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Buckling Analysis of Rectangular Plates with Variable Thickness Resting on Elastic Foundation

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Abstract. Buckling of rectangular plates of variable thickness resting in elastic foundation is analysed using a quintic spline approximation technique. The thickness of the plate varies in the direction of one edge and the variations are assumed to be linear, exponential and sinusoidal. The plate is subjected to in plane load of two opposite edges. The buckling load and the mode shapes of buckling are computed from the eigenvalue problem that arises. Detailed parametric studies are made with different boundary conditions and the results are presented through the diagram and discussed.

1. Introduction

The buckling characteristics of non-uniform plates are utilized in many areas including aerospace engineering, engineering design and earthquake resistant structures. Significant studies have been made by '[1-3]' on buckling of plates of variable thickness. A spline function approximation was also used for such a study by '[4-7]' .The plate was homogenous with sinusoidal variation in thickness in the direction parallel to one edge. In the study of '[8]' a circular plate was considered with the plate thickness varying along the radius and the method of solution used a Frobenius series approximation. '[9]' studied on buckling loads of braced beam resting on elastic foundation using analytical method. However, not much information is available on buckling of plates of variable thickness resting on elastic foundation.

In the present work the buckling of thin rectangular plate of varying thickness, resting on elastic foundation is studied using spline function approximation technique. The plate is fully attached to the foundation. The plate thickness varies in the direction of an edge. Linear, exponential and sinusoidal variations in thickness are all considered in one formulation.

The solution of the differential equation characterising the deflection is assumed in a separable form. A closed form solution is generally not possible. A spline function numerical technique is used, exploiting the attractive features as excellent interpolating and approximation functions. '[10]' first presented it as an efficient tool to solve two-point boundary value problems, bringing out its computational superiority to other schemes like Hermite interpolation technique.

A pair of opposite edges of the rectangular plate are subjected to compressive uniform load. Two cases of boundary conditions are considered for these edges: Clamped-Clamped and Clamped- Simply supported. The other edges are simply supported. The deflection equation yields an eigenvalue problem solving which the critical loads and the modes shapes of buckling are obtained.

Parametric studies of the variation of the critical load with respect to aspect ratio, foundation coefficient and variation of thickness of the plate are made. Selected mode shapes of buckling are also presented.

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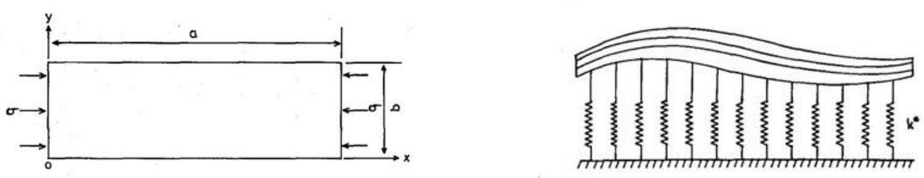
2. Formulation of the Problem

Consider a thin rectangular plate bounded by $x=0$, $x=a$, $y=0$ and $y=b$ as shown in 'figure 1'. The plate is subjected to an in-plane compressive force N_x acting on and normal to the edge $x=0$ and $x=a$. Its transverse deflection $w(x,y)$ is governed by the differential equation

$$\nabla^2(D\nabla^2 w)+(\upsilon-1)\Diamond^4(w,D)=q \tag{1}$$

where ∇^2 is Laplace's operator, υ is poisson ratio, $D=D(x,y)=Eh^3(x,y)/12(1-\upsilon)^2$ is the variable flexural rigidity, $h=h(x,y)$ is the variable thickness, E is Young's modulus, q is the distributed normal load and

$$\Diamond^4(D,w)=\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \tag{2}$$



a) Distributed Normal Load $=q_x$ b) Foundation Modulus $=k^*$

Figure 1 Rectangular plate of variable thickness on elastic foundation a) Geometry and load b) Description of elasticfoundation

The plate is fully attached to the elastic foundation of elastic coefficient k^*. The intensity of reaction is proportional to k^*w. Then

$$q_x=-N_x\frac{\partial^2 w}{\partial x^2}-K^*w-\rho h\frac{\partial^2 w}{\partial t^2} \tag{3}$$

The vibration of the plate is not considered and hence the last term of 'equation (3)' will be absent. Let the edge $y=0$ and $y=b$ of the plate be simply supported, then the transverse deflection can be assumed in the separable form

$$w(x,y)=\overline{W}(x)\sin\frac{n\pi y}{b} \tag{4}$$

'Equation (2-4)' are used in 'equation (1)'. The thickness of the plate is assumed as a function of x only, given by

$$h(x)=h_o\left[1+C_\ell\frac{x}{a}+C_e e^{x/a}+C_s\sin\frac{n\pi}{a}\right] \tag{5}$$

If C_ℓ, C_e and C_s are all zero, then the plate thickness becomes constant. If C_e and C_s are zero, then the thickness varies linearly. If C_ℓ and C_s vanish then the thickness variation is exponential. If C_ℓ and C_e are zero then the thickness variation is sinusoidal. We impose the conditions $C_\ell>-1$, $C_e<-1/e$ and $C_s>-1$, which ensure that the thickness is always positive.

Assume also

$$N_x=\sigma h_o,\quad X=\frac{x}{a},\quad Y=\frac{y}{b},\quad W=W(x)=\frac{\overline{W}}{a},\ H=\frac{h_o}{a},\quad \phi=\frac{q}{b},\quad f=n\phi\pi \tag{6}$$

A buckling coefficient K and a foundation coefficient L are defined by

$$K = \frac{12(1-\upsilon^2)\sigma}{\pi^2 EH^2\phi^2}, \qquad L = \frac{k^* a}{EH^3} \tag{7}$$

For the least critical value of K, n is least '[11]' and hence we set $n = 1$. The differential 'equation (1)' becomes

$$W^{iv}(x) + A_0 W''' + A_1 W'' + A_2 W' + A_3 W = 0 \tag{8}$$

where

$$A_0 = \frac{6(C_\ell + C_e e^X + C_s \pi q)}{(1 + C_\ell X + C_e e^X + C_s p)}$$

$$A_1 = \frac{6(C_\ell + C_e e^X + C_s \pi q)^2}{(1 + C_\ell X + C_e e^X + C_s p)^2} + \frac{3(C_e e^X - C_s \pi^2 p)}{(1 + C_\ell X + C_e e^X + C_s p)} - 2f^2 + \frac{Kf^2}{(1 + C_\ell X + C_e e^X + C_s p)^3}$$

$$A_2 = \frac{6f^2(C_\ell + C_e e^X + C_s \pi q)}{(1 + C_\ell X + C_e e^X + C_s p)}$$

$$A_3 = f^4 - 3\upsilon f^2 \left\{ \frac{2(C_\ell + C_e e^X + C_s \pi q)^2}{(1 + C_\ell X + C_e e^X + C_s p)} + \frac{3(C_e e^X - C_s \pi^2 p)}{(1 + C_\ell X + C_e e^X + C_s p)} \right\} + \frac{12(1-\upsilon^2)L}{(1 + C_\ell X + C_e e^X + C_s p)}$$

(9)

$$p = \sin \pi X \quad \text{and} \quad q = \cos \pi X \tag{10}$$

The 'equation (8)' is a linear differential equation with variable coefficients. Boundary conditions are imposed on the edges, $X = 0$ and $X = 1$. Thus the 'equation (8)' constitutes as well-defined boundary value problem for $0 \le X \le 1$.

3 Method of Solution

Since a closed form solution for the differential 'equation (8)' does not exist, we hence to resort to numerical methods for solving it. The collocation procedure, using a quintic spline approximation for $W(X)$ is used because the 'equation (8)' is of order four.

Therefore, the approximating spline function is assume to be

$$W^*(X) = \sum_{i=0}^{4} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^5 H(X - X_j) \tag{11}$$

In which $H(X - X_j)$ is the Heaviside function and a_i, b_j are the unknown coefficients.

This function has very attractive characteristics compared to many others from the point of view of convergence, accuracy and elegance of usage on computer and efficiency for computational work.

Let us assume that the subintervals are all equal in length. The knots are at

$$X = X_s = \frac{S}{N}, \qquad s = 0, 1, 2, \ldots, N.$$

Imposing the condition that the 'equation (11)' satisfies 'equation (8)' at each $X = X_s$, we have the following system of $N+1$ equations in the $N+5$ spline coefficients : a_0, a_1, a_2, a_3, a_4, b_0, b_1, …, b_{N-1}.

4 Boundary conditions

The boundary conditions of the plate considered are as follows:

i. (C – C) : Clamped at both the edges $X = 0$ and $X = 1$
ii. (C – S) : Clamped at $X = 0$ and Simply supported at $X = 1$

The pairs of boundary conditions that should be satisfied at a clamped edge, a simply supported edge and a free edge are respectively,

$$w = \frac{\partial w}{\partial x} = 0 \text{ (Clamped)}, \quad w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ (simply supported)} \tag{12}$$

Each of these cases gives four more equations on the spline coefficients. Combining these equations with 'equations (8)' we have a generalized eigenvalue problem in the form of

$$[M]\{c\} = K[P]\{c\} \tag{13}$$

Where $[M]$ and $[P]$ are the square matrices of order $(N+5)$ and $\{c\}$ is a column vector whose elements are the spline coefficients. The eigenparameter is the buckling coefficient K, whose least value only is of interest since buckling occurs for the corresponding minimum value of σ, the critical load. After obtaining an initial approximation to the least value of K, its finer approximation is obtained using Regular-Falsi method. The eigenvector $\{c\}$ corresponding to this least K determines the spline function $W^*(X)$ from which the mode shapes of buckling of the plate can be constructed.

5 Results and Discussion

'Table 1' shows the convergence study on K with respect to the number of knot $N = 6(2)18$. Consistent improvement was observed in the value of K with the increase in the number of knots. It has been found that it sufficient to keep $N = 12$ since further increase in N improves K by less than 0.16%.

Table 1 Convergence study

N	$C_e = -0.2, L = 1000, \phi = 0.5$ C-S		$C_\ell = -0.5, L = 200, \phi = 0.5$ C-C	
	K	% change	K	% change
6	37.70713	-	29.26319	-
8	37.83680	0.3439	29.33973	0.2410
10	37.90233	0.1732	29.38955	0.1698
12	37.96188	0.1571	29.43378	0.1505
14	38.01875	0.1498	29.47737	0.1481
16	38.06821	0.1301	29.51858	0.1398
18	38.12082	0.1382	29.55642	0.1282

The $K-\phi$ relations obtained for $L = 0$ (not attached foundation) and $C_\ell = C_e = 0$ (sinusoidal thickness variation) are seen to completely agree with the results of '[12]'. The $K-\phi$ relations obtained for $L = 0$ and $C_\ell = C_e = C_s = 0$ (constant thickness) are matching perfectly with those of '[13]'. The other results are new.

The following parametric values are considered to analyse the problem.

$$\begin{aligned} &\upsilon = 0.3,\ \phi = 0.5(0.25)2.5, C_\ell = -0.5(0.25)0.5, C_e = -0.2(0.1)0.2, \\ &C_s = -0.5(0.25)0.5, L = 0(200)1000 \end{aligned} \tag{16}$$

The nature of variation of the least value of the buckling coefficient K with respect to each of the five variable parametric values $(\phi, C_\ell, C_e, C_s$ and $L)$, with the other four parametric values being fixed, is studied and depicted in the 'figures (2-4)'. Some typical mode shapes of buckling of the plate are presented in 'figure 5'.

The rate of change of K with ϕ may be discontinuous in some cases when $L = 0$ ('[2]'and '[12]'). It is now found to be so in some cases even when $L \neq 0$.

'Figures (2-4)' depict the nature of variations of the value of the buckling coefficient K with respect to the increase of the aspect ratio ϕ for plates whose thickness varies linearly $(C_e = C_s = 0)$, exponentially $(C_s = C_\ell = 0)$ and sinusoidally $(C_\ell = C_e = 0)$ for different values of the thickness variation parameter C_ℓ, C_e and C_s respectively, for two fixed values of the foundation coefficient, $L = 200$ and $L = 1000$. They show clearly that the value of K in general reduces with the increasing ϕ for fixed values of the other parameters for all the type of boundary conditions considered. Again the K - values are found to be higher for higher values of the thickness variation parameters C_ℓ , C_e and C_s . Further K is found to be considerably higher for the higher value of the foundation coefficient $(L = 1000)$ than for its lower value $(L = 200)$. These can be clearly attributed to the facts that the stiffness of the plate decreases with increasing ϕ and increases with C_ℓ , C_e , C_s and L . It is also seen from the same figures that the K values are the least, in general, for the C-S boundary conditions and a higher for the C-C boundary conditions.

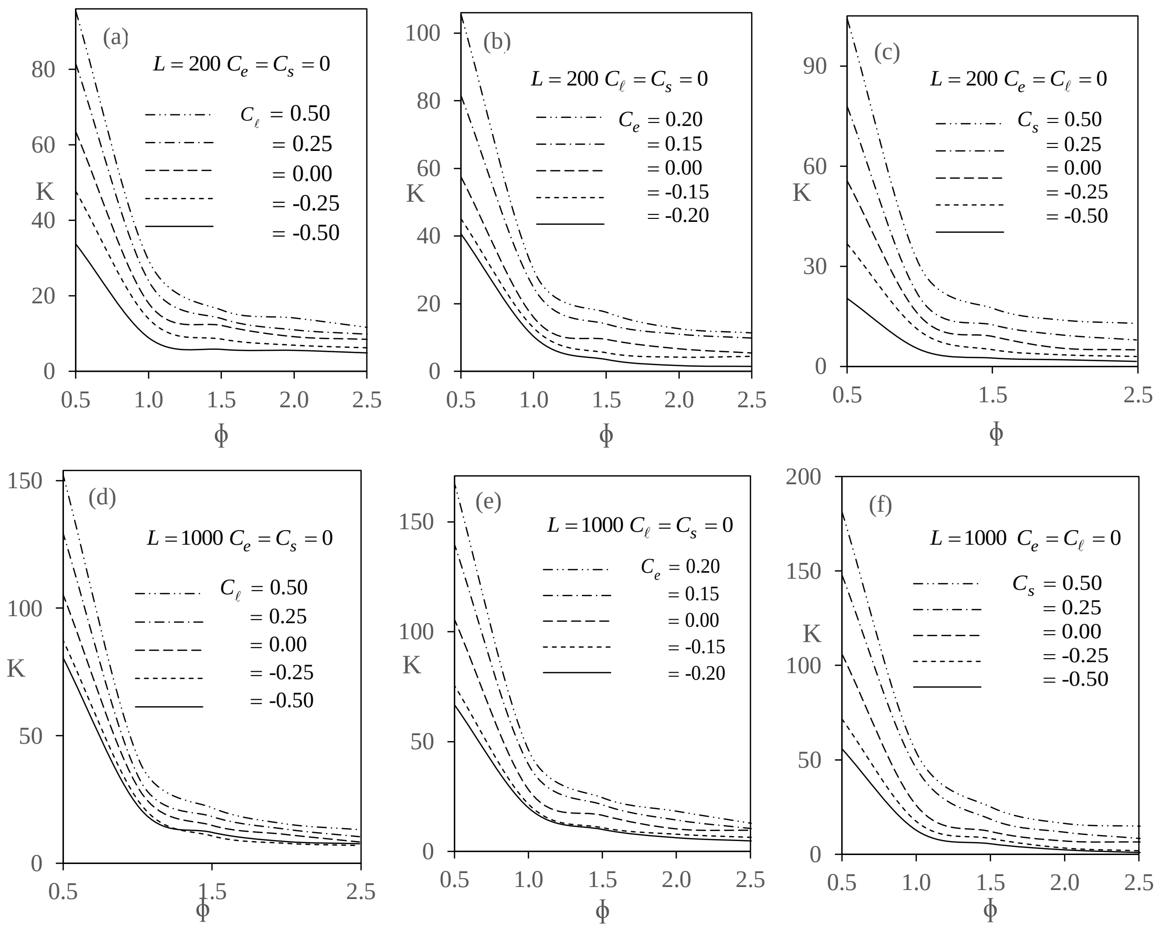


Figure 2 $K-\phi$ Diagrams for C-C plates

'Figure 4' presents the nature of change of K with respect to L for two cases of aspect ratio, $\phi = 1.5$ and $\phi = 2$ and the C-S boundary condition. As before, the linear $(C_e = C_s = 0)$, exponential $(C_s = C_\ell = 0)$ and sinusoidal $(C_\ell = C_e = 0)$ variations in thickness are considered. With increasing L , the values of K are seen, in all cases, to increase, almost steadily for long ranges of values of L . The other characteristics of variation of K agree with those studied from 'figures (2-4)'.

Qualitatively similar characteristics are observed in the variation of K with L for the considered other boundary conditions also. Some mode shapes of buckling are presented in 'figure 5'. They can be viewed as the buckled shapes of the central line $y = b/2$ of the rectangular plate considered. The two rows of diagrams correspond to the two types of boundary conditions: C-C and C-S. The two columns corresponds to the three types of thickness variation: linear (three cases), exponential (two cases) and sinusoidal (two cases). The case 3 of diagrams in (a), (d) and (g) corresponds to constant thickness $(C_\ell = C_e = C_s = 0)$ and therefore is not shown in the other diagram to avoid repetition.

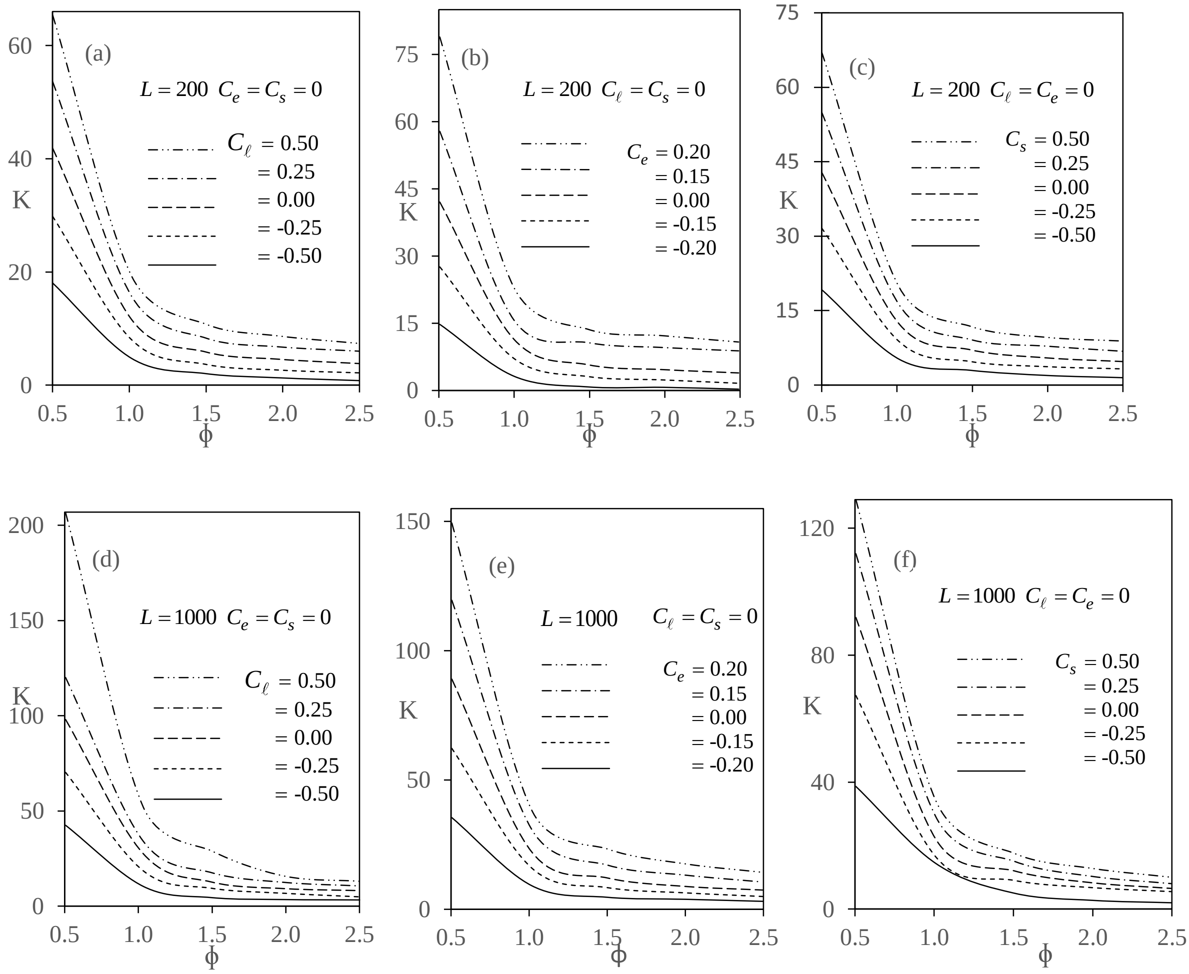


Figure 3 $K-\phi$ Diagrams for C-S plates

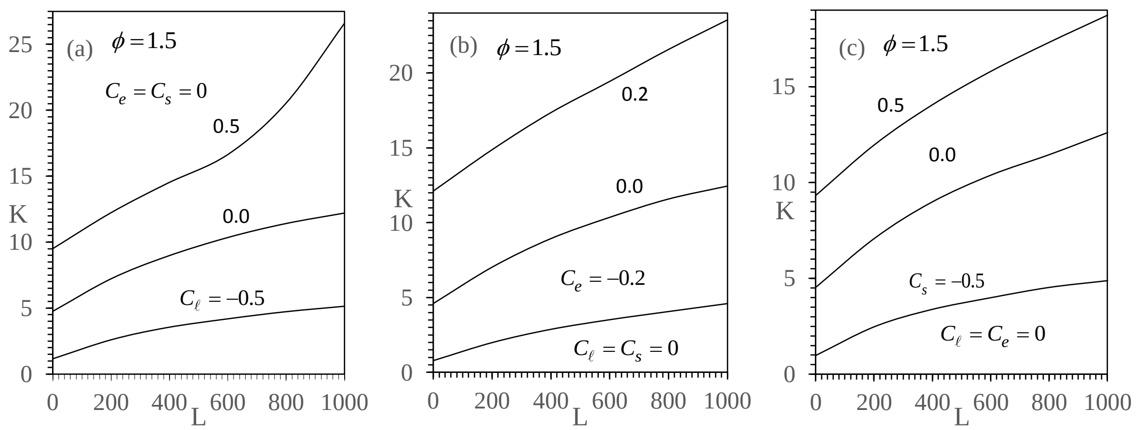


Figure 4 $K-L$ Diagrams for C-S plates

Two types of foundations are considered. The plate is not a square with $\phi = 0.5$. The deflections have been normalized with respect to the numerically largest deflections. It can be observed that the two mode shapes marked 3 in 'figure 5 (a)' and all mode shapes in 'figure 5 (c)' are either symmetric or antisymmetric since they corresponds to symmetric geometric and boundary conditions (plates of constant thickness and sinusoidally varying thickness, under C-C conditions). The mode shapes in general, are interesting and informative. The buckling mode numbers for corresponding buckling loads, for example, can be easily inferred from the mode shapes.

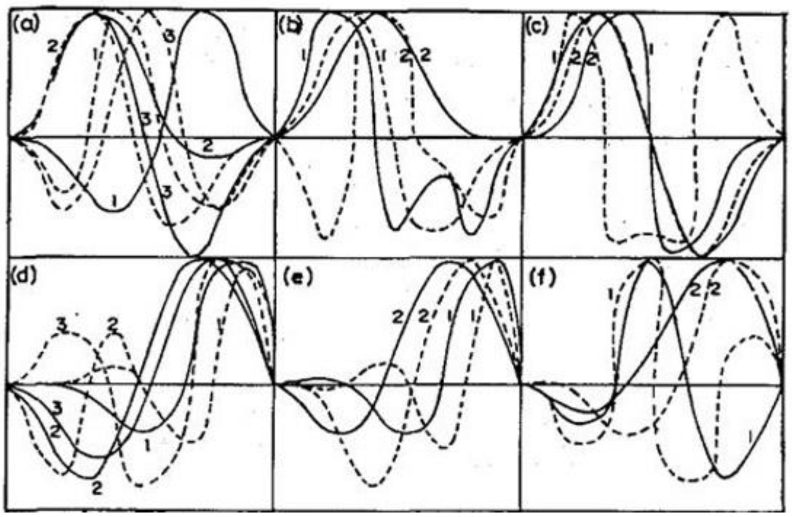


Figure 5 Mode shapes of Buckling.

Boundary conditions: (a), (b), (c): C-C, (d), (e), (f): C-S.

Thickness Variation : (a), (d): $C_e = C_s = 0$; $1: C_\ell = 0.5$; $2: C_\ell = -0.5$; $3: C_\ell = 0.0$

(b), (e): $C_s = C_\ell = 0$; $1: C_e = -0.2$; $2: C_e = 0.2$.

(c), (f): $C_\ell = C_e = 0$; $1: C_s = 0.5$; 2: $C_s = 0.5$.

Foundation Modulus: ——: $L = 200$; - - - - : $L = 1000$.

6 Conclusion

Buckling of thin plates of linearly, exponentially and sinusoidally varying thickness and subject to a constant in plane load has been studied. The plate is resting on an elastic foundation and is subject to two types of boundary conditions on a pair of opposite edges with the other edges simply are supported. An elegant spline function approximation for the unknown transverse deflection is used and the buckling loads and mode shapes of buckling are computed. Extensive parametric studies are made.

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