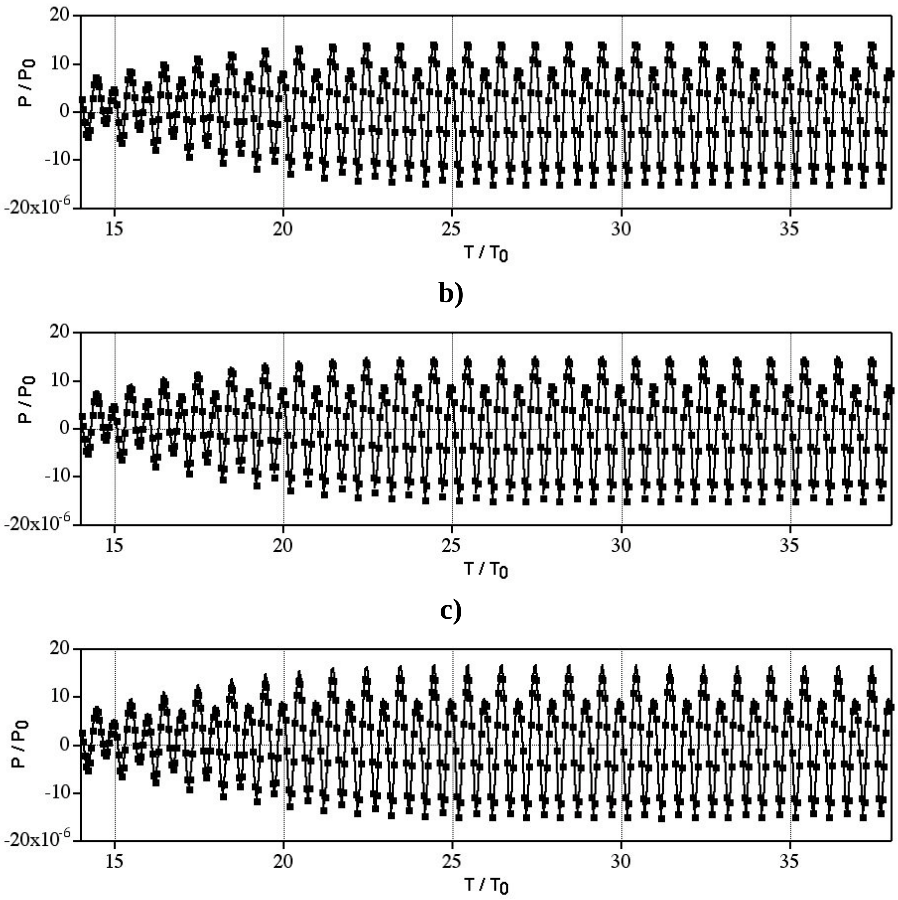


Figure 7.49 - *Case* **B2bis**, *Eq.***(1)** *:* **Limit cycle**

a) Exact nonlinear ■ **dedt** —— sum of terms
b) Minimum nonlinear ■ **dedt** —— sum of terms
c) Minimum linear ■ $\mathbf{dedt}_l$ —— sum of terms

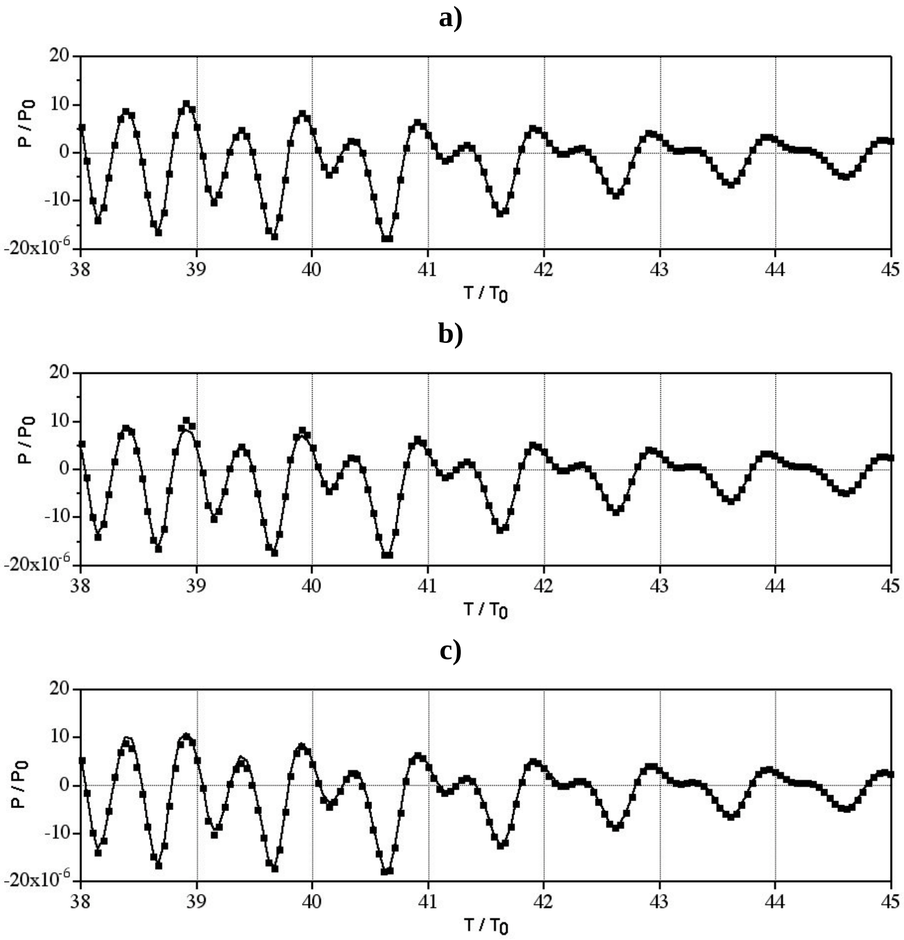


Figure 7.50 - *Case* **B2bis**, *Eq.***(1)** *:* **Relaxation**

a) Exact nonlinear ▪ **dedt** —— sum of terms
b) Minimum nonlinear ▪ **dedt** —— sum of terms
c) Minimum linear ▪ $\mathbf{dedt}_l$ —— sum of terms

Equation 2[Eq.(6.18)]

Figs.(7.54), (7.55) and (7.56) present the balance of the nonlinear conservation equation for the disturbance energy in entropy fluctuations. As for Eq.(**1**)[Eq.(6.10)], the evolution of the balance is split in three steps. The analysis of the balance of Eq.(**2**)[Eq.(6.18)] is first done for the onset of the instability, then the limit cycle is studied and finally the relaxation phase is analyzed.

- Onset of the instability

  Figs.(7.54) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**2**) during the onset of the instability. As for Eq.(**1**)[Eq.(6.10)], the time derivative term shows a much stronger 1700Hz component than for the case **B2**. The maximum absolute amplitude reached by this term represents 3% of the mean power of the flame after 14 periods of instability. This value should be compared to the 6% reached at the same instant in case **B2**.
  As for Eq.(**1**)[Eq.(6.10)], the maximum amplitude of the time derivative term is smaller in **B2bis** than in **B2**. The arguments given for the pressure and velocity fluctuations still apply here for entropy fluctuations. Locally, the mean value of the entropy already takes into account a part of the fluctuations compared to the steady field. Therefore, as an entropy disturbance occurs at one place, its amplitude is artificially lowered of an amount equal to the mean value of the fluctuations happening at the same place during all the cycle : excitation-limit cycle-relaxation.

  In the present case, the major terms of the balance are the same as in **B2**. Yet, one has to take into account the mean-correlations term displayed in Fig.(7.51).This term becomes positive one period after the beginning of the instability. It remains strictly positive and therefore adds power into the entropy disturbance energy during 7 periods. Then its minimum goes above zero and this term also periodically dissipates disturbance energy. Note that after 14 periods, its maximum absolute amplitude is 6% of the mean power of the flame which is two times the amplitude of $\mathbf{de_s dt}$ at the same moment.

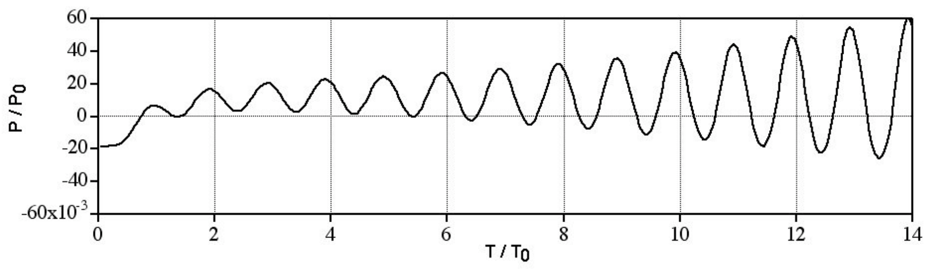


Figure 7.51 - *Case* **B2bis**, *Eq.*(**2**) *:* **Onset of the instability** —— *mean-correlations term*

  Fig.(7.54) **c)** shows that no linearization of Eq.(**2**)[Eq.(6.18)] in possible in this case.

- Limit cycle

  Figs.(7.55) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**2**) during the limit cycle. The time derivative term maximum amplitude is almost constant after 20 periods of oscillation and represents 6% of the mean power released by the flame which should be compared to the 10% reached in case **B2**. Besides the difference in maximum amplitudes, Figs.(7.55) **a)** and **b)** also show that the 1700 Hz component of the term $\mathbf{de_s dt}$ is much stronger in **B2bis** than in **B2** (see Fig.(7.35)).

Fig.(7.52) shows the mean-correlations term during the limit cycle. Its maximum absolute amplitude reaches 8.5% of the mean power of the flame and is therefore really significant in the balance. Note that in contrary to the mean-correlations term for Eq.(**1**)[Eq.(6.10)] during the limit cycle, it does not oscillates around zero. Even during this phase (which shows no specific growth of disturbance energy), this term tends to produce (after averaging over one period) entropy disturbance energy.

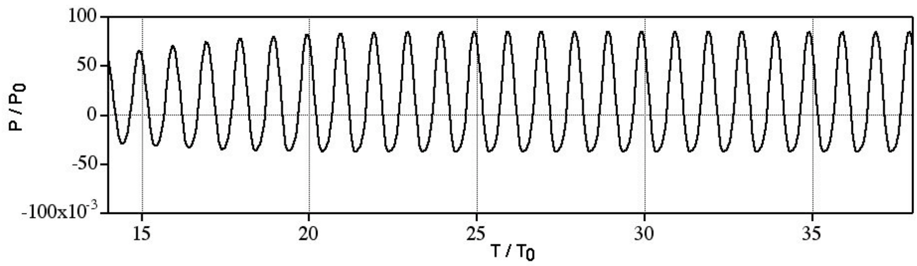


Figure 7.52 - *Case* **B2bis**, *Eq.***(2)** *:* **Limit Cycle** —— *mean-correlations term*

- Relaxation

  Fig.(7.56) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**2**) during the relaxation. In contradiction with **B2**, the 856Hz frequency component does not vanishes in the $\mathbf{de_s dt}$ signal.

  The mean-correlations term shows an interesting behavior as the acoustic reflection coefficient is released (see Fig.(7.53)). Less than one period after this, its maximum absolute amplitude decreases but not its minimum one which tends to make of it a sink term for the instability (after averaging over one period of instability). Its amplitude goes back to its value before the onset of the instability.

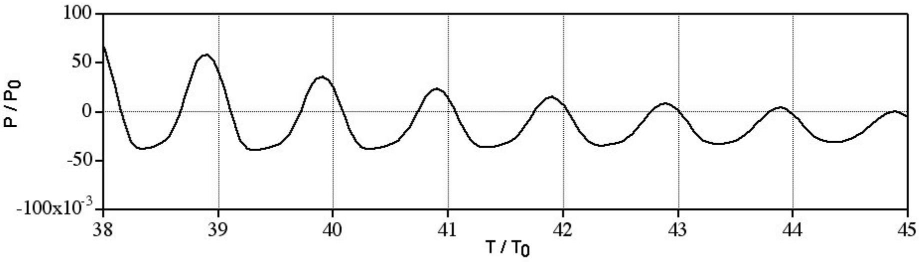
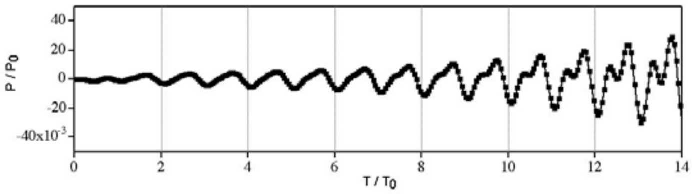


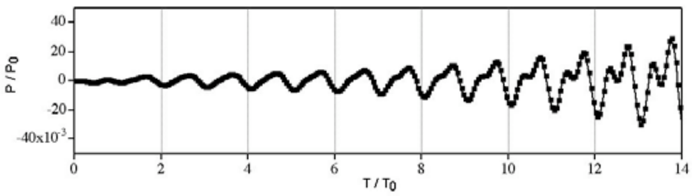
Figure 7.53 - *Case* **B2bis**, *Eq.***(2)** *:* **Relaxation** —— *mean-correlations term*

a)



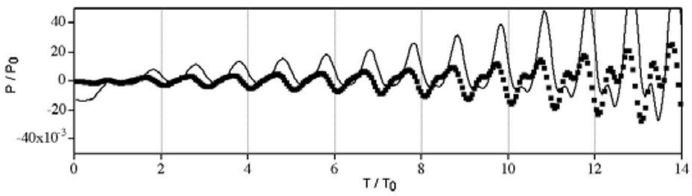
Maximum amplitude of time derivative : 6.10^{-2}
ALL PHASES

b)



$$\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right] + \frac{s'}{c_p}\left[\dot{\omega}_T - \frac{\rho T}{\overline{\rho}\overline{T}}\overline{\dot{\omega}_T}\right] + \frac{s'}{c_p}\left[-\vec{\nabla}.\vec{q} + \frac{\rho T}{\overline{\rho}\overline{T}}\vec{\nabla}.\overline{\vec{q}}\right] - \frac{s'}{c_p}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] + \frac{s'}{c_p}\frac{\rho T}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] - \frac{ps'}{rc_p}cons_s$$

c)

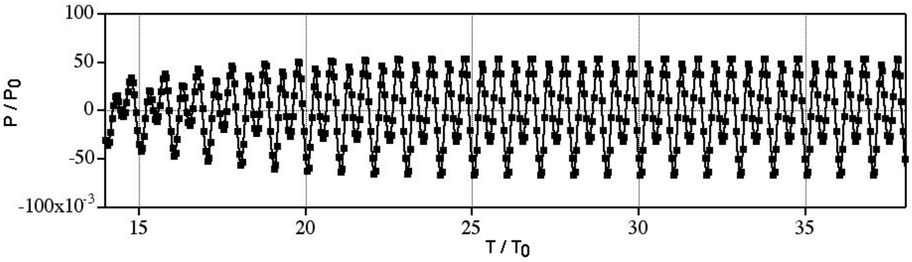


$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} = -\frac{p_0s_1}{r_0c_{p0}}\vec{u_1}.\vec{\nabla}s_0 + \frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q_1}\right] - \frac{p_0s_1}{r_0c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2} + \frac{T_1}{{T_0}^2}\right]\left[\dot{\omega}_{T0} - \vec{\nabla}.\vec{q_0}\right] - \frac{s_1}{c_{p0}}\sum_k g_{sk1}\left[\dot{\omega}_{k0} - \vec{\nabla}.\vec{q_{k0}}\right] - \frac{s_1}{c_{p0}}\sum_k g_{sk0}\left[\dot{\omega}_{k1} - \vec{\nabla}.\vec{q_{k1}}\right] + \frac{p_0s_1}{r_0c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2} + \frac{T_1}{{T_0}^2}\right]\sum_k g_{sk0}\left[\dot{\omega}_{k0} - \vec{\nabla}.\vec{q_{k0}}\right] - \frac{p_0s_1}{r_0c_{p0}}cons_s$$

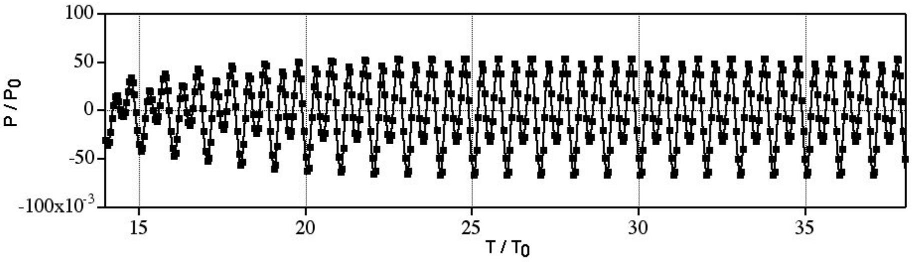
Figure 7.54 - *Case* **B2bis**, *Eq.*(**2**) *:* **Onset of the instability**

a) Exact nonlinear ▪ $\mathbf{de_sdt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{de_sdt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{de_sdt}_l$ —— sum of terms

a)



b)



c)

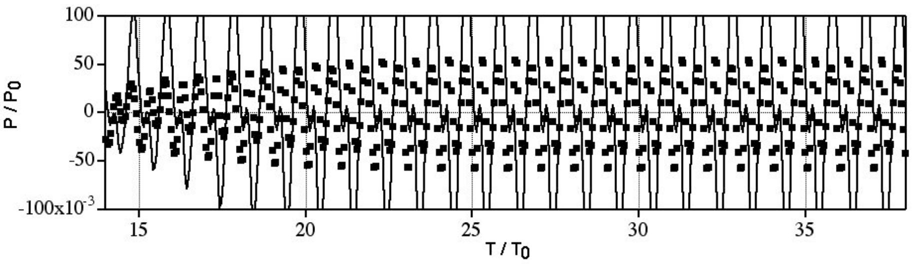


Figure 7.55 - *Case* **B2bis**, *Eq.***(2)** *:* **Limit cycle**

a) Exact nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{de_s dt}$ —— sum of terms
c) Minimum linear ▪ $\mathbf{de_s dt}_l$ —— sum of terms

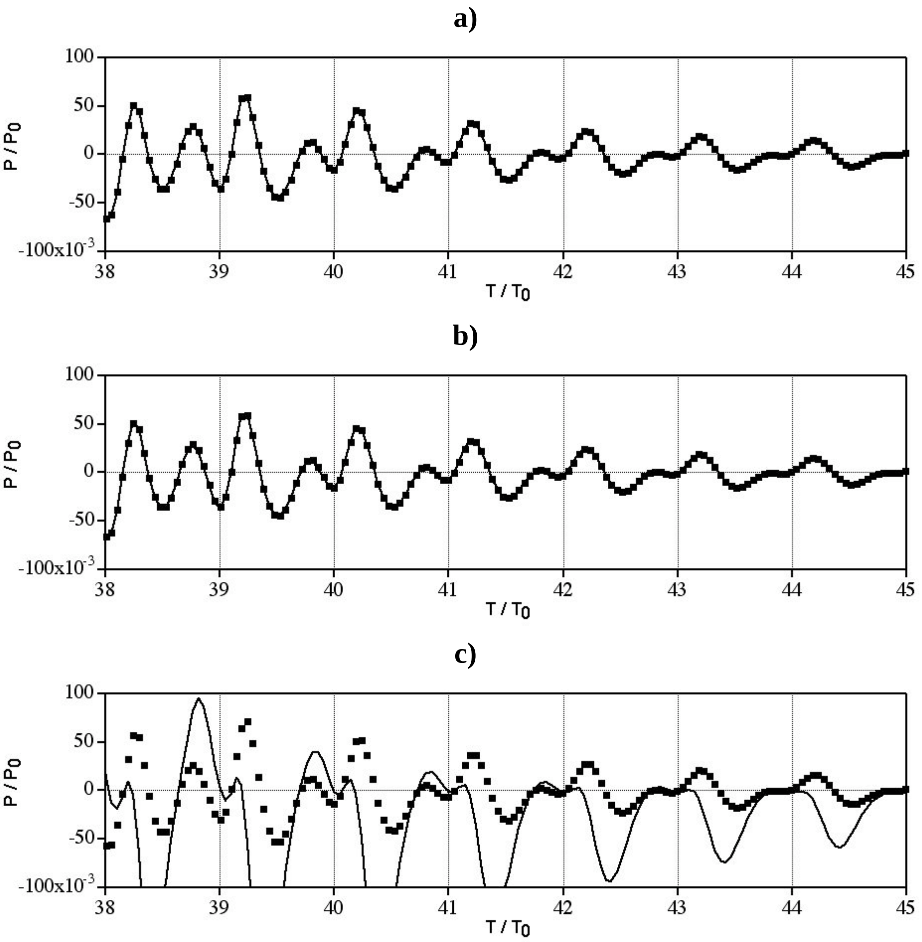


Figure 7.56 - *Case* **B2bis**, *Eq.***(2)** *:* **Relaxation**

a) Exact nonlinear ■ $\mathbf{de_s dt}$ —— sum of terms

b) Minimum nonlinear ■ $\mathbf{de_s dt}$ —— sum of terms

c) Minimum linear ■ $\mathbf{de_s dt}_l$ —— sum of terms

Equation 3[Eq.(6.37)]

Figs.(7.60), (7.61) and (7.62) present the balance of the nonlinear conservation equation for the nonlinear disturbance energy. As for Eqs.(**1**) and (**2**), the evolution of the balance is split in three steps. The nature of the new terms added here in comparison to case **B2** is different from the additional terms that appear in Eq.(**1**)[Eq.(6.10)] or (**2**). These terms still correspond to mean correlations of fluctuations but they are not multiplied by any variable. Therefore, as it is shown in the following, their amplitude and sign remain the same whether the instability is growing or decreasing. Finally, note that no linearization of Eq.(**3**)[Eq.(6.37)] is provided.

- Onset of the instability

  Fig.(7.60) **a)** shows the balance of the exact nonlinear Eq.(**3**)[Eq.(6.37)] during the onset of the instability. The maximum amplitude of the time derivative term grows from zero to 4% of the mean flame power in 10 periods of instability. This level should be compared to the 10% reached at the same instant by $\mathbf{dE_d dt}$.

  Fig.(7.60) **b)** shows the balance closure of the minimum nonlinear Eq.(**3**)[Eq.(6.37)]. The major terms are:

  – $\mathbf{D_s}$: $-s'\vec{m}'\vec{\nabla}\bar{T} + \bar{\vec{m}}s'\vec{\nabla}T'$
  – $\mathbf{D_Q}$: $T'Q'$
  – $\mathbf{D_Y}$: $\vec{m}'.\vec{\psi^{\star}}' + T'Q^{\star\prime} - \sum_{k=1}^{n} g'_{sk}\Omega'_k + \sum_{k=1}^{n} g'_{sk}\overline{Y_k}\vec{\nabla}.\vec{m}' + \sum_{k=1}^{n} \overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m}'$
  – $\overline{\mathbf{D_s}}$: $-\bar{\vec{m}}.\overline{T'\vec{\nabla}s'} - \overline{(\vec{m}'s')}.\vec{\nabla}\bar{T}$
  – $\overline{\mathbf{D_Q}}$: $\overline{T'Q'}$

  $\overline{\mathbf{D_Q}}$ is a constant sink term for the instability. Its amplitude is -8.8% of the mean power of the flame. $\overline{\mathbf{D_s}}$ is a source term with an amplitude of 10.1%. The global influence of these terms on the balance of disturbance energy is therefore positive and measures the influence of the stationary hypothesis in this case.

  Fig.(7.57) shows the major unsteady terms of the balance of Eq.(**3**)[Eq.(6.37)]. There are many differences between the terms represented here and the same terms for case **B2** (see Fig.(7.37)). The first one is that at the beginning of the instability $\mathbf{D_Y}$, $\mathbf{D_Q}$ and $\mathbf{D_s}$ are not zero. This comes from the fact that this instant is not very much different from another one when defining the fluctuations using the averaged mean flow. The fluctuation level is not zero when the instability starts.

- Limit cycle

  Fig.(7.61) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**3**)[Eq.(6.37)] during the limit cycle. After 20 periods of instability, the limit cycle is reached. The absolute maximum amplitude reached by $\mathbf{dE_d dt}$ represents 20% of the mean power of the flame (30% for **B2**).

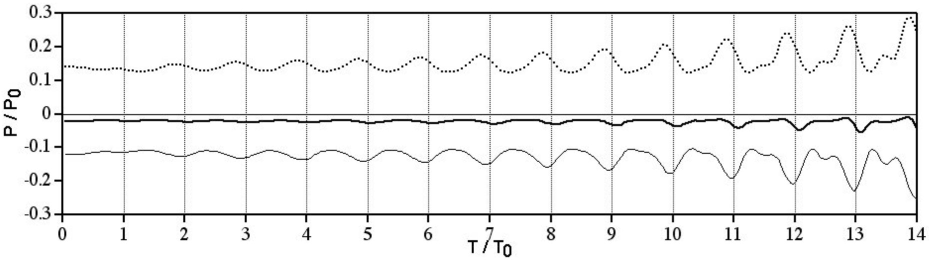


Figure 7.57 - *Case* **B2**, *Eq.***(3)** *:* **Onset of the instability** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

Fig.(7.58) shows the major terms of the minimum Eq.(**3**)[Eq.(6.37)]. The main difference between Fig.(7.58) and Fig.(7.38) is that terms $\mathbf{D_s}$ and $\mathbf{D_Q}$ never get close to zero, therefore always creating or dissipating disturbance energy.

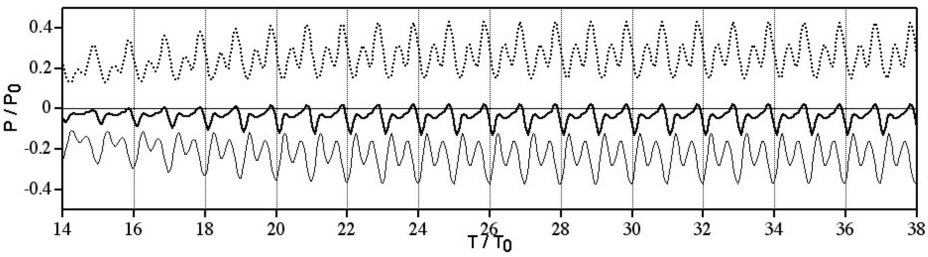


Figure 7.58 - *Case* **B2**, *Eq.***(3)** *:* **Limit cycle** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

- Relaxation

  Fig.(7.62) **a)** and **b)** respectively show the balance of the exact and minimum nonlinear Eq.(**3**)[Eq.(6.37)] during the relaxation. The similarity between Fig.(7.62) and Fig.(7.42) is striking. One may say that the nonlinear disturbance energy relaxes the same way whatever the reference field is.
  The fluctuations of the main terms also look like in Fig.(7.59) and in Fig.(7.39). Yet, the main difference is that here $\mathbf{D_Q}$, $\mathbf{D_s}$ and $\mathbf{D_Y}$ do not return to zero amplitude but to their initial values (steady state flow).

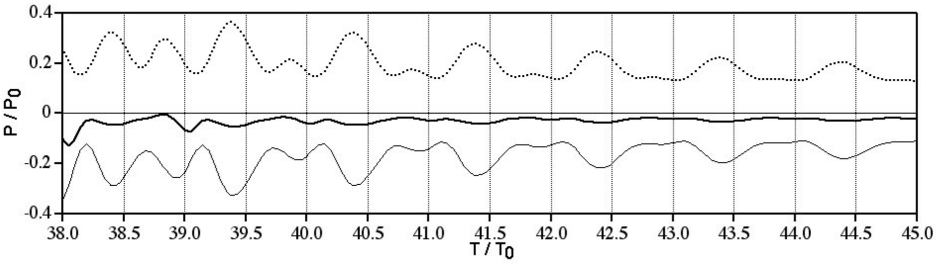
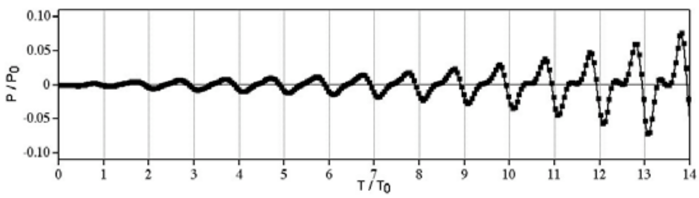


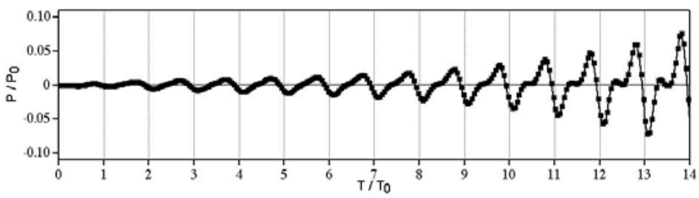
Figure 7.59 - *Case* **B2**, *Eq.***(3)** *:* **Relaxation** *(Representative terms)*
—— $\mathbf{D_Y}$ —— $\mathbf{D_Q}$ ········ $\mathbf{D_s}$

a)



Maximum amplitude of time derivative : 2.10^{-1}
ALL PHASES

b)



$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}}s'\vec{\nabla}T' + T'Q' + \vec{m}'.\vec{\psi^{\star}}' + T'Q^{\star\prime} - \sum_{k=1}^{n} g'_{sk}\Omega'_k + \sum_{k=1}^{n} g'_{sk}\overline{Y_k}\vec{\nabla}.\vec{m'} + \sum_{k=1}^{n} \overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m'} - \bar{\vec{m}}.\overline{T'\vec{\nabla}s'} - \overline{(\vec{m'}s')}.\vec{\nabla}\bar{T} + \overline{T'Q'}$$

Figure 7.60 - *Case* **B2bis**, *Eq.***(3)** *:* **Onset of the instability**
a) Exact nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms
b) Minimum nonlinear ▪ $\mathbf{dE_d dt}$ —— sum of terms

a)

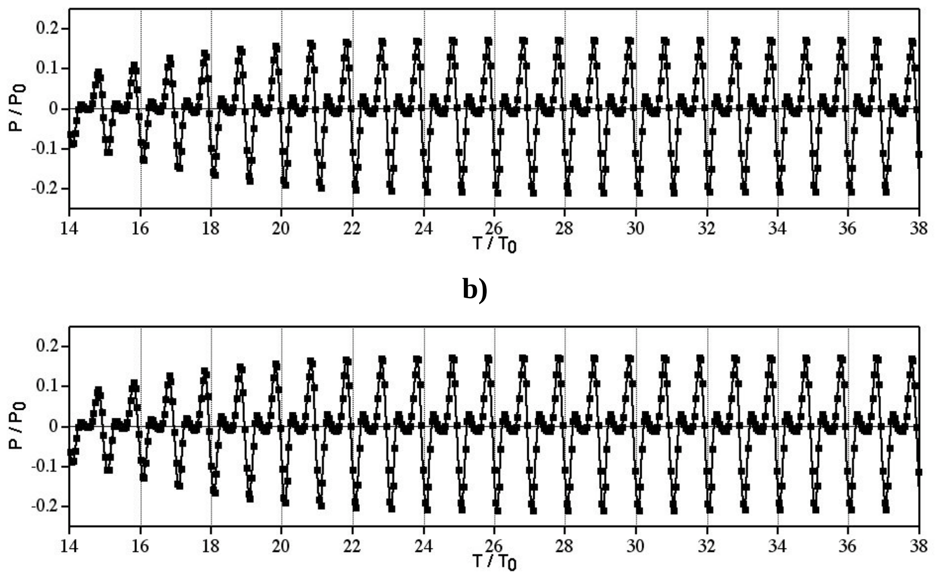


Figure 7.61 - *Case* **B2bis**, *Eq.***(3)** *:* **Limit cycle**

a) Exact nonlinear ■ $\mathbf{dE_d dt}$ —— sum of terms

b) Minimum nonlinear ■ $\mathbf{dE_d dt}$ —— sum of terms

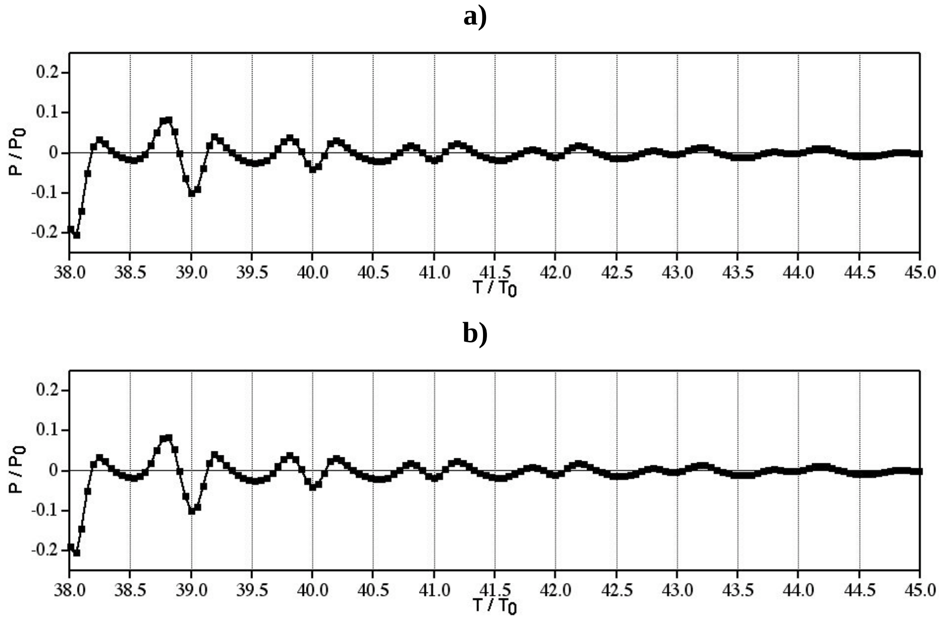


Figure 7.62 - *Case* **B2bis**, *Eq.***(2)** *:* **Relaxation**

a) Exact nonlinear ■ $\mathbf{dE_d dt}$ —— sum of terms

b) Minimum nonlinear ■ $\mathbf{dE_d dt}$ —— sum of terms

7.3 Summary of the results

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** . **Note that ”minimum” and ”minimum linear” equations, deriving from the balance closure of Eqs.1, 2 and 3, depend on the case studied and therefore will not be found in these tables.**

7.3.1 Influence of mean Mach number terms

The minimum linear Eqs.(**1**), (**2**) and the minimum Eq.(**3**) involve terms related to the mean Mach number that are reported here. Case **B2** is taken as a reference for the present discussion. Note that the mean Mach number is 0.01 in this case.
No mean heat release can be sustained in a combustor if the mean velocity is zero. Therefore, the presence of terms related to the mean heat release in Eq.(**1**), (**2**) and (**3**) reveals the influence of the mean velocity field on the level of disturbance energies.

- Minimum linear Eq.(**1**) which refers to the Pressure-Velocity (PV) disturbance energy[2] involves a term which is linked to the mean Mach number :

$$\frac{\gamma_1 p_1}{\gamma_0 p_0}(p_0 \vec{\nabla}.\vec{u}_0 + \dot{\omega_{T0}})$$

  The ratio $\frac{\gamma_1 p_1}{\gamma_0 p_0}$ is very small and its maximum mean value during the instability process is equal to $3,5.10^{-4}$. Yet, it must be multiplied by the mean heat release to know its influence on the PV disturbance energy. In this case, the mean heat release per unit volume is $6.10^6 W.m^{-3}$. The product of the two gives a maximum value per unit volume equal to $2100 W.m^{-3}$ which represents 60% of the maximum power of the disturbance energy per unit volume.

- Minimum linear Eq.(**2**) which refers to entropy disturbance energy, involves the following terms related to the mean Mach number

$$-\frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2}+\frac{T_1}{{T_0}^2}\right]\dot{\omega}_{T0} - \frac{s_1}{c_{p0}}\sum_k g_{sk1}\dot{\omega}_{k0} + \frac{p_0 s_1}{r_0 c_{p0}}\left[\frac{\rho_1}{{\rho_0}^2}+\frac{T_1}{{T_0}^2}\right]\sum_k g_{sk0}\dot{\omega}_{k0}$$

  As for the minimum linear Eq.(**1**), these terms include the mean heat release which therefore has an influence on the balance of the entropy disturbance energy. Yet, since no closure of the minimum linear Eq.(**2**) can be reached, a discussion on the exact influence of those terms may be somewhat hazardous.

- Minimum Eq.(**3**) which refers to nonlinear disturbance energy involves the following term which is directly related to the mean Mach number : $\overline{\vec{m}s'\vec{\nabla}T'}$. This term states that a fluctuation of entropy, positively correlated to a fluctuation of the gradient of temperature may create nonlinear

[2]in homogeneous mixing, isentropic and small mean Mach flow, this disturbance energy is the acoustic energy

disturbance energy provided that the mean Mach number is not zero.
Also note that the term $T'Q'$ contains the influence of the mean heat release since it linearizes as

$$\frac{1}{T_0}\left[\dot{\omega}_{T1} - \vec{\nabla}.\vec{q}_1 + \phi_1\right] - \frac{T_1}{{T_0}^2}\left[\dot{\omega}_{T0} - \vec{\nabla}.\vec{q}_0 + \phi_0\right]$$

7.3.2 Influence of chemical related terms

In the exact nonlinear equations **1**,**2** and **3** considered here, chemical related terms can be divided in two groups.

- The first one involves the direct creation or dissipation of heat due to the presence of the flame. This group therefore takes into account the fluctuations of heat release $\dot{\omega_T}'$ and heat flux $\vec{\nabla}.\vec{q'}$.
- The other group is more linked with the fluctuations of the chemical products concentrations and thermodynamic quantities due to the flame. It therefore involves the fluctuations of the gradients of mass fractions $\vec{\nabla}Y_k'$ and of such quantities as g_{sk}', γ' and r', etc...

Case **A** tells us that the influence of the second group on the disturbance energy is negligible in a 1D flame case. Yet, the first group may influence the level of the disturbance energy as soon as the frequency of the forcing is small enough. This comes from the fact that as the frequency decreases, it gives more time to the flame front to react to the incoming perturbation. Without changing the shape of the flame front, it changes the volume concerned by the fluctuations of heat release and heat flux and therefore leads to the observed result.

Case **B** points out the influence of the two groups on disturbance energies even for higher frequencies. Both cases **B1** and **B2** require introducing the first group of chemical terms in minimum equations to match the evolution of disturbance energies. It means that in case **B** the velocity fluctuation in front of the flame is big enough to make it move significantly. One can expect that it would also be the case for turbulent flames.
The influence of the second group of terms mainly depends on the maximum absolute amplitude of the time derivative term. For Eq.(**1**), this group lies at the limit (10%) of neglecting. It is therefore kept for **B1** and not for **B2** although it almost has the same amplitude in both cases. As far as Eqs.(**2**) and (**3**) are concerned, this group of terms should remain.

Derivations of disturbance energies in reacting flows often make the assumption that the second group of terms is negligible ("homogeneous mixing"), this study shows that it is not always the case and therefore that great care should be taken when doing such an assumption in reacting configurations.
Note that for Eq.(**1**), even when the second group is negligible, the linearization of the term

$$\frac{p'}{\gamma p}((\gamma-1)\dot{\omega_T})'$$

leads to a term that writes

$$\frac{p_1\gamma_1}{p_0\gamma_0}\dot{\omega}_{T0}$$

which is not negligible because of high local values of the mean heat release.

7.3.3 The issue of the linearization

The issue of the linearization only concerns Eqs.(**1**) and (**2**). To describe the evolution of disturbance energies, these conservation equations need to be linearized. The 1D flame case (**A**) shows no specific issue concerning the linearization for both conservation equations as long as all major terms are kept.
In case **B**, the equation giving the evolution of the ”pressure-velocity” disturbance energy (Eq.(**1**)[Eq.(6.10)]) linearizes quite nicely. The balance of the minimum linear Eq.(**1**) shows that nonlinearities only become important when the rate of change of the disturbance energy is higher than a certain level. The conclusions are quite different for Eq.(**2**)[Eq.(6.18)] which gives the evolution of the ”entropy” disturbance energy. The linearization of Eq.(**2**)[Eq.(6.18)] is impossible as soon as the flame front is perturbed.

Since case **B** is much closer to a realistic reacting configuration, it implies that the linearization of Eq.(**2**)[Eq.(6.18)] might be problematic in those cases. The reason for that is that Eq.(**2**)[Eq.(6.18)] directly involves terms such as

$$\left[\frac{\dot{\omega_T}}{\rho T}\right]'$$

which cannot be linearized as

$$\left[\frac{\dot{\omega_{T1}}}{\rho_0 T_0}\right] - \left[\frac{\rho_1}{\rho_0^2 T_0} + \frac{T_1}{\rho_0 T_0^2}\right]\dot{\omega_{T0}}$$

because $T_1 \not\ll T_0$ in the influence area of the flame.

7.3.4 Influence of the reference field

The influence of the reference field is studied here by comparing the results obtained in cases **B2** and **B2bis**. Both cases represent the same physical phenomenon of an instability in a combustor. The same snapshops are used to compute the balances of Eqs.(**1**), (**2**) and (**3**). The differences between the two cases therefore come from the choice of the reference field. The values of the reference variables directly influence the amplitude of the fluctuations inside the domain. Yet, all terms are not modified in the same way as shown by Fig.(7.43). The maximum power reached by the disturbance energy is smaller for case **B2bis**. It represents 75% of case **B2** for Eq.(**1**)[Eq.(6.10)] and respectively 60% and 62% for Eq.(**2**)[Eq.(6.18)] and (**3**). Actually, choosing the averaged mean field as the baseline flow minimizes the level of fluctuations in the flow.

Also note that the change of reference field influences the shape of the time derivative of disturbance energies by introducing more of the 1700 Hz component into it.
Finally, the sign of the mean correlation term for Eq.(**1**)[Eq.(6.10)] and (**2**) seems to follow the instability. It is positive (in average over a period) during the onset and negative during the relaxation. These terms measure the influence of the stationary hypothesis in that these terms are zero is the reference flow is stationary.

7.3.5 Influence of numerical corrections

Three kind of numerical corrections are considered here:

- **AVI** : which comes from the use of artificial numerical viscosity.
- **BND** : which represents the influence of boundary nodes on the variables of the code.
- **LW** : which represents the correction added to reach second order of spatial accuracy.

AVI is the only correction that the user can directly influence. Here, thanks to sufficient mesh refinement, only small artificial viscosity is needed, so that this term is always negligible when not zero.

BND, the influence of the boundaries (which is not a ”numerical correction” is the sense that it is part of the physical problem), has small influence on the disturbance energies studied here. This term is therefore neglected in minimum equations. This should be the case as long as the ratio between the total volume of the boundary nodes over the entire volume of the geometry remains small enough.

LW correction appears when fluxes of quantities such as mass, momentum, total energy are not precise enough when given by a first spatial order divergence operator. This term corrects these fluxes in order to reach second order accuracy. In this study, the **LW** correction only remains in minimum Eq.(**1**) and not in Eq.(**2**)$_{\text{[Eq.(6.18)]}}$ and (**3**). Note that the disturbance levels for Eq.(**1**)$_{\text{[Eq.(6.10)]}}$ are much smaller than for the two other disturbance energies. The ”pressure-velocity” disturbance energy is therefore more influenced by numerical terms even for highly discretized meshes such as configuration **B**[3]. This last issue will have to be addressed and solved when trying to close budgets of disturbance energies in industrial cases.

[3] $\Delta_x = 0.03mm$ in the flame. There is therefore more than ten mesh points across the flame front.

Chapter 8

Stability criteria in reacting flows

This chapter focuses on the implications of the previous results in terms of stability criterion for each type of disturbances energy.

Section 8.1 points out the similarities between the disturbance energies studied in this work and gives the evolution of their rate of change as the instability occurs.

Section 8.2 presents the results given by the criteria of Rayleigh and Chu applied to case **B2**.

Section 8.3 presents three types of stability criterion corresponding to the minimum equations derived in Chapter 7. These criteria are tested on the instability occurring in case **B2**.

Section 8.4 presents concluding remarks for this part as well as some prospects that this study implies.

8.1 Evolution of disturbance energies

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** .

Time derivatives of disturbance energies are integrated over time. Figs.(8.1), (8.2) and (8.3) show, for each kind of disturbance energy, the increase or loss during each period of oscillation. Results are normalized by the energy released by the flame during the same amount of time (E_0=4.61 $J.m^{-1}$).

The gain amplitudes respect the order of the maximum amplitude of time derivative terms seen in Chapter 7. The nonlinear disturbance energy is the most important. Then comes the entropy disturbance energy and finally the pressure-velocity disturbance energy which is far smaller.
Yet, even if the levels they reach are very different, they all show the same trends as the instability grows

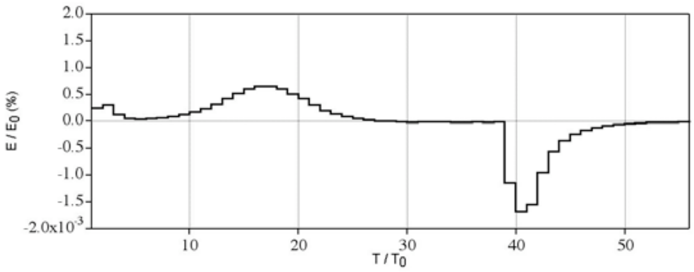


Figure 8.1 - **Energy 1** *Gain per period of oscillation (F_0=856Hz)*
(in percent of the energy released by the flame during one period (4.61$J.m^{-1}$))

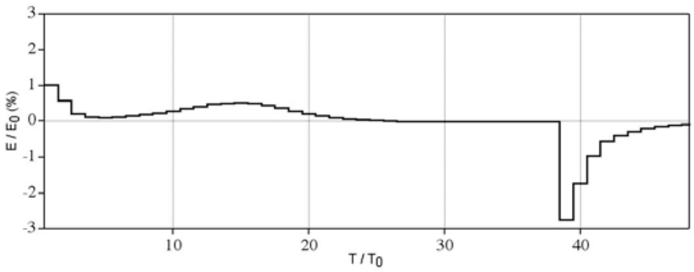


Figure 8.2 - **Energy 2** *Gain per period of oscillation (F_0=856Hz)*
(in percent of the energy released by the flame during one period (4.61$J.m^{-1}$)

and decays. First, the disturbance energy grows rapidly during the first three periods which is the time needed by an acoustic fluctuation to go from the outlet to the inlet and come back. After period 6, a second growth phase lasting approximately 20 periods occurs before the limit cycle is reached.
The outlet reflection coefficient is released at the beginning of period 38. The gain shrinks immediately and the system undergoes a relaxation phase during at least 10 periods.

The striking point is that the three disturbance energies although representing different types of fluctuations (pressure-velocity (Eq.(**1**)[Eq.(6.10)]), entropy (Eq.(**2**)[Eq.(6.18)]) and nonlinear (Eq.(**3**)[Eq.(6.37)])) exactly show the same behavior. With a closer look, one can however notice that the pressure-velocity disturbance energy is slightly shifted, late of approximately one and a half period compared to the other disturbance energies.

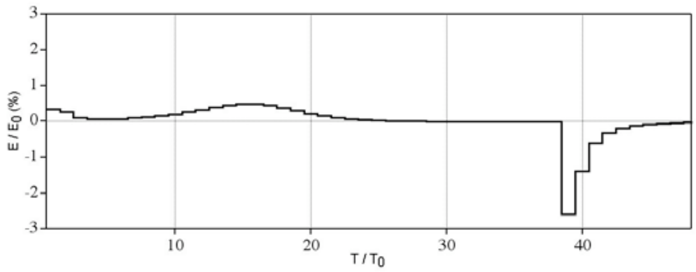


Figure 8.3 - **Energy 3** *Gain per period of oscillation (F_0=856Hz)*
(in percent of the energy released by the flame during one period (4.61 $J.m^{-1}$))

8.2 Linear criteria for stability : Rayleigh Criteria, Chu criterion

8.2.1 The classical and extended Rayleigh criteria

The classical Rayleigh criterion writes

$$\int_{T_0}\int_V p_1 \dot{\omega_{T1}} dvdt \ > \ 0$$

It states that if the fluctuation of heat release and the fluctuation of pressure positively correlate (so that the previous integral is positive over one period of oscillation), an acoustic instability may grow inside the combustor.

The extended Rayleigh criterion, derived by Nicoud and Poinsot in [93] applies to the acoustic energy and writes :

$$\int_{T_0}\int_V \frac{(\gamma_0 - 1)p_1}{\gamma_0 p_0} \dot{\omega_{T1}} dvdt - \int_{T_0}\int_S p_1 \vec{u}_1 . \vec{n} dsdt \ > \ 0$$

It states that the acoustic disturbance energy may grow if the source term per unit volume involving heat release and heat flux (due to the unsteady combustion process) is greater than the losses due to the acoustic fluxes across the domain.

These two criteria are applied to case **B2** and the results are shown on Fig.(8.4). In order to give a comparison point and also because these two criteria apply to the acoustic stability, the gain of the PV disturbance energy is also provided. Both criteria should therefore be positive during the growth phase and negative during the relaxation. It is not the case. These criteria correctly detect the onset of the instability but remain positive during the limit cycle and even when the instability vanishes. They therefore do not give satisfactory results by missing the detection of the relaxation phase (especially the classical Rayleigh criterion).
Note that both criteria always remain over the gain of pressure-velocity disturbance energy. They therefore tend to overestimate the instability occurring in the combustor.

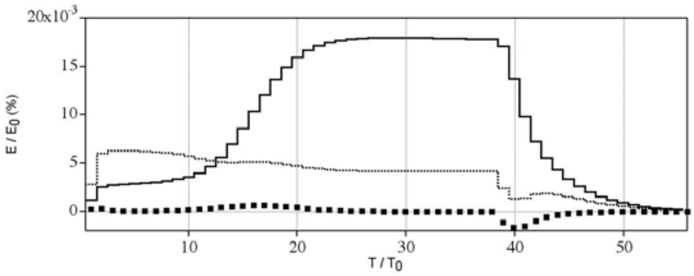


Figure 8.4 - *Gain per period of oscillation (F_0=856Hz)*
(percent of the energy released by the flame during one period (4.61J))
■ **Energy 1** —— Classical Rayleigh ········ Extended Rayleigh

8.2.2 The Chu criterion

Chu derives a disturbance energy conservation equation in reacting cases [23]. His disturbance energy involves the sum of the disturbance energies defined by linear Eq.(**1**)[Eq.(6.10)] and (**2**). For the cases enclosed in this study, entropy disturbance energy levels are much bigger than levels of PV disturbance energy. A criterion applying to entropy disturbance energy can therefore be derived from his work as already written in [93].
This criterion writes :

$$\int_{T_0}\int_V \frac{T_1}{T_0}\dot{\omega_{T1}}dvdt - \int_{T_0}\int_S p_1\vec{u}_1.\vec{n}dsdt \ > \ 0$$

This linear criterion for stability states that the growth of the disturbance energy is due to the positive correlation between the fluctuation of temperature and the fluctuations of heat release and heat flux over a period of instability. Note that this work shows that the acoustic flux term composing the Chu criterion has an amplitude which is almost three orders of magnitude lower than the term $\int_{T_0}\int_V \frac{T_1}{T_0}\dot{\omega_{T1}}dvdt$.

Fig.(8.5) shows the expected gain given by this criterion and compares it to the gain of the entropy disturbance energy in case **B2**.
The Chu criterion correctly detects the onset of the instability as well as the limit cycle. Yet, it increases when the relaxation starts as if a second phase of instability growth occurred. Because of this last point, this criterion cannot be considered as giving satisfactory results as far as the detection of the main trends of the instability are concerned.

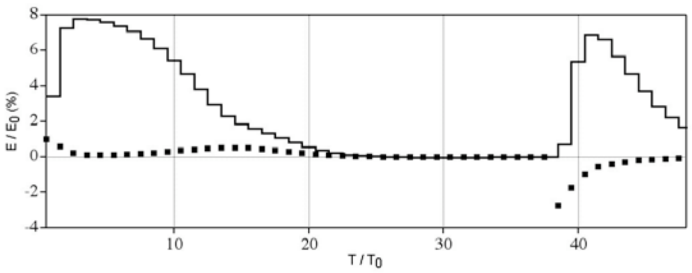


Figure 8.5 - *Gain per period of oscillation (F_0=856Hz)*
(percent of the energy released by the flame during one period (4.61J))
■ **Energy 2** —— Chu

8.3 Deriving stability criteria from Eqs.(1), (2) and (3)

Tables 6.3, 6.4 and 6.5 summarize Eqs.**1**, **2** and **3** .

Energy 1 refers to $\frac{p_1^2}{2\gamma_0 p_0} + \frac{\rho_0 u_1^2}{2}$ defined in linear Eq.(**1**)

Energy 2 refers to $\frac{p_0}{2r_0 c_{p0}} s_1^2$ defined in linear Eq.(**2**)

Energy 3 refers to $\rho\left(H - \overline{H}\right) - \rho\bar{T}\left(s - \overline{s}\right) - \overline{\vec{m}}.(\vec{u} - \overline{\vec{u}}) - (p - \overline{p}) - \sum_{k=1}^{n} \overline{g_{sk}}\rho(Y_k - \overline{Y_k})$ defined in Eq.(**3**)

8.3.1 Pressure-Velocity (PV) disturbance energy (Eq.1).

Following the results obtained in case **B2** in chapter 7, linear minimum Eq.(**1**) writes

$$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} + \frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} = -\vec{\nabla}.(p_1\vec{u}_1) + \frac{\gamma_0 - 1}{\gamma_0 p_0}p_1(\dot{\omega_{T1}} - \vec{q}_1) + \frac{\gamma_1 p_1}{\gamma_0 p_0}(\dot{\omega_{T0}} - \vec{\nabla}.\vec{q}_0 - \vec{\nabla}.\vec{u}_0) + \vec{u}_1 : \vec{\nabla}\tau_1 \quad (8.1)$$

This linear minimum equation still involves too many terms to provide a simple stability criterion. If one only keeps the terms that have an amplitude bigger than 50% of the maximum absolute amplitude of the time derivative term, minimum Eq.(8.1) leads to the following simple linear stability criterion :

$$\int_{T_0}\left[+\int_V \frac{\gamma_0 - 1}{\gamma_0 p_0}p_1(\dot{\omega_{T1}} - \lambda_0\Delta T_1)dv - \int_S p_1\vec{u}_1.\vec{n}dS\right]dt \; > \; 0 \quad (8.2)$$

If this relation is satisfied, the disturbance energy $e_1 = \frac{1}{2\gamma_0 p_0}p_1^2 + \frac{\rho_0}{2}u_1^2$ will grow inside the domain. This criterion extends the Rayleigh criterion by adding to it the influence of the fluctuation of the heat flux.

This criterion is tested in the case **B2**. Fig.(8.6) shows the expected gain given by the Pressure-Velocity (PV) criterion and compares it to the real gain of e_1 per period of oscillation. This criterion succeeds in detecting both the onset of the instability as well as the relaxation of the system (changing its sign at the right moment). The predictions given by this Pressure-Velocity (PV) criterion are therefore trustful in this case and could be tested on an instability occurring in a real combustor.

Note that in case **B2bis**, one must include the mean-correlations term to this criterion to be able to predict the occurring of the instability. Fig.(8.7) shows the results of the previous criterion without

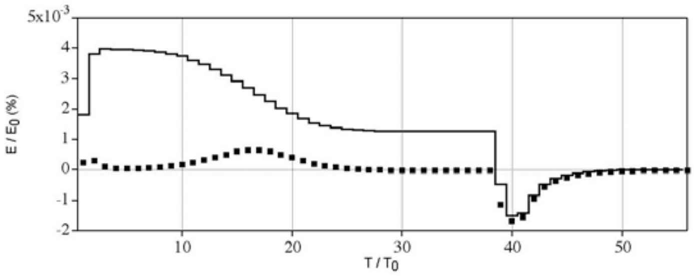


Figure 8.6 - *Gain per period of oscillation (F_0=856Hz)*
(percent of the energy released by the flame during one period (4.61$J.m^{-1}$))
▪ **Energy 1** —— Criterion1

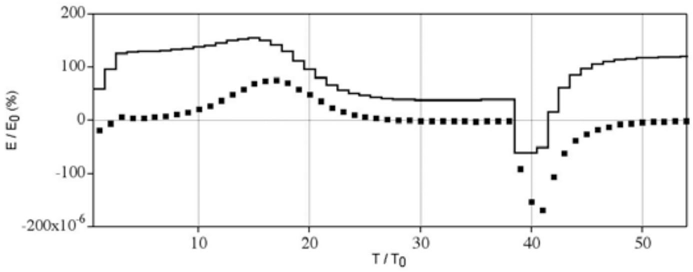


Figure 8.7 - **Case B2bis** *Gain per period of oscillation (F_0=856Hz)*
(percent of the energy released by the flame during one period (4.61$J.m^{-1}$))
▪ **Energy 1** —— Criterion1

taking into account the mean-correlation term in case **B2bis**. The criterion detects the relaxation by becoming negative, but also artificially predicts an other growing phase. This is due to the missing mean-correlations term which has a significant importance during the relaxation. This result shows that when using the average field as the reference flow, mean correlations have an influence on the stability of the system. Obviously, more work is needed to define a simple criterion valid for turbulent cases (where the mean average field is the only possible reference flow).

8.3.2 Entropy disturbance energy (Eq.2).

As already known, in a flame, large entropy fluctuations occur and chapter 7 shows that the amplitude of the time derivative term related to the quantity s'^2 is far more important than the one related to fluctuations of pressure and velocity. In all cases studied here, the maximum absolute amplitude of the time derivative term $\mathbf{de_sdt}$ is from 2.10^3 to 10^4 times bigger than $\mathbf{dedt}$. It shows that the flame creates a lot more entropy disturbance energy than PV disturbance energy as it becomes unstable.
Yet, chapter 7 also shows that no linearization of Eq.(**2**)[Eq.(6.18)] is possible here. Because of that, no linear criterion for stability can be derived for entropy disturbance energy within the framework of this study.

8.3.3 Nonlinear disturbance energy (Eq.3).

This impossibility of deriving a linear criterion for entropy disturbance energy gives another reason for deriving a nonlinear stability criterion. Considering the results obtained in chapter 7 and keeping only the major terms of Eq.(**3**)[Eq.(6.37)], this equation can be simplified to:

$$\frac{\partial E_d}{\partial t} = -s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}}s'\vec{\nabla}T' + T'Q' + \mathbf{D_Y} \tag{8.3}$$

An instability criterion can be derived from this equation which writes:

$$\int_{T_0} \left(\begin{array}{c} -\int_v s'\vec{m}'\vec{\nabla}\bar{T}dv + \int_v \overline{\vec{m}}s'\vec{\nabla}T'dv \\ +\int_v T'Q'dv + \int_v \mathbf{D_Y}dv \end{array} \right) dt > 0 \tag{8.4}$$

If the relation 8.4 is true, the nonlinear disturbance energy in the flow should grow from one period of instability to the other.
The results given by this criterion are shown in Fig.(8.8) and compared to the gain of nonlinear disturbance energy per period of oscillation. Results are normalized by the energy released by the flame during the same period of time and expressed in percent. The two curves have exactly the same behavior. The criterion detects the two growth phases. Yet, it underestimates the growth and therefore is already negative during the limit cycle. For this reason its sign does not change as the relaxation starts. However, it shrinks during the relaxation following the gain of disturbance energy.

Since its sign does not change during the relaxation, only its shrink in amplitude detects the end of the instability. Obviously here the nonlinear criterion seems to have the potential to predict both the occurence of the instability but also its amplitude. Yet, a source term present during the limit cycle is missing to make this nonlinear (NL) criterion perfect.
The author sees three possible reasons for the observed discrepancy :

- There might be some small inconsistencies in the code when computing the entropy of the reacting species because of the tabulation of thermodynamic variables (every 100K). It makes for example that c_{pk} only changes every 100K.
- Note also that no numerical corrections have been considered for Eq.(**3**)[Eq.(6.37)]. The discrepancy might come from those neglected terms.
- It might also be due to the precision of the POSTTIT tool which cannot give the closure of the exact equation with hardly less than 1% error. Since the maximum amplitude of the time derivative term is almost a third of the mean power of the flame, this error could lead to the observed discrepancy.

Note that in case **B2bis**, one must include the mean-correlations terms to this criterion to be able to predict the occurring of the instability. Fig.(8.9) shows the results of the previous criterion without these terms. Since mean correlations are positive during all the computation, the criterion largely underpredicts the instability in case **B2bis**.

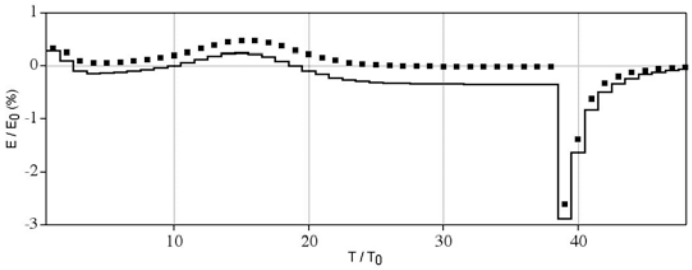


Figure 8.8 - *Gain per period of oscillation (F_0=856Hz)*
(percent of the energy released by the flame during one period (4.61J))
▪ **Energy 3** —— Criterion3

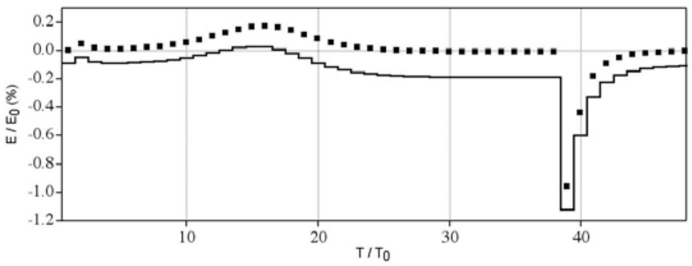


Figure 8.9 - **Case B2bis** *Gain per period of oscillation (F_0=856Hz)*
(percent of the energy released by the flame during one period (4.61J))
▪ **Energy 3** —— Criterion3

8.4 Conclusion and prospects

8.4.1 Conclusions

The closure of balance equations of three different disturbance energies are studied in 1D and 2D laminar reacting cases.
First a linear conservation equation (Eq.(**1**)) for PV disturbance energy is derived.
Then, the balance closure of a linear conservation equation (Eq.(**2**)) for entropy disturbance energy is checked.
Finally, a conservation equation (Eq.(**3**)[Eq.(6.37)]) for a nonlinear disturbance energy in flow is derived and its balance analysed.

Some interesting remarks can be made on the levels of the three different types of disturbance energies (here in case **B2**) :

- Fig.(8.10) shows the quantity (in percent) of PV disturbance energy contained in the nonlinear disturbance energy versus time. It shows that PV disturbance energy only represents less than 0.5% of the nonlinear disturbance energy. Other types of disturbance energies such as entropy disturbance energy are therefore needed to explain the global level of disturbances in the flow.
- Fig.(8.11) shows the quantity (in percent) of entropy disturbance energy in the nonlinear disturbance energy. This type of energy represents in average 60 to 70% of the nonlinear disturbance energy. Note that the curve goes beyond 100%. It means that for a reason which is not given here, the level of entropy disturbance energy is higher than the level of the nonlinear energy although it should be included into it.
  The remaining percent (between 100% and the level of entropy disturbance energy) decreases significantly at the beginning of the instability and during the relaxation. This supports the idea that this part of the nonlinear disturbance energy is indeed linked with nonlinear effects which are smaller during these two phases.

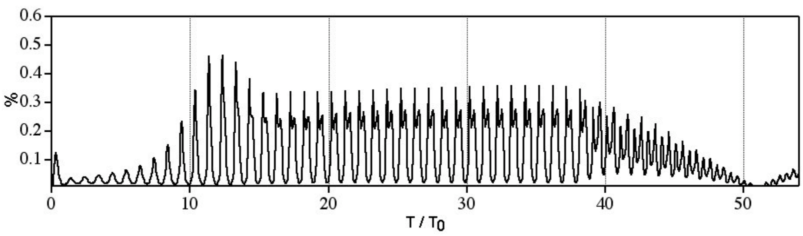


Figure 8.10 - *Quantity (%) of PV disturbance energy in the nonlinear disturbance energy. (Case* **B2***)*

One of the main outputs of this work is to point out the influence of mean Mach number terms on the level of disturbance energies. This includes mean heat release which, multiplied by $\frac{\gamma' p'}{\gamma p}$ enters in the

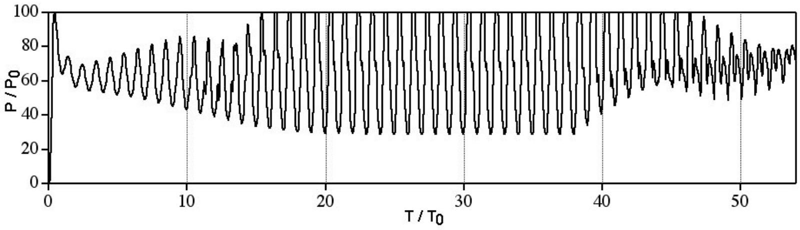


Figure 8.11 - *Quantity (%) of entropy disturbance energy in the nonlinear disturbance energy. (Case* **B2***)*

balance of the PV disturbance energy.
The influence of chemical related terms other than the heat release is also pointed out. Especially, the fluctuation of thermodynamic quantities such as r cannot always be neglected.
It also appears that linearizing Eq.(**2**)[Eq.(6.18)] is impossible in the cases studied here. This is due to the high amplitude of the temperature fluctuation which forbids to consider that

$$\left(\frac{1}{T}\right)_1 = -\frac{T_1}{{T_0}^2}.$$

One of the consequences is that no linear stability criterion can be derived for the entropy disturbance energy.
The influence of the reference field is studied here as a first step toward the procedure that should be used in turbulent cases. It reveals no major issue but mainly shows a decrease of the amplitude of the fluctuations and disturbance energies when using the average mean field as the baseline flow. Note that the ratio of the amplitudes of the different terms constituting the disturbance energies equation changes. For example, terms such as the second type of chemical related terms are not negligible anymore because they are almost not affected by the change of the reference flow though the major term related to the heat release fluctuation decreases much.
Finally, this study reveals that because of the small amplitudes of the PV disturbance energy, the closure of the balance of minimum linear Eq.(**1**) depends on the numerical correction introduced by the spatial numerical scheme. Acoustic and turbulent power density levels are often small compared to the heat released by the flame. This study therefore shows that high precision operators are needed to get these levels correctly.

Two stability criteria in reacting flows are successfully tested on case **B2**. The first one is linear and applies to the PV disturbance energy. It extends the Rayleigh stability criterion by introducing the influence of the fluctuation of the heat flux. The second one is nonlinear and applies to the nonlinear disturbance energy defined by Eq.(**3**)[Eq.(6.37)]. Note that these criterion do not detect the instability in case **B2bis**, meaning that more work will be needed when trying to derive simple criteria in turbulent configurations.

Part of the work (derivation (with averaged mean reference field) and balance closure of Eq.(**3**)[Eq.(6.37)] in a 2-D combustor) was done during the Stanford CTR Summer Program and is published in the proceedings of CTR Summer Program 2006 (here in Appendix F). A publication (derivation of Eq.(**3**)[Eq.(6.37)] with steady reference field) has been submitted to Combustion and Flame and may be found in Appendix G.

8.4.2 Prospects

The main prospect of this study is the analysis of the balance of the nonlinear disturbance energy derived here, in 3-D fully turbulent cases. Extending the present work to such cases would answer the following questions:

- Is the nonlinear stability criterion still valid in turbulent cases ?
- What is the influence of the turbulence on the global balance of disturbance energy?
- What is the amplitude of the nonlinear part of the nonlinear disturbance energy in this case?
- Is acoustic disturbance energy always negligible compared to entropy disturbance energy in reacting flows?

Note that the PV linear criterion could also be tested on such flows with some interests.

General Conclusion

The general objective of this thesis is to extend the understanding of combustion instabilities by testing models, physical concepts and numerical procedures, and by providing new numerical post-processing tools to do so. Two main aspects of the issue of combustion instabilities are studied and lead to many outputs.

Four different methods for the determination of Flame Transfer Functions (FTFs) in LES have been tested. **HF-FFT** method, based on harmonic flame forcing and FFT post-processing gives results of global combustion delays that compare well with the experiments. This method also provides the local FTF which gives valuable information about the amplitudes, delays and locations of the local flame response. This method should reveal useful for configurations where the response of the flame is not compact compared to the characteristic wavelength of the excitation, as it is in distributed reaction cases. These results have been published in Journal of Turbulence [47].
A new **WN-WH** method based on filtered-white noise forcing and post-processing using the Wiener-Hopf relation is successfully compared to **HF-FFT**. Though this method should be handled with care, its main advantage is that it gives access to the frequency spectrum of the local FTF with no additional computational cost.
An important aspect of this study is its link with stability analysis of combustors. Obviously, FTFs do have an influence on the frequency and amplification rates of modes in the numerical methods used for combustor stability [9, 91]. The present results show that methods like **WN-WH** and **HF-FFT** provide slightly different local FTF maps. Whether these differences will affect or not significantly the frequencies and amplification rates of modes remains to be studied. What this study has shown is how to construct FTFs which is the important "brick" of acoustic analysis : further studies are needed to determine which method will be the most precise. Note also that experimental results on global FTFs are available but that no-one has studied local FTFs yet. This is obviously a required step for the future.

Disturbance energies and stability criteria in reacting flows have been studied. Following the work of Myers [88] for non-reacting flows, a new nonlinear conservation equation for a disturbance energy in gaseous reacting flows is derived. The derivation of this equation as well as an order of magnitude analysis of its R-H-S terms have been submitted to Combustion and Flame[13].
A new modular post-processing tool is used here to check the balance closure of disturbance energies on laminar 1D and 2D flames. A first step towards 3D turbulent configurations is also provided thanks to a discussion on the choice of the reference field. This tool gives access to all the physical and numerical terms responsible for the evolution of disturbance energies in the flow.

For each equation, major terms are identified and this work proposes two stability criteria for reacting flows. These criteria are validated on the case of an instability developing in a 2D reacting configuration. The first criterion extends the linear Rayleigh criterion by taking into account the influence of the fluctuation of the heat flux. This work therefore gives a relevant linear tool for the study of the stability of combustion chambers.

The second criterion is nonlinear and includes the influence of the entropy disturbance energy on the global stability (this part of the disturbance energy has been identified by Nicoud and Poinsot [93] as a missing part in usually considered disturbance energies). This criterion aims at giving relevant information on the stability when no linearization of the flow is possible. Note that these criterion do not detect the instability in case **B2bis**, meaning that more work will be needed when trying to derive simple criteria in turbulent configurations. A significant part of these developments has been made during the CTR Summer program at Stanford. A publication summarizing these is in press in the Proceedings of the CTR Summer Program[46].

This work also shows that the entropy disturbance energy cannot be linearized in reacting flows because of the local amplitude of temperature fluctuations. Finally, this study reveals that because of the small amplitudes that disturbance energies can have, the closure of their conservation equations may depends on the numerical correction introduced by the spatial numerical scheme. Acoustic and turbulent power density levels are often small compared to the heat released by the flame. This work therefore shows that high precision operators are needed to get these levels correctly.

Further studies are required to test the validity of the two criteria proposed here in 3D fully turbulent cases. These studies would answer the question of what the main source terms of thermo-acoustic instabilities in ”real” industrial combustors are.

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Appendix

Appendix A

Linear conservation equation for Pressure-Velocity (PV) disturbance energy

The derivation of a conservation equation for pressure uses the conservation equations for mass, velocity and total energy:

$$\frac{\partial \rho}{\partial t} + \vec{u}.\vec{\nabla}(\rho) = 0 \quad \text{(A.1)}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \left(\vec{u} \times \vec{\nabla}\right)(\vec{u}) = -\vec{\nabla}(p) + \vec{\nabla}(\tau) \quad \text{(A.2)}$$

$$\rho \frac{\partial E}{\partial t} + \rho \vec{u}.\vec{\nabla}(E) = \dot{\omega_T} - \vec{\nabla}.(\vec{q}) - \vec{u} : \vec{\nabla}(\sigma) \quad \text{(A.3)}$$

with :

$$\dot{\omega_T} = -\sum_{k=1}^{N} \Delta h_{f,k}^0 \dot{\omega_k} \quad \text{(A.4)}$$

$$\vec{q} = -\lambda \vec{\nabla}(T) + \rho \sum_{k=1}^{N} h_k Y_k \vec{V_k} \quad \text{(A.5)}$$

Using the same notations, the conservation equation for the sensible enthalpy can be obtained[1]:

$$\frac{D\rho h_s}{Dt} = \dot{\omega_T} + \frac{Dp}{Dt} - \vec{\nabla}.(\vec{q}) + \tau : \vec{\nabla}(\vec{u}) \quad \text{(A.6)}$$

Using the relation :

$$\rho \frac{Dh_s}{Dt} = \rho \sum_{k=1}^{N} h_{sk} \frac{DY_k}{Dt} + \rho \overline{C_p} \frac{DT}{Dt}, \quad \text{(A.7)}$$

[1] detailed derivation can be found in ”Theoretical and Numerical combustion” by T.Poinsot and D.Veynante, p16-18

one obtains the following conservation equation for the temperature:

$$\rho\overline{C_p}\frac{DT}{Dt} = \dot{\omega_T} - \vec{\nabla}.\left(\vec{q}\right) + \frac{Dp}{Dt} - \sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} + \tau : \vec{\nabla}\left(\vec{u}\right). \tag{A.8}$$

Dividing the previous equation by $\rho\overline{C_p}T$ and using the state equation $p = \rho r T$, one obtains the following equation for $ln(p)$

$$\frac{1}{\gamma}\frac{Dln\left(p\right)}{Dt} + \nabla.\vec{u} = \frac{\gamma - 1}{\gamma p}\left[\dot{\omega_T} - \vec{\nabla}.\left(\vec{q}\right) - \sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} + \tau : \vec{\nabla}\left(\vec{u}\right)\right] + \frac{1}{r}\frac{Dr}{Dt}. \tag{A.9}$$

It is quite direct to derive from this equation a conservation equation for the pressure:

$$\frac{\partial p}{\partial t} = -\gamma p\vec{\nabla}.\vec{u} - \vec{u}.\vec{\nabla}p + (\gamma - 1)\left[\dot{\omega_T} - \vec{\nabla}.\left(\vec{q}\right) - \sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} + \tau : \vec{\nabla}\left(\vec{u}\right)\right] + \frac{\gamma p}{r}\frac{Dr}{Dt}. \tag{A.10}$$

When needed, all variables will be written as being a sum of their mean part (i.e. $\overline{\rho}$) and their fluctuating part (i.e. ρ'). Eq.(A.10) is used to derive the following mean equation for pressure:

$$\begin{aligned} &cons_p - \overline{\gamma p}\vec{\nabla}.\overline{\vec{u}} - \overline{\vec{u}}.\vec{\nabla}\overline{p} + \\ (\overline{\gamma} - 1)&\left[\overline{\dot{\omega_T}} - \vec{\nabla}.\overline{\vec{q}} - \sum_{k=1}^{N} \overline{h_{sk}}\overline{\rho}\frac{D\overline{Y_k}}{Dt} + \overline{\tau} : \vec{\nabla}\left(\overline{\vec{u}}\right)\right] + \frac{\overline{\gamma p}}{\overline{r}}\frac{D\overline{r}}{Dt} = 0. \end{aligned} \tag{A.11}$$

where $cons_p$ contains all mean cross-correlations of fluctuations (ie.$\overline{\gamma' p'}$, $\overline{\gamma' \vec{u}'}$,etc...).

Substracting Eq.(A.11) to Eq.(A.10) therefore gives an exact form for the conservation equation of the fluctuating pressure (p'):

$$\begin{aligned} \frac{\partial p'}{\partial t} = -\gamma p\vec{\nabla}.\vec{u} - \vec{u}.\vec{\nabla}p + (\gamma - 1)&\left[\dot{\omega_T} - \frac{\partial q_i}{\partial x_i} - \sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} + \tau : \vec{\nabla}\left(\vec{u}\right)\right] \\ &+\frac{\gamma p}{r}\frac{Dr}{Dt} + \overline{\gamma p}\vec{\nabla}.\overline{\vec{u}} + \overline{\vec{u}}.\vec{\nabla}\overline{p} - \\ (\overline{\gamma} - 1)&\left[\overline{\dot{\omega_T}} - \vec{\nabla}.\overline{\vec{q}} - \sum_{k=1}^{N} \overline{h_{sk}}\overline{\rho}\frac{D\overline{Y_k}}{Dt} + \overline{\tau} : \vec{\nabla}\left(\overline{\vec{u}}\right)\right] - \frac{\overline{\gamma p}}{\overline{r}}\frac{D\overline{r}}{Dt} - cons_p \end{aligned} \tag{A.12}$$

Rearranging this equation, multiplying it by p', and dividing it by γp gives :

$$\begin{aligned}
\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t} = -\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \overline{\gamma p}\vec{\nabla}.\overline{\vec{u}}\right] - \frac{p'}{\gamma p}\left[\vec{u}.\vec{\nabla}p - \overline{\vec{u}}.\vec{\nabla}\overline{p}\right] \\
+\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right] - \frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q} - (\overline{\gamma}-1)\,\vec{\nabla}.\overline{\vec{q}}\right] \\
-\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N} h_{sk}\rho\frac{DY_k}{Dt} - (\overline{\gamma}-1)\sum_{k=1}^{N}\overline{h_{sk}}\overline{\rho}\frac{D\overline{Y_k}}{Dt}\right] \\
+\frac{p'}{\gamma p}\left[(\gamma-1)\left(\tau : \vec{\nabla}\left(\vec{u}\right)\right) - (\overline{\gamma}-1)\left(\overline{\tau} : \vec{\nabla}\left(\overline{\vec{u}}\right)\right)\right] \\
+\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt} - \frac{\overline{\gamma p}}{\overline{r}}\frac{D\overline{r}}{Dt}\right] - cons_p\frac{p'}{\gamma p}
\end{aligned} \tag{A.13}$$

Eq.(A.13) is a nonlinear equation for the fluctuations of pressure to the square. Considering the complexity of Eq.(A.13), the linearization is done term by term. All terms of order higher than second order are neglected. It gives:

- $$\begin{aligned}
-\frac{p_1}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \gamma_0 p_0\vec{\nabla}.\vec{u_0}\right] = -\frac{p_1}{\gamma_0 p_0}\left[\gamma p\vec{\nabla}.\vec{u_1} + (\gamma_0 p_1 + \gamma_1 p_0)\,\vec{\nabla}.\vec{u_0}\right] \\
-\frac{p_1}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u} - \gamma_0 p_0\vec{\nabla}.\vec{u_0}\right] = -p_1\vec{\nabla}.\vec{u_1} - \frac{p_1^2}{p_0}\vec{\nabla}.\vec{u_0} - \frac{p_1\gamma_1}{\gamma_0}\vec{\nabla}.\vec{u_0}
\end{aligned} \tag{A.14}$$

- $$\begin{aligned}
-\frac{p_1}{\gamma p}\left[\vec{u}.\vec{\nabla}p - \vec{u_0}.\vec{\nabla}p_0\right] = -\frac{p_1}{\gamma_0 p_0}\left[\vec{u_0}.\vec{\nabla}p_1 + \vec{u_1}.\vec{\nabla}p_0\right] \\
-\frac{p_1}{\gamma p}\left[\vec{u}.\vec{\nabla}p - \vec{u_0}.\vec{\nabla}p_0\right] = -\frac{\vec{u_0}}{2\gamma_0 p_0}.\vec{\nabla}p_1^2 - \frac{p_1}{\gamma_0 p_0}\vec{u_1}.\vec{\nabla}p_0
\end{aligned} \tag{A.15}$$

- $$\begin{aligned}
+\frac{p_1}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\gamma_0-1)\,\dot{\omega_{T0}}\right] = \frac{p_1}{\gamma p}\left[(\gamma_0-1)\,\dot{\omega_{T1}} + \gamma_1\dot{\omega_{T0}}\right] \\
+\frac{p_1}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T} - (\gamma_0-1)\,\dot{\omega_{T0}}\right] = \frac{\gamma_0-1}{\gamma_0 p_0}p_1\dot{\omega_{T1}} + \frac{\gamma_1 p_1}{\gamma_0 p_0}\dot{\omega_{T0}}
\end{aligned} \tag{A.16}$$

- $$-\frac{p_1}{\gamma p}\left[(\gamma-1)\vec{\nabla}.\vec{q}-(\gamma_0-1)\vec{\nabla}.\vec{q_0}\right]=-\frac{p_1}{\gamma p}\left[(\gamma_0-1)\vec{\nabla}.\vec{q_1}+\gamma_1\vec{\nabla}.\vec{q_0}\right]$$
$$-\frac{p_1}{\gamma p}\left[(\gamma-1)\vec{\nabla}.\vec{q}-(\gamma_0-1)\vec{\nabla}.\vec{q_0}\right]=-\frac{\gamma_0-1}{\gamma_0 p_0}p_1\vec{\nabla}.\vec{q_1}-\frac{\gamma_1 p_1}{\gamma_0 p_0}\vec{\nabla}.\vec{q_0} \tag{A.17}$$

- $$-\frac{p_1}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N}h_{sk}\rho\frac{DY_k}{Dt}-(\gamma_0-1)\sum_{k=1}^{N}h_{sk0}\rho_0\frac{DY_{0k}}{Dt}\right]=$$
$$-\frac{p_1\gamma_1}{\gamma_0 p_0}\sum_{k=1}^{N}h_{sk0}\rho_0\frac{DY_{0k}}{Dt}-\frac{p_1(\gamma_0-1)}{\gamma_0 p_0}\sum_{k=1}^{N}\left[h_{sk1}\rho_0\frac{DY_{0k}}{Dt}+h_{sk0}\rho_1\frac{DY_{0k}}{Dt}+h_{sk0}\rho_0\frac{DY_{k1}}{Dt}\right] \tag{A.18}$$

- $$\frac{p_1}{\gamma p}\left[(\gamma-1)\left(\tau:\vec{\nabla}(\vec{u})\right)-(\gamma_0-1)\left(\tau_0:\vec{\nabla}(\vec{u_0})\right)\right]=$$
$$\frac{p_1\gamma_1}{\gamma_0 p_0}\tau_0:\vec{\nabla}(\vec{u_0})+\frac{p_1(\gamma_0-1)}{\gamma_0 p_0}\left[\tau_1:\vec{\nabla}(\vec{u_0})+\tau_0:\vec{\nabla}(\vec{u_1})\right] \tag{A.19}$$

- $$\frac{p_1}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt}-\frac{\gamma_0 p_0}{r_0}\frac{Dr_0}{Dt}\right]=\frac{p_1}{\gamma_0 p_0}\left[\frac{\gamma_1 p_0}{r_0}\frac{Dr_0}{Dt}+\frac{\gamma_0 p_1}{r_0}\frac{Dr_0}{Dt}+\frac{\gamma_0 p_0}{r_0}\frac{Dr_1}{Dt}-\frac{\gamma_0 p_0 r_1}{r_0^2}\frac{Dr_0}{Dt}\right]$$
$$\frac{p_1}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt}-\frac{\gamma_0 p_0}{r_0}\frac{Dr_0}{Dt}\right]=\frac{p_1}{r_0}\frac{Dr_1}{Dt}+\frac{Dr_0}{Dt}\left[\frac{\gamma_1 p_1}{\gamma_0 p_0}+\frac{p_1^2}{r_0 p_0}-\frac{p_1 r_1}{r_0^2}\right] \tag{A.20}$$

Adding all these terms gives a linearized equation at second order for the disturbance energy contained in quadratic pressure fluctuations.

$$\begin{aligned}\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t} = -p_1\vec{\nabla}.\vec{u_1} - \frac{p_1^2}{p_0}\vec{\nabla}.\vec{u_0} - \frac{p_1\gamma_1}{\gamma_0}\vec{\nabla}.\vec{u_0} \\ -\frac{\vec{u_0}}{2\gamma_0 p_0}.\vec{\nabla}p_1^2 - \frac{p_1}{\gamma_0 p_0}\vec{u_1}.\vec{\nabla}p_0 + \frac{\gamma_0 - 1}{\gamma_0 p_0}p_1\dot{\omega_{T1}} + \frac{\gamma_1 p_1}{\gamma_0 p_0}\dot{\omega_{T0}} \\ -\frac{\gamma_0 - 1}{\gamma_0 p_0}p_1\vec{\nabla}.\vec{q_1} - \frac{\gamma_1 p_1}{\gamma_0 p_0}\vec{\nabla}.\vec{q_0} \\ -\frac{p_1\gamma_1}{\gamma_0 p_0}\sum_{k=1}^{N} h_{sk0}\rho_0\frac{DY_{0k}}{Dt} - \frac{p_1(\gamma_0 - 1)}{\gamma_0 p_0}\sum_{k=1}^{N}\left[h_{sk1}\rho_0\frac{DY_{0k}}{Dt} + h_{sk0}\rho_1\frac{DY_{0k}}{Dt} + h_{sk0}\rho_0\frac{DY_{k1}}{Dt}\right] \\ +\frac{p_1\gamma_1}{\gamma_0 p_0}\tau_0 : \vec{\nabla}(\vec{u_0}) + \frac{p_1(\gamma_0 - 1)}{\gamma_0 p_0}\left[\tau_1 : \vec{\nabla}(\vec{u_0}) + \tau_0 : \vec{\nabla}(\vec{u_1})\right] \\ +\frac{p_1}{r_0}\frac{Dr_1}{Dt} + \frac{Dr_0}{Dt}\left[\frac{\gamma_1 p_1}{\gamma_0 p_0} + \frac{p_1^2}{r_0 p_0} - \frac{p_1 r_1}{r_0^2}\right] \\ -cons_p\frac{p_1}{\gamma_0 p_0}\end{aligned} \quad \text{(A.21)}$$

The same reasoning is applied to the velocity to get an exact nonlinear conservation equation for the disturbance energy contained in quadratic velocity fluctuations. The mean conservation equation for velocity is :

$$co\vec{n}s_v - \overline{\rho}\left(\overline{\vec{u}} \times \vec{\nabla}\right)\overline{\vec{u}} - \vec{\nabla}\overline{p} + \vec{\nabla}\overline{\tau} = 0 \quad \text{(A.22)}$$

where $co\vec{n}s_v$ contains all mean cross-correlations of fluctuations (ie.$\overline{\vec{u'}.\vec{u'}}$, etc...).
Therefore, a conservation equation for the fluctuating velocity is:

$$\begin{aligned}\rho\frac{\partial\vec{u'}}{\partial t} = -\rho\left(\vec{u} \times \vec{\nabla}\right)\vec{u} - \vec{\nabla}p + \vec{\nabla}\tau \\ +\overline{\rho}\left(\overline{\vec{u}} \times \vec{\nabla}\right)\overline{\vec{u}} + \vec{\nabla}\overline{p} - \vec{\nabla}\overline{\tau} - co\vec{n}s_v,\end{aligned} \quad \text{(A.23)}$$

and gives:

$$\rho\frac{\partial\vec{u'}}{\partial t} = -\rho\left(\vec{u} \times \vec{\nabla}\right)\vec{u'} - \vec{\nabla}p' + \vec{\nabla}\tau' + \left[\left[\overline{\rho}\overline{\vec{u}} - \rho\vec{u}\right] \times \vec{\nabla}\right]\overline{\vec{u}} - co\vec{n}s_v \quad \text{(A.24)}$$

taking the scalar product of Eq.(A.24) by $\vec{u'}$ gives a nonlinear conservation equation for u'^2:

$$\frac{\rho}{2}\frac{\partial u'^2}{\partial t} = -\rho\vec{u}.\vec{\nabla}\left(\frac{u'^2}{2}\right) - \vec{u'}.\vec{\nabla}p' + \vec{u'} : \vec{\nabla}\tau' + \vec{u'}.\left[\left[\overline{\rho}\overline{\vec{u}} - \rho\vec{u}\right] \times \vec{\nabla}\right]\overline{\vec{u}} - \vec{u'}.co\vec{n}s_v \quad \text{(A.25)}$$

Linearizing Eq.(A.25) to second order gives:

$$\begin{aligned}\frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t} = -\rho_0\vec{u_0}.\vec{\nabla}\left(\frac{u_1^2}{2}\right) - \vec{u_1}.\vec{\nabla}p_1 + \vec{u_1} : \vec{\nabla}\tau_1 \\ -\rho_1\vec{u_1}.\vec{\nabla}\left(\frac{u_0^2}{2}\right) - \rho_0 u_1^2\vec{\nabla}.\vec{u_0} - \vec{u_1}.co\vec{n}s_v\end{aligned} \quad \text{(A.26)}$$

If Eqs.(A.13) and Eq.(A.25) are combined, one obtains the following exact equation in p'^2 and u'^2 :

$$\begin{aligned}
\frac{1}{2\gamma p}\frac{\partial p'^2}{\partial t}+\frac{\rho}{2}\frac{\partial u'^2}{\partial t} &= -\rho\vec{u}.\vec{\nabla}\left(\frac{u'^2}{2}\right)-\vec{u'}.\vec{\nabla}p'+\vec{u'}:\vec{\nabla}\tau' \\
&+\vec{u'}.\left[\left[\overline{\rho}\vec{\overline{u}}-\rho\vec{u}\right]\times\vec{\nabla}\right]\vec{\overline{u}}-\vec{u'}.co\vec{n}s_v \\
&-\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u}-\overline{\gamma p}\vec{\nabla}.\vec{\overline{u}}\right]-\frac{p'}{\gamma p}\left[\vec{u}.\vec{\nabla}p-\vec{\overline{u}}.\vec{\nabla}\overline{p}\right] \\
&+\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T}-(\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right]-\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q}-(\overline{\gamma}-1)\,\vec{\nabla}.\vec{\overline{q}}\right] \\
&-\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N}h_{sk}\rho\frac{DY_k}{Dt}-(\overline{\gamma}-1)\sum_{k=1}^{N}\overline{h_{sk}}\overline{\rho}\frac{D\overline{Y_k}}{Dt}\right] \\
&+\frac{p'}{\gamma p}\left[(\gamma-1)\left(\tau:\vec{\nabla}\vec{u}\right)-(\overline{\gamma}-1)\left(\overline{\tau}:\vec{\nabla}\vec{\overline{u}}\right)\right] \\
&+\frac{p'}{\gamma p}\left[\frac{\gamma p}{r}\frac{Dr}{Dt}-\frac{\overline{\gamma p}}{\overline{r}}\frac{D\overline{r}}{Dt}\right]-cons_p\frac{p'}{\gamma p}
\end{aligned} \tag{A.27}$$

Combining Eqs.(A.21) and Eq.(A.26) gives the following linear equation:

$$\begin{aligned}
&\frac{1}{2\gamma_0 p_0}\frac{D_0 p_1^2}{Dt}+\frac{\rho_0}{2}\frac{D_0 u_1^2}{Dt}+\vec{\nabla}.\left(p_1\vec{u_1}\right)= \\
&\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\dot{\omega_{T1}}-\vec{\nabla}.\vec{q_1}-\sum_{k=1}^{N}\left(h_{sk1}\rho_0\frac{D_0Y_{0k}}{Dt}+h_{sk0}\rho_1\frac{D_0Y_{0k}}{Dt}+h_{sk0}\rho_0\frac{D_0Y_{k1}}{Dt}\right)\right] \\
&+\frac{\gamma_0-1}{\gamma_0 p_0}p_1\left[\tau_1:\vec{\nabla}\vec{u_0}+\tau_0:\vec{\nabla}\vec{u_1}\right] \\
&+\frac{p_1\gamma_1}{\gamma_0 p_0}\left[-p_0\vec{\nabla}.\vec{u_0}+\dot{\omega_{T0}}-\vec{\nabla}.\vec{q_0}-\sum_{k=1}^{N}\left(h_{sk0}\rho_0\frac{D_0Y_{0k}}{Dt}+\tau_0:\vec{\nabla}\vec{u_0}\right)\right] \\
&-\frac{p_1^2}{p_0}\vec{\nabla}.\vec{u_0}-\frac{p_1}{\gamma_0 p_0}\vec{u_1}.\vec{\nabla}p_0-\rho_1\vec{u_1}.\vec{\nabla}\left(\frac{u_0^2}{2}\right)-\rho_0 u_1^2\vec{\nabla}.\vec{u_0} \\
&+\frac{p_1}{r_0}\frac{D_0 r_1}{Dt}+\frac{D_0 r_0}{Dt}\left[\frac{\gamma_1 p_1}{\gamma_0 p_0}+\frac{p_1^2}{r_0 p_0}-\frac{p_1 r_1}{r_0^2}\right]+\vec{u_1}:\vec{\nabla}\tau_1 \\
&-cons_p\frac{p_1}{\gamma_0 p_0}-\vec{u_1}.co\vec{n}s_v
\end{aligned} \tag{A.28}$$

$\frac{D_0}{Dt}$ denotes the total derivative with a spatial component relying only on the mean velocity (i.e $\frac{D_0 f}{Dt}=\frac{\partial f}{\partial t}+\vec{u_0}.\vec{\nabla}f$). Eq.(A.28) is a conservation equation for the quantity $\frac{1}{2\gamma_0 p_0}p_1^2+\frac{\rho_0}{2}u_1^2$ which represents the acoustic energy in a laminar, isentropic flow. In other cases, this quantity a priori contains part of the acoustic, turbulent and entropic disturbance energies.

Appendix B

Linear conservation equation for Entropy disturbance energy

Here is the derivation of a conservation equation for entropy, valid in any reacting flow.
T, p, Y_k is the set of independent variables used for this analysis. The following relations will be used in the analysis :

$$
\begin{aligned}
dg = \left(\frac{\partial g}{\partial Y_k}\right)_{T,p,Y_{j\neq k}} dY_k + \left(\frac{\partial g}{\partial T}\right)_{p,Y_k} dT + \left(\frac{\partial g}{\partial p}\right)_{T,Y_k} dp \\
g = h - Ts = \frac{1}{\rho}\sum \rho Y_k g_k = \frac{1}{\rho}\sum_k \rho Y_k \left(h_k - Ts_k\right) \\
h_k = h_k^{st} + \int_{T^{st}}^{T} c_{pk} dT \\
s_k = s_k^{st} + \int_{T^{st}}^{T} \frac{c_{pk}}{T} dT - \frac{R}{W_k} ln\left(\frac{p_k}{p^{st}}\right) \\
p_k = \rho R T \frac{Y_k}{W_k} = p\overline{W}\frac{Y_k}{W_k}
\end{aligned}
\tag{B.1}
$$

the term $\left(\frac{\partial g}{\partial Y_k}\right)_{T,p,Y_{j\neq k}}$ is primarily evaluated.

$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{Y_k}}\right)_{\mathbf{T},\mathbf{p},\mathbf{Y_{j\neq k}}} = \frac{1}{\rho}\left[\frac{\partial \rho g}{\partial Y_k} - g\frac{\partial g}{\partial Y_k}\right]_{T,p,Y_{j\neq k}}$$
$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{Y_k}}\right)_{\mathbf{T},\mathbf{p},\mathbf{Y_{j\neq k}}} = \frac{1}{\rho}\sum_{j=1}^{N}\frac{\partial}{\partial Y_k}\left(\rho Y_j\left(h_j - Ts_j\right)\right)_{T,p,Y_{j\neq k}} - \frac{1}{\rho}g\left(\frac{\partial \rho}{\partial Y_k}\right)_{T,p,Y_{j\neq k}}$$
$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{Y_k}}\right)_{\mathbf{T},\mathbf{p},\mathbf{Y_{j\neq k}}} = \frac{1}{\rho}\sum_{j=1}^{N} Y_j g_j\left(\frac{\partial \rho}{\partial Y_k}\right)_{T,p,Y_{j\neq k}} + \frac{1}{\rho}\sum_{j=1}^{N}\rho g_j\left(\frac{\partial Y_j}{\partial Y_k}\right)_{T,p,Y_{j\neq k}} - \frac{1}{\rho}g\left(\frac{\partial \rho}{\partial Y_k}\right)_{T,p,Y_{j\neq k}}$$
$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{Y_k}}\right)_{\mathbf{T},\mathbf{p},\mathbf{Y_{j\neq k}}} = \frac{1}{\rho}\left[\sum_{j=1}^{N} Y_j g_j - g\right]\left(\frac{\partial \rho}{\partial Y_k}\right)_{T,p,Y_{j\neq k}} + \frac{1}{\rho}\sum_{j=1}^{N}\rho g_j\left(\frac{\partial Y_j}{\partial Y_k}\right)_{T,p,Y_{j\neq k}}$$
$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{Y_k}}\right)_{\mathbf{T},\mathbf{p},\mathbf{Y_{j\neq k}}} = g_k \qquad \text{(B.2)}$$

Then the term $\frac{\partial g}{\partial T}_{p,Y_j}$ is evaluated :

$$\frac{\partial \mathbf{g}}{\partial \mathbf{T}}_{\mathbf{p},\mathbf{Y_j}} = \frac{1}{\rho}\left[\left(\frac{\partial \rho g}{\partial T}\right)_{p,Y_j} - g\left(\frac{\partial \rho}{\partial T}\right)_{p,Y_j}\right]$$
$$\frac{\partial \mathbf{g}}{\partial \mathbf{T}}_{\mathbf{p},\mathbf{Y_j}} = \frac{1}{\rho}\left[\sum_{j=1}^{N}\left(\frac{\partial \rho Y_j\left(h_j - Ts_j\right)}{\partial T}\right)_{p,Y_j} - g\left(\frac{\partial \rho}{\partial T}\right)_{p,Y_j}\right]$$
$$\frac{\partial \mathbf{g}}{\partial \mathbf{T}}_{\mathbf{p},\mathbf{Y_j}} = \frac{1}{\rho}\left[\sum_{j=1}^{N} Y_j g_j - g\right]\left(\frac{\partial \rho}{\partial T}\right)_{p,Y_j} + \frac{1}{\rho}\sum_{j=1}^{N}\rho Y_j\left(\frac{\partial\left(h_j - Ts_j\right)}{\partial T}\right)_{p,Y_j}$$
$$\frac{\partial \mathbf{g}}{\partial \mathbf{T}}_{\mathbf{p},\mathbf{Y_j}} = \sum_{j=1}^{N} Y_j\left(c_{pj} - T\frac{c_{pj}}{T} - s_j\right)$$
$$\frac{\partial \mathbf{g}}{\partial \mathbf{T}}_{\mathbf{p},\mathbf{Y_j}} = -s \qquad \text{(B.3)}$$

The term $\frac{\partial g}{\partial p}_{T,Y_j}$ is evaluated :

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}_{\mathbf{T},\mathbf{Y_j}} = \frac{1}{\rho}\left[\left(\frac{\partial \rho g}{\partial p}\right)_{T,Y_j} - g\left(\frac{\partial \rho}{\partial p}\right)_{T,Y_j}\right]$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}_{\mathbf{T},\mathbf{Y_j}} = \frac{1}{\rho}\left[\sum_{j=1}^{N}\left(\frac{\partial \rho Y_j\left(h_j - Ts_j\right)}{\partial p}\right)_{T,Y_j} - g\left(\frac{\partial \rho}{\partial p}\right)_{T,Y_j}\right]$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}_{\mathbf{T},\mathbf{Y_j}} = \frac{1}{\rho}\left[\sum_{j=1}^{N} Y_j g_j - g\right]\left(\frac{\partial \rho}{\partial p}\right)_{T,Y_j} + \frac{1}{\rho}\sum_{j=1}^{N}\rho Y_j\left(\frac{\partial\left(h_j - Ts_j\right)}{\partial p}\right)_{T,Y_j}$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}_{\mathbf{T},\mathbf{Y_j}} = \sum_{j=1}^{N} Y_j T\left(\frac{\partial s_j}{\partial p}\right)_{T,Y_j}$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}_{\mathbf{T},\mathbf{Y_j}} = TR\sum_{j=1}^{N}\frac{Y_j}{W_j p}$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}_{\mathbf{T},\mathbf{Y_j}} = \frac{1}{\rho} \tag{B.4}$$

From Eqs.(B.2, B.3) and Eq.(B.4) the following relation can be derived:

$$dg = \sum_{k=1}^{N} g_k dY_k - sdT + \frac{1}{\rho}dp \tag{B.5}$$

And as $g = h - Ts = e + \frac{p}{\rho} - Ts$,

$$de = \sum_{k=1}^{N} g_k dY_k + Tds - pd\left(\frac{1}{\rho}\right)$$

$$\rho Tds = \rho de - \frac{p}{\rho}d\rho - \sum_{k=1}^{N}\rho g_k dY_k \tag{B.6}$$

where e is the sensible-chemical energy. Using the conservation equations for e, ρ and Y_k, the conservation equation for entropy in a multicomponent, reacting medium writes :

$$\frac{\partial s}{\partial t} + \vec{u}.\vec{\nabla}s = \frac{1}{\rho T}\left[\phi - \vec{\nabla}.\vec{q} - \sum_k \Delta h^0_{f,k}\vec{\nabla}.\dot{\vec{\omega}}_k\right] - \frac{1}{\rho T}\sum_k g_k\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] \tag{B.7}$$

The mean conservation equation for entropy writes :

$$-\vec{\bar{u}}.\vec{\nabla}\bar{s} + \frac{1}{\overline{\overline{\rho T}}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\vec{\overline{q}}\right]$$
$$-\frac{1}{\overline{\overline{\rho T}}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] + cons_s = 0 \tag{B.8}$$

where

$$cons_s = \overline{\vec{u'}.\vec{\nabla}s'} - \overline{\frac{1}{\rho T}\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right]} + \frac{1}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right]$$
$$+\overline{\frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right]} - \frac{1}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] \quad \text{(B.9)}$$

Taking the difference between Eq.(B.7) and Eq.(B.8) gives:

$$\frac{\partial s'}{\partial t} = -\vec{u}.\vec{\nabla}s + \frac{1}{\rho T}\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right]$$
$$-\frac{1}{\rho T}\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] + \overline{\vec{u}}.\vec{\nabla}\overline{s} - \frac{1}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right]$$
$$+\frac{1}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right] - cons_s \quad \text{(B.10)}$$

Multiplying the previous equation by $\frac{ps'}{rc_p}$ gives a nonlinear conservation equation for s'^2 :

$$\frac{p}{2rc_p}\frac{\partial s'^2}{\partial t} = -\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla}s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right]$$
$$+\frac{s'}{c_p}\left[\left[\dot{\omega}_T + \phi - \vec{\nabla}.\vec{q}\right] - \frac{\rho T}{\overline{\rho}\overline{T}}\left[\overline{\dot{\omega}_T} + \overline{\phi} - \vec{\nabla}.\overline{\vec{q}}\right]\right]$$
$$-\frac{s'}{c_p}\left[\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] - \frac{\rho T}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]\right]$$
$$-\frac{ps'}{rc_p}cons_s \quad \text{(B.11)}$$

Linearizing Eq.(B.11) to second order gives :

$$\frac{p_0}{2r_0c_{p0}}\frac{\partial s_1^2}{\partial t} = -\frac{p_0 s_1}{r_0 c_{p0}}\left(\vec{u_1}.\vec{\nabla}\left(s_0\right) + \vec{u_0}.\vec{\nabla}\left(s_1\right)\right)$$
$$+\frac{s_1}{c_{p0}}\left[\dot{\omega}_{T1} + \phi_1 - \vec{\nabla}.\left(\vec{q_1}\right)\right]$$
$$+\frac{s_1}{c_{p0}}\left[\left(-\frac{\rho_1}{\rho_0} - \frac{T_1}{T_0}\right)\left(\dot{\omega}_{T0} + \phi_0 - \vec{\nabla}.\left(\vec{q_0}\right)\right)\right]$$
$$-\frac{\rho_0 s_1}{c_{p0}}\left[-\frac{T_1}{T_0}\sum_k g_{sk0}\frac{DY_{k0}}{Dt} + \sum_k g_{sk1}\frac{DY_{k0}}{Dt} + \sum_k g_{sk0}\frac{\partial Y_{k1}}{\partial t}\right]$$
$$-\frac{p_0 s_1}{r_0 c_{p0}}cons_s \quad \text{(B.12)}$$

Appendix C

Exact conservation equation for a nonlinear disturbance energy

Since the derivation of the fluctuating energy equation requires Crocco's formulations for the entropy and momentum transport equations, details are given on the specific transformations of these equations before starting the desired derivation.

- **Entropy conservation equation**

  The entropy conservation equation can be written as follows :

$$\frac{\partial s}{\partial t} + \vec{u}.\vec{\nabla} s = \frac{1}{\rho T}\left[\dot{\omega}_T - \vec{\nabla}.\vec{q} + \phi\right] - \frac{1}{\rho T}\sum_k g_k \left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right], \tag{C.1}$$

  and can therefore be simplified to :

$$\frac{\partial s}{\partial t} + \vec{u}.\vec{\nabla} s = \frac{Q}{\rho} - \frac{1}{\rho T}\sum_k g_{sk} \left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] \tag{C.2}$$

- **Momentum conservation equation**

  Concerning the momentum equation the following transformation can be done :

$$\rho\frac{\partial \vec{u}}{\partial t} + \rho\vec{u}.\vec{\nabla}\vec{u} + \vec{\nabla}p - \rho\vec{\psi} = 0 \tag{C.3}$$

  with $\psi_j = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_i}$.
  Stagnation enthalpy (Crocco formulation) is introduced in Eq.(C.3) using Gibbs relation.

$$dh - Tds - \sum_{k=1}^{N} g_k dY_k = \frac{1}{\rho}dP \tag{C.4}$$

which gives:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}.\vec{\nabla}\vec{u} + \vec{\nabla}h - T\vec{\nabla}s - \sum_{k=1}^{N} g_k \vec{\nabla}Y_k - \vec{\psi} = 0 \tag{C.5}$$

Using the fact that $\vec{u}.\vec{\nabla}\vec{u} = \vec{\nabla}(\frac{\vec{u}.\vec{u}}{2}) + (\vec{\nabla} \otimes \vec{u}) \otimes \vec{u}$, that $g_k = g_{sk} + \Delta h_{f,k}$ and that $h = h_s + \sum_{k=1}^{N} \Delta h_{f,k} Y_k$, it yields:

$$\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}(\frac{\vec{u}.\vec{u}}{2}) + (\vec{\nabla} \otimes \vec{u}) \otimes \vec{u} + \vec{\nabla}h_s + \sum_{k=1}^{N} \Delta h_{f,k} \vec{\nabla}Y_k - T\vec{\nabla}s$$
$$-\vec{\psi^\star} - \sum_{k=1}^{N} \Delta h_{f,k} \vec{\nabla}Y_k - \vec{\psi} = 0 \tag{C.6}$$

which can be simplified in :

$$\frac{\partial \vec{u}}{\partial t} + (\vec{\nabla} \otimes \vec{u}) \otimes \vec{u} + \vec{\nabla}H - T\vec{\nabla}s - \vec{\psi^\star} - \vec{\psi} = 0 \tag{C.7}$$

Conservation equations for unsteady and mean flow

The set of Eqs.(C.8-C.11) therefore fully describe any reactive turbulent flow:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{m} = 0 \tag{C.8}$$
$$\frac{\partial \vec{u}}{\partial t} + \vec{\zeta} + \vec{\nabla}H - T\vec{\nabla}s = \vec{\psi} + \vec{\psi^\star} \tag{C.9}$$
$$\frac{\partial \rho Y_k}{\partial t} + \vec{\nabla}.(\vec{m}Y_k + \vec{q_k}) = \omega_k,\ for\ k\ =\ 1, 2, ..., n \tag{C.10}$$
$$\frac{\partial \rho s}{\partial t} + \vec{\nabla}.\left(\vec{m}s\right) = Q + Q^\star \tag{C.11}$$

where $Q^\star = -\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right]/T$ and $\vec{\psi^\star} = \sum_{k=1}^{N} g_{sk}\vec{\nabla}Y_k$, with g_{sk} being the sensible free enthalpy of species k, $\dot{\omega_k}$ the volumic mass rate of consumption for species k and $\vec{q_k}$ the species flux defined as : $\vec{q_k} = \rho Y_k \vec{V_k}$. All other variables correspond to the analysis of Myers [88], except that in this reacting case derivation: $Q = \frac{\dot{\omega_T} - \vec{\nabla}.\vec{q} + \Phi}{T}$.
The corresponding equations for the turbulent-perturbed mean fields are:

$$\vec{\nabla}.\overline{\vec{m}} = 0 \tag{C.12}$$
$$\overline{\vec{\zeta}} + \vec{\nabla}\overline{H} - \bar{T}\vec{\nabla}\bar{s} - \overline{\vec{\psi^\star}} - \overline{\vec{\psi}} - \overline{T'\vec{\nabla}s'} = 0 \tag{C.13}$$
$$\vec{\nabla}.\overline{\vec{m}}\overline{Y_k} + \vec{\nabla}.\overline{\vec{m}'Y_k'} + \vec{\nabla}.\overline{\vec{q_k}} = \overline{\omega_k} \tag{C.14}$$
$$\vec{\nabla}.\left(\overline{\vec{m}}\bar{s}\right) - \bar{Q} - \bar{Q}^\star + \vec{\nabla}.\overline{\left(\vec{m}'s'\right)} = 0 \tag{C.15}$$

The derivation of a disturbance energy will also be using the specific stagnation enthalpy transport instantaneous and mean equations:

$$\frac{\partial}{\partial t}\left(\rho H - p\right) + \vec{\nabla}.\left(\vec{m}H\right) - \vec{m}.\vec{\psi} - TQ = 0$$
$$\vec{\nabla}.\left(\overline{\vec{m}}\overline{H}\right) - \overline{\vec{m}}.\overline{\vec{\psi}} - \bar{T}\bar{Q} + \vec{\nabla}.\overline{\left(\vec{m}'H'\right)} - \overline{\vec{m}'.\vec{\psi}'} - \overline{T'Q'} = 0 \tag{C.16}$$

Derivation of a disturbance energy conservation equation

The sum of $\left(\overline{H} - \bar{T}\overline{s} - \sum_{k=1}^{n} \overline{g_{sk}}\overline{Y_k}\right)$ times Eq.(C.8), $\bar{T}$ times Eq.(C.11), $\overline{g_{sk}}$ times Eq.(C.10) and the scalar product of $\overline{\vec{m}}$ with Eq.(C.9) are substracted from Eq.(C.16). Furthermore, the correlation term $\overline{\rho' e_s'}$ is substracted to ensure the positive definitiveness of the disturbance energy (E_d). Doing so it gives:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho H - p - \overline{\rho' e_s'}\right) - \left(\overline{H} - \bar{T}\overline{s}\right)\frac{\partial \rho}{\partial t} - \overline{\vec{m}}.\frac{\partial \vec{u}}{\partial t} - \bar{T}\frac{\partial \rho s}{\partial t} - \sum_{k=1}^{n}\overline{g_{sk}}\frac{\partial \rho(Y_k - \overline{Y_k})}{\partial t} \\ + \bar{T}Q^\star - \left(\overline{H} - \bar{T}\overline{s}\right)\vec{\nabla}.\vec{m} - \overline{\vec{m}}.\vec{\zeta} - \\ \overline{\vec{m}}.\vec{\nabla}H + T\overline{\vec{m}}.\vec{\nabla}s + \overline{\vec{m}}.\vec{\psi}^\star + \overline{\vec{m}}.\vec{\psi} \\ -\bar{T}\vec{\nabla}.\left(\vec{m}s\right) + \bar{T}Q + \vec{\nabla}.\left(\vec{m}H\right) - \vec{m}.\vec{\psi} - TQ \\ -\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k + \vec{q_k}) + \sum_{k=1}^{n}\overline{g_{sk}}\omega_k + \sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m} = 0\end{aligned} \tag{C.17}$$

which can be rearranged into :

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho\left(H - \overline{H}\right) - \rho\bar{T}\left(s - \overline{s}\right) - \overline{\vec{m}}.\vec{u} - p - \sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k - \overline{Y_k}) - \overline{\rho' e_s'}\right] \\ +\vec{\nabla}.\left[\vec{m}H - \vec{m}\left(\overline{H} - \bar{T}\overline{s}\right) - \vec{m}s\bar{T} - \overline{\vec{m}}H + \overline{\vec{m}}Ts\right] \\ \underbrace{+\,\vec{m}.\vec{\nabla}\left(\overline{H} - \bar{T}\overline{s}\right) + s\vec{m}.\vec{\nabla}\bar{T} + H\vec{\nabla}.\overline{\vec{m}} - s\vec{\nabla}.\left(T\overline{\vec{m}}\right) - \vec{m}.\left(\overline{\vec{\psi}} - \overline{\vec{\zeta}}\right)}_{A} \\ -\left(\vec{m} - \overline{\vec{m}}\right).\vec{\psi} - \left(T - \bar{T}\right)Q + \vec{m}.\left(\overline{\vec{\psi}} - \overline{\vec{\zeta}}\right) - \overline{\vec{m}}.\vec{\zeta} \\ +\bar{T}Q^\star + \overline{\vec{m}}.\vec{\psi}^\star \\ -\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k + \vec{q_k}) + \sum_{k=1}^{n}\overline{g_{sk}}\omega_k + \sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m} = 0\end{aligned} \tag{C.18}$$

A can be rewritten using the set of mean Eqs.(C.12-C.15):

$$A = \vec{m}.\left[\vec{\nabla}\overline{H} - \bar{T}\vec{\nabla}\overline{s} - \overline{\vec{\psi}} + \overline{\vec{\zeta}}\right] + s\vec{m}.\vec{\nabla}\bar{T}$$
$$-\overline{s}\vec{m}.\vec{\nabla}\bar{T} + H\vec{\nabla}.\overline{\vec{m}} - s\vec{\nabla}.\left(T\overline{\vec{m}}\right) \tag{C.19}$$

$$A = \vec{m}.\left[\overline{\vec{\psi}^{\star}} + \overline{T'\vec{\nabla}s'}\right] + s\vec{m}.\vec{\nabla}\bar{T}$$
$$-\overline{s}\vec{m}.\vec{\nabla}\bar{T} + \underbrace{(H - sT)\,\vec{\nabla}.\overline{\vec{m}}}_{=0} - s\overline{\vec{m}}.\vec{\nabla}T \tag{C.20}$$

$$A = \vec{m}.\left[\overline{\vec{\psi}^{\star}} + \overline{T'\vec{\nabla}s'}\right] + (s - \overline{s})\,\vec{m}.\vec{\nabla}\bar{T} - s\overline{\vec{m}}.\vec{\nabla}T \tag{C.21}$$

Using Eq.(C.21), Eq.(C.18) can be rewritten:

$$\frac{\partial}{\partial t}\left[\rho\left(H - \overline{H}\right) - \rho\bar{T}\,(s - \overline{s}) - \overline{\vec{m}}.\vec{u} - p - \sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k - \overline{Y_k}) - \overline{\rho' e'_s}\right]$$
$$+\underbrace{\vec{\nabla}.\left[\vec{m}H - \vec{m}\left(\overline{H} - \bar{T}\overline{s}\right) - \vec{m}s\bar{T} - \overline{\vec{m}}H + \overline{\vec{m}}Ts\right]}_{B}$$
$$+\vec{m}.\left[\overline{\vec{\psi}^{\star}} + \overline{T'\vec{\nabla}s'}\right] + (s - \overline{s})\,\vec{m}.\vec{\nabla}\bar{T}$$
$$-s\overline{\vec{m}}.\vec{\nabla}T - \left(\vec{m} - \overline{\vec{m}}\right).\vec{\psi} - \left(T - \bar{T}\right)Q + \vec{m}.\left(\overline{\vec{\psi}} - \overline{\vec{\zeta}}\right) - \overline{\vec{m}}.\vec{\zeta}$$
$$+\bar{T}Q^{\star} + \overline{\vec{m}}.\vec{\psi}^{\star}$$
$$-\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k + \vec{q_k}) + \sum_{k=1}^{n}\overline{g_{sk}}\omega_k + \sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m} = 0 \tag{C.22}$$

The term B can be rewritten as follows:

$$B = \vec{\nabla}.\left[\left(\vec{m} - \overline{\vec{m}}\right)\left(H - \overline{H}\right) - \vec{m}\bar{T}\,(s - \overline{s}) + \overline{\vec{m}}T\,(s - \overline{s})\right]$$
$$-\vec{\nabla}.\left(\overline{\vec{m}}\overline{H}\right) + \vec{\nabla}.\left(\overline{\vec{m}}T\overline{s}\right) \tag{C.23}$$

$$\vec{\nabla}.\left(\overline{\vec{m}}\overline{H}\right) = -\overline{\vec{m}}.\overline{\vec{\zeta}} + \bar{T}\overline{\vec{m}}.\vec{\nabla}\overline{s}$$
$$+\overline{\vec{m}}.\overline{\vec{\psi}^{\star}} + \overline{\vec{m}}.\overline{\vec{\psi}} + \overline{\vec{m}}.\overline{T'\vec{\nabla}s'} \tag{C.24}$$

$$\vec{\nabla}.\left(\overline{\vec{m}}\overline{s}\right) = \bar{Q} + \overline{Q^{\star}} - \overline{\vec{\nabla}.\left(\vec{m}'s'\right)} \tag{C.25}$$

$$-\vec{\nabla}.\left(\overline{\vec{m}}\overline{H}\right) = \overline{\vec{m}}.\overline{\vec{\zeta}} - \bar{T}\bar{Q} - \bar{T}\overline{Q^{\star}} - \overline{\vec{m}}.\overline{T'\vec{\nabla}s'}$$
$$-\overline{\vec{m}}.\overline{\vec{\psi}} - \overline{\vec{m}}.\overline{\vec{\psi}^{\star}} + \bar{T}\overline{\vec{\nabla}.\left(\vec{m}'s'\right)} \tag{C.26}$$

$$\vec{\nabla}.\left(\overline{\vec{m}}T\overline{s}\right) = \overline{s}\overline{\vec{m}}.\vec{\nabla}T + T\bar{Q} + T\overline{Q^{\star}} - T\overline{\vec{\nabla}.\left(\vec{m}'s'\right)} \tag{C.27}$$

$$\Longrightarrow B = \vec{\nabla}.\left[\left(\vec{m} - \overline{\vec{m}}\right)\left(H - \overline{H}\right) - \vec{m}\bar{T}\,(s - \overline{s}) + \overline{\vec{m}}T\,(s - \overline{s})\right]$$
$$+\overline{\vec{m}}.\overline{\vec{\zeta}} - \bar{T}\bar{Q} - \bar{T}\overline{Q^{\star}} - \overline{\vec{m}}.\overline{\vec{\psi}} - \overline{\vec{m}}.\overline{\vec{\psi}^{\star}} + \overline{s}\overline{\vec{m}}.\vec{\nabla}T$$
$$+T\bar{Q} + T\overline{Q^{\star}} - \overline{\vec{m}}.\overline{T'\vec{\nabla}s'} - (T - \bar{T})\overline{\vec{\nabla}.\left(\vec{m}'s'\right)} \tag{C.28}$$

Introducing Eq.(C.28) into Eq.(C.22) gives:

$$\frac{\partial}{\partial t}\left[\rho\left(H-\overline{H}\right)-\rho\bar{T}\left(s-\overline{s}\right)-\overline{\vec{m}}.\vec{u}-p-\sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k-\overline{Y_k})-\overline{\rho' e'_s}\right]$$
$$+\vec{\nabla}.\left[\left(\vec{m}-\overline{\vec{m}}\right)\left(H-\overline{H}\right)-\vec{m}\bar{T}\left(s-\overline{s}\right)+\overline{\vec{m}}T\left(s-\overline{s}\right)\right]$$
$$+\overline{\vec{m}}.\overline{\vec{\zeta}}-\bar{T}\bar{Q}-\bar{T}\overline{Q^\star}-\overline{\vec{m}}.\overline{\vec{\psi}}-\overline{\vec{m}}.\overline{\vec{\psi}^\star}+\overline{s}\overline{\vec{m}}.\vec{\nabla}T+T\bar{Q}$$
$$+T\overline{Q^\star}+\bar{T}Q^\star+\overline{\vec{m}}.\vec{\psi}^\star+\vec{m}.\overline{\vec{\psi}^\star}+\left(s-\overline{s}\right)\vec{m}.\vec{\nabla}\bar{T}-s\overline{\vec{m}}.\vec{\nabla}T$$
$$-\left(\vec{m}-\overline{\vec{m}}\right).\vec{\psi}-\left(T-\bar{T}\right)Q+\vec{m}.\left(\overline{\vec{\psi}}-\overline{\vec{\zeta}}\right)-\overline{\vec{m}}.\vec{\zeta}$$
$$-\overline{\vec{m}}.\overline{T'\vec{\nabla}s'}-(T-\bar{T})\overline{\vec{\nabla}.\left(\vec{m}'s'\right)}+\vec{m}.\overline{T'\vec{\nabla}s'}$$
$$-\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k+\vec{q_k})+\sum_{k=1}^{n}\overline{g_{sk}}\omega_k+\sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m}=0 \quad \text{(C.29)}$$

Using the fact that $\overline{\vec{u}}$ and $\overline{p}$ are independent of t, $\vec{m}.\vec{\zeta}=\overline{\vec{m}}.\overline{\vec{\zeta}}=0$ it follows that :

$$\frac{\partial}{\partial t}\left[\rho\left(H-\overline{H}\right)-\rho\bar{T}\left(s-\overline{s}\right)-\overline{\vec{m}}.(\vec{u}-\overline{\vec{u}})-(p-\overline{p})-\sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k-\overline{Y_k})-\overline{\rho' e'_s}\right]$$
$$+\vec{\nabla}.\left[\left(\vec{m}-\overline{\vec{m}}\right)\left[\left(H-\overline{H}\right)-\bar{T}(s-\overline{s})\right]-\overline{\vec{m}}(T-\bar{T})\left(s-\overline{s}\right)-T\overline{\left(\vec{m}'s'\right)}\right]$$
$$(\vec{m}-\overline{\vec{m}})(\vec{\zeta}-\overline{\vec{\zeta}})+\overline{\vec{m}'.\vec{\zeta}'}$$
$$(s-\overline{s})(\vec{m}-\overline{\vec{m}})\vec{\nabla}\bar{T}-\overline{\vec{m}}(s-\overline{s})\vec{\nabla}(T-\bar{T})$$
$$-\overline{\vec{m}}.\overline{T'\vec{\nabla}s'}+\bar{T}\overline{\vec{\nabla}.\left(\vec{m}'s'\right)}+\vec{m}.\overline{T'\vec{\nabla}s'}+\overline{(\vec{m}'s')}.\vec{\nabla}T$$
$$-(T-\bar{T})(Q-\bar{Q})$$
$$-(T-\bar{T})(Q^\star-\overline{Q^\star})+TQ^\star$$
$$-(\vec{m}-\overline{\vec{m}}).(\vec{\psi}-\overline{\vec{\psi}})$$
$$-(\vec{m}-\overline{\vec{m}}).(\vec{\psi}^\star-\overline{\vec{\psi}^\star})+\vec{m}.\vec{\psi}^\star$$
$$-\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k+\vec{q_k})+\sum_{k=1}^{n}\overline{g_{sk}}\omega_k+\sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m}=0 \quad \text{(C.30)}$$

Eq.(C.13) and Eq.(C.16) can be combined to get :

$$\overline{\vec{m}}.\overline{\vec{\zeta}}-\overline{\vec{m}}.\bar{T}\vec{\nabla}\overline{s}-\overline{\vec{m}}.\overline{\vec{\psi}^\star}-\overline{\vec{m}}.\overline{T'\vec{\nabla}s'}=-\bar{T}\bar{Q}+\vec{\nabla}.\overline{\left(\vec{m}'H'\right)}-\overline{\vec{m}'.\vec{\psi}'}-\overline{T'Q'}, \quad \text{(C.31)}$$

using furthermore Eq.(C.15) gives:

$$\bar{T}\vec{\nabla}.\overline{\left(\vec{m}'s'\right)}-\overline{\vec{m}}.\overline{T'\vec{\nabla}s'}=-\bar{T}\bar{Q}+\vec{\nabla}.\overline{\left(\vec{m}'H'\right)}-\overline{\vec{m}'.\vec{\psi}'}-\overline{T'Q'}$$
$$+\bar{T}\bar{Q}+\bar{T}\bar{Q}^\star+\overline{\vec{m}}.\overline{\vec{\psi}^\star} \quad \text{(C.32)}$$

Introducing Eq.(C.32) in Eq.(C.30) gives:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\rho\left(H-\overline{H}\right)-\rho\bar{T}\left(s-\overline{s}\right)-\overline{\vec{m}}.(\vec{u}-\overline{\vec{u}})-(p-\overline{p})-\sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k-\overline{Y_k})-\overline{\rho' e_s'}\right]\\
&+\vec{\nabla}.\left[\left(\vec{m}-\overline{\vec{m}}\right)\left[\left(H-\overline{H}\right)-\bar{T}(s-\overline{s})\right]-\overline{\vec{m}}(T-\bar{T})\left(s-\overline{s}\right)+\overline{\left(\vec{m}'H'\right)}-T\overline{\left(\vec{m}'s'\right)}\right]\\
&\qquad+(\vec{m}-\overline{\vec{m}})(\vec{\zeta}-\overline{\vec{\zeta}})+\overline{\vec{m}'.\vec{\zeta}'}\\
&+(s-\overline{s})(\vec{m}-\overline{\vec{m}})\vec{\nabla}\bar{T}-\overline{\vec{m}}(s-\overline{s})\vec{\nabla}(T-\bar{T})+\vec{m}.\overline{T'\vec{\nabla}s'}+\overline{(\vec{m}'s')}.\vec{\nabla}T\\
&\qquad-(T-\bar{T})(Q-\bar{Q})-\overline{T'Q'}\\
&\qquad-(T-\bar{T})(Q^\star-\overline{Q^\star})\\
&\qquad-(\vec{m}-\overline{\vec{m}}).(\vec{\psi}-\overline{\vec{\psi}})-\overline{\vec{m}'.\vec{\psi}'}\\
&\qquad-(\vec{m}-\overline{\vec{m}}).(\vec{\psi}^\star-\overline{\vec{\psi}^\star})\\
&\qquad\underbrace{+\vec{m}.\vec{\psi}^\star+\overline{\vec{m}}.\overline{\vec{\psi}^\star}+TQ^\star+\bar{T}\overline{Q^\star}}_{D}\\
&\qquad\underbrace{-\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k+\vec{q_k})+\sum_{k=1}^{n}\overline{g_{sk}}\omega_k+\sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m}}_{E}=0
\end{aligned}\tag{C.33}$$

The goal of the derivation is to derive an equation for a "disturbance energy" which is explicitly second order when linearized. It means that the source and flux terms should be written as $\alpha'\beta'$, α and β being any primitive variable. At this stage, a few terms in the derivation don't verify this criterion and therefore have to be rewritten. They are grouped under the terms D and E. The following paragraph focuses on those terms.

$$\begin{aligned}
D+E=&-\sum_{k=1}^{n}\overline{g_{sk}}\vec{\nabla}.(\vec{m}Y_k+\vec{q_k})+\sum_{k=1}^{n}\overline{g_{sk}}\omega_k+\sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m}\\
&+\vec{m}.\vec{\psi}^\star+\overline{\vec{m}}.\overline{\vec{\psi}^\star}+TQ^\star+\bar{T}\overline{Q^\star}
\end{aligned}\tag{C.34}$$

$$\begin{aligned}
D+E=&-\sum_{k=1}^{n}(g_{sk}-\overline{g_{sk}})\vec{\nabla}.(\omega_k-\vec{q_k})-\sum_{k=1}^{n}\overline{g_{sk}}\vec{m}.\vec{\nabla}Y_k\\
&-\sum_{k=1}^{n}\overline{g_{sk}}Y_k\vec{\nabla}.\vec{m}+\sum_{k=1}^{n}\overline{g_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m}'\\
&+\vec{m}.\vec{\psi}^\star+\overline{\vec{m}}.\overline{\vec{\psi}^\star}+\bar{T}\overline{Q^\star}
\end{aligned}\tag{C.35}$$

$$D+E=-\sum_{k=1}^{n} g'_{sk}\frac{\partial \rho Y_k}{\partial t}-\sum_{k=1}^{n} g'_{sk}Y_k\vec{\nabla}.\vec{m'}-\sum_{k=1}^{n}\overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m'}$$
$$+\overline{\vec{m}}.\overline{\vec{\psi^\star}}+\bar{T}\overline{Q^\star} \tag{C.36}$$

Eq.(C.10) can be written as

$$-\sum_{k=1}^{n}\rho g_{sk}\frac{\partial Y_k}{\partial t}=Q^\star T+\vec{m}.\vec{\psi^\star}, \tag{C.37}$$

therefore:

$$\overline{Q^\star}\bar{T}+\overline{\vec{m}}.\overline{\vec{\psi^\star}}=-\overline{Q^{\star\prime}T'}-\overline{\vec{m'}.\vec{\psi^{\star\prime}}}-\overline{\sum_{k=1}^{n}\rho g_{sk}\frac{\partial Y_k}{\partial t}} \tag{C.38}$$

Using the assumption that $\overline{\left(\frac{\partial}{\partial t}\right)}=0$, it gives

$$\overline{Q^\star}\bar{T}+\overline{\vec{m}}.\overline{\vec{\psi^\star}}=-\overline{Q^{\star\prime}T'}-\overline{\vec{m'}.\vec{\psi^{\star\prime}}}-\overline{\sum_{k=1}^{n}(\rho g_{sk})'\frac{\partial Y'_k}{\partial t}} \tag{C.39}$$

$$\overline{Q^\star}\bar{T}+\overline{\vec{m}}.\overline{\vec{\psi^\star}}=-\overline{Q^{\star\prime}T'}-\overline{\vec{m'}.\vec{\psi^{\star\prime}}}-\overline{\sum_{k=1}^{n} g'_{sk}[\omega'_k-\vec{\nabla}.\vec{q'_k}-\vec{\nabla}.(\vec{m}Y_k)']}$$
$$-\overline{\sum_{k=1}^{n}(g_{sk}Y_k)'\vec{\nabla}.\vec{m'}} \tag{C.40}$$

Introducing Eq.(C.40) in Eq.(C.36) gives:

$$D+E=-\sum_{k=1}^{n} g'_{sk}\frac{\partial (\rho Y_k)'}{\partial t}-\sum_{k=1}^{n} g'_{sk}Y_k\vec{\nabla}.\vec{m'}-\sum_{k=1}^{n}\overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m'}$$
$$-\overline{Q^{\star\prime}T'}-\overline{\vec{m'}.\vec{\psi^{\star\prime}}}-\overline{\sum_{k=1}^{n} g'_{sk}[\omega'_k-\vec{\nabla}.\vec{q'_k}-\vec{\nabla}.(\vec{m}Y_k)']}$$
$$-\overline{\sum_{k=1}^{n}(g_{sk}Y_k)'\vec{\nabla}.\vec{m'}} \tag{C.41}$$

By using Eq.(C.10) it also can be written as:

$$D+E=-\sum_{k=1}^{n} g'_{sk}\Omega'_k-\sum_{k=1}^{n} g'_{sk}Y_k\vec{\nabla}.\vec{m'}-\sum_{k=1}^{n}\overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m'}$$
$$-\overline{Q^{\star\prime}T'}-\overline{\vec{m'}.\vec{\psi^{\star\prime}}}-\overline{\sum_{k=1}^{n} g'_{sk}\Omega'_k}-\overline{\sum_{k=1}^{n}(g_{sk}Y_k)'\vec{\nabla}.\vec{m'}} \tag{C.42}$$

where $\Omega_k = [\omega_k' - \vec{\nabla}.\vec{q_k'} - \vec{\nabla}.(\vec{m}Y_k)']$. Introducing Eq.(C.42) into Eq.(C.33) gives the following equation for a disturbance energy (E_d) which is explicitly second order when linearized.

$$\frac{\partial}{\partial t}\underbrace{\left[\rho\left(H-\overline{H}\right)-\rho\bar{T}\left(s-\overline{s}\right)-\overline{\vec{m}}.(\vec{u}-\overline{\vec{u}})-(p-\overline{p})-\sum_{k=1}^{n}\overline{g_{sk}}\rho(Y_k-\overline{Y_k})-\overline{\rho' e_s'}\right]}_{E_d}$$

$$+\vec{\nabla}.\underbrace{\left[\left(\vec{m}-\overline{\vec{m}}\right)\left[\left(H-\overline{H}\right)-\bar{T}(s-\overline{s})\right]-\overline{\vec{m}}(T-\bar{T})\left(s-\overline{s}\right)+\overline{\left(\vec{m'}H'\right)}-T\overline{\left(\vec{m'}s'\right)}\right]}_{W}$$

$$\underbrace{+(\vec{m}-\overline{\vec{m}})(\vec{\zeta}-\overline{\vec{\zeta}})+\overline{\vec{m'}.\vec{\zeta'}}}_{D_\zeta}$$

$$\underbrace{+(s-\overline{s})(\vec{m}-\overline{\vec{m}})\vec{\nabla}\bar{T}-\overline{\vec{m}}(s-\overline{s})\vec{\nabla}(T-\bar{T})+\vec{m}.\overline{T'\vec{\nabla}s'}+\overline{(\vec{m'}s')}.\vec{\nabla}T}_{D_s}$$

$$\underbrace{-(T-\bar{T})(Q-\bar{Q})-\overline{T'Q'}}_{D_Q}$$

$$\underbrace{-(T-\bar{T})(Q^\star-\overline{Q^\star})-\overline{Q^{\star\prime}T'}}_{D_{Q^\star}}$$

$$\underbrace{-(\vec{m}-\overline{\vec{m}}).(\vec{\psi}-\overline{\vec{\psi}})-\overline{\vec{m'}.\vec{\psi'}}}_{D_{\vec{\psi}}}$$

$$\underbrace{-(\vec{m}-\overline{\vec{m}}).(\vec{\psi^\star}-\overline{\vec{\psi^\star}})-\overline{\vec{m'}.\vec{\psi^{\star\prime}}}}_{D_{\vec{\psi^\star}}}$$

$$\underbrace{-\sum_{k=1}^{n}g_{sk}'\Omega_k'-\sum_{k=1}^{n}g_{sk}'Y_k\vec{\nabla}.\vec{m'}-\sum_{k=1}^{n}\overline{g_{sk}}Y_k'\vec{\nabla}.\vec{m'}-\overline{\sum_{k=1}^{n}g_{sk}'\Omega_k'}-\overline{\sum_{k=1}^{n}(g_{sk}Y_k)'\vec{\nabla}.\vec{m'}}}_{D_{Yk}}=0 \quad \text{(C.43)}$$

Linearization

Retention of all second-order terms in the exact flux vector and source terms results in a rather complex disturbance energy equation, where much of the complexity is contained in viscous stress, dissipation, and heat conduction terms that can be argued to be negligible in most combusting flows. Ignoring such terms as well as the vorticity terms results in the following linearized disturbance energy equation,

$$\frac{\partial E_{d2}}{\partial t}+\vec{\nabla}.\vec{W}_2=D_2, \quad \text{(C.44)}$$

where the disturbance energy density E_{d2}, flux vector $\vec{W}_2$, and source D_2 terms are

$$E_{d2} = \frac{p_1^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 \vec{u}_1.\vec{u}_1 + \rho_1 \vec{u}_0.\vec{u}_1 + \frac{\rho_0 T_0 s_1^2}{2c_{p0}} + E_{Y2}, \tag{C.45}$$

$$\vec{W}_2 = (p_1 + \rho_0 \vec{u}_0.\vec{u}_1)\left(\vec{u}_1 + \frac{\rho_1}{\rho_0}\vec{u}_0\right) + \vec{m}_0 T_1 s_1, \tag{C.46}$$

and

$$\begin{aligned} D_2 &= -\overline{\vec{m}_1 s_1}.\vec{\nabla} T_0 - \vec{m}_0.\overline{T_1 \vec{\nabla} s_1} \\ &- s_1 \vec{m}_1.\vec{\nabla} T_0 + s_1 \vec{m}_0.\vec{\nabla} T_1 \\ &+ \frac{T_1}{T_0}(\omega_{T1} - \vec{\nabla}.\vec{q}_1) + T_1(\omega_{T0} - \vec{\nabla}.\vec{q}_0)\left(\frac{1}{T} - \frac{1}{T_0}\right) \\ &+ T_1 Q_1^* + \vec{m}_1.\vec{\psi}_1^* \\ &+ \sum_{k=1}^{n} g_{sk1}\Omega_{k1} + \sum_{k=1}^{n}(g_{sk1}Y_{k1} + g_{sk0}Y_{k1})\,\vec{\nabla}.\vec{m}_1. \end{aligned} \tag{C.47}$$

The E_{Y2} term in Eq.(C.45) is the contribution from the $\rho g_{sk0}(Y_k - Y_{k0})$ terms in Eq.(C.43) and is equal to

$$\begin{aligned} E_{Y2} &= \frac{p_0}{2}\sum_{k=1}^{n}\left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right)\frac{g_{sk} - e_{sk}}{C_v T} + \frac{W}{W_k}\left(1 + \frac{1}{Y_k}\right)\right] Y_{k1}^2 \\ &+ p_0 \sum_{k=1}^{n}\sum_{j\neq k}\left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right)\frac{g_{sj} - e_{sj}}{C_v T} + \frac{W}{W_j}\right] Y_{k1}Y_{j1} \\ &+ \frac{\rho}{C_v}\sum_{k=1}^{n}(g_{sk} - e_{sk})\, s_1 Y_{k1} + \sum_{k=1}^{n}\left[(\gamma - 1)(g_{sk} - e_{sk}) + \frac{RT}{W_k}\right]\rho_1 Y_{k1}. \end{aligned} \tag{C.48}$$

Appendix D

Evaluation of numerical corrections for Eqs.(1), (2) and (3)

As mentioned in Section 5.2, numerical corrections can be split in three main components.

- a boundary correction denoted $\mathbf{BND}$
- a correction due to the Lax-Wendroff scheme denoted **LW**
- a correction due to the use of artificial viscosity denoted $\mathbf{AVI}$

Using explicit time advancement, these corrections are all added to the update function at the end of the iteration to get the final result. In the following, $\mathbf{BND}$ correction will be taken as a model since the general principles and equations are the same for all numerical corrections.
During the iteration, one only has access to $\mathbf{BND_{\boldsymbol{\rho}u}}$, $\mathbf{BND_{\boldsymbol{\rho}E}}$ and $\mathbf{BND_{\boldsymbol{\rho}Y_k}}$.
$\mathbf{BND_{E_1}}$ and $\mathbf{BND_{E_2}}$ should therefore be calculated using those terms. This is done by considering the differential equations linking $\mathbf{E_1}$ and $\mathbf{E_2}$ to the primitive variables ($\rho\vec{u}$, ρE and ρY_k) advanced in the code. When needed, the term $\sum_k \mathbf{d}\boldsymbol{\rho}\mathbf{Y_k}$ will be changed in $\mathbf{d}\boldsymbol{\rho}$.

Numerical terms for Eq.(1)

After some algebra, $\mathbf{dE_1}$ writes :

$$\mathbf{dE_1} = \beta\frac{p'}{\gamma p}\left(\mathbf{d}\boldsymbol{\rho}\mathbf{E} - \vec{u}.\mathbf{d}\boldsymbol{\rho}\tilde{\mathbf{u}} + \alpha\mathbf{d}\boldsymbol{\rho}\right) + \vec{u'}.(\mathbf{d}\boldsymbol{\rho}\tilde{\mathbf{u}} - \vec{u}\mathbf{d}\boldsymbol{\rho}) + \frac{p'}{\gamma p}\sum_k Xi_k \mathbf{d}\boldsymbol{\rho}\mathbf{Y_k} \tag{D.1}$$

where $\beta = \gamma - 1$, $\alpha = \frac{u^2}{2}$ and $Xi_k = r_k T - \beta es_k$
The following equation therefore defines $\mathbf{BND_{E_1}}$:

$$\mathbf{BND_{E_1}} = \beta \frac{p'}{\gamma p} \left(\mathbf{BND_{\boldsymbol{\rho} E}} - \vec{u}.\mathbf{BND_{\boldsymbol{\rho}\tilde{u}}} + \alpha \mathbf{BND_{\boldsymbol{\rho}}} \right)$$

$$+\vec{u'}.(\mathbf{BND_{\boldsymbol{\rho}\tilde{u}}} - \vec{u}\mathbf{BND_{\boldsymbol{\rho}}}) + \frac{p'}{\gamma p} \sum_k Xi_k \mathbf{BND_{\boldsymbol{\rho} Y_k}} \tag{D.2}$$

Numerical terms for Eq.(2)

After some algebra, $\mathbf{dE_2}$ writes :

$$\mathbf{dE_2} = \frac{s'}{C_p} \left[\mathbf{d\boldsymbol{\rho} E} - E\mathbf{d\boldsymbol{\rho}} - \rho\vec{u}.\mathbf{d\boldsymbol{\rho}\tilde{u}} - \frac{P}{\rho}\mathbf{d\boldsymbol{\rho}} + \sum_k \rho g_k (\mathbf{d\boldsymbol{\rho} Y_k} - Y_k \mathbf{d\boldsymbol{\rho}}) \right] \tag{D.3}$$

The following equation therefore defines $\mathbf{BND_{E_2}}$:

$$\mathbf{BND_{E_2}} = \frac{s'}{C_p} \left[\mathbf{BND_{\boldsymbol{\rho} E}} - E\mathbf{BND_{\boldsymbol{\rho}}} - \rho\vec{u}.\mathbf{BND_{\boldsymbol{\rho}\tilde{u}}} \right]$$

$$+\frac{s'}{C_p} \left[-\frac{P}{\rho}\mathbf{BND_{\boldsymbol{\rho}}} + \sum_k \rho g_k (\mathbf{BND_{\boldsymbol{\rho} Y_k}} - Y_k \mathbf{BND_{\boldsymbol{\rho}}}) \right] \tag{D.4}$$

Numerical terms for Eq.(3)

In the present work, no numerical corrections are considered for Eq.(**3**).

Appendix E

Short notations for balance closure analysis (Chapter 7).

E.1 Equation 1

$\mathbf{deudt}$	$\frac{\rho_0}{2}\frac{\partial u_1^2}{\partial t}$
$\mathbf{depdt}$	$\frac{1}{2\gamma_0 p_0}\frac{\partial p_1^2}{\partial t}$
$\mathbf{dedt}$	$\mathbf{deudt}+\mathbf{depdt}$
$\mathbf{Ac_1}$	$-\frac{p'}{\gamma p}\left[\gamma p\vec{\nabla}.\vec{u}-\overline{\gamma p}\vec{\nabla}.\vec{\overline{u}}\right]$
$\mathbf{Ac_2}$	$-\vec{u'}.\vec{\nabla}p'$
$\mathbf{Ac_3}$	$\frac{p'}{\gamma p}\left[(\gamma-1)\,\dot{\omega_T}-(\overline{\gamma}-1)\,\overline{\dot{\omega_T}}\right]$
$\mathbf{Ac_4}$	$-\frac{p'}{\gamma p}\left[(\gamma-1)\,\vec{\nabla}.\vec{q}-(\overline{\gamma}-1)\,\vec{\nabla}.\vec{\overline{q}}\right]$
$\mathbf{Ac_5}$	$\vec{u'}:\vec{\nabla}\tau'$
$\mathbf{Ac_6}$	$-\frac{p'}{\gamma p}\left[(\gamma-1)\sum_{k=1}^{N}h_{sk}\rho\vec{u}.\vec{\nabla}Y_k-(\overline{\gamma}-1)\sum_{k=1}^{N}\overline{h_{sk}}\overline{\rho}\vec{\overline{u}}.\vec{\nabla}\overline{Y_k}\right]$
$\mathbf{Ac_7}$	$\frac{p'}{r}\frac{\partial r}{\partial t}$
$\mathbf{LW_{E_1}}$	Lax-Wendroff correction term
$\mathbf{BND_{E_1}}$	BouNDary correction term
$\mathbf{AVI_{E_1}}$	Artificial VIscosity correction term

E.2 Equation 2

$\mathbf{de_s dt}$	$\frac{p_0}{2r_0 c_{p0}} \frac{\partial s_1^2}{\partial t}$
$\mathbf{En_1}$	$-\frac{ps'}{rc_p}\left[\vec{u}.\vec{\nabla} s - \overline{\vec{u}}.\vec{\nabla}\overline{s}\right]$
$\mathbf{En_2}$	$\frac{s'}{c_p}\left[\dot{\omega}_T - \frac{\rho T}{\overline{\rho}\overline{T}}\overline{\dot{\omega}_T}\right]$
$\mathbf{En_3}$	$\frac{s'}{c_p}\left[-\vec{\nabla}.\vec{q} + \frac{\rho T}{\overline{\rho}\overline{T}}\vec{\nabla}.\overline{\vec{q}}\right]$
$\mathbf{En_4}$	$-\frac{s'}{c_p}\left[\sum_k g_{sk}\left[\dot{\omega}_k - \vec{\nabla}.\vec{q_k}\right] - \frac{\rho T}{\overline{\rho}\overline{T}}\sum_k \overline{g_{sk}}\left[\overline{\dot{\omega}_k} - \vec{\nabla}.\overline{\vec{q_k}}\right]\right]$
$\mathbf{LW_{E_2}}$	Lax-Wendroff correction term
$\mathbf{BND_{E_2}}$	BouNDary correction term
$\mathbf{AVI_{E_2}}$	Artificial VIscosity correction term

E.3 Equation 3

$\mathbf{D_s}$	$-s'\vec{m}'\vec{\nabla}\bar{T} + \overline{\vec{m}s'}\vec{\nabla}T'$
$\mathbf{D_Q}$	$T'Q'$
$\mathbf{D_Y}$	$\vec{m}'.\vec{\psi^{\star}}' + T'Q^{\star'} - \sum_{k=1}^{n} g'_{sk}\underline{\Omega'_k} + \sum_{k=1}^{n} \underline{g'_{sk}}\overline{Y_k}\vec{\nabla}.\vec{m}' + \sum_{k=1}^{n}\overline{g_{sk}}Y'_k\vec{\nabla}.\vec{m}'$
$\overline{\mathbf{D_s}}$	$-\bar{\vec{m}}.T'\vec{\nabla}s' - (\vec{m}'s').\vec{\nabla}\bar{T}$
$\overline{\mathbf{D_Q}}$	$\overline{T'Q'}$

Appendix F

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Budget of disturbance energy in gaseous reacting flows

By **A. Giauque†, T. Poinsot‡, M. Brear¶, AND F. Nicoud‖**

This paper presents an energy analysis of the disturbances that occur in gaseous combustion. It builds upon the previous work of Myers (1991) by including species and heat release terms, thus extending Myers' exact and linearized energy corollaries to combusting flows. These energy corollaries identify additional and significant energy density, flux, and source terms, thereby generalizing the recent results of Nicoud and Poinsot (2005) to include non-zero mean flow quantities, large amplitude disturbances, and varying specific heats. The associated stability criterion is therefore significantly different from the Rayleigh criterion in several ways. The closure of the exact equation is performed on an oscillating 2-D laminar flame. Results show that in this case the general equation can be substantially simplified by considering only entropy, heat release, and heat flux terms. The first one behaves as source term whereas the latter two dissipate the disturbance energy. Moreover, terms associated with the non-zero baseline flow are found to be important for the global closure of the balance even though the mean Mach number is small.

1. Introduction

Combustion stability has received sustained attention in both the academic and industrial communities, particularly over the last fifty years. During this time, the literature on this issue has grown enormously, and now spans numerous applications, including rockets (Flandro 1985; Culick 2001), afterburners (Bloxsidge *et al.* 1988), gas turbines (Dowling and Stow 2003; Poinsot and Veynante 2001), and industrial burners (Putnam 1971). The "Rayleigh criterion" is the most common argument for explaining combustion stability. While Rayleigh himself only first stated this criterion in prose form (Lord Rayleigh 1878), it is often written as

$$\int_{\Omega} \overline{p'\omega_T'} d\mathbf{x} > L, \tag{1.1}$$

where p', ω_T', L, and Ω are the static pressure and heat release rate disturbances at a point in space, the losses from the combustor, and the combustor volume, respectively. $(\bar{\ })$ denotes the time average. This criterion states that the combustor is unstable when the relative phase of the pressure and heat release disturbances over the combustor volume are such that the integral is larger than the (at present unspecified) losses.
Despite the "Rayleigh criterion" having its origin well over a century ago, a recent paper by Nicoud and Poinsot (2005) suggested that it is still at the very least unclear under what conditions this criterion can be derived from the equations governing combusting

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fluid motion. The earlier works of Chu (1956, 1965), Bloxsidge *et al.* (1988), and Dowling (1997) attempted to show precisely this, using progressively more general definitions of acoustic or disturbance energies that have appeared over the last fifty or so years. These works illustrate that deriving the Rayleigh criterion from the governing equations first requires a valid conservation equation for the energy contained in the disturbances, whether these disturbances are considered acoustic or otherwise.

In combusting flows, any equation stating disturbance energy conservation must start from equations of motion that at least include non-zero mean flow quantities and entropy variation. To ignore either the mean flow or entropy variation causes conceptual problems, discussed later in this paper. It appears that only the energies defined by Morfey (1971) and Myers (1991) do this. Viscous dissipation and heat conduction, while included in these works, are not essential and are usually small. The derivation in Dowling (1997) of an acoustic energy conservation equation for combusting flows extended the approach of Morfey (1971) for non-combusting flows. In Morfey's analysis, any entropy disturbances are shifted into the source term. Myers (1991) allows entropy disturbances to remain in both the energy density and flux terms. Myers' equation was consistent with those developed earlier by Chu (1965) and Pierce (1981) for zero-mean flow. Nicoud and Poinsot (2005) rederived the fluctuating energy equation of Chu (1965) and argued that the Rayleigh criterion is an incomplete description of the significant sources of fluctuating energy in combustion. In the limit of small disturbance amplitude, a source term proportional to $T_1\omega_{T1}$ was found where ω_{T1} and T_1 are the first terms in the heat release and static temperature asymptotic expansions. This term is analogous but significantly different to the Rayleigh term in Eq. (1.1). Entropy disturbances through the flame were also argued to be a significant source of disturbance energy, but Nicoud and Poinsot's formulation was conceptually problematic because they assumed zero-mean flow quantities. Bloxsidge *et al.* (1988) and Dowling (1997) also showed that terms other than the Rayleigh term existed for their differently defined acoustic energy equation, but both argued that these terms were small in practice.

This difference of opinion on such a fundamental and practically important problem needs to be resolved. This can be achieved by first deriving a general equation for disturbance energy, as this paper does, and then studying numerically the magnitudes of all the identified source terms. The basic equation should not be linear as it is often the case when dealing with acoustics. Indeed the temperature, entropy, and velocity disturbances in particular can be large within flames and non-linear effects are already known to be significant in the acoustic energy analysis of solid rocket combustion (Flandro 1985; Culick 2001). Since Myers (1991) presented both exact and linearized disturbance energy equations, comparison of the two would determine the applicability of the linearized equations on a given combustor if the generalization of Myers' approach to combusting flows can be handled appropriately.

This paper supports the questioning of the validity of the Rayleigh criterion (Nicoud and Poinsot 2005) in common combusting flows. It draws heavily on Myers' exact and linearized energy corollaries (Myers 1991) , and extends Nicoud and Poinsot's results to non-zero mean Mach numbers, large amplitude disturbances, and varying specific heats. Preliminary testing of the proposed exact disturbance energy equation is then performed by post-processing numerical simulation of a 2-D laminar oscillating flame.

2. Formulation

2.1. *What is a "disturbance energy" ?*

The notion of a disturbance energy Ed is somewhat vague; we first present a set of properties that we believe this quantity should meet in order to be useful when analyzing combustion stability. Many of these properties are obtained when expanding E_d or any other quantity as $f(\mathbf{x}, t) = f_0(\mathbf{x}) + \sum \epsilon^i f_i(\mathbf{x}, t)$, where ϵ is a small parameter.

- **P0**: E_d should be zero when there are no fluctuations, that is $E_{d0} = 0$,
- **P1**: E_d should be quadratic in the primitive variable fluctuations, that is $E_{d1} = 0$, and reduce properly to the well-established energies derived earlier for small amplitude disturbances (Chu 1965; Myers 1991; Nicoud and Poinsot 2005) when proper assumptions are made. **P1** is also a pre-requisite for **P3**,
- **P2**: The leading order term of E_d, viz. E_{d2}, should only depend on the first order term of the primitive variable fluctuations ρ_1, p_1, $\mathbf{u}_1$, etc.
- **P3**: E_{d2} should be definite positive so that it increases with the amplitude of the fluctuations. The disturbance energy itself should remain positive even for large amplitude fluctuations.

While **P0** is an obvious statement, **P1** is enforced for consistency with previous works. Noting E the sensible stagnation energy of the mixture, **P1** disqualifies $E - E_0$ as a disturbance energy since $(E - E_0)_1 = C_v T_1$ for a calorifically perfect mixture at rest. Property **P2** is required for practical use: if E_{d2} were depending on both p_1 and p_2, for instance, one would have to define and handle two different pressure fluctuations when computing/analysing E_{d2}. Eventually, let us assume a conservation equation of the form

$$\frac{\partial E_d}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{W} = D \tag{2.1}$$

for E_d, where $\mathbf{W}$ and D stand for the flux vector and source of the disturbance energy. If property **P3** is satisfied, a stability criterion can easily be derived by integrating Eq. (2.1) over the flow domain Ω bounded by the surface S:

$$\text{STABILITY} \qquad \Leftrightarrow \qquad \int_\Omega D d\mathbf{x} - \int_S \mathbf{W} \cdot \mathbf{n} dS < 0, \tag{2.2}$$

where $\mathbf{n}$ is the outward normal vector.

At this point, it is unclear whether a disturbance energy satisfying **P0-P3** exists and if it is unique. A potential candidate for which we can show that **P0-P2** and at least partly **P3** are satisfied is discussed in the rest of this paper.

2.2. *Basic equations*

The exact disturbance energy conservation equation is derived from statements of mass conservation, species mass conservation, momentum transport, energy conservation, and entropy transport for a mixture of n gaseous species,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{m} = 0, \tag{2.3}$$

$$\frac{\partial \rho Y_k}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{m} + \mathbf{q}_k) = \omega_k, \quad \text{for} \quad k = 1, 2, ..., n \tag{2.4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\zeta} + \boldsymbol{\nabla} H - T\boldsymbol{\nabla} s = \boldsymbol{\psi} + \boldsymbol{\psi}^*, \tag{2.5}$$

$$\frac{\partial}{\partial t}(\rho H - p) + \boldsymbol{\nabla} \cdot (\mathbf{m}H) - \mathbf{m} \cdot \boldsymbol{\psi} = TQ, \tag{2.6}$$

$$\frac{\partial \rho s}{\partial t} + \boldsymbol{\nabla} \cdot (s\mathbf{m}) = Q + Q^*, \tag{2.7}$$

In Eqs. (2.3)-(2.7), ρ is the volumetric density, $\mathbf{u}$ is the velocity vector, $\mathbf{m} = \rho\mathbf{u}$ is the momentum, $\boldsymbol{\xi} = \boldsymbol{\nabla} \times \mathbf{u}$ is the vorticity and $\boldsymbol{\zeta} = \boldsymbol{\xi} \times \mathbf{u}$, $H = h_s + \mathbf{u} \cdot \mathbf{u}/2$ is the sensible stagnation enthalpy, T is the temperature, s is the sensible entropy, $\boldsymbol{\psi}$ is such that $\psi_{ij} = (1/\rho)(\partial\tau_{ij}/\partial x_i)$ where τ_{ij} is the ij^{th} component of the viscous stress tensor. Noting Y_k, ω_k, h_{sk}, g_{sk}, $\mathbf{V}_k$ and $\mathbf{q}_k = \rho Y_k \mathbf{V}_k$ the mass fraction, volumic rate of production, sensible enthalpy, sensible Gibbs free energy, diffusion velocity and diffusion mass flux of the k^{th} species respectively, one also defines $\boldsymbol{\psi}^* = \sum_{k=1}^{n} g_{sk} \boldsymbol{\nabla} Y_k$, $TQ^* = -\sum_{k=1}^{n} g_{sk} (\omega_k - \boldsymbol{\nabla} \cdot \mathbf{q}_k)$ and $TQ = -\boldsymbol{\nabla} \cdot \mathbf{q} + \phi + \omega_T$ where $\mathbf{q} = -\lambda \boldsymbol{\nabla} T + \rho \sum_{k=1}^{n} Y_k h_{sk} \mathbf{V}_k$ is the heat flux, ϕ is the viscous dissipation, and ω_T is the heat release per unit volume. Splitting each quantity () into time-averaged ($\bar{}$) and fluctuating components $()'$, Eqs. (2.3)-(2.7) can be time averaged to give:

$$\boldsymbol{\nabla} \cdot \bar{\mathbf{m}} = 0, \tag{2.8}$$

$$\boldsymbol{\nabla} \cdot (\bar{Y}_k \bar{\mathbf{m}}) + \boldsymbol{\nabla} \cdot \bar{\mathbf{q}}_k + \boldsymbol{\nabla} \cdot \overline{\mathbf{m}' Y_k'} = \bar{\omega}_k, \tag{2.9}$$

$$\bar{\boldsymbol{\zeta}} + \boldsymbol{\nabla} \bar{H} - \bar{T} \boldsymbol{\nabla} \bar{s} - \overline{T' \boldsymbol{\nabla} s'} = \bar{\boldsymbol{\psi}} + \bar{\boldsymbol{\psi}}^*, \tag{2.10}$$

$$\boldsymbol{\nabla} \cdot (\bar{\mathbf{m}} \bar{H}) + \boldsymbol{\nabla} \cdot \overline{\mathbf{m}' H'} - \bar{\mathbf{m}} \cdot \bar{\boldsymbol{\psi}} - \overline{\mathbf{m}' \cdot \boldsymbol{\psi}'} = \bar{T}\bar{Q} + \overline{T' Q'}, \tag{2.11}$$

$$\boldsymbol{\nabla} \cdot (\bar{\mathbf{m}} \bar{s}) + \boldsymbol{\nabla} \cdot \overline{\mathbf{m}' s'} = \bar{Q} + \bar{Q}^*. \tag{2.12}$$

Note that Eqs. (2.8)-(2.12) have been obtained by assuming that the time averages of time derivatives are zero, which implies averaging either over a very long period of time or over a finite number of periods of oscillation. For variables $\mathbf{u}$, p, $\mathbf{m}$, Y_k, $\mathbf{q}_k$, ω_k, $\boldsymbol{\zeta}$, H, T, s, $\boldsymbol{\psi}$, $\boldsymbol{\psi}^*$, Q, Q^*, e_s, g_{sk} and $\Omega_k = \omega_k - \boldsymbol{\nabla} \cdot \mathbf{q}_k - \boldsymbol{\nabla} \cdot (\mathbf{m} Y_k)$, $f(\mathbf{x}, t) = \bar{f}(\mathbf{x}) + \sum \epsilon^i f_i(\mathbf{x}, t)$ where $\bar{f}(\mathbf{x})$ defines the baseline flow. Note that f' is therefore not equivalent to f_1 as defined in section 2.1. It is exactly $\sum_{i=1}^{+\infty} f_i(\mathbf{x}, t)\epsilon^i$ and thus equals ϵf_1 to order ϵ^2. Note also that Myers (1991) defined his disturbances around a laminar base flow, which has little meaning in many combusting cases, particularly those undergoing strong limit cycle oscillations. Nonetheless, if the disturbances are sufficiently small, the time averages of the products of the disturbances in Eqs. (2.8)-(2.12) are negligible and the laminar equations are recovered.

2.3. *Non-linear disturbance energy*

Myers' approach (1991) consists of subtracting from Eq. (2.6) an appropriate linear combination of Eqs. (2.3), (2.5), and (2.7) in order to obtain an *exact* conservation equation for a disturbance energy. The same approach has been followed in this study, including combustion and mixing effects by including the species mass conservation Eq. (2.3) into the linear combination that is removed from (2.6). A guide to determine a proper linear combination of these equations is to eventually obtain a disturbance energy that satisfies properties **P0-P3**. One shows that a proper choice is to multiply Eq. (2.3) by $(\bar{H} - \bar{T}\bar{s} - \sum_{k=1}^{n} \overline{g_{sk}}\bar{Y}_k)$, Eq. (2.7) by $\bar{T}$, Eq. (2.5) by $\bar{\mathbf{m}}$, and Eq. (2.4) by $\bar{g_{sk}}$ and then subtract all these from (2.6). Making use of Eq.s 2.8, 2.9, 2.11, and 2.12, one eventually obtains an equation of the form 2.1, where the disturbance energy, flux vector, and source terms are defined respectively as

$$E_d = \rho[H - \bar{H} - \bar{T}(s - \bar{s})] - \bar{\mathbf{m}} \cdot (\mathbf{u} - \bar{\mathbf{u}}) - (p - \bar{p}) - \sum_{k=1}^{n} \overline{g_{sk}}\rho(Y_k - \bar{Y}_k) - \overline{\rho' e_s'} \quad (2.13)$$

$$\begin{aligned} \mathbf{W} = (\mathbf{m} - \bar{\mathbf{m}})[(H - \bar{H}) - \bar{T}(s - \bar{s})] + \bar{\mathbf{m}}(T - \bar{T})(s - \bar{s}) \\ + \overline{\mathbf{m}' H'} - T\overline{\mathbf{m}' s'} \end{aligned} \quad (2.14)$$

and

$$D = D_\xi + D_s + D_Q + D_{Q^*} + D_\psi + D_{\psi^*} + D_{Y_k}, \quad (2.15)$$

where

$$\begin{aligned}
D_\xi &= -(\mathbf{m} - \bar{\mathbf{m}}) \cdot (\boldsymbol{\zeta} - \bar{\boldsymbol{\zeta}}) - \overline{\mathbf{m}' \cdot \boldsymbol{\xi}'}, \\
D_s &= -(\mathbf{m} - \bar{\mathbf{m}}) \cdot (s - \bar{s})\boldsymbol{\nabla}\bar{T} + (s - \bar{s})\bar{\mathbf{m}} \cdot \boldsymbol{\nabla}(T - \bar{T}) \\
&\quad - \overline{\mathbf{m}' s'} \cdot \boldsymbol{\nabla} T - \mathbf{m} \cdot \overline{T' \boldsymbol{\nabla} s'} \\
D_Q &= (T - \bar{T})(Q - \bar{Q}) + \overline{T' Q'} \\
D_{Q^*} &= (T - \bar{T})(Q^* - \bar{Q^*}) + \overline{T' Q^{*\prime}} \\
D_{\boldsymbol{\psi}} &= (\mathbf{m} - \bar{\mathbf{m}}) \cdot (\boldsymbol{\psi} - \bar{\boldsymbol{\psi}}) + \overline{\mathbf{m}' \cdot \boldsymbol{\psi}'} \\
D_{\boldsymbol{\psi}^*} &= (\mathbf{m} - \bar{\mathbf{m}}) \cdot (\boldsymbol{\psi}^* - \bar{\boldsymbol{\psi}^*}) + \overline{\mathbf{m}' \cdot \boldsymbol{\psi}^{*\prime}} \\
D_{Y_k} &= \sum_{k=1}^{n} g_{sk}' \Omega_k' + \sum_{k=1}^{n} (g_{sk}' Y_k + \overline{g_{sk}} Y_k') \, \boldsymbol{\nabla} \cdot \mathbf{m}' \\
&\quad + \sum_{k=1}^{n} \overline{g_{sk}' \Omega_k'} + \overline{g_s' \boldsymbol{\nabla} \cdot \mathbf{m}'},
\end{aligned}$$

where $\Omega_k = \omega_k - \boldsymbol{\nabla} \cdot \mathbf{q}_k - \boldsymbol{\nabla} \cdot (\mathbf{m} Y_k)$. Note that the correlation term in Eq. (2.13) as been introduced so that the leading order term of the disturbance energy does not contain a constant contribution and is positive. Also note that the disturbance energy contains the fluctuation of the turbulent kinetic energy, $E_k = \bar{\rho}\frac{1}{2}(u_i' u_i' - \overline{u_i' u_i'})$.

2.4. *Linearization*

It is obvious from Eq. (2.13) that E_d satisfies property **P0**. Disturbances of the form $()' = () - (\bar{\ }) = \sum_{i=1}^{\infty} \epsilon^i ()_i$ are then substituted into the exact Eq. (2.1), and only the lowest order terms in ϵ are retained. In keeping with the other studies of disturbance energy in isentropic and homentropic flows (Morfey 1971; Cantrell and Hart 1964; Pierce 1981), the remaining terms are of second order in the disturbances, meaning that E_d also satisfies **P1**. Retention of all second-order terms in the exact flux vector and source terms results in a rather complex disturbance energy equation, where much of the complexity is contained in viscous stress, dissipation, and heat conduction terms that can be argued to be negligible in most combusting flows. Ignoring such terms as well as the vorticity terms results in the following linearized disturbance energy equation,

$$\frac{\partial E_{d2}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{W}_2 = D_2, \tag{2.16}$$

where the disturbance energy density E_{d2}, flux vector $\mathbf{W}_2$, and source D_2 terms are

$$E_{d2} = \frac{p_1^2}{2\bar{\rho}\bar{c}^2} + \frac{1}{2}\bar{\rho}\mathbf{u}_1 \cdot \mathbf{u}_1 + \rho_1 \bar{\mathbf{u}} \cdot \mathbf{u}_1 + \frac{\bar{\rho}\bar{T} s_1^2}{2\bar{c}_p} + E_{Y2}, \tag{2.17}$$

$$\mathbf{W}_2 = (p_1 + \bar{\rho}\bar{\mathbf{u}} \cdot \mathbf{u}_1)\left(\mathbf{u}_1 + \frac{\rho_1}{\bar{\rho}}\bar{\mathbf{u}}\right) + \bar{\mathbf{m}} T_1 s_1 + \overline{\mathbf{m}' H'} - T\overline{\mathbf{m}' s'}, \tag{2.18}$$

and

$$\begin{aligned} D_2 = & -\overline{\mathbf{m}' s'} \cdot \boldsymbol{\nabla}\bar{T} - \bar{\mathbf{m}} \cdot \overline{T' \boldsymbol{\nabla} s'} \\ & - s_1 \mathbf{m}_1 \cdot \boldsymbol{\nabla}\bar{T} + s_1 \bar{\mathbf{m}} \cdot \boldsymbol{\nabla} T_1 \\ & + \left(\frac{\omega_1 T_1}{\bar{T}} - \frac{\bar{\omega_T} T_1^2}{(\bar{T})^2}\right) + \overline{T' \omega_T'} \\ & + T_1 Q_1^* + \overline{T' Q^{*\prime}} + \mathbf{m}_1 \cdot \boldsymbol{\psi}_1^* + \overline{\mathbf{m}' \cdot \boldsymbol{\psi}^{*\prime}} \\ & + \sum_{k=1}^{n} g_{sk1} \Omega_{k1} + \sum_{k=1}^{n} \left(g_{sk1} \bar{Y_k} + \bar{g_{sk}} Y_{k1}\right) \boldsymbol{\nabla} \cdot \mathbf{m}_1 \\ & + \sum_{k=1}^{n} \overline{g_{sk1} \Omega_{k1}} + \overline{g_s' \boldsymbol{\nabla} \cdot \mathbf{m}'}. \end{aligned} \tag{2.19}$$

The E_{Y2} term in Eq. (2.17) is the contribution from the $\rho\bar{g_{sk}}(Y_k - \bar{Y_k})$ terms in Eq. (2.13) and is equal to

$$\begin{aligned} E_{Y2} = & \frac{\bar{p}}{2} \sum_{k=1}^{n} \left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right) \frac{g_{sk} - e_{sk}}{C_v T} + \frac{W}{W_k}\left(1 + \frac{1}{Y_k}\right)\right] Y_{k1}^2 \\ & + \bar{p} \sum_{k=1}^{n} \sum_{j \neq k} \left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right) \frac{g_{sj} - e_{sj}}{C_v T} + \frac{W}{W_j}\right] Y_{k1} Y_{j1} \\ & + \frac{\rho}{C_v} \sum_{k=1}^{n} (g_{sk} - e_{sk}) s_1 Y_{k1} + \sum_{k=1}^{n} \left[(\gamma - 1)(g_{sk} - e_{sk}) + \frac{RT}{W_k}\right] \rho_1 Y_{k1}. \end{aligned} \tag{2.20}$$

From Eqs. (2.17)-(2.20), the proposed disturbance energy also satisfies property **P2** since its leading order term only depends on first-order quantities. Note also that Eq. (2.16) simplifies to other, existing acoustic energy conservation equations under the condition of homentropic flow and homogeneous mixture. The energy density E_{d2} and flux $\mathbf{W}_2$ terms then become those defined by Cantrell and Hart (1964) for acoustic propagation in a non-stationary medium. Under the zero Mach number flow assumption and calorific perfection, Eq. (2.16) reduces to the form given in Nicoud and Poinsot (2005). The last three lines in Eq. (2.19) as well as E_{Y2} are related to mixture inhomogeneities over space. These terms do not seem to have been reported elsewhere and require further investigation. Although not negligible, they are not necessary to obtain a reasonable closure of the disturbance energy budget (see Section 3.2).

2.5. *Definite positivity of* E_{d2}

It is not evidenced that E_{d2} as defined in Eq. (2.17) is a positive definite quadratic form (property **P3**). Even in the case of a homogeneous mixture where all the mass fraction fluctuations are zero, this property is not clearly established because of the $\rho_1 \bar{\mathbf{u}} \cdot \mathbf{u}_1$ term (Hanifi *et al.* 1996). We propose in the following a simple proof that E_{d2} is indeed definite positive in the case of a flow without mass fraction fluctuations ($E_{Y2} = 0$) at small enough mean Mach number. Making use of $\rho_1/\bar{\rho} = p_1/(\gamma\bar{p}) - s_1/C_p$, which is true only at chemical equilibrium, and then rewriting E_{d2} in the following matrix form:

$$E_{d2} = F^t A F, \tag{2.21}$$

where F is the reduced first order fluctuation vector $F^t = [p_1/(\bar{\rho}\bar{c}^2) \;\; ||\mathbf{u}_1||/\bar{c} \;\; s_1/C_p]$ and A is the following matrix:

$$A = \frac{\bar{\rho}\bar{c}^2}{2} \begin{bmatrix} 1 & M\mathbf{n}_0 \cdot \mathbf{n}_1 & 0 \\ M\mathbf{n}_0 \cdot \mathbf{n}_1 & 1 & -M\mathbf{n}_0 \cdot \mathbf{n}_1 \\ 0 & -M\mathbf{n}_0 \cdot \mathbf{n}_1 & 1/(\gamma - 1) \end{bmatrix} = \frac{\bar{\rho}\bar{c}^2}{2} B, \tag{2.22}$$

where $\mathbf{n}_0$ and $\mathbf{n}_1$ are $\bar{\mathbf{u}}/||\bar{\mathbf{u}}||$ and $\mathbf{u}_1/||\mathbf{u}_1||$ respectively. Note that B being symmetric, it admits three real eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$ and that the eigenvalues of A are then simply $\lambda_i = \bar{\rho}\bar{c}^2 \mu_i/2$. Thus E_{d2} is positive definite as soon as $\lambda_3 > 0$, viz. $\mu_3 > 0$. From Eq. (2.22), the characteristic polynomial of B is proportional to

$$P(\mu) = (\gamma - 1)\mu^3 - (2\gamma - 1)\mu^2 + (\gamma + 1)\mu - (1 - \gamma M^2 (\mathbf{n}_0 \cdot \mathbf{n}_1)^2)$$

and its roots are such that $\mu_1 + \mu_2 + \mu_3 = (2\gamma - 1)/(\gamma - 1) > 0$ and $\mu_1\mu_2\mu_3 = (1 - \gamma M^2 (\mathbf{n}_0 \cdot \mathbf{n}_1)^2)/(\gamma - 1)$, which is strictly positive as soon as $M < 1/\sqrt{\gamma} \leq 1/(|\mathbf{n}_0 \cdot \mathbf{n}_1|\sqrt{\gamma})$. Under this latter restriction, it follows that the eigenvalues of A are either **(a)** all positive or **(b)** such that $\mu_1 > 0$ and $\mu_3 \leq \mu_2 < 0$. Since $\gamma > 1$, it is obvious that the two roots of the derivative of $P(\mu)$, viz. $P'(\mu) = 3(\gamma - 1)\mu^2 - 2(2\gamma - 1)\mu + \gamma + 1$, are both strictly positive so that at least two roots of $P(\mu)$ are positive. Thus only the **(a)** choice is acceptable and $\mu_3 > 0$ as long as $M < 1/\sqrt{\gamma}$, which is not a restrictive condition in many combusting flows. In other words, the linearized disturbance energy can be recast under the form:

$$E_{d2} = \lambda_1 \theta_1^2 + \lambda_2 \theta_2^2 + \lambda_3 \theta_3^2,$$

where the λ_i's are all positive and the θ_i's are linear combinations of $\bar{\mathbf{u}} \cdot \mathbf{u}_1$, p_1, and s_1. Further work is required to investigate whether this **P3** property still holds in the case where mass fraction fluctuations are accounted for ($E_{Y2} \neq 0$).

3. Numerical Results

3.1. *Configuration*

The 2-D laminar oscillating flame configuration considered for testing the closure of the budget of the disturbance energy is adapted from an experiment described in Le Helley (1994). The burner consists of a ducted premixed propane-air flame that is stabilized thanks to a perforated plate with multiple holes. For certain operation modes this burner features small laminar Bunsen tip flames behind each hole and no turbulence effects

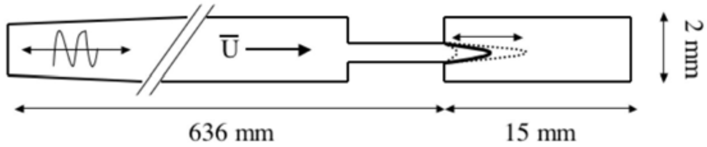

FIGURE 1. Schematic of the computational domain.

FIGURE 2. Heat release when the flame is the shortest (most left) and the longest (second on the left) during the cycle of oscillation. The above illustrates time averaged heat release (first on the right) and temperature (most right).

are present. We limit our study to this regime. The 2-D planar computational mesh has been chosen such that chemistry and thermodynamic effects are resolved on the mesh (50,000 nodes). Chemistry is modeled using a single-step Arrhenius law involving 5 gaseous species. The pre-exponential factor, mass fraction exponent, and activation temperature are fitted to produce the proper flame speed in the lean regime. The main characteristics of the computational flow domain are depicted in Fig. 1 (not to scale). The velocity and temperature are imposed at the inlet while static pressure is prescribed at the outlet. In both cases partially reflecting characteristic boundary conditions (Kaufmann *et al.* 2002; Selle *et al.* 2004) have been used for numerical stability reasons. For the time average mass flow rate ($4.1\mathbf{x}10^{-3}kg.s^{-1}$) concerned and at stoichiometric equivalence ratio, the flame is self-excited at a frequency close to 820 Hz, which corresponds to the third longitudinal acoustic mode of the combustor including the air feeding line.

The extreme positions of the flame over the cycle are shown in FIg. 2, where the time averaged temperature and heat release fields in the flame are shown. Recall that the time averaged solution is the baseline flow and corresponds to no fluctuation and zero disturbance energy in the present analysis.

The numerical tool used in this section is the unstructured combustion code AVBP developed at CERFACS (AVBP 2006). AVBP solves the complete Navier-Stokes equations including chemistry in two and three spatial dimensions. The unstructured approach allows computing not only the combustor but also the whole air feeding line as well as the exhaust system. This code was selected because it solves the complete compressible Navier-Stokes equations under a form that is mathematically equivalent to Eqs. (2.3), (2.4), (2.5), and (2.6). Its ability to reproduce the unsteady behavior of the Le Helley's flame has been demonstrated elsewhere (Kaufmann *et al.* 2002).

3.2. *Energy budget*

The disturbance energy and all the sources and flux terms in Eq. (2.1) have been computed by post-processing 40 fields over two periods of the limit cycle of the flame depicted in Fig. 2. These quantities have subsequently been integrated over the computational domain to obtain

$$\mathcal{E}_d = \int_\Omega E_d d\Omega \qquad \mathcal{W} = \int_S \mathbf{W}\cdot\mathbf{n}dS \qquad \mathcal{D} = \int_\Omega D d\Omega.$$

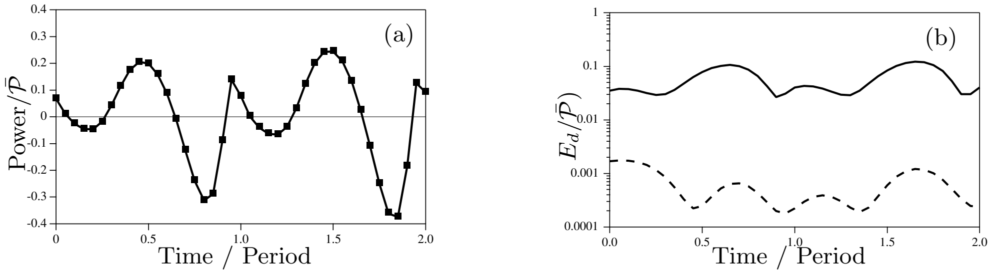

FIGURE 3. Time evolution of **(a)** the time derivative of the total disturbance energy dE_d/dt (■) and spatial terms $D - W$ (———) and **(b)** the disturbance energy E_d for $3.4\text{x}10^{-3} < |\mathbf{m}| < 4.8\text{x}10^{-3}$ kg/s. (present flame, ———) and $1.1\text{x}10^{-3} < |\mathbf{m}| < 1.3\text{x}10^{-3}$ kg/s (– – – –).

As shown in Fig. 3a, the global budget $d\mathcal{E}_d/dt = \mathcal{D} - \mathcal{W}$ closes nicely when all the terms are included. Note that the scaling is by the time averaged total heat release of the flame $\bar{\mathcal{P}} = \int_\Omega \bar{q} d\Omega$. The net power curve shows a near derivative discontinuity at time $t \approx 0.9T$. It corresponds to the shrinking of the flame front, which quickly creates disturbance energy. Note that the time sampling shown on Fig. 3 is not the one used for the assessment of the time derivative of E_d. Fig. 3b depicts the time evolution of the disturbance energy for the flame of Fig. 2 as well as for a calculation with smaller mean and oscillating mass flow rate ($1.1\text{x}10^{-3} < |\mathbf{m}| < 1.3\text{x}10^{-3}$ kg/s instead of $3.4\text{x}10^{-3} < |\mathbf{m}| < 4.8\text{x}10^{-3}$ kg/s). In both cases the disturbance energy remains positive (property **P4**), E_d being larger for larger oscillating mass flow rate.

Order of magnitude analysis suggests that the viscous stress, thermal diffusion, viscous dissipation (D_ψ) terms are usually small in combustion. The vorticity term (D_ξ) should also be insignificant in most combusting flows although several orders of magnitude larger than the other small terms (D_ψ). This is confirmed by the numerical results from the 2-D oscillating laminar flame, as shown in Fig. 4. Some terms are 1000 times (or more) smaller than $d\mathcal{E}/dt$ (see Fig. 4a) and do not contribute to the global budget. Note that besides the vorticity and viscous dissipation terms, the boundary terms ($\mathcal{W}$) belong to this category. This result seems to contradict with the findings of Martin *et al.* (2004), where the boundary terms were balancing the first-order Rayleigh term $p'\omega'_T$. The difference comes from the fact that only the acoustic part of the fluctuating energy was considered in Martin *et al.* (2004), while E_d also contains the entropy fluctuations. Although the boundary terms are still of the same order as $p'\omega'_T$ (not shown), they are much smaller than the first-order term in the total disturbance energy balance, viz. $T'\omega'_T$. Indeed, the Rayleigh term is approximately $1.5\text{x}10^{-4}$ $\bar{\mathcal{P}}$, while the $T'\omega'_T$ term is roughly $2\text{x}10^{-1}$ $\bar{\mathcal{P}}$ (see Fig. 5).

Other terms in the energy balance are only a few percent of $d\mathcal{E}_d/dt$ and only contribute slightly to the global budget. As shown in Fig. 4b, these terms are the contributions of the D_{Q^*}, D_{ψ^*}, and D_{Y_k} terms in Eq. (2.15) and are related to the mixture inhomogeneities and mass fraction fluctuations. Eventually, the first-order term in the energy balance are the contributions from D_Q and D_s (see Fig. 4c). Since $Q = (-\nabla \cdot \mathbf{q} + \phi + \omega_T)/T$ is almost equal to $\omega_T/T - \nabla \cdot \mathbf{q}/T$, the entropy, heat release, and heat flux terms are the most important terms in the energy balance. Figure 4d shows that keeping only these large terms leads to a reasonable closure of the disturbance energy equation.

Figure 5a,b shows the contributions of the different terms in the definition of D_Q and D_s (see Eq. (2.15)). In both cases, all the terms have approximately the same magnitude,

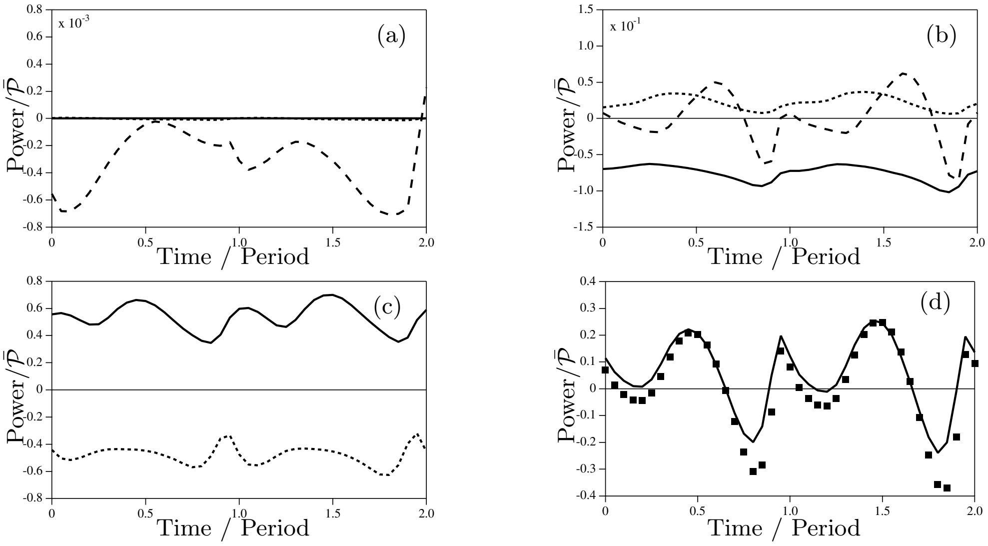

FIGURE 4. Time evolution of the volume integrals of **(a)** the boundary ($\mathcal{W}$, – – – –), D_ξ (————) and D_ψ (········) terms, **(b)** the D_{Q^*} (————), D_{ψ^*} (········) and D_{Y_k} (– – – –) terms, **(c)** the D_Q (········) and D_s (————) terms. Approximate budget **(d)** based on the heat release (D_Q) and entropy (D_s) terms only: exact time derivative of the total disturbance energy (■) and approximate spatial terms (————)

meaning that no further simplification can be made in the energy budget. Noticeably, the correlation terms are important and cannot be neglected in the disturbance energy balance. Recall that these correlation terms arise from the choice of the time averaged solution as the baseline flow. Although it is natural that the definition of the no-fluctuation state appears in the disturbance energy equation, it seems that the importance of these correlation terms has not been reported elsewhere. Figure 5c also shows that only the heat release and heat flux terms contribute in the $T'Q'$ term, the viscous dissipation based term being negligible. The heat release term can be further split in three terms by using the identity

$$+T'\left(\frac{\omega_T}{T}\right)' = T'\frac{\omega'_T}{T} + \bar{\omega_T}T'\left[\frac{1}{T} - \overline{\left(\frac{1}{T}\right)}\right] - T'\overline{\left(\omega'_T\left(\frac{1}{T}\right)'\right)},$$

where the three contributions on the RHS have similar amplitude as shown in Fig. 5d. Note however that the time average of $T'\overline{\omega'_T(1/T)'}$ is zero so that this term does not contribute to the (in)stability of the flow. The first term in this decomposition, viz. $T'\omega'_T/T$, is a *source* term. Note that assuming small amplitude fluctuations, this term becomes $T_1\omega_{T1}/\bar{T}$, the corrected Rayleigh term of Nicoud and Poinsot (2005). The second term in the decomposition of $T'(\omega_T/T)'$ is proportional to $\bar{\omega_T}$ and can be linearized as $-\bar{\omega_T}(T_1/\bar{T})^2$ for small amplitude fluctuations, indicating that this term tends to dissipate disturbance energy. Figure 5d indeed shows that the $\bar{\omega_T}$ term is a *sink* term in which amplitude is comparable to the corrected Rayleigh *source* term so that the heat flux term eventually contributes more than the heat release terms (Fig. 5c). Note that under the zero Mach number assumption used by Nicoud and Poinsot (2005), the time averaged heat release $\bar{\omega_T}$ is null and only the positive contribution of $T'(\omega_T/T)'$, viz. $T_1\omega_{T1}/\bar{T}$, is present. This is another output of the present analysis that non-zero Mach number terms

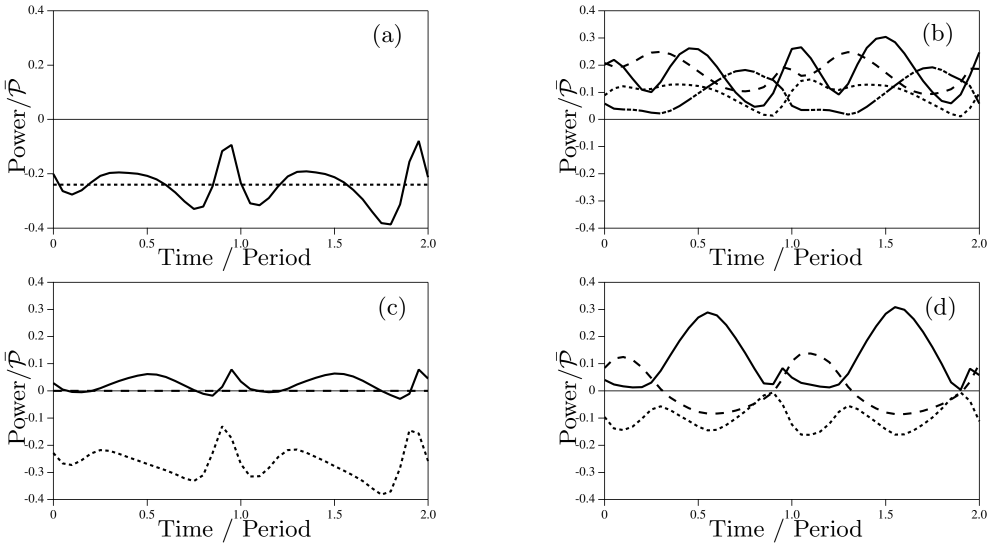

FIGURE 5. Time evolution of the volume integrals of **(a)** the terms in D_Q, viz. $T'Q'$ (———) and $\overline{T'Q'}$ (········) and **(b)** the terms in D_s, viz. $-\mathbf{m}' \cdot s'\nabla\bar{T}$ (———), $s'\bar{\mathbf{m}} \cdot \nabla T'$ (········), $-\overline{\mathbf{m}'s'} \cdot \nabla T$ (– – – –) and $-\mathbf{m} \cdot \overline{T'\nabla s'}$ (—·-), **(c)** the terms in $T'Q'$, viz. $T'(\omega_T/T)'$ (———), $-T'(\nabla\cdot\mathbf{q}/T)'$ (········) and $T'(\phi/T)'$ (– – – –), **(d)** the terms in $T'(\omega_T/T)'$, viz. $T'\omega_T'/T$ (———), $\bar{\omega_T}T'(1/T - \overline{1/T})$ (········) and $-T'\overline{\omega_T'(1/T)'}$ (– – – –)

can have a significant contribution in the energy balance, even though the mean Mach number is very small (of order 3x10^{-2} for the flame considered).

The production terms for the disturbance energy equation are related to entropy as shown in Fig. 5b. In the low Mach number limit, $\nabla\bar{p} = 0$ and $\nabla\bar{T} \propto \nabla\bar{s}$ so that the $-\mathbf{m}' \cdot s'\nabla\bar{T}$ term is proportional to the classical production term $-\mathbf{m}' \cdot s'\nabla\bar{s}$ for scalars. Assuming that acoustic fluctuations are negligible in the reaction zone, one obtains that $s' \propto T'$ so that the $s'\bar{\mathbf{m}} \cdot \nabla T'$ term in D_s is proportional to $\bar{\mathbf{m}} \cdot \nabla(s'^2)$, which is most likely positive since $\bar{\mathbf{m}}$ is from the fresh to the burnt gas and the entropy fluctuations are generated in the flame region. This is indeed also confirmed by Fig. 5b. The remaining entropy terms are based on time averaged correlations and require further investigation.

4. Conclusions

The exact transport equation derived for the disturbance energy from the basic governing equations for a combusting gaseous mixture can be used for generating the most general stability criterion if E_{d2} is indeed positive definite. Although the positive definiteness of E_{d2} has only been shown analytically in cases where mass fraction fluctuations can be neglected, the numerical results obtained for a 2-D laminar flame suggest that it might hold also in the case of large amplitude fluctuations and variable mass fractions. Moreover, the numerical results suggest that the time evolution of the global fluctuating energy is mostly governed by the heat release, heat flux, and entropy source terms, the contribution arising from the mixture changes over space and time being the largest of the negligible terms. Previous classical energy forms for homentropic flows are recovered as special cases of the fluctuating energy defined in this study. It is also shown that the terms proportional to the mean Mach number can be significant even if the baseline flow speed is very small ($M = 3\mathbf{x}10^{-2}$ in the present study).

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Appendix G

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Transport of energy by disturbances in gaseous combustion

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Abstract

This paper presents an analysis of the energy transported by disturbances in gaseous combustion. It builds upon the previous work of Myers [J. Fluid Mech. 226 (1991) 383–400] by including species and heat release terms, thus extending Myers' exact and linearised 'energy corollaries' to gaseous combustion.

These energy corollaries identify additional energy density, flux and source terms, and generalise the recent work of Nicoud and Poinsot [Combust. Flame 142 (2005) 153–159] to include non-zero mean flow quantities, small or large amplitude disturbances, varying specific heats and chemical non-equilibrium. The source terms are significantly different to that stated in the Rayleigh criterion in several ways, even for linear disturbances in a non-diffusive flow. An order of magnitude analysis then suggests that linear arguments of energy transport by disturbances are unlikely to apply in turbulent combusting flows in particular. Finally, a linear stability criterion is proposed for laminar combusting gases.

Key words: disturbance energy, combustion stability, thermoacoustic instability, Rayleigh criterion

1 Introduction

Combustion stability has received sustained attention in both the academic and industrial communities over the last fifty years in particular. During this time, the literature on this problem has grown enormously, and now spans numerous applications including rockets [1,2], afterburners [3], gas turbines [4,5] and industrial burners [6]. The sustained research on this problem is primarily because manufacturers still rely heavily on in situ testing and tuning of the complete, operating device to avoid instability. In the author's view, this continued reliance on testing has several causes, including incompleteness in our fundamental understanding of the problem, as argued recently by Nicoud and Poinsot [7].

The 'Rayleigh criterion' is the most common argument for explaining combustion stability. Whilst Rayleigh himself only first stated this criterion in prose form [8], it is usually written mathematically as

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$$\iiint_V \overline{p'\omega'} > L, \tag{1}$$

where $p'(Pa)$, $\omega'(W/m^3)$, L and $V(m^3)$ are harmonic disturbances in the static pressure and heat release rate at a point in space, the 'losses' from the combustor and the combustor volume respectively. The overbar $\overline{()}$ denotes the time average over one cycle. This criterion states that the combustor is unstable when the relative phase of the pressure and heat release disturbances over the combustor volume are such that the integral is larger than the (at present unspecified) losses.

Despite the Rayleigh criterion having its origin well over a century ago, a recent paper by Nicoud and Poinsot [7] suggested that it still at the very least unclear under what conditions this criterion can be derived from the equations governing combusting fluid motion. The earlier works of Chu [9,10], Bloxsidge et al. [3], and Dowling [11] attempted to show precisely this, using progressively more general definitions of acoustic or disturbance energies that have appeared over the last fifty or so years. These works illustrate that deriving the Rayleigh criterion from the governing equations first requires a valid conservation equation for the energy contained in the disturbances, whether these disturbances are considered 'acoustic' or otherwise.

In combusting flows, any equation stating disturbance energy conservation must start from equations of motion that at least include non-zero mean flow quantities and entropy variation. To ignore either the mean flow or entropy variation causes conceptual problems, as is discussed later in this paper. It appears that only the energies defined by Morfey [12] and Myers [13] do this. Viscous dissipation and heat conduction, whilst included in these works, are not essential to the problem.

Dowling's [11] derivation of an acoustic energy conservation equation for combusting flows extended Morfey's [12] approach for non-combusting flows. Morfey considered a viscous, heat conducting fluid, and first split the disturbance velocity field into irrotational and solenoidal components which were defined as the acoustic and unsteady vortical motions respectively. He then applied the definitions of acoustic energy density and flux proposed by Cantrell and Hart [14] for inviscid, non heat conducting flows to his acoustic field. Any resulting entropy disturbances in Morfey's analysis were then shifted into the source term. However, the definition of all irrotational velocity disturbances as 'acoustic' is problematic in combusting flows which feature significant irrotational velocity disturbances due to heat addition. These disturbances can also be expressed as entropy disturbances and are therefore convective rather than acoustic [15].

The earlier approach of Bloxsidge et al. [3] was also based on Morfey's [12] analysis. However, Bloxsidge et al. did not separate the disturbance velocity field into irrotational and solenoidal components, but presented a one dimensional analysis. Once again, entropy disturbances were considered as part of a relatively complex source term.

Myers [13] took a different approach in deriving a disturbance energy conservation equation. He allowed entropy disturbances to remain in both the energy density and flux terms, and did not split the velocity field. Myers' equation was consistent with those developed earlier by Chu [10] and Pierce [16] for zero mean flow. However, because the energy density and flux terms contained entropy disturbances, the resulting energies are not 'acoustic' and are properly called a ‘disturbance energy’ [10,13].

Nicoud and Poinsot [7] used Chu’s [10] fluctuating energy equation, and argued that the Rayleigh criterion is an incomplete description of the significant sources of ‘fluctuating’ energy in combustion. A source term proportional to $T'\omega'$ was found where T' is the static temperature disturbance. This term is analogous but significantly different to the ’Rayleigh term’ in equation 1. Entropy disturbances through the flame were also argued to be a significant source of disturbance energy, but Nicoud and Poinsot’s formulation was conceptually problematic because they assumed zero mean flow quantities. Bloxsidge et al. [3] and Dowling [11] also showed that terms other than the Rayleigh term existed for their differently defined acoustic energy equation, but both argued that these terms were small in practice.

This difference of opinion on such a fundamental and practically important problem needs resolving. This can be achieved by first generalising Nicoud and Poinsot’s results to include mean flow quantities, as this paper does. Later numerical and experimental studies can then determine the magnitudes of all the identified source terms.

A further, fundamental issue in these cited studies concerns the use of linearised equations of motion. This may be acceptable for determining linear stability and for classical acoustics, but it is questionable within flames where temperature, entropy and velocity disturbances in particular can be large. Indeed, nonlinear effects are already known to be significant in the acoustic energy analysis of solid rocket combustion [1,2]. Since Myers [13] presented both exact and linearised disturbance energy equations, comparison of the two will determine the applicability of the linearised equations on a given combustor.

Myers' [13] exact energy equation also justifies the inclusion of entropy disturbances in the energy density and flux terms. Large entropy or pressure disturbances can be accompanied by convectively and sonically travelling disturbances in the pressure and entropy fields respectively [15]. Thus, a useful disturbance energy should incorporate pressure, velocity and entropy disturbances for nonlinear studies. Myers' linearised disturbance energy density and flux terms can then be viewed as formulations that are consistent with those required for nonlinear studies.

This paper supports Nicoud and Poinsot's [7] questioning of the validity of the Rayleigh criterion in common combusting flows. It draws on Myers' [13] exact and linearised energy corollaries, and generalises Nicoud and Poinsot's results by removing a conceptual difficulty of their analysis and also presenting a complete definition of exact and linearised disturbance energy density, flux and source terms for combusting flows. It is then shown that the Rayleigh criterion can only be derived from the equations of combusting fluid motion under assumptions that are inappropriate for turbulent combustion in particular, and most likely gaseous combustion in general.

2 Results and discussion

2.1 Governing equations

Myers [13] commenced his derivation of the exact disturbance energy conservation equation from statements of mass conservation, momentum transport, energy conservation and entropy transport. Transport of $(n-1)$ species is added in the present analysis which, together with mass conservation, guarantees conservation of the n^{th} species. Appendix A derives these equations for a combusting gas with body forces absent,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.\mathbf{m} = 0, \tag{2}$$

$$\sum_{k=1}^{n-1}[\frac{\partial \rho Y_k}{\partial t} + \boldsymbol{\nabla}.(\mathbf{m}Y_k) = \omega_k - \boldsymbol{\nabla}.(\rho \mathbf{V_k} Y_k)], \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\zeta} + \boldsymbol{\nabla} H - T\boldsymbol{\nabla} s = \boldsymbol{\psi} + \boldsymbol{\psi}^*, \tag{4}$$

$$\frac{\partial}{\partial t}(\rho H - p) + \boldsymbol{\nabla}.(\mathbf{m}H) - \mathbf{m}.\boldsymbol{\psi} = TQ, \tag{5}$$

$$\frac{\partial \rho s}{\partial t} + \boldsymbol{\nabla}.(\mathbf{m}s) = Q + Q^*, \tag{6}$$

In equations 2 to 6, ρ (kg/m^3) is the density, $\mathbf{u}$ (m/s) is the velocity vector, $\mathbf{m} = \rho\mathbf{u}$ is the mass flux vector, Y_k, ω_k and $\mathbf{V_k}$ are the mass fraction, reaction rate and diffusion velocity of species k, $\boldsymbol{\xi}$ $(1/s) = \boldsymbol{\nabla}\times\mathbf{u}$ is the vorticity and $\boldsymbol{\zeta} = \boldsymbol{\xi}\times\mathbf{u}$, H (J/kg) is the sensible stagnation enthalpy, T (K) is the temperature, s (J/kgK) is the entropy, $\boldsymbol{\psi}$ is such that $\psi_{ij} = (1/\rho)(\partial\tau_{ij}/\partial x_i)$ where τ_{ij} (Pa) is the ij^{th} component of the viscous stress tensor, $\boldsymbol{\psi}^* = \sum_{k=1}^{n-1} g_k \boldsymbol{\nabla} Y_k$ where g_k is the sensible Gibbs free energy of the k^{th} species, TQ is directly proportional to the combustion heat release ω and also contains thermal and species diffusion terms and the viscous dissipation. All of these terms are defined in Appendix A.

Equations 2 to 6 have steady solution,

$$\begin{aligned}
\boldsymbol{\nabla}.\mathbf{m_0} &= 0, && (7)\\
\sum_{k=1}^{n-1}[\boldsymbol{\nabla}.(\mathbf{m_0}Y_{k0}) &= \omega_{k0} - \boldsymbol{\nabla}.(\rho_0\mathbf{V_{k0}}Y_{k0})], && (8)\\
\boldsymbol{\zeta_0} + \boldsymbol{\nabla}H_0 - T_0\boldsymbol{\nabla}s_0 &= \boldsymbol{\psi_0} + \boldsymbol{\psi_0}^*, && (9)\\
\boldsymbol{\nabla}.(\mathbf{m_0}H_0) - \mathbf{m_0}.\boldsymbol{\psi_0} &= T_0Q_0, && (10)\\
\boldsymbol{\nabla}.(\mathbf{m_0}s_0) &= Q_0 + Q_0^*. && (11)
\end{aligned}$$

Rather than the steady equations 7 to 11, the time average of an unsteady flow can also be chosen as the base flow. The resulting energy corollaries are more complex and more useful for post-processing experimental and numerical data, but do not add significant insight to the problem at hand.

2.1.1 Extending Myers' [13] exact disturbance energy equation to gaseous combustion

Equations 2, 3 and 6 can be multiplied by $(H_0 - T_0s_0 - g_0)$, g_{k0} and T_0 respectively and added to the scalar product of equation 4 and $\mathbf{m_0}$. This sum is subtracted from equation 5 and, after considerable algebra that utilises equations 7 to 11, results in an *exact* conservation equation for a 'disturbance energy' of the form

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}.\mathbf{W} = D. \tag{12}$$

In equation 12, the disturbance energy density E (J/m^3) and $\mathbf{W}$ (W/m^2) flux vector terms are respectively

$$E = \rho(H' - T_0s') - \mathbf{m_0}.\mathbf{u}' - p' - \rho\sum_{k=1}^{n-1} g_{k0}Y_k', \tag{13}$$

and

$$\mathbf{W} = \mathbf{m}'(H' - T_0 s') + \mathbf{m_0} T' s', \tag{14}$$

where disturbances defined as $()' = () - ()_0$ have been used. The exact source term D (W/m^3) is

$$D = D_\xi + D_s + D_Q + D_{Q^*} + D_\psi + D_{\psi^*} + D_{Y_k}, \tag{15}$$

where

$$\begin{aligned}
D_\xi &= -\mathbf{m}'.[\boldsymbol{\xi} \times \mathbf{u} - \boldsymbol{\xi_0} \times \mathbf{u_0}], \\
D_s &= -\mathbf{m}'.s'\boldsymbol{\nabla} T_0 + s'\mathbf{m_0}.\boldsymbol{\nabla} T', \\
D_Q &= T'Q', \\
D_{Q^*} &= T'Q^{*\prime}, \\
D_\psi &= \mathbf{m}'.\boldsymbol{\psi'}, \\
D_{\psi^*} &= \mathbf{m}'.\boldsymbol{\psi^{*\prime}}, \\
D_{Y_k} &= g'\boldsymbol{\nabla}.\mathbf{m}' + \sum_{k=1}^{n-1} g'_k \Omega'_k,
\end{aligned}$$

and $\Omega_k = \omega_k - \boldsymbol{\nabla}.(\rho \mathbf{V_k} Y_k) - \boldsymbol{\nabla}.(\mathbf{m} Y_k)$.

Order of magnitude analysis discussed later in subsection 2.2 suggests that the vorticity term D_ξ should be insignificant in most combusting flows, but it is retained nonetheless. Neglecting viscous stress, viscous dissipation, thermal diffusion and species diffusion terms, several terms in equation 15 can also be simplified to

$$Q = \frac{\omega}{T}, \;\; Q^* = -\sum_{k=1}^{n-1} g_k \omega_k / T, \;\; \boldsymbol{\psi} = 0, \tag{16}$$

with the other terms defined as above.

2.1.2 *The linearised disturbance energy equation for equilibrium chemistry*

Previous studies of acoustic or disturbance energy transport in combustion usually make the assumption of equilibrium chemistry (e.g. [3], [9], [11], [1], [7]). In this case species transport can be ignored, simplifying equation 12 and its linear equivalent. In particular, the mass fraction disturbances Y_k' in equation 13 and the terms D_{Q^*}, D_{ψ^*} and D_{Y_k} in equation 15 are all zero.

Disturbances of the form $()' = () - ()_0$ are substituted into the exact equation 12, and only the lowest order terms are retained. In keeping with the other studies of disturbance energy in isentropic and homentropic flows, e.g. [12,14,16], the remaining terms are all of second order in the disturbances. Retention of all second order terms in the exact flux vector and source terms results in a rather complex disturbance energy equation, where much of the complexity is contained in viscous stress, dissipation and heat conduction terms. Ignoring these terms results in the following linearised disturbance energy equation,

$$\frac{\partial E_2}{\partial t} + \boldsymbol{\nabla}.\mathbf{W_2} = D_2, \tag{17}$$

where the linearised disturbance energy density E_2, flux vector $\mathbf{W_2}$ and source D_2 terms are

$$E_2 = \frac{p'^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u'^2 + \rho' \mathbf{u_0}.\mathbf{u}' + \frac{\rho_0 T_0 s'^2}{2c_{p0}}, \tag{18}$$

$$W_2 = (p' + \rho_0 \mathbf{u_0}.\mathbf{u}')\left(\mathbf{u}' + \frac{\rho'}{\rho_0}\mathbf{u_0}\right) + \mathbf{m_0} T' s', \tag{19}$$

and

$$\begin{aligned} D_2 = {} & \rho_0 \mathbf{u_0}.(\boldsymbol{\xi'} \times \mathbf{u}') + \rho' \mathbf{u}'.(\boldsymbol{\xi_0} \times \mathbf{u_0}) \\ & - s'\mathbf{m}'.(\boldsymbol{\nabla} T_0) + s'\mathbf{m_0}.(\boldsymbol{\nabla} T') \\ & + \left(\frac{\omega' T'}{T_0} - \frac{\omega_0 T'^2}{T_0^2}\right). \end{aligned} \tag{20}$$

It is noted that the equilibrium results $(H' - T_0 s') = p'/\rho_0 + \mathbf{u_0}.\mathbf{u}'$ and $p' = c_0^2 \rho_0 s'/c_{p0} + c_0^2 \rho'$ have been used to derive equations 18 and 19.

As discussed later, the vorticity terms in equation 20 may be small in some cases, permitting further simplification. Equation 17 simplifies to other, existing acoustic energy conservation equations under the condition of homentropic flow. The energy density E_2 and flux $\mathbf{W_2}$ terms then become those defined by Cantrell and Hart [14] for acoustic propagation in a non-stationary medium. In this case only the vorticity source terms remain and represent the acoustic energy generated by unsteady vortical flow if it is assumed that sound features irrotational velocity disturbances [11].

2.1.3 Comparison with Nicoud and Poinsot's [7] disturbance energy equation

The recent work of Nicoud and Poinsot [7] derived a linearised equation for the disturbance energy in a combusting flow, but made the significant simplification that all mean flow quantities were zero. Their equations bear a close resemblance to those of Myers [13], but were derived independently from the

Table 1

Myers' [13], 'Extended Myers' (EM)' and Nicoud and Poinsot's (NP) [7] definitions of the linearised energy density E_2, flux vector $\mathbf{W_2}$ and source term D_2.

Myers	$E_2 = \frac{p'^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u'^2 + \rho' \mathbf{u_0}.\mathbf{u}' + \frac{\rho_0 T_0 s'^2}{2c_{p0}}$
NP	$E_2 = \frac{p'^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u'^2 + \quad 0 \quad + \frac{\rho_0 T_0 s'^2}{2c_p}$
Myers	$\mathbf{W_2} = (p' + \rho_0 \mathbf{u_0}.\mathbf{u}')(\mathbf{u}' + \frac{\rho'}{\rho_0}\mathbf{u_0}) + \mathbf{m_0} T' s' +$ terms in τ_{ij}, ϕ and $\mathbf{q}$
NP	$\mathbf{W_2} = p'u' \quad + \quad 0 \quad + \quad 0$
EM	$D_2 = -s'\mathbf{m}'.(\boldsymbol{\nabla} T_0) + s'\mathbf{m_0}.(\boldsymbol{\nabla} T') + \frac{\omega' T'}{T_0} - \frac{\omega_0 T'^2}{T_0^2} +$ terms in τ_{ij}, ϕ and $\mathbf{q}$
NP	$D_2 = -\frac{p_0}{R c_p} s'\mathbf{u}'.(\boldsymbol{\nabla} s_0) + \quad 0 \quad + \frac{\omega' T'}{T_0} - \quad 0 \quad +$ terms in τ_{ij}, ϕ and $\mathbf{q}$

earlier work of Chu [10]. Table 1 summarises the equations found in both works, with vorticity terms ignored. It can be seen that the difference between the linearised energy density and flux terms arises primarily due to Nicoud and Poinsot's assumption of zero mean flow quantities. Table 1 shows another, less significant difference in the last term in the energy density. Nicoud and Poinsot assumed calorific perfection, hence their use of c_p. Myers did not assume this, and so linearised disturbances in entropy properly include the specific heat of the mean state c_{p0}.

Table 1 also highlights a conceptual problem with Nicoud and Poinsot's [7] formulation. Equation 6 showed that entropy disturbances convect with the flow. Thus, to make the assumption of zero mean flow means that there can be no flux of entropy in or out of the domain when thermal and viscous diffusion are ignored, as Nicoud and Poinsot's flux term in table 1 shows. This then means that a bounded steady state value of the disturbance energy density E_2 can only be achieved by disturbances in heat addition ω' around a zero steady state $\omega_0 = 0$, i.e. some part of the disturbance is *cooling* the flow. This highlights the inadequacy of the assumption of zero mean flow, something discussed in some detail by Chu [10]. By permitting non-zero mean flow quantities, Myers [13] allows flux of entropy disturbances in and out of the domain, and so avoids this problem. Myers' formulation can then accommodate a positive mean heat release rate ω_0 around which steady state disturbances ω' can occur without ever cooling the domain. This is a correct formulation for combustion.

Table 1 summarises the source terms of 'extended Myers' as well as Nicoud and Poinsot [7] for the terms expected to be significant in most combusting flows. The phrase 'extended Myers' is applied to the disturbance energy source term with the terms added to account for combustion ω_0, ω'. Nicoud and Poinsot's source term $-p_0 s' \mathbf{u'}.(\boldsymbol{\nabla} s_0)/(Rc_p)$ becomes $-\rho_0 s' \mathbf{u'}.(\boldsymbol{\nabla} T_0)$ under their assumptions of zero mean flow, calorific perfection and ideal gas behaviour, since then $\boldsymbol{\nabla} s_0 = c_p \boldsymbol{\nabla} T_0 / T_0$. Myers' formulation contains added source terms due to entropy disturbances, $-\rho' s' \mathbf{u_0}.(\boldsymbol{\nabla} T_0)$ (part of the first term) and $s' \mathbf{m_0}.(\boldsymbol{\nabla} T')$, both of which contain non-zero mean flow quantities and which are later argued to have significant magnitude. It is interesting to note that the terms containing entropy disturbances may interfere either constructively or destructively, so care must be taken in understanding their individual and combined effects.

Nicoud and Poinsot's [7] assumption of zero mean flow also means that they could not have obtained the vorticity terms in equation 20. Of course, table 1 also contains source terms involving heat addition. The extended Myers as well as Nicoud and Poinsot formulations both contain a term $\omega' T'/T_0$. The significance of this is discussed later in section 2.3.

Table 1 also shows that the extended Myers source term contains a term $\omega_0 T'^2/T_0^2$. This term is expected to have significant magnitude and is always positive and so stabilising. As with the entropy terms above, destructive interference with the term $\omega' T'/T_0$ could occur. Once again, Nicoud and Poinsot [7] did not find this term because of their assumption of zero mean flow quantities, and the present authors have not seen any other work in the literature that puts forward an analogous term.

2.1.4 The linearised disturbance energy equation for non-equilibrium chemistry

In the exact analysis above, g_0 and g_{k0} multiplied by equations 2 and 3 respectively are additional to Myers [13] original approach, and allow derivation of an exact disturbance energy equation for non-equilibrium flows that is entirely of second order in the disturbances if a steady base flow is considered. Linearisation of the exact equation 12 nonetheless becomes substantially more complex for a non-equilibrium flow with n species since there are now $(n+1)$ independent variables in which to expand the exact equation. The exact source term 15 is already second order in the disturbances, and so can be easily written neglecting the vorticity (later argued to be small), viscous stress, dissipation and heat conduction terms as

$$
\begin{aligned}
D_2 = &\rho_0 \mathbf{u_0}.(\boldsymbol{\xi'} \times \mathbf{u}') + \rho' \mathbf{u}'.(\boldsymbol{\xi_0} \times \mathbf{u_0}) \\
&- s'\mathbf{m}'.(\boldsymbol{\nabla} T_0) + s'\mathbf{m_0}.(\boldsymbol{\nabla} T') \\
&+ \left(\frac{\omega' T'}{T_0} - \frac{\omega_0 T'^2}{T_0^2} \right) \\
&- \frac{T'}{T_0} \sum_{k=1}^{n-1} (g_{k0}\omega_k' + g_k' \omega_{k0}) \\
&+ \mathbf{m}'.[\sum_{k=1}^{n-1} (g_k' \boldsymbol{\nabla} Y_{k0} + g_{k0} \boldsymbol{\nabla} Y_k')] \\
&+ g' \boldsymbol{\nabla}.\mathbf{m}' + \sum_{k=1}^{n-1} g_k' [\omega_k' - \boldsymbol{\nabla}.(\mathbf{m_0} Y_k' + \mathbf{m}' Y_{k0})].
\end{aligned} \tag{21}
$$

Linearisation of the flux vector is also relatively straightforward. Making use of Gibbs equation in terms of the enthalpy of the mixture,

$$
dh = T ds + \frac{1}{\rho} dp + \sum_{k=1}^{n-1} g_k dY_k,
$$

the flux vector is

$$
\begin{aligned}
W_2 = &\mathbf{m}'(H' - T_0 s') + \mathbf{m_0} T' s', \\
= &\left(p' + \rho_0 \sum_{k=1}^{n-1} g_{k0} Y_k' + \rho_0 \mathbf{u_0}.\mathbf{u}' \right) \left(\mathbf{u}' + \frac{\rho'}{\rho_0} \mathbf{u_0} \right) + \mathbf{m_0} T' s',
\end{aligned} \tag{22}
$$

where the term containing Y_k' can be considered the flux of energy associated with mass fraction disturbances.

The approach to the linearised energy density term is in keeping with the approach of Myers [13], where expansion in the sensible entropy and density is used, with a further $(n-1)$ mass fractions Y_k added. Appendix B derives the energy density

$$
\begin{aligned}
E_2 = & \frac{p'^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u'^2 + \rho' \mathbf{u_0}.\mathbf{u}' + \frac{\rho_0 T_0 s'^2}{2c_{p0}} \\
& - \frac{p'}{\rho_0 c_0^2} \sum_{k=1}^{(n-1)} \Big[\frac{\rho_0 R T_0}{W_{k0}} + \frac{p_0}{c_{v0} T_0}(g_{k0} - e_{k0})\Big] Y_k', \\
& + \frac{1}{2\rho_0 c_0^2} \sum_{j=1}^{(n-1)} \sum_{k=1}^{(n-1)} \Big[\frac{\rho_0 R T_0}{W_{j0}} + \frac{p_0}{c_{v0} T_0}(g_{j0} - e_{j0})\Big] \Big[\frac{\rho_0 R T_0}{W_{k0}} + \frac{p_0}{c_{v0} T_0}(g_{k0} - e_{k0})\Big] Y_j' Y_k', \\
& + \sum_{j=1}^{(n-1)} \sum_{k=1}^{(n-1)} \frac{\rho_0}{c_{v0}} \Big(c_{pj0} - c_{vj0} + \frac{R}{W_{j0}} - s_{j0}\Big) \Big[(g_{j0} - e_{j0}) + \frac{R}{Y_{j0} W_{j0}}\Big] Y_{j \neq k}' Y_k', \\
& + \sum_{k=1}^{(n-1)} \frac{\rho_0}{2c_{v0}} \Big(c_{pk0} - c_{vk0} + \frac{R}{W_{k0}} - s_{k0}\Big) \Big[(g_{k0} - e_{k0}) + \frac{R}{Y_{k0} W_{k0}}\Big] Y_k'^2, \\
& + \sum_{k=1}^{(n-1)} \Big[\frac{R T_0}{W_{k0}} + (\gamma_0 - 1)(g_{k0} - e_{k0})\Big] \rho' Y_k', \\
& + \sum_{k=1}^{(n-1)} \frac{\rho_0}{c_{v0}} (g_{k0} - e_{k0}) s' Y_k'.
\end{aligned}
\tag{23}
$$

As required, equation 23 reduces to the disturbance energy density proposed by Myers [13] for a gas containing one species. Nonetheless, the terms involving mass fraction disturbances are very complex and are not clearly positive definite. It is therefore difficult to see how such an expression can provide physical insight, even if they have significant magnitude.

2.2 Order of magnitude analysis for chemical equilibrium

The order of magnitude of each source term in the linearised disturbance energy equation 20 can be estimated using several assumptions and basic reasoning. First, the flow is assumed to be of low Mach number such that $s' \sim c_{p0} ln(T'/T_0 + 1)$ and $\rho'/\rho_0 \sim T'/T_0$. The mean heat release is $\omega_0 \sim \rho_0 u_0 c_{p0} T'/l_0$ from the mean energy equation, where l_0 (m) is a mean flow lengthscale and the temperature disturbance is assumed to scale with the mean temperature change over the flame. It is also reasoned that the fluctuating heat release should be $\omega' \sim \omega_0$ by the hydrodynamic motion of the flame.

Denoting turbulent fluctuations by velocity u_t and length l_t scales and the flame thickness by l_f results in the following scalings for the most significant terms in equation 20;

$$
\begin{aligned}
D_{2\xi} &\sim \frac{\rho_0 u_0^3}{l_0}\frac{u_t}{u_0}\Big(\frac{l_0}{l_t}\frac{u_t}{u_0} + \frac{T'}{T_0}\Big),\\
D_{2s} &\sim \frac{c_{p0}\rho_0 u_0 T_0}{l_0}\ \frac{T'}{T_0}\ ln\Big(\frac{T'}{T_0}+1\Big)\Big[\Big(\frac{u_t}{u_0}+\frac{T'}{T_0}\Big)+\frac{l_0}{l_f}\Big],\\
D_{2Q} &\sim \frac{c_{p0}\rho_0 u_0 T_0}{l_0}\Big(\frac{T'}{T_0}\Big)^2\Big(1+\frac{T'}{T_0}\Big).
\end{aligned}
$$

Several results from these scalings are noteworthy. First, the term l_0/l_f in D_{2s} arises from the $\boldsymbol{\nabla} T'$ term in equation 20 because the disturbance temperature gradient should scale with the flame thickness. If this is the case and the disturbances are favourably phased, this term dominates the other term in D_{2s} which is proportional to $\boldsymbol{\nabla} T_0$. Second, the term T'/T_0 in D_{2Q} arises from the $\omega_0 T'^2/T_0^2$ term in equation 20. Surprisingly, given favourable phasing once again, this new term should dominate the term $\omega' T'/T_0$ found by Nicoud and Poinsot [7] and which was earlier argued to be analogous to the traditional Rayleigh term.

Using estimates for all quantities suggests that D_{2s} should be at most an order of magnitude larger than D_{2Q}. This supports Nicoud and Poinsot's [7] argument that entropy disturbances should significantly affect combustion stability, although in the present case this is due to the term $s'\mathbf{m_0}.(\boldsymbol{\nabla} T')$ in equation 20, which Nicoud and Poinsot did not find. Clearly, such uncertainty can only be resolved by experimental and numerical validation. Further, $D_{2\xi}$ is expected to be several orders of magnitude smaller than D_{2Q}, and so should be ignored in most cases. This clearly distinguishes the fundamental mechanisms involved in combustion stability and combustion generated noise from those in aerodynamically generated sound [17].

The anticipated large amplitude disturbances T'/T_0 and ω'/ω_0 also suggest that the linearised disturbance energy equation (equation 17) may not close on a typical flames. The term D_Q in equation 15 does not have finite order in the disturbances, unlike D_s which is of second order exactly. Whilst this must be investigated further, if it is true, then the utility of linear studies of combustion stability is limited.

Further, whilst order of magnitude analysis of the terms associated with chemical non-equilibrium in equation 21 is more complex, these terms look to be significant. Closure of these equations on various test cases will determine this conclusively.

2.3 Towards a general stability criterion for gaseous combustion

The equations presented earlier featured several significant terms not found in the Rayleigh criterion. First, the 'extended Myers' source term, like that of Nicoud and Poinsot [7], contains a term $\omega' T'/T_0$ (table 1). Nicoud and Poinsot correctly argue that this term is analogous *but not equivalent to* the traditional 'Rayleigh term' $\omega' p'$ in equation 1. For example, a calorifically perfect, inviscid, and non-heat conducting flow can feature linearised temperature disturbances that contain both acoustic and convective (due to entropy disturbances) terms [15]. It is only the first of these terms that couples with the linearised pressure disturbance such that $T' = p'/(c_{p0}\rho_0)$ iff $s' = 0$.

This calls into question whether the 'Rayleigh term' $\omega' p'$ should appear at all in a stability criterion of combusting flows. The temperature disturbances in a turbulent flame are strongly dependent on hydrodynamic motion. In this case the temperature and heat release disturbances should be strongly correlated, causing significant $\omega' T'/T_0$ even though $\omega' p'$ could be small.

Equations 15 and 20 also contained apparently significant source terms containing entropy fluctuations and a previously unreported term that was proportional to the mean heat release. This is further strong evidence against the completeness of a combustion stability criterion that features only the traditional Rayleigh term $\omega' p'$ as a source term. Indeed, this paper formally shows that ignoring viscous and diffusive effects, the Rayleigh term is only exact for linear disturbances in a stationary and homentropic mean flow at chemical equilibrium.

This paper therefore supports Nicoud and Poinsot's [7] argument for a new combustion stability criterion. However, it has also been argued that nonlinearity is expected to be significant in most combusting flows, especially in the heat release terms, meaning that an appropriate equation on which to base this criterion may be equation 12. Application of Gauss' theorem shows that $\partial E/\partial t > 0$ when

$$\iiint_V D \; dV > \iint_A \mathbf{W}.d\mathbf{A}, \qquad (24)$$

where $V(m^3)$ and $A(m^2)$ are the volume and surface area vector of the combustion chamber and the chamber surface outward normal respectively. However,

E is not positive definite, and so such a criterion has no clear physical meaning. In cases where nonlinearity can be ignored, such as perhaps the stability of some laminar flames, this problem can be avoided; terms that are not positive definite in equation 23 can simply be moved to the source term. It remains to be seen whether such an approach is useful in practice.

2.4 Combustion stability and thermoacoustic stability

Because the proposed disturbance energy densities contain entropy and species disturbances, a general stability criterion based on these equations cannot be considered 'thermoacoustic' even in the linear limit. This paper therefore deliberately discusses 'combustion stability' in which instability may feature growth of any or all of the pressure, velocity, entropy or species disturbances. It is conceivable, for example, that this form of combustion instability results in increased turbulence within the combustor but little change in the acoustics. Whether this form of unstable combustion is also thermoacoustically unstable can only be determined by further analysis based on an acceptable definition of acoustic energy for (usually nonlinear) combusting flows. As argued in the Introduction, derivation of such an energy is an outstanding problem.

3 Conclusions

An analysis of the energy transported the disturbances in gaseous combustion has been presented. This built upon the previous work of Myers [J. Fluid Mech. 226 (1991) 383–400], who derived exact and linearised conservation equations for a 'disturbance energy' in a viscous, rotational and heat conducting fluid with varying entropy and specific heats but without combustion. This paper added to Myers' results by including species and chemical reaction, and so derived exact and linearised equations describing the transport of disturbance energy in gaseous combustion. In particular:

(1) Simplification of the general, linearised disturbance energy equation recovered the equation of Nicoud and Poinsot [Combust. Flame 142 (2005) 153–159]. By including a non-zero mean flow, the presented equations resolved conceptual problems of Nicoud and Poinsot's formulation in combusting flows, and provided several additional terms in a complete disturbance energy conservation equation.
(2) An order of magnitude analysis supported Nicoud and Poinsot's claim that source terms containing entropy disturbances are significant, and could even be the main source of disturbance energy in combusting flows. This in turn suggested a new criterion for combustion stability, with the newly identified terms rendering a linear criterion complete.
(3) Terms that are analogous to the traditional 'Rayleigh term' were presented. When linearised, a new term proportional to the mean heat release was found and which does not appear to have been previously identified. The proposed linear combustion stability criterion only became the Rayleigh criterion if all mean flow quantities were zero and the flow was homentropic. These conditions are likely to be poor approximations in turbulent combustion in particular.
(4) When post-processing data, the difference between the exact, simplified and linearised disturbance energy equations on a given flame or combustion chamber allows objective evaluation of the relative importance of all terms. Nonlinearity in particular is expected to be significant in

most cases, and this may limit the utility of linear stability arguments on practical combustors.

4 Achnowledgements

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A Equations of motion

The following equations of motion ignore body forces. The momentum equation can be derived from the more common form,

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial p}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_i},$$

where u_i (m/s) is the component of the velocity vector $\mathbf{u}$ in direction x_i, ρ (kg/m^3) is the density, p (Pa) is the static pressure and τ_{ij} (Pa) is the ij^{th} component of the viscous stress tensor. It is first noted that the convective acceleration can be written as

$$u_i \frac{\partial u_j}{\partial x_i} = \boldsymbol{\nabla}(\frac{1}{2}|u|^2) + \boldsymbol{\xi} \times \mathbf{u},$$

where $\boldsymbol{\xi}$ $(1/s) = \boldsymbol{\nabla} \times \mathbf{u}$ is the vorticity. Gibbs' equation for a reacting mixture of n species can be written as

$$de = Tds - pd(\frac{1}{\rho}) + \sum_{k=1}^{n-1} g_k dY_k,$$

where e (J/kg) is the internal energy, s (J/kgK) is the entropy whilst g_k (J/kg) and Y_k are the Gibbs free energy and mass fraction of the k^{th} species. It is noted that g_k is equal to the chemical potential μ_k of that species [18]. Gibbs' equation and the definition of the enthalpy $h = e + p/\rho$, can be combined to show that

$$\frac{\nabla \boldsymbol{p}}{\rho} = \nabla h - T\nabla s - \sum_{k=1}^{n-1} g_k \nabla Y_k.$$

Combined with the momentum equation and the vector form of the convective acceleration above, this results in a modified form of Crocco's theorem for an unsteady, viscous and combusting gas,

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\zeta} + \nabla H - T\nabla s = \boldsymbol{\psi} + \boldsymbol{\psi}^*, \tag{A.1}$$

where $\boldsymbol{\zeta} = \boldsymbol{\xi} \times \mathbf{u}$, $\boldsymbol{\psi} = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_i}$ and $\boldsymbol{\psi}^* = \sum_{k=1}^{(n-1)} n g_k \nabla Y_k$. It is relatively straight forward algebraically to show that the enthalpy, entropy and Gibb's free energy in this equation can be considered as either sensible or total quantities.

The energy equation can be written [5]

$$\frac{\partial}{\partial t}(\rho H - p) + \nabla.(\mathbf{m}H) - \mathbf{m}.\boldsymbol{\psi} = TQ, \tag{A.2}$$

where $\mathbf{m} = \rho\mathbf{u}$ is the mass flux vector, H (J/kg) is the sensible stagnation enthalpy and T (K) is the static temperature. Q includes the thermal and species diffusion flux vector $\mathbf{q}$ (W/m^2), viscous dissipation ϕ (W/m^3) and combustion heat release ω (W/m^3) terms. Assuming Fourier's law of conduction with thermal conductivity λ (W/mK), Q can be written as

$$Q = \frac{1}{T}(\boldsymbol{\nabla}.\mathbf{q} + \phi + \omega),$$

where

$$\begin{aligned}\boldsymbol{\nabla}.\mathbf{q} &= -\boldsymbol{\nabla}.(\lambda \boldsymbol{\nabla} T) + \rho \frac{\partial}{\partial x_i} \sum_{i=1}^{n} h_k Y_k V_{k,i}, \\ \phi &= \tau_{ij} \frac{\partial u_j}{\partial x_i}, \\ \omega &= -\sum_{i=1}^{n} \Delta h_{f,k}^0 \omega_k.\end{aligned}$$

The entropy transport equation also starts with Gibbs' equation above, now written with the substantial derivative,

$$\rho T \frac{Ds}{Dt} = \rho \frac{De}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt} - \rho \sum_{k=1}^{n-1} g_k \frac{DY_k}{Dt}.$$

Here, the first term on the right hand side is the energy equation in terms of the internal energy. The second term can be shown to be equal to $-p(\boldsymbol{\nabla}.\mathbf{u})$. Application of the species transport equation to the last term then finally yields the sensible entropy transport equation

$$\frac{D(\rho s)}{Dt} = Q + Q^*, \tag{A.3}$$

where

$$Q^* = -\sum_{k=1}^{n-1} g_k\omega_k + \sum_{k=1}^{n-1} g_k \frac{\partial \rho V_{k,i} Y_k}{\partial x_i}$$

and g_k is the sensible Gibbs free energy of the k^{th} species. All the equations required for extending Myers' [13] to gaseous combustion are now in an appropriate form.

B The linearised energy density for non-equilibrium chemistry

Equation 13 states the exact energy density,

$$\begin{aligned} E = & \rho[H' - T_0 s'] - \mathbf{m_0}.\mathbf{u}' - p' - \rho \sum_{k=1}^{n-1} g_{k0} Y_k', \qquad \text{(B.1)} \\ = & \rho e + \frac{1}{2}\rho \mathbf{u}.\mathbf{u} - \rho e_0 - \frac{\rho}{\rho_0} p_0 - \frac{1}{2}\rho \mathbf{u_0}.\mathbf{u_0} - \rho T_0 s' - \mathbf{m_0}.\mathbf{u}' - p_0 \\ & -\rho \sum_{k=1}^{n-1} g_{k0} Y_k', \end{aligned}$$

where e (J/kg) is the sensible internal energy. The term ρe is then expanded as a $(n+1)$ dimensional Taylor series in ρ', s' and $Y_1' \dots Y_{n-1}'$ to second order,

$$\begin{aligned} \rho e = & \rho_0 e_0 + \Big(\frac{\partial \rho e}{\partial \rho}\Big)_0 \rho' + \Big(\frac{\partial \rho e}{\partial s}\Big)_0 s' + \sum_{k=1}^{n-1} \Big(\frac{\partial \rho e}{\partial Y_k}\Big)_0 Y_k' \qquad \text{(B.2)} \\ & + \Big(\frac{\partial^2 \rho e}{\partial \rho^2}\Big)_0 \frac{\rho'^2}{2} + \Big(\frac{\partial^2 \rho e}{\partial s^2}\Big)_0 \frac{s'^2}{2} + \sum_{k=1}^{n-1} \Big(\frac{\partial^2 \rho e}{\partial Y_k^2}\Big)_0 \frac{Y_k'^2}{2} + \Big(\frac{\partial^2 \rho e}{\partial \rho \partial s}\Big)_0 \rho' s' \\ & + \sum_{k=1}^{n-1} \Big[\Big(\frac{\partial^2 \rho e}{\partial \rho \partial Y_k}\Big)_0 \rho' Y_k' + \Big(\frac{\partial^2 \rho e}{\partial s \partial Y_k}\Big)_0 s' Y_k' + \sum_{j=1}^{n-1} \Big(\frac{\partial^2 \rho e}{\partial Y_j \partial Y_k}\Big)_0 Y_{j\neq k}' Y_k' \Big]. \end{aligned}$$

Evaluation of the partial derivatives in B.2 first requires deviation of several exact differentials. Those derivatives not involving Y_k are presented in Myers [13] and are true at chemical equilibrium, whilst derivatives involving Y_k

require use of non-equilibrium differentials given in appendix C,

$$\Big(\frac{\partial \rho e}{\partial \rho}\Big)_0 = h_0, \Big(\frac{\partial \rho e}{\partial s}\Big)_0 = \rho_0 T, \Big(\frac{\partial \rho e}{\partial Y_k}\Big)_0 = \rho_0 g_{k0}, \Big(\frac{\partial^2 \rho e}{\partial \rho^2}\Big)_0 = \frac{c_0^2}{\rho_0}, \tag{B.3}$$
$$\Big(\frac{\partial^2 \rho e}{\partial s^2}\Big)_0 = \frac{\rho_0 T_0}{c_{v0}},$$
$$\Big(\frac{\partial^2 \rho e}{\partial Y_k^2}\Big)_0 = \frac{\rho_0}{c_{v0}}\Big(c_{pk0} - c_{vk0} + \frac{R}{W_{k0}} - s_{k0}\Big)\Big[(g_{k0} - e_{k0}) + \frac{R}{Y_{k0}W_{k0}}\Big],$$
$$\Big(\frac{\partial^2 \rho e}{\partial \rho \partial s}\Big)_0 = \gamma_0 T_0,$$
$$\Big(\frac{\partial^2 \rho e}{\partial \rho \partial Y_k}\Big)_0 = \Big[\frac{RT_0}{W_{k0}} + \gamma_0 g_{k0} - (\gamma_0 - 1)e_{k0}\Big],$$
$$\Big(\frac{\partial^2 \rho e}{\partial s \partial Y_k}\Big)_0 = \frac{\rho_0}{c_{v0}}(g_{k0} - e_{k0}),$$
$$\Big(\frac{\partial^2 \rho e}{\partial Y_j \partial Y_k}\Big)_0 = \frac{\rho_0}{c_{v0}}\Big(c_{pj0} - c_{vj0} + \frac{R}{W_{j0}} - s_{j0}\Big)\Big[(g_{j0} - e_{j0}) + \frac{R}{Y_{j0}W_{j0}}\Big].$$

Substitution of equation B.2 into equation B.1 then yields the linearised energy density which contains only second order terms,

$$E_2 = \frac{p'^2}{2\rho_0 c_0^2} + \frac{1}{2}\rho_0 u'^2 + \rho' \mathbf{u_0}.\mathbf{u}' + \frac{\rho_0 T_0 s'^2}{2c_{p0}} \tag{B.4}$$
$$- \frac{p'}{\rho_0 c_0^2}\sum_{k=1}^{(n-1)}\Big[\frac{\rho_0 R T_0}{W_{k0}} + \frac{p_0}{c_{v0}T_0}(g_{k0} - e_{k0})\Big]Y_k',$$
$$+ \frac{1}{2\rho_0 c_0^2}\sum_{j=1}^{(n-1)}\sum_{k=1}^{(n-1)}\Big[\frac{\rho_0 R T_0}{W_{j0}} + \frac{p_0}{c_{v0}T_0}(g_{j0} - e_{j0})\Big]\Big[\frac{\rho_0 R T_0}{W_{k0}} + \frac{p_0}{c_{v0}T_0}(g_{k0} - e_{k0})\Big]Y_j' Y_k',$$
$$+ \sum_{j=1}^{(n-1)}\sum_{k=1}^{(n-1)}\frac{\rho_0}{c_{v0}}\Big(c_{pj0} - c_{vj0} + \frac{R}{W_{j0}} - s_{j0}\Big)\Big[(g_{j0} - e_{j0}) + \frac{R}{Y_{j0}W_{j0}}\Big]Y_{j\neq k}' Y_k',$$
$$+ \sum_{k=1}^{(n-1)}\frac{\rho_0}{2c_{v0}}\Big(c_{pk0} - c_{vk0} + \frac{R}{W_{k0}} - s_{k0}\Big)\Big[(g_{k0} - e_{k0}) + \frac{R}{Y_{k0}W_{k0}}\Big]Y_k'^2,$$
$$+ \sum_{k=1}^{(n-1)}\Big[\frac{RT_0}{W_{k0}} + (\gamma_0 - 1)(g_{k0} - e_{k0})\Big]\rho' Y_k',$$
$$+ \sum_{k=1}^{(n-1)}\frac{\rho_0}{c_{v0}}(g_{k0} - e_{k0}) s' Y_k'.$$

C Exact differentials for non-equilibrium chemistry

Gibbs equation states an exact differential for the sensible internal energy e in terms of the entropy, density and mass fractions,

$$de = Tds + \frac{p}{\rho^2}d\rho + \sum_{k=1}^{n-1} g_k dY_k.$$

Since

$$e = \int_{T_0}^{T} c_v dT - \frac{RT_0}{W}, \tag{C.1}$$

where R is the ideal gas constant and $W = 1/(\Sigma_{k=1}^{n-1} Y_k/W_k)$ is the molecular mass, de can also be written as

$$de = c_v dT + \sum_{k=1}^{n-1} e_k dY_k. \tag{C.2}$$

Equating equations C.3 and C.2 then gives an exact differential for the temperature

$$dT = \frac{1}{c_v}[Tds + \frac{p}{\rho^2}d\rho + \sum_{i=1}^{n-1}(g_k - e_k)dY_k].$$

Similar working yields exact differentials for the pressure, enthalpy of the mixture and the Gibb's free energy of species k,

$$dp = \frac{p}{c_v} ds + \frac{\gamma p}{\rho} d\rho + \sum_{i=1}^{n-1} [\frac{\rho RT}{W_k} + \frac{\rho R}{W c_v} (g_k - e_k)], \tag{C.3}$$

$$dh = T ds + \frac{\gamma p}{\rho^2} d\rho + \sum_{i=1}^{n-1} [\frac{RT}{W_k} + \gamma g_k - (\gamma - 1) e_k] dY_k,$$

$$dg_k = \Big(c_{pk} - c_{vk} + \frac{R}{W_k} - s_k \Big) \frac{T}{c_v} ds$$

$$+ \frac{RT}{\rho W} \Big[\frac{W}{c_v} (c_{pk} - c_{vk} + \frac{R}{W_k} - s_k) + \frac{W}{W_k} \Big] d\rho \tag{C.4}$$

$$+ \frac{1}{c_v} \Big(c_{pk} - c_{vk} + \frac{R}{W_k} - s_k \Big) \sum_{k=1}^{(n-1)} \Big[(g_k - e_k) + \frac{R}{Y_k W_k} \Big] dY_k. \tag{C.5}$$