MATEMATIKA, 2014, Volume 30, Number 1a, 8–20

Conformal Mapping and Periodic Cubic Spline Interpolation

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Abstract Previous studies have shown computation of conformal mapping in which the exact parameterization of the boundary of the region is assumed known. However there are regions whose boundaries have no known exact parameterization. Periodic cubic spline interpolation had been introduced to approximate and obtain the parameterization. We present a numerical procedure to generate periodic cubic spline from the boundary of a 2-dimensional object by using Mathematica software. First we obtain Cartesian coordinates points from the boundary of this 2-dimensional object. Then we convert them into polar coordinates form. Finally the cubic spline is generated based on this polar coordinate points. Some results of our numerical experiments are presented.

Keywords Conformal Mapping; Integral Equation; Periodic Cubic Spline.

2010 Mathematics Subject Classification 30C30, 30E25, 65E05

1 Introduction

Conformal mapping is a mapping with the property that magnitude and direction of the angle between two curves are preserved. A complex analytic function with nonzero derivatives at any points on a region D is said to be conformal in D. The idea of conformal mapping is to transform of a complicated boundary to a simpler and more manageable boundary. Conformal mapping can be applied to various fields such as image processing, mechanics, electrostatics, heat flow, aerodynamics and others [1, 2]. Integral equation method is an effective method for numerical conformal mapping besides other methods such as expansion method, osculation methods, Cauchy-Riemann equation methods [1, 3, 4].

Numerical conformal mappings via integral equations have been discussed in [3–9], in which the numerical examples involved known exact parameterizations of the boundaries. However, in applications, there are regions whose boundaries have no known exact parameterizations. O'Donnel and Rokhlin [10] have resampled a given boundary into equispaced nodes using periodic splines under tension as the basis for interpolation but no details were given. This paper describes in detail the use of periodic cubic spline for numerical conformal mapping for regions with no known exact parameterizations.

2 Theoretical Framework

2.1 Integral Equation

Kerzman-Stein-Trummer integral equation is a second kind integral equation useful for computing interior conformal mapping [5, 6]. The kernel of the integral equation is the

Kerzman-Stein kernel. Murid et al. [4] have constructed an integral equation with the Kerzman-Stein kernel to compute conformal mapping $\tilde{R}(z)$ from an unbounded simply connected region with smooth counter-clockwise boundary Γ onto the exterior of a disk. The integral equation is given by

$$U(z) - \int_\Gamma A(z,w)U(w)|dw| = 1, \tag{1}$$

where

$$A(z,w) = \begin{cases} \dfrac{1}{2\pi \mathrm{i}}\left[\dfrac{\overline{T(z)}}{\overline{w}-\overline{z}} - \dfrac{T(w)}{w-z}\right], & w,z \in \Gamma, w \neq z \\ 0, & w,z \in \Gamma, w = z \end{cases}, \tag{2}$$

and

$$U(z) = \sqrt{\gamma \tilde{R}'(z)}. \tag{3}$$

The kernel $A(z,w)$ is known as the Kerzman-Stein kernel, $U(z)$ is the solution of the integral equation, and $T(z)$ is the unit tangent to Γ at z. The mapping function $\tilde{R}$ can be expressed in terms of U by [4]

$$\tilde{R}(z) = -\mathrm{i}T(z)\frac{U(z)^2}{|U(z)|^2}, \quad z \in \Gamma. \tag{4}$$

2.2 Cauchy's Integral Formula

Let the interior region of Γ be denoted as D^+ while the exterior region denoted as D^-. If $f(z)$ is analytic in D^-, $f(z)$ can be expressed as [11]

$$f(z) = f(\infty) - \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{f(\tau)}{\tau - z}\, d\tau, \qquad z \in D^-. \tag{5}$$

Since $\tilde{R}(z)$ is the exterior mapping function, then

$$\tilde{R}(\infty) = \lim_{z\to\infty} \tilde{R}(z) = \infty.$$

The Laurant series of $\tilde{R}$ at ∞ has the form of [12]

$$\tilde{R}(z) = \gamma^{-1}z + k_0 + k_1 z^{-1} + \cdots, \gamma > 0, z \in D^-, \tag{6}$$

where k_i is a constant for each $i = 0, 1, 2, \ldots$. We take

$$f(z) = \frac{\tilde{R}(z)}{z}. \tag{7}$$

From (6), we observe that $\tilde{R}'(\infty) = \gamma^{-1}$. Therefore

$$f(\infty) = \gamma^{-1}. \tag{8}$$

Substitute (7) and (8) into (5) gives

$$\tilde{R}(z) = \gamma^{-1}z - \frac{z}{2\pi \mathrm{i}}\int_\Gamma \frac{\tilde{R}(\tau)}{\tau(\tau - z)}\, d\tau. \tag{9}$$

The formula (9) is suitable to compute $\tilde{R}(z)$ for z in the exterior region D^-. If z is located closed to the boundary Γ, the integration is nearly singular. To overcome this problem, we use the fact that $\frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{1}{\tau - z}\,d\tau = 0$ for $z \in D^-$. Thus (9) can be written as

$$\tilde{R}(z) = \frac{\gamma^{-1}z - \frac{z}{2\pi \mathrm{i}}\int_\Gamma \frac{\tilde{R}(\tau)}{\tau(\tau-z)}\,d\tau}{1 - \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{1}{\tau-z}\,d\tau}. \tag{10}$$

Assume the boundary Γ is parameterized by $\tau = z(t)$, $0 \le t \le \beta$. Since the image of $\tilde{R}(z)$ describe the unit circle, then $\tilde{R}(z(t))$ has the form

$$\tilde{R}(z(t)) = e^{\mathrm{i}t}, \qquad 0 \le t \le \beta. \tag{11}$$

Then (10) becomes

$$\tilde{R}(z) = \frac{\gamma^{-1}z - \frac{z}{2\pi \mathrm{i}}\int_0^\beta \frac{e^{\mathrm{i}t}z'(t)}{z(t)(z(t)-z)}\,dt}{1 - \frac{1}{2\pi \mathrm{i}}\int_0^\beta \frac{z'(t)}{z(t)-z}\,dt}. \tag{12}$$

In numerical computation, the denominator in this formula compensates for the error in the numerator [13]. This approach gives better approximation for z near to the boundary Γ.

2.3 Cubic Spline Interpolation

In this section cubic spline interpolation, which is most common piecewise-polynomial approximation, is discussed [14].

Definition 1 Given a function f defined on $[x_0, x_n]$ and a set of nodes $x_0 < x_1 < x_2 < \cdots < x_n$, a cubic spline interpolant S for f satisfies the following conditions:

(i) *$S(x)$ is a cubic polynomial, denoted $S_i(x)$, on the subinterval $[x_i, x_{i+1}]$ for each $i = 0, 1, 2, \ldots, n-1$;*

(ii) *$S_i(x_i) = f(x_i)$ and $S_i(x_{i+1}) = f(x_{i+1})$ for each $i = 0, 1, 2, \ldots, n-1$;*

(iii) *$S_{i+1}(x_{i+1}) = S_i(x_{i+1})$ for each $i = 0, 1, 2, \ldots, n-2$;*

(iv) *$S'_{i+1}(x_{i+1}) = S'_i(x_{i+1})$ for each $i = 0, 1, 2, \ldots, n-2$;*

(v) *$S''_{i+1}(x_{i+1}) = S''_i(x_{i+1})$ for each $i = 0, 1, 2, \ldots, n-2$;*

Definition 1 shows the general condition of cubic spline polynomial. The additional conditions must be imposed in order to generate a unique cubic spline. Given subinterval $[x_i, x_{i+1}]$, where $i = 0, 1, \ldots, n-1$, the general cubic spline polynomial function is given by

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \tag{13}$$

where a_i, b_i, c_i, d_i are constants that need to be determined. In this paper, since Γ is a simple closed curve, periodic cubic spline is applied. Periodic cubic spline satisfies periodic end conditions [15]:

$$f(x_0) = f(x_n), \tag{14}$$

$$f'(x_0) = f'(x_n), \tag{15}$$

$$f''(x_0) = f''(x_n). \tag{16}$$

Since $S(x)$ is periodic, $S(x)$ can be expressed as

$$S(x) = S(x + kp), \qquad k \in \mathbb{Z}, p = b - a. \tag{17}$$

3 Numerical Implementation

3.1 Cubic Spline Interpolation

Parameterization of Γ need to be determined before we can compute conformal mapping. Some coordinate points (x, y) which lie on a simple Jordan curve Γ were assumed known. We convert them into polar coordinates (r, θ). The points are then arranged such that $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$. The task is to find the relationship between r and θ. We assume the relationship can be expressed in the form similar to (13). For every subinterval $[\theta_i, \theta_{i+1}]$, where $i = 0, 1, , n-1$, $r(\theta)$ can be expressed as a piecewise polynomial function which passes through the set of polar coordinates

$$\{(r(\theta_0), \theta_0), (r(\theta_1), \theta_1), \ldots, (r(\theta_n), \theta_n)\},$$

where $r(\theta_n) = r(\theta_0)$ and $\theta_n - \theta_0 = 2\pi$. The relationship between $r(\theta)$ and θ is given by

$$r(\theta) = S_i(\theta) = a_i + b_i(\theta - \theta_i) + c_i(\theta - \theta_i)^2 + d_i(\theta - \theta_i)^3. \tag{18}$$

The task is to determine the values of a_i, b_i, c_i, d_i for the cubic polynomials S_i. From (18), we have

$$S_i(\theta_i) = a_i. \tag{19}$$

From condition (iii) of Definition 2.1 and (19), for each $i = 0, 1, \ldots, n-2$,

$$a_{i+1} = a_i + b_i(\theta_{i+1} - \theta_i) + c_i(\theta_{i+1} - \theta_i)^2 + d_i(\theta_{i+1} - \theta_i)^3. \tag{20}$$

By setting $h_i = \theta_{i+1} - \theta_i$ for each $i = 0, 1, \ldots, n-1$ and $a_n = r(\theta_n)$, (20) becomes

$$a_{i+1} = a_i + b_i h_i + c_i h_i^2 + d_i h_i^3. \tag{21}$$

From (21), differentiate $S_i(\theta)$ with respect of θ gives

$$S_i'(\theta) = b_i + 2c_i(\theta - \theta_i) + 3d_i(\theta - \theta_i)^2. \tag{22}$$

This implies $S_i'(\theta_i) = b_i$. By setting $S_{n-1}'(\theta_n) = b_n$, for each $i = 0, 1, \ldots, n-1$, applying condition (iv) gives

$$b_{i+1} = b_i h_i + c_i h_i^2 + d_i h_i^3. \tag{23}$$

From (22), differentiate $S_i'(\theta)$ with respect of θ give

$$S_i''(\theta) = 2c_i + 6d_i(\theta - \theta_i). \tag{24}$$

This implies $S_i'(\theta_i) = 2c_i$. By setting $S_{n-1}''(\theta_n) = 2c_n$ for each $i = 0, 1, \ldots, n-1$, applying condition (v) gives

$$c_{i+1} = c_i + 3d_i h_i, \tag{25}$$

and express d_i as

$$d_i = \frac{c_{i+1} - c_i}{3h_i}. \tag{26}$$

Substitute (26) into (21) gives

$$a_{i+1} = a_i + b_i h_i + c_i h_i^2 + \frac{c_{i+1} - c_i}{3} h_i^2. \tag{27}$$

Thus b_i is given by

$$b_i = \frac{1}{h_i}(a_{i+1} - a_i) + \frac{h_i}{3}(2c_i + c_{i+1}). \tag{28}$$

Replacing the index i by $i-1$, (28) becomes

$$b_{i-1} = \frac{1}{h_{i-1}}(a_i - a_{i-1}) + \frac{h_{i-1}}{3}(2c_{i-1} + c_i). \tag{29}$$

Substitute (26) to (23) and replace i by $i-1$, we get

$$b_i = b_{i-1} + h_{i-1}(c_{i-1} + c_i), \qquad i = 1, 2, \ldots, n-1. \tag{30}$$

Substitute (29) and (29) into (30) gives, for each $i = 1, 2, \ldots, n-1$, the system of linear equations

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1}). \tag{31}$$

Since $x(p)$ is periodic, we have

$$a_n = a_0, \tag{32}$$

$$a_{n+1} = a_1, \tag{33}$$

$$b_n = b_0, \tag{34}$$

$$b_{n+1} = b_1, \tag{35}$$

$$c_n = c_0, \tag{36}$$

$$c_{n+1} = c_1, \tag{37}$$

$$h_n = h_1. \tag{38}$$

Condition (ii) of Definition 1 implies

$$a_i = S_i(p_i) = x_i. \tag{39}$$

Linear equation (31) with $i = 1$ gives

$$h_0 c_0 + 2(h_0 + h_1)c_1 + h_1 c_2 = \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0). \tag{40}$$

Apply (36) to (40) gives

$$2(h_0 + h_1)c_1 + h_1 c_2 + h_0 c_n = \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0). \tag{41}$$

Linear equation (31) with $i = n$ gives

$$h_{n-1}c_{n-1} + 2(h_{n-1} + h_n)c_n + h_n c_{n+1} = \frac{3}{h_n}(a_{n+1} - a_n) - \frac{3}{h_{n-1}}(a_n - a_{n-1}). \tag{42}$$

Substitute (32), (33),(37) and (38) into (42) gives

$$h_0 c_1 + h_{n-1}c_{n-1} + 2(h_{n-1} + h_0)c_n = \frac{3}{h_0}(a_1 - a_0) - \frac{3}{h_{n-1}}(a_0 - a_{n-1}). \tag{43}$$

From (31) with $i = 1, 3, \ldots, n-1$, (41) and (43), the linear system is constructed as $M\mathbf{c} = \mathbf{g}$, where

$$M = \begin{pmatrix} 2(h_0+h_1) & h_1 & 0 & \cdots & 0 & h_0 \\ h_1 & 2(h_1+h_2) & h_2 & & & 0 \\ 0 & h_2 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & h_{n-2} & 0 \\ 0 & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ h_0 & 0 & \cdots & 0 & h_{n-1} & 2(h_{n-1}+h_1) \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{g} = \begin{pmatrix} \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_3}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_0}(a_1 - a_0) - \frac{3}{h_{n-1}}(a_0 - a_{n-1}) \end{pmatrix}.$$

The matrix M is the $n \times n$ cyclically tridiagonal matrix. The linear system with cyclically tridiagonal matrix can be solved by using the algorithm in [16] or *Mathematica* solver.

Once the solution c_i has been computed, b_i is calculated by using (28) and d_i is calculated by using (26). Relationship of r and θ is obtained in the form of

$$r(\theta) = \begin{cases} S_0(\theta), & \theta_0 \le \theta \le \theta_1 \\ S_1(\theta), & \theta_1 \le \theta \le \theta_2 \\ \vdots & \\ S_{n-1}(\theta), & \theta_{n-1} \le \theta \le \theta_n \end{cases}. \tag{44}$$

The domain of this function is $\theta \in [\theta_0, \theta_n]$. In our numerical computation, we want to work with the domain $\theta \in [0, 2\pi]$. So we rewrite (44) as

$$r(\theta) = \begin{cases} S_{n-1}(\theta + 2\pi), & 0 \le \theta \le \theta_0 \\ S_0(\theta), & \theta_0 \le \theta \le \theta_1 \\ S_1(\theta), & \theta_1 \le \theta \le \theta_2 \\ \vdots & \\ S_{n-1}(\theta), & \theta_{n-1} \le \theta \le 2\pi \end{cases}. \tag{45}$$

3.2 Numerical Conformal Mapping

Suppose we have $z(r,\theta) = r(\theta)e^{\mathrm{i}\theta}$ where $r(\theta)$ is given as in (45). Replacing θ by t, then z can be expressed as

$$z(t) = r(t)e^{\mathrm{i}t}, \tag{46}$$

Conformal mapping of a region exterior to a smooth Jordan curve in the complex plane onto the exterior of the unit disk is computed by using (1). Using the parametric representation $z(t)$ of Γ, (1) becomes

$$\phi(t) - \int_0^{2\pi} K(t,s)\phi(s)\,ds = \psi(t), \tag{47}$$

where

$$\begin{gathered} \phi(t) = |z'(t)|^{1/2}U(z(t)), \\ U(z(t)) = \sqrt{\gamma \tilde{R}'(z(t))}, \\ K(t,s) = |z'(t)|^{1/2}|z'(s)|^{1/2}A(z(t),z(s)), \\ \tilde{\psi}(t) = |z'(t)|^{1/2}. \end{gathered}$$

Since the functions ϕ, K, ψ are $2\pi-$periodic, an appealing numerical method for solving (47) is using the Nystrom's method with trapezoidal rule [17]. Trapezoidal rule will give most accurate values for integration of periodic functions [18]. Taking n equidistant collocation points $t_i = (i-1)2\pi/n$ for $i = 1,2,\ldots,n$ and Nystrom's method with trapezoidal rule to discretize (47), we obtain

$$\phi(t_i) - \frac{2\pi}{n}\sum_{j=1}^{n} K(t_i,t_j)\phi(t_j) = \psi(t_i), \tag{48}$$

Defining the $n \times n$ matrix B where $B_{ij} = 2\pi K(t_i,t_j)/n$ and $x_i = \phi(t_i)$, $y_i = \psi(t_i)$, take $\mathbf{x}$ and $\mathbf{y}$ to be the vectors with elements x_i and y_i respectively, where $i = 1,2,\ldots,n$. Hence (48) can be rewritten as an $n \times n$ system

$$(I - B)\mathbf{x} = \mathbf{y}. \tag{49}$$

Since (47) has a unique solution, then (49) also has a unique solution, as long as n is sufficiently large [17]. Once $x_i = \phi(t_i)$ have been computed, the mapping function $\tilde{R}(z(t))$, the boundary corresponding function $\tilde{\theta}(t)$ and the capacity γ can be computed by using the following formulas [4]:

$$\begin{gathered} \tilde{R}(z(t)) = -\mathrm{i}T(z(t))\tfrac{\phi^2(t)}{|\phi^2(t)|}, \\ \tilde{\theta}(t) = \mathrm{Arg}(-\mathrm{i}\phi^2(t)z'(t)), \\ \gamma = \textstyle\int_0^{2\pi} |\phi(t)|^2 dt. \end{gathered}$$

To compute the interior values of the conformal mapping function in the region D^- we use (12). Since the integrand is $2\pi-$periodic, trapezoidal rule will give most accurate solution. Taking n equidistant collocation points $t_i = (i-1)2\pi/n$ for $i = 1,2,\ldots,n$ and Nystrom's method with trapezoidal rule to discretize (12), we obtain

$$RMC(z) = \frac{\gamma^{-1}z - \frac{z}{n\mathrm{i}}\sum_{j=1}^{n}\frac{e^{\mathrm{i}t_j}z'(t_j)}{z(t_j)(z(t_j)-z)}}{1 - \frac{1}{n\mathrm{i}}\sum_{j=1}^{n}\frac{z'(t_j)}{z(t_j)-z}}, \tag{50}$$

where $z \in D^-$ and $RMC(z)$ is an approximation of $\tilde{R}(z)$.

4 Numerical Examples

In this section, we consider two test regions.

4.1 Exterior of Oval of Cassini

The region is represented by $|z-1||z+1| > \alpha^2$, $\alpha > 1$, and the complex parametric representation of an oval of Cassini is given by

$$\Gamma : z(t) = \sqrt{\cos(2t) + \sqrt{\alpha^4 - \sin^2(2t)}}e^{\mathrm{i}t}, 0 \leq t \leq 2\pi. \tag{51}$$

The exact mapping function is given by [19]

$$\tilde{R}(z) = \frac{\sqrt{z^2-1}}{\alpha}. \tag{52}$$

The boundary points of oval of Cassini for $n = 128$, $\alpha = 1.11$ are shown in Figure 1. Suppose the points of the exterior region are denoted as (r_k, θ_k) where $k = 1, 2, \ldots, m$. Since the parameterization of boundary can be expressed in the form given by (45), the points in exterior region can be determined using the inequality $r_k > r(\theta_k)$. Figure 2 shows exterior region of oval of Cassini. Figure 3 shows the transformed image from Figure 2.

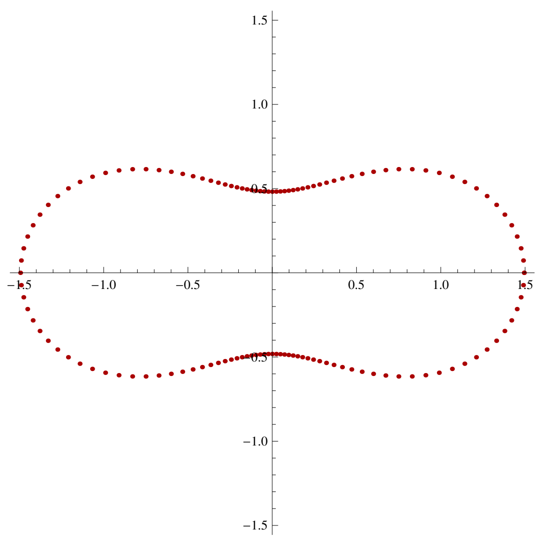

Figure 1: Boundary Points of Oval of Cassini

We list the sup-norm error $\|\tilde{\theta}(t) - \tilde{\theta}_{n_1}(t)\|_\infty$ and $\|\tilde{\theta}(t) - \tilde{\theta}_{n_2}(t)\|_\infty$, where $\tilde{\theta}(t)$ is the exact boundary correspondence function, and $\tilde{\theta}_{n_1}(t)$ is the approximation obtained at the collocation points by using the exact parameterization and $\tilde{\theta}_{n_2}(t)$ is the approximation obtained at the collocation points by using the approximated parameterization (periodic cubic spline). Table 1 shows the numerical results for the errors.

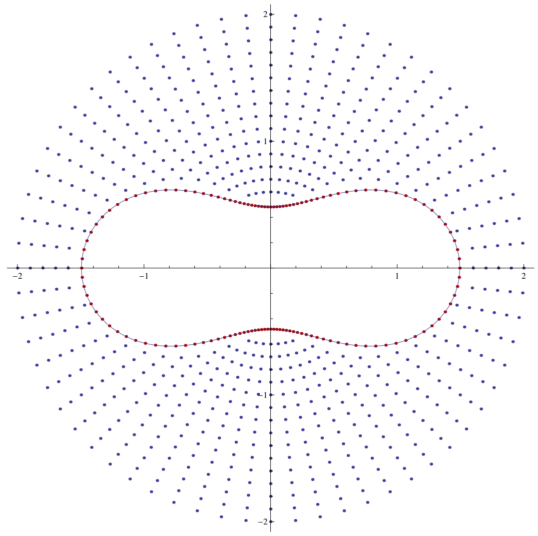

Figure 2: Exterior Region of Oval of Cassini

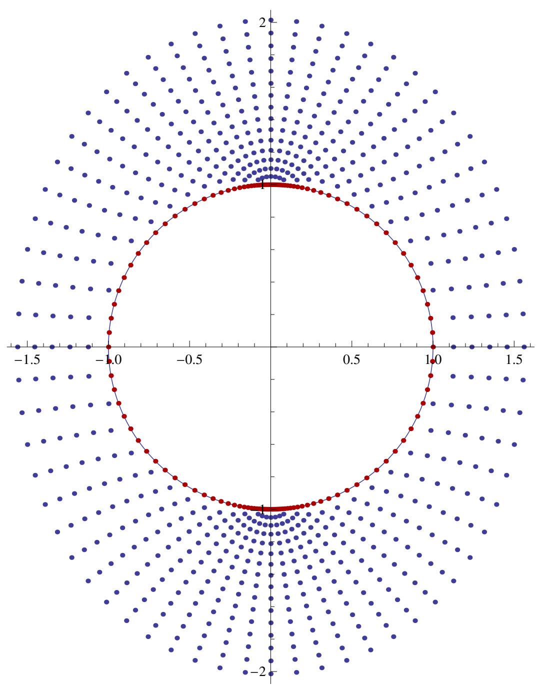

Figure 3: The Transformed Image From Figure 2

Table 1: The error norm for $\alpha = 1.11$

n	$\|\tilde{\theta}(t) - \tilde{\theta}_{n_1}(t)\|_\infty$	$\|\tilde{\theta}(t) - \tilde{\theta}_{n_2}(t)\|_\infty$
32	5.97(-6)	2.4(-4)
64	9.8(-11)	7.4(-6)
128	8.9(-16)	2.7(-7)
256	-	9.3(-9)

4.2 Exterior of a Boat

We inserted the boat image into *Mathematica* and obtain the boundary points of the boat by using the drawing tools of *Mathematica*. Figure 4 shows the boat image and cubic spline that interpolates all points on the boundary. The way of selected points for exterior region is similar as in Section 4.1. Figure 5 shows the exterior region of a boat. Figure 6 shows the transformed image from Figure 5.

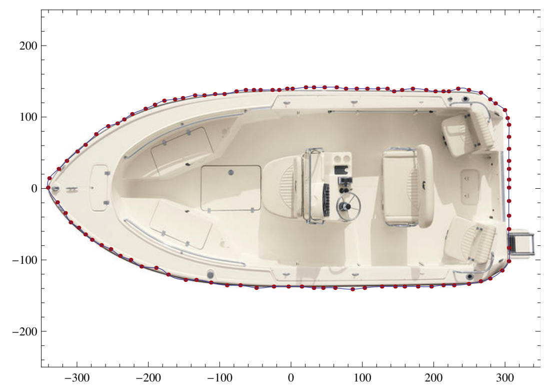

Figure 4: Boat Image

We compute the fluid flow around the boat by using conformal mapping. Take $w = \tilde{R}(z(t))$ as the conformal mapping function that maps the exterior of the boat onto the exterior of the unit disk in the $w-$plane. Let $W = g(w) = w + 1/w$ be the mapping function that maps the exterior region of the unit disk in the $w-$plane to the whole $W-$plane. We assume the flow is horizontal in the $W-$plane, i.e. $\mathrm{Im}(W) = k$, where k is any given constant. The fluid flow on the $w-$plane can be represented by

$$\mathrm{Im}(g(w)) = k. \tag{53}$$

Therefore the flow around the boat has the representation

$$\mathrm{Im}(g(\tilde{R}(z(t)))) = k. \tag{54}$$

Figure 7 shows the computed fluid flow around the boat.

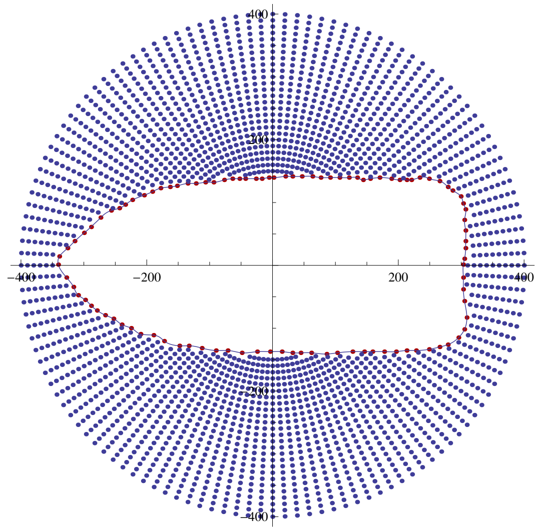

Figure 5: Exterior Region of a Boat

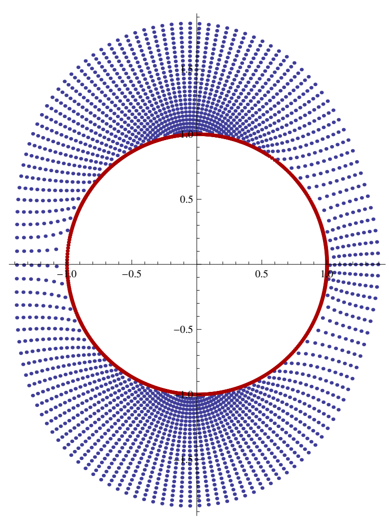

Figure 6: The Transformed Image From Figure 5

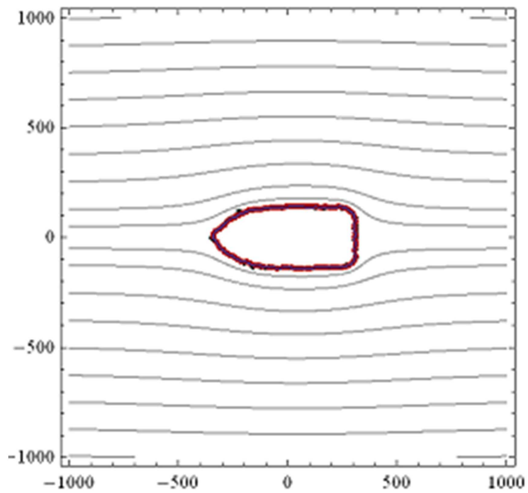

Figure 7: Fluid Flows Around the Boat

5 Conclusion

This study has presented a method for numerical conformal mapping of unbounded simply connected region by means of integral equation and approximating the boundary of the region using periodic cubic spline. The presented method is useful for numerical conformal mapping for regions with no known exact parameterizations. The numerical results have shown that accuracy increases as more nodes are chosen on the boundary.

Acknowledgemnts

This work was supported in part by the Malaysian Ministry of Education (MOE) through the Research Management Centre (RMC), Universiti Teknologi Malaysia (GUPQ.J130000.2526.04H62). This support is gratefully acknowledged.

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