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Approximations in Credit Risk Models

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Abstract

In this thesis we present the intensity-based approach to consider default in a general local-stochastic volatility model with stochastic interest rate. In this setting we describe, as in [14], a technique to find approximate solutions of the corresponding partial differential equations and we provide numerical examples in the particular case of JDCEV and Vasicek model, respectively, for the dynamics of the asset and the short rate. Finally, we introduce a formula for the par CDS spreads and applying the approximation method we calibrate our intensity model to credit data finding the model parameters matching the default probabilities implicit in CDS prices (by bootstrapping) to the default probabilities implied by the model itself.

Sommario

In questa tesi presentiamo l'approccio per modellizzare la probabilità di default basato sull'intensità di un processo di Poisson in un modello a volatilità locale e stocastica per la dinamica di un singolo asset con tasso di volatilità stocastico. In questa configurazione, come in [14], descriviamo una tecnica per trovare soluzioni approssimate delle corrispettive equazioni alle derivate parziali fornendo esempi numerici nel caso particolare dei modelli JDCEV e Vasicek, rispettivamente, per la dinamica del sottostante e del tasso d'interesse. Infine, introduciamo una formula per i CDS spreads e applichiamo il metodo di approssimazione per calibrare il modello scelto ai dati di mercato, trovando così i parametri che fanno coincidere le probabilità di default implicite nei CDS spreads (tramite bootstrapping) con le probabilità di default implicate dal modello stesso.

Introduction

In this work we address the problem of finding explicit formulas for efficient and reliable analytical approximation for the price of European-style options, in the context of local stochastic volatility models where we consider the possibility of default for the underlying asset. We model the default under the intensity-based approach as in [11], then we choose JDCEV model of Carr and Linetsky [6] for the underlying asset and Vasicik model for the stochastic interest rate.

The aim of this work comes from financial mathematics, where the options pricing problem is reduced to the calculation of an expected value or equivalently, of a solution of a partial differential equation. The speed of computation of prices and calibration procedures is a very strong operational constraint and we attempt to provide real-time tools (or at least more competitive than Monte Carlo simulations, in the case of multi-dimensional diffusion) to meet these needs.

We follow the method presented by Lorig, Pagliarini and Pascucci in [13] and in [14] based on an expansion of the differential operator associated with the local stochastic volatility dynamics, finding explicit expressions for the approximate prices that do not require any special or computationally expensive functions. First, we apply this method in our setting and we compute the price of a defaultable Zero Coupon Bond, then, after calibrating the Vasicek model to its affine term structure, we use model from Brigo in [3] and in [5] to get a formula for the par CDS spreads. Finally, thanks to the expansion method, we calibrate the model to the market CDS spreads finding the parameters matching the implicit default probabilities (by bootstrapping) and the default probabilities implied by the model.

In Chapter 1 we introduce general results related to the analysis of random times and their filtrations in the intensity-based approach to consider default in market models. We see how to deal with the associated enlargement of filtration and, giving the definition of survival probability, we see how default intensity has the same structure as discount factors.

In Chapter 2 we present the general local stochastic volatility model for a defaultable asset with stochastic interest rate and we find the classical drift condition for the log-price process. Then, we compute the price for a generic defaultable European-style option and we show that its value can be seen, thanks to Feynman-Kač Theorem, as a solution of a partial differential equation. Hence, we focus on the work of Lorig, Pagliarini and Pascucci describing the expansion method and providing the main idea of the proof in the 2-dimensional case.

In Chapter 3 we initially study the property of the JDCEV and Visicek models independently. First of all, we implement codes that allows to verify the consistency of the approximation method computing the survival probabilities in the one-dimensional model and comparing the results with the exact formula as in [17]. Secondly, we present the affine structure property for the stochastic rate models and we find an explicit solution for the defaul-free bond that will be necessary to calibrate the model. Finally we describe the two-dimensional model and we perform simulations for the price of a defaultable Zero Coupon Bond, testing the method in comparison to the results obtained by Monte Carlo.

In conclusion, in Chapter 4 we use the approximation technique to calibrate the model. We initially calibrate on the term structure the Vasicek model to the market default-free bond price, then by replacing the parameters, in order to calibrate the JDCEV model, we provide explicit formula for par CDS spreads finding implicit default probabilities by bootstrapping. First, we find an approximation for the default probabilities and we calibrate the model to the bootstrapping values. Secondly, we compute explicit formulas for CDS spreads with the expansion method and we calibrate the model to real market spreads, obtaining the parameters that match the default probabilities implicit in CDS prices to the ones implied by the model.

Contents

Chapter 1

Default Risk & Hazard Process

In this chapter we present general results related to the analysis of random times and their filtrations in the intensity-based approach to consider default in market models. In particular, we see how to deal with the default time and the associated enlargement of filtration.
Let $\tau : \Omega \to \mathbb{R}$ be a non-negative random variable on a probability space $(\Omega, \mathscr{G}, P)$ such that

- $P(\tau = 0) = 0$
- $P(\tau > t) > 0$, for every $t \in \mathbb{R}$.

Definition 1.1. The stochastic process D

$$D_t = \mathbb{1}_{\{\tau \leq t\}} \quad t \in \mathbb{R}^+$$

is called *jump process* associated with the random variable τ. The process D has right-continuous sample paths, each one equal to 0 before random time τ and equal to 1 for $t \geq \tau$. We denote by $\mathbb{D} = (\mathscr{D}_t)_{t \geq 0}$ the filtration generated by D, that is $\mathscr{D}_t = \sigma(D_u; u \leq t)$.

Let $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ be a given filtration on $(\Omega, \mathscr{G}, P)$ such that $\mathbb{G} := \mathbb{F} \vee \mathbb{D}$, i.e. $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{D}_t$ for every $t \in \mathbb{R}^+$. From financial point of view, the filtration $\mathbb{G}$ represents the whole flow of observations available to the investors, it describes the default-free market variables up to t (by filtration $\mathscr{F}_t$) and tells whether default occurs

before t (by filtration $\mathscr{D}_t$).
All the filtration are assumed to satisfy the "usual conditions" of right continuity and completeness.
By definition, the process D is $\mathbb{G}$-adapted, since $\mathscr{D}_t \subset \mathscr{G}_t$ for any t; in other words τ is a stopping time with respect to $\mathbb{G}$ (it may fail to be a $\mathbb{F}$-stopping time).

Definition 1.2. We write $F_t := P(\tau \leq t|\mathscr{F}_t)$, for every $t \in \mathbb{R}^+$, and we call

$$1 - F_t = P(\tau > t|\mathscr{F}_t)$$

the $\mathbb{F}$-*survival process* of τ.

Remark 1.1. It results by definition that $1-F_t$ is non-negative and $\mathbb{F}$-supermartingale. Moreover, since $\mathscr{F}_0$ contains no information, we have:

$$1 - F_0 = P(\tau > 0) > 0.$$

Definition 1.3. We define the $\mathbb{F}$-*hazard process* of τ, denoted by Γ, through the formula

$$1 - F_t = e^{-\Gamma_t} \quad \text{for every } t \in \mathbb{R}^+.$$

It is assumed that $F_t < 1$ hold for every t, and thus the process Γ exists.

Remark 1.2. It is important to observe that $\mathscr{G}_t \subset \mathscr{G}_t^*$, with

$$\mathscr{G}_t^* := \{A \in \mathscr{G}_t \,|\, \exists B \in \mathscr{F}_t \quad A \cap \{\tau > t\} = B \cap \{\tau > t\}\}.$$

In fact, we have $\mathscr{G}_t = \mathscr{D}_t \vee \mathscr{F}_t = \sigma(\mathscr{D}_t, \mathscr{F}_t) = \sigma(\{\tau \leq u\}, u \leq t, \mathscr{F}_t)$. It is also easy to see that $\mathscr{G}_t^*$ is a sub-σ-algebra of $\mathscr{G}$. Therefore, it is enough to check the following possibilities:

i) if $A = \{\tau \leq u\}$ for some $u \leq t$, then $\exists B \in \mathscr{F}_t$ s.t. $A \cap \{\tau > t\} = B \cap \{\tau > t\}$, with $B = \emptyset$;

ii) if $A \in \mathscr{F}_t$, then $\exists B \in \mathscr{F}_t$ s.t. $A \cap \{\tau > t\} = B \cap \{\tau > t\}$, with $B = A$.

We want to substitute the conditional expectation with respect to $\mathscr{G}_t$ with the conditioning relative to the σ-algebra $\mathscr{F}_t$.

Lemma 1.1. *For any $\mathscr{G}$-measurable random variable Y we have, for any $t \in \mathbb{R}^+$*

$$\mathbb{E}[\mathbb{1}_{\{\tau>t\}}Y|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[\mathbb{1}_{\{\tau>t\}}Y|\mathscr{F}_t]}{P(\tau>t|\mathscr{F}_t)} = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}[\mathbb{1}_{\{\tau>t\}}Y|\mathscr{F}_t] \tag{1.1}$$

Proof. Let us fix $t \in \mathbb{R}^+$. In view of Remark (1.2), on the set $\{\tau > t\}$, any $\mathscr{G}_t$-measurable random variable coincides with some $\mathscr{F}_t$-measurable random variable. Therefore,

$$\mathbb{E}[\mathbb{1}_{\{\tau>t\}}Y|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}[Y|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}X,$$

where X is a $\mathscr{F}_t$-measurable random variable. Taking conditional expectation with respect to $\mathscr{F}_t$, we have

$$\mathbb{E}[\mathbb{1}_{\{\tau>t\}}Y|\mathscr{F}_t] = P(\tau>t|\mathscr{F}_t)X.$$

The second equality holds from the definition of Γ. □

Proposition 1.1. *Let Z be a bounded $\mathbb{F}$-predictable process. Then for any $t < s \le \infty$*

$$\mathbb{E}[\mathbb{1}_{\{t<\tau\le s\}}Z_\tau|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}[\int_{]t,s]} Z_u dF_u|\mathscr{F}_t]. \tag{1.2}$$

Proof. We start by assuming that Z is a stepwise $\mathbb{F}$-predictable process,i.e.

$$Z_u = \sum_{i=0}^{n} Z_{t_i}\mathbb{1}_{]t_i,t_{i+1}]}(u),$$

with $t_0 = t < \ldots < t_{n+1} = s$ and Z_{t_i} $\mathscr{F}_i$-measurable for every $i = 0,\ldots,n$. In view of (1.1) it holds

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{t_i<\tau\le t_{i+1}\}}Z_\tau|\mathscr{G}_t] &= \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}[\mathbb{1}_{\{t_i<\tau\le t_{i+1}\}}Z_{t_i}|\mathscr{F}_t]\\ &= \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}[(F_{t_{i+1}}-F_{t_i})Z_{t_i}|\mathscr{F}_t].\end{aligned}$$

We conclude by approximating an arbitrary $\mathbb{F}$-predictable process by a sequence of stepwise F-predictable processes. □

Remark 1.3. Proposition (1.1) remains valid if $\mathbb{F} = \mathbb{G}$, i.e. when τ is an $\mathbb{F}$-stopping time, but in this case does not provide a significant formula.
Indeed, since $F_t = \mathbb{1}_{\{\tau\le t\}}$, the random variable e^{Γ_t} is equal to 1 on the set $\{\tau > t\}$, and thus left and right-hand sides of the equation are equal.

Moreover, recalling that any $\mathbb{G}$-predictable process coincides with a $\mathbb{F}$-predictable process up to time τ, there is no point in dealing in Proposition (1.1) with $\mathbb{G}$-predictable processes.

Corollary 1.1. *Let Y be a $\mathscr{G}$-measurable random variable. Then, for $t \leq s$,*

$$\mathbb{E}[\mathbb{1}_{\{\tau>s\}}Y|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_t}Y|\mathscr{F}_t]. \tag{1.3}$$

Furthermore, for any $\mathscr{F}_s$-measurable random variable Y it holds

$$\mathbb{E}[\mathbb{1}_{\{\tau>s\}}Y|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}[Ye^{\Gamma_t-\Gamma_s}|\mathscr{F}_t]. \tag{1.4}$$

If F is continuous and increasing, then for every $\mathbb{F}$-predictable process Z we have

$$\mathbb{E}[\mathbb{1}_{\{t<\tau\leq s\}}Z_\tau|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\int_t^s Z_u e^{\Gamma_t-\Gamma_u}d\Gamma_u|\mathscr{F}_t]. \tag{1.5}$$

Proof. Thanks to (1.1), to prove that (1.3) holds, it is enough to see that $\mathbb{1}_{\{\tau>s\}}\mathbb{1}_{\{\tau>t\}} = \mathbb{1}_{\{\tau>s\}}$, for every $s \geq t$. For (1.4), by virtue of (1.3), we obtain

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{\tau>s\}}Y|\mathscr{G}_t] &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_t}Y|\mathscr{F}_t] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\tau>s\}}|\mathscr{F}_s]e^{\Gamma_t}Y|\mathscr{F}_t] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[P(\tau>s|\mathscr{F}_s)e^{\Gamma_t}Y|\mathscr{F}_t] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[Ye^{\Gamma_t-\Gamma_s}|\mathscr{F}_t].
\end{aligned}$$

Last equality is a direct consequence of (1.2). Indeed, if F is increasing $dF_u = e^{-\Gamma_u}d\Gamma_u$. □

1.1 Survival Probability

In this section we give the definition of survival probability and we see how it have the same structure as discount factors. It turns out, in fact, that the default intensity plays the same role as interest rates.

We assume that the process F is absolutely continuous. Hence, we can write

$$1-F_t = P(\tau>t|\mathscr{F}_t) = 1-\int_0^t f_u du = e^{-\Gamma_t} = \exp(-\int_0^t \gamma_u du). \tag{1.6}$$

so that the *intensity of default* γ and the *Hazard process* Γ satisfy respectively $\gamma_u = f_u(1-F_u)^{-1}$, and $\Gamma_t = -\int_0^t \gamma_u du$. In fact, by Ito's formula we have:

$$-f_u du = -\gamma_u e^{-\Gamma_u} du \tag{1.7}$$

$$\Rightarrow f_u = \gamma_u(1-F_u). \tag{1.8}$$

Under this assumption we can give the definition of *survival probability.*

Definition 1.4. First, we can observe that

$$\mathbb{E}[\mathbb{1}_{\{\tau>T\}}|\mathscr{G}_t] = \mathbb{E}[\mathbb{1}_{\{\tau>T\}}|\mathscr{F}_t \vee \mathscr{D}_t] \tag{1.9}$$

$$= P(\tau > T|\mathscr{F}_t, \tau > t), \tag{1.10}$$

clarifying that this value represents the probability that default occurs after maturity time T, with the information at time t. Now, in view of (1.4), we have

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{\tau>T\}}|\mathscr{G}_t] &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[e^{\Gamma_t-\Gamma_T}|\mathscr{F}_t] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[e^{-\int_t^T \gamma(s,X_s)ds}|\mathscr{F}_t].\end{aligned} \tag{1.11}$$

Therefore, we define the risk-neutral *survival probability* Q at time t by

$$Q(t, \tau > T) := \mathbb{E}[e^{-\int_t^T \gamma(s,X_s)ds}|\mathscr{F}_t]. \tag{1.12}$$

This is why survival probabilities can be interpreted as zero coupon bonds and intensities γ as instantaneous credit spreads.

Chapter 2

Pricing Approximations for Models with Default

2.1 General Local-Stochastic Volatility Models with Default

In this chapter we will present the general local-stochastic volatility models with default and stochastic short interest.
We consider a frictionless market, no arbitrage and no dividends. We assume, as given, an equivalent martingale measure[1] P chosen by the market on a complete filtered probability space $(\Omega, \mathscr{G}, \{\mathscr{G}_t, t \geq 0\}, P)$. The filtration $\mathbb{G}$ is defined as $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$, where $\mathbb{F}$ is the σ-algebra generated by X (the log-price process) and $\mathbb{D}$ the one generated by the default time τ. We consider a defaultable asset S whose

[1] see [18] Definition (10.24)

risk-neutral dynamics are given by

$$\begin{aligned} S_t &= \mathbb{1}_{\{\tau>t\}} e^{X_t}, \\ dX_t &= \mu(t,X_t,r_t)dt + \sigma(t,X_t,r_t)dW_t^1, \\ dr_t &= \alpha(t,X_t,r_t)dt + \beta(t,X_t,r_t)dW_t^2, \\ \rho(t,X_t,r_t)dt &= d<W^1,W^2>_t, \\ \tau &= \inf\{t \geq 0 : \int_0^t \gamma(s,X_s)ds \geq \epsilon\}, \end{aligned} \tag{2.1}$$

with $|\rho| < 1$ and $\epsilon \sim Exp(1)$, independent of $W = (W_1, W_2)$, 2-dimensional correlated Brownian motion.

Equations (2.1) include virtually all local volatility models, all one-factor stochastic volatility models and all one-factor local-stochastic volatility models. In this work we will present a method to price European-style options for a generic model represented by (2.1); however, in the applications, we will focus in a particular stochastic volatility model in order to analyse all the outcomes that could rise from this specific case.

Remark 2.1. It results that τ is a random time, indeed it's obvious that $P(\tau = 0) = 0$, we have to prove that $P(\tau > t) > 0$.

Since $\{\tau > t\} = \{\int_0^t \gamma(s,X_s)ds < \epsilon\}$, we have

$$P(\tau > t) = E[P(\int_0^t \gamma(s,X_s)ds < \epsilon | \mathscr{F}_t)].$$

Thanks to Lemma (A.18) in [18] we have, since ϵ is independent of $\mathbb{F}$ and $\int_0^t \gamma(s,X_s)ds$ is $\mathscr{F}_t$-measurable,

$$P(\tau > t) = E[h(\int_0^t \gamma(s,X_s)ds)] > 0,$$

where $h(x) = P(\epsilon > x)$. This means that there exists a positive probability that τ is greater then t, i.e. that S_t can't be indistinguishable from the zero process.

Thanks to the EMM it is possible to find a relation between the coefficients of the underlying asset. In fact, in the absence of arbitrage there exists an equivalent martingale measure such that the discounted asset price $\tilde{S}_t = e^{-\int_0^t r_s ds} S_t$ is a $\mathbb{G}$-martingale. This property allows to find the following drift condition.

Proposition 2.1. *In the absence of arbitrage, we have*

$$\mu(t,X_t,r_t) = r_t + \gamma(t,X_t) - \frac{1}{2}\sigma^2(t,X_t,r_t)$$

Proof. By assumption, $\tilde{S}_t = e^{-\int_0^t r_s ds} S_t$ is a $\mathbb{G}$-martingale, i.e.

$$\begin{aligned} e^{-\int_0^t r_s ds} S_t = \mathbb{E}[e^{-\int_0^T r_s ds} S_T | \mathscr{G}_t] &\Leftrightarrow S_t = \mathbb{E}[e^{-\int_t^T r_s ds} S_T | \mathscr{G}_t] \\ &\Leftrightarrow e^{X_t} \mathbb{1}_{\{\tau > t\}} = \mathbb{E}[e^{-\int_t^T r_s ds + X_T} \mathbb{1}_{\{\tau > T\}} | \mathscr{G}_t] \\ &\Leftrightarrow e^{X_t} \mathbb{1}_{\{\tau > t\}} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}[e^{-\int_t^T r_s ds + X_T} e^{\Gamma_t - \Gamma_T} | \mathscr{F}_t] \\ &\Leftrightarrow e^{X_t - \int_0^t r_s ds - \int_0^t \gamma(s,X_s) ds} = \mathbb{E}[e^{X_T - \int_0^T r_s ds - \int_0^T \gamma(s,X_s) ds} | \mathscr{F}_t], \end{aligned}$$

where in the third step we have used (1.4). Therefore $\tilde{S}_t$ is a $\mathbb{G}$-martingale if and only if $Y_t := e^{X_t - \int_0^t r_s ds - \int_0^t \gamma(s,X_s) ds}$ is a $\mathbb{F}$-martingale. We can prove the drift condition by applying the Ito's formula to the process Y_t as follows:

$$\begin{aligned} dY_t &= Y_t(-r_t - \gamma(t,X_t))dt + Y_t dX_t + \frac{1}{2} Y_t d < X >_t \\ &= Y_t((-r_t - \gamma(t,X_t)) + \mu(t,X_t,r_t) + \frac{1}{2}\sigma^2(t,X_t,r_t))dt + \sigma(t,X_t,r_t) dW_t, \end{aligned}$$

and setting the drift term equal to zero. □

2.2 Valuation of Contingent Claims

We denote by V the no-arbitrage price of a European derivative expiring at time T with payoff of the form

$$H(X_T)\mathbb{1}_{\{\tau > T\}} + G(X_T)\mathbb{1}_{\{\tau \le T\}} = (H(X_T) - G(X_T))\mathbb{1}_{\{\tau > T\}} + G(X_T),$$

where $H(X_T)$, $G(X_T) \in \mathscr{F}_T$. Here $G(X_T)$ represents the recovery payoff in case of default prior to maturity T.

Proposition 2.2. *The no-arbitrage price V_t is given by*

$$V_t = \mathbb{E}\left[e^{-\int_t^T r_s ds} G(X_T) | \mathscr{F}_t\right] + \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[e^{-\int_t^T (r_s + \gamma(s,X_s)) ds} \left(H(X_T) - G(X_T)\right) | \mathscr{F}_t\right]. \tag{2.2}$$

Proof. The value V_t of the option at time t, applying risk-neutral pricing, is the conditional expectation of the discounted payoff. Our goal is to switch the conditional expectation w.r.t. $\mathscr{G}_t$ and the one w.r.t. $\mathscr{F}_t$. In virtue of (1.4), we have

$$\begin{aligned} V_t &= \mathbb{E}\left[e^{-\int_t^T r_s ds}((H(X_T)-G(X_T))\mathbb{1}_{\{\tau>T\}}+G(X_T))|\mathscr{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T r_s}e^{\Gamma_t-\Gamma_T}\left(H(X_T)-G(X_T)\right)|\mathscr{F}_t\right]+\mathbb{E}\left[e^{-\int_t^T r_s ds}G(X_T)|\mathscr{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T (r_s+\gamma(s,X_s))ds}\left(H(X_T)-G(X_T)\right)|\mathscr{F}_t\right]+\mathbb{E}\left[e^{-\int_t^T r_s ds}G(X_T)|\mathscr{F}_t\right]. \end{aligned}$$

□

Example 2.1. The price at time t of a default-free bond paying 1 at maturity T is

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}|\mathscr{G}_t\right].$$

Now, if a defaultable zero-coupon bond with zero recovery payoff is traded in the market, then its price is

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}|\mathscr{G}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T (r_s+\gamma(s,X_s))ds}|\mathscr{F}_t\right].$$

From Proposition (2.2), we see that, to value a European option, we must compute functions of the form

$$u(t,x,r) = \mathbb{E}\left[e^{-\int_t^T (r_s+\gamma(s,X_s))ds}\varphi(X_T)|X_t = x, r_t = r\right]. \tag{2.3}$$

By applying the Feynman-Kač representation theorem[2], the function $u(t,x,r)$ is the classical solution (if exists) of the Kolmogorov backward Cauchy problem

$$\begin{cases} (\partial_t+\mathcal{A})u(t,x,r) = 0, & t<T, (x,r)\in\mathbb{R}^2 \\ u(T,x,r) = \varphi(x,r), & (x,r)\in\mathbb{R}^2 \end{cases} \tag{2.4}$$

where $\mathcal{A}$ is the second order elliptic differential operator with variable coefficients

$$\begin{aligned} \mathcal{A}(t,x,r) = {} & \frac{1}{2}\partial_x^2\sigma^2(t,x,r)+\partial_x\partial_r\rho(t,x,r)\sigma(t,x,r)\beta(t,x,r)+ \\ & +\frac{1}{2}\partial_r^2\beta^2(t,x,r)+\partial_x\mu(t,x,r)+\partial_r\alpha(t,x,r)+(-\gamma(t,x,r)-r). \end{aligned} \tag{2.5}$$

[2]see [18], Theorem (9.45)

The operator $\bar{\mathcal{A}} = \mathcal{A} + (r_t + \gamma(t,x,r))$ is called *characteristic operator* of (X_t, r_t). One of the main consequences of the Feynman-Kač Theorem is that if this operator has a fundamental solution $p(t,x,r;T,\xi,\eta)$, then this function turns out to be the *transition density* of the process, i.e. the density of the random variable $(X_T, r_T)^{t,(x,r)}$.

This trivial result allows to understand the deep connection between PDEs and SDEs, meaning that, in order to compute the price of an option, we need to solve PDE in (2.5). Conversely, this means that we can use Monte Carlo-type methods to find an approximation for a solution of the Dirichlet problem.

2.3 Pricing Approximation: Analytical expansions

In this section we will discuss a method to find an approximation of the analytical solution of the Cauchy Problem associated with a general parabolic PDE. In this way, we will obtain an explicit formula for the approximate prices, thanks to which it will be possible to compute them efficiently. Let us consider the following Cauchy Problem

$$\begin{cases} (\partial_t + \mathcal{A})u(t,x) = 0, & t \in [0,T[, x \in \mathbb{R}^d \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^d \end{cases} \tag{2.6}$$

where

$$\mathcal{A} = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{d} a_i(t,x)\partial_{x_i} + a(t,x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d. \tag{2.7}$$

In general, one is interested in the *fundamental solution* $\Gamma(t,x;T,y)$, from which the solution u can be obtained via convolution. However, when there are x-dependent coefficients, the fundamental solution is not available in closed-form and one usually seeks an approximation.

The method we will discuss is based on the expansion technique and it has been introduced in Pagliarani and Pascucci (2012), where $\mathcal{A}$ is a differential operator corresponding to the generator of a scalar diffusion. These ideas have been extended in Pagliarani et al. (2013) and Lorig et al. (2013a) to the case where $\mathcal{A}$ may be an integro-differential operator corresponding to the generator of a scalar Lévy-type process, and lately generalized to the d-dimensional case in [14].

We will operate under the standard assumptions for the coefficients of *uniform ellipticity* and *regularity* on $[0,\overline{T}]\times\mathbb{R}^d$, $\overline{T}\geq T$, to ensure the existence of a classical solution for (2.6) for any $\varphi\in L^\infty$. From now on, it will be useful to choose the following notations for (2.7):

$$\mathcal{A} := \sum_{|\alpha|\leq 2} a_\alpha(t,x)D_x^\alpha, \quad t\in\mathbb{R}, x\in\mathbb{R}^d, \tag{2.8}$$

where, as usual,

$$\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d, \quad |\alpha|=\sum_{i=1}^d \alpha_i, \quad D_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}. \tag{2.9}$$

The method consists in expanding formally the operator $\mathcal{A}$ as an infinite sum, $\mathcal{A}=\sum_{n\geq 0}\mathcal{A}_n$, after a polynomial expansion of the coefficients. Precisely, we introduce the following

Definition 2.1. Let $(a_{\alpha,n})_{n\geq 0}$ be a sequence of continuous functions on $[0,\overline{T}]\times\mathbb{R}^d$. We call $(a_{\alpha,n})_{0\leq n\leq N}$ an *N-th order polynomial expansion* if

- $a_{\alpha,n}(t,\cdot)$ for any $t\in[0,\overline{T}]$, with $a_{\alpha,0}(t,\cdot)=a_{\alpha,0}(t)$,
- we have convergence (pointwise or in some norm) of the partial sums $\sum_{n=0}^N a_{\alpha,n}(t,\cdot)$ to the coefficients $a_\alpha(t,z)$.

There exist many examples of polynomial expansions. For our model, however, we will use in the applications only time-dependent Taylor expansion.

Example 2.2. Time-dependent Taylor polynomial expansion
Fixed $\overline{x}:\mathbb{R}_+\to\mathbb{R}^d$, let $a_{\alpha,n}$ be the n-th order term of the Taylor expansion of a_α around $\overline{x}$. More precisely,

$$a_{\alpha,n}(\cdot,x)=\sum_{|\beta|=n}\frac{D^\beta a_\alpha(\cdot,\overline{x}(\cdot))}{\beta!}(x-\overline{x}(\cdot))^\beta, \quad 0\leq n\leq N, \quad |\alpha|\leq 2, \tag{2.10}$$

here $\beta! = \beta_1! \cdots \beta_d!$ and $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$. This choice will be helpful to set up an initial value that will result in a highly accurate approximation for the option pricing. Indeed, $\mathcal{A}_0$ will be the generator of a process X^0, pointing at a natural choice for $\overline{x}(t)$, i.e. $\overline{x}(t) = \mathbb{E}[X_t^0]$.

As mentioned before, we formally write the operator $\mathcal{A}$ as

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n, \qquad \mathcal{A}_n = \sum_{|\alpha| \le 2} a_{\alpha,n}(t,x) D_x^\alpha. \tag{2.11}$$

The idea is the same as in the perturbation theory, that is to expand the solution of (2.6) as

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.12}$$

This allows to insert (2.11) and (2.12) and find that the functions $(u_n)_{n \ge 0}$ satisfy the following sequence of nested Cauchy Problems

$$\begin{cases} (\partial_t + \mathcal{A}_0) u_0(t,x) = 0, & t \in [0,T[, x \in \mathbb{R}^d \\ u_0(T,x) = \varphi(x), & x \in \mathbb{R}^d \end{cases} \tag{2.13}$$

$$\begin{cases} (\partial_t + \mathcal{A}_0) u_n(t,x) = -\sum_{h=1}^{n} \mathcal{A}_h u_{n-h}(t,x), & t \in [0,T[, x \in \mathbb{R}^d \\ u_n(T,x) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{2.14}$$

The advantage is that, since the functions $a_{\alpha,0}$ depend only on t, the operator $\mathcal{A}_0$ is elliptic with only time-dependent coefficients. Therefore it is convenient to write the operator $\mathcal{A}_0$ in the following form:

$$\mathcal{A}_0 = \frac{1}{2} \sum_{i,j=1}^{d} C_{ij}(t) \partial_{x_i x_j} + \langle m(t), \nabla_x \rangle + \gamma(t), \qquad \langle m(t), \nabla_x \rangle = \sum_{i=1}^{d} m_i(t) \partial_{x_i}. \tag{2.15}$$

Under the assumptions, the $d \times d$ matrix C is positive definite and m is a d-vector of scalar functions.

2.3.1 Expression for u_0

The model presented in the previews section corresponds to the case of $d=2$. For this reason we write the full expression of u_0 in this case. We have:

$$C = \begin{pmatrix} 2a_{(2,0),0} & a_{(1,1),0} \\ a_{(1,1),0} & 2a_{(0,2),0} \end{pmatrix}, \qquad m = \left(a_{(1,0),0}, a_{(0,1),0}\right), \qquad \gamma = a_{(0,0),0}.$$

The solution of the problem (2.13) is given by

$$u_0(t,x) = e^{\int_t^T a_{(0,0),0}(s)ds} \int_{\mathbb{R}^d} \Gamma_0(t,x;T,y)\varphi(y)dy, \quad t<T, x \in \mathbb{R}^d. \tag{2.16}$$

In fact $\mathcal{A}_0$ has fundamental solution Γ_0, where

$$\Gamma_0(t,x;T,y) = \frac{1}{\sqrt{(2\pi)^d |\mathbf{C}(t,T)|}} \exp\left(-\frac{1}{2}\langle \mathbf{C}^{-1}(t,T)(y-x-\mathbf{m}(t,T)),(y-x-\mathbf{m}(t,T))\rangle\right) \tag{2.17}$$

is the d-dimensional Gaussian density with covariance matrix

$$\mathbf{C}(t,T) = \int_t^T C(s)ds, \tag{2.18}$$

and mean vector $x+\mathbf{m}(t,T)$, where

$$\mathbf{m}(t,T) = \int_t^T m(s)ds. \tag{2.19}$$

Remark 2.2. We could also find the same results observing that $\mathcal{A}_0$ is the *characteristic operator* of a 2-dimensional stochastic process X^0. Thanks to Ito's formula then, it would be sufficient to find the expression of the characteristic function

$$\hat{p}_0(t,;T,\xi) := \mathbb{E}[e^{i\xi X_T^0}|X_t = x] = \int_{\mathbb{R}} e^{i\xi y} p_0(t,x;T,y)dy,$$

proving that the transition density of X^0 is $p_0(t,x;T,y) = \Gamma_0(t,x;T,y)$, i.e.

$$X^0 \sim \mathcal{N}(x+\mathbf{m}(t,T), \mathbf{C}(t,T)).$$

2.3.2 Expression for u_n

As in [14] we present an explicit formula for each u_n that requires only a normal CDF to be computed. There are, in fact, many local volatility (LV), stochastic volatility (SV) and local-stochastic volatility(LSV) models for which European option prices can be found explicitly, but usually special functions or numerically highly oscillatory functions are required. Moreover, it's important to observe that this general result works in the d-dimensional case, for which, often, explicit solutions are harder to be computed.

Theorem 2.1. *For any $n \geq 1$, the n-th term u_n in (2.14) is given by*

$$u_n(t,x) = \mathcal{L}_n^x(t,T)u_0(t,x), \qquad t < T, x \in \mathbb{R}^d, \tag{2.20}$$

where $\mathcal{L}_n^x(t,T)$ is the differential operator acting on x and defined as

$$\mathcal{L}_n^x(s_0,T) := \sum_{h=1}^{n} \int_{s_0}^{T} ds_1 \int_{s_1}^{T} ds_2 \dots \int_{s_{h-1}}^{T} ds_h \sum_{i \in I_{n,h}} \mathcal{G}_{i_1}^x(s_0,s_1)\cdots\mathcal{G}_{i_h}^x(s_0,s_h), \tag{2.21}$$

with

$$I_{n,h} = \{i = (i_1,\dots,i_h) \in \mathbb{N}^h | i_1 + \cdots + i_h = n\}, \qquad 1 \leq h \leq n. \tag{2.22}$$

The operator $\mathcal{G}_n^x(t,s)$ is defined as

$$\mathcal{G}_n^x(t,s) := \mathcal{A}_n(s, \mathcal{M}^x(t,s)) = \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}^x(t,s))D_x^\alpha, \tag{2.23}$$

and

$$\mathcal{M}^x(t,s) = x + \mathbf{m}(t,s) + \mathbf{C}(t,s)\nabla_x. \tag{2.24}$$

Remark 2.3. We observe that, if the fundamental solution $\Gamma_0(t,x;T,y)$ is explicitly available, then to obtain u_n one can apply the operator $\mathcal{L}_n^x(t,T)$ directly to $\Gamma_0(t,x;T,y)$ in (2.16) .

We present the main idea to get the expression for u_n by studying the case $d = 2$, the Cauchy problem (2.14) with $n = 1$ and choosing the Taylor polynomial expansion. We will use only general properties of transition densities such as the Chapman-Kolmogorov equation and the standard Duhamel's principle.
To simplify the notations we write $x = (x,y) \in \mathbb{R}^2$ and we denote by $\tilde{\mathcal{A}}$ the formal

adjoint of the operator $\mathcal{A}$, obtained by integrating by parts. Moreover, we set $\mathcal{A} = \mathcal{A}^{(x,y)}(t)$ to clarify that $\mathcal{A}$ acts on the variables (x,y) and takes t as an argument.

Let $\varphi(x,y) = \delta_{(X,Y)}(x,y)$ so that $u_0(t,x,y) = \Gamma_0(t,x,y;T,X,Y)$ thanks to (2.16). We need to solve:

$$\begin{cases}(\partial_t + \mathcal{A}_0)u_1(t,x,y) = -\mathcal{A}_1 u_0(t,x,y), & t \in [0,T[, (x,y) \in \mathbb{R}^2 \\ u_1(T,x,y) = 0, & (x,y) \in \mathbb{R}^2.\end{cases} \tag{2.25}$$

First, we need a Lemma to compute the Gaussian derivatives.

Lemma 2.1. *Let $\Gamma_0 = \Gamma_0(t,x,y;s,\eta,\omega)$ be the 2-dimensional Gaussian function as in (2.17) . Then we have*

$$\tilde{\mathcal{A}}_1^{(\eta,\omega)}(s)\Gamma_0(t,x,y;s,\eta,\omega) = \mathcal{G}_1^{(x,y)}(t,s)\Gamma_0(t,x,y;s,\eta,\omega). \tag{2.26}$$

Proof. A direct computation shows that

$$\begin{aligned}&\partial_\eta^n \partial_\omega^m (\eta - \bar{x})^h (\omega - \bar{y})^k \Gamma_0(t,x,y;s,\eta,\omega)) \\ &= (-1)^{n+m} \left(\mathcal{M}_1^{(x,y)}(t,s)\right)^h \left(\mathcal{M}_2^{(x,y)}(t,s)\right)^k \partial_\eta^n \partial_\omega^m \Gamma_0(t,x,y;s,\eta,\omega)),\end{aligned} \tag{2.27}$$

where $\mathcal{M}_1^{(x,y)}(t,s)$ and $\mathcal{M}_2^{(x,y)}(t,s)$ are the component of $\mathcal{M}^{(x,y)}(t,s)$ defined in (2.24) . In the case of Taylor polynomial expansion we can write

$$\mathcal{A}_1^{(x,y)}(s) = \sum_{|\alpha| \le 2} (\partial_x a_\alpha(s,\bar{x},\bar{y})(x-\bar{x}) + \partial_y a_\alpha(s,\bar{x},\bar{y})(y-\bar{y})) D^\alpha_{(x,y)},$$

so, by definition of $\tilde{\mathcal{A}}_1$ and in view of (2.27) , we have

$$\tilde{\mathcal{A}}_1^{(\eta,\omega)}(s)\Gamma_0(t,x,y;s,\eta,\omega) = \mathcal{G}_1^{(x,y)}(t,s)\Gamma_0(t,x,y;s,\eta,\omega). \tag{2.28}$$

□

Now we are able to find the expression for u_1. We have:

$$
\begin{aligned}
&u_1(t,x,y)e^{-\int_t^T a_{(0,0),0}(s)ds}\\
&= \int_t^T ds \int_{\mathbb{R}^2} d\eta d\omega \Gamma_0(t,x,y;s,\eta,\omega)\mathcal{A}_1^{(\eta,\omega)}(s)\Gamma_0(s,\eta,\omega;T,X,Y) && \text{(by Duhamel's principle)}\\
&= \int_t^T ds \int_{\mathbb{R}^2} d\eta d\omega \left(\tilde{A}_1^{(\eta,\omega)}(s)\Gamma_0(t,x,y;s,\eta,\omega)\right)\Gamma_0(s,\eta,\omega;T,X,Y) && \text{(by parts)}\\
&= \int_t^T ds \mathcal{G}_1^{(x,y)}(t,s)\int_{\mathbb{R}^2} d\eta d\omega \Gamma_0(t,x,y;s,\eta,\omega)\Gamma_0(s,\eta,\omega;T,X,Y) && \text{(by Lemma (2.1) and (2.24))}\\
&= \int_t^T ds \mathcal{G}_1^{(x,y)}(t,s)\Gamma_0(t,x,y;T,X,Y) && \text{(by Chapman-Kolmogorov).}
\end{aligned}
$$

In view of (2.16) , multiplying both sides by $e^{\int_t^T a_{(0,0),0}(s)ds}$ it holds

$$u_1(t,x,y) = \mathcal{L}_1^{(x,y)} u_0(t,x,y), \qquad \mathcal{L}_1^{(x,y)} = \int_t^T ds \mathcal{G}_1(t,s). \tag{2.29}$$

Remark 2.4. (Accuracy of the pricing approximation).
Asymptotic convergence results were proved in Pagliarani et al. (2013); Lorig et al. (2013a) and more precisely in [14]. Under the assumptions on the coefficients of (2.7) , if $(\bar{x},\bar{y}) = (x,y)$, then for any $N \in \mathbb{N}$,

$$u(t,x,y) = \sum_{n=0}^{N} u_n(t,x,y) + \mathcal{O}\left((T-t)^{\frac{N+1}{2}}\right) \quad \text{as } t \to T^-. \tag{2.30}$$

It's important to stress that (2.30) is an asymptotic estimate for small times that does not imply convergence as N goes to infinity. In fact, the constant C in the right hand side of (2.30) turns out to depend on N. However, adding some hypotheses on the regularity of the derivatives of final datum φ would allow to use some relevant functions very common in financial applications (for instance the Call payoff function).

Chapter 3

A Model with Default

In this chapter we initially present some details about the model chosen for the log of the price process X and for the short rate r, respectively the JDCEV and the Vasicek model. Then we present the main aspects of Monte Carlo Method, and how it can be used, together with the Euler scheme, to estimate the price of a defaultable zero coupon bond in our model. Finally we compare the results of the approximation method presented in (2.3) to the numerical results from Monte Carlo in terms of accuracy and computing times.

3.1 JDCEV

We model the log of the price process by the JDCEV model as in Carr and Linetsky (2006) [6]. The JDCEV, or Jump to Default Constant Elasticity of Variance model, is a stochastic volatility model which attempts to capture on one hand stochastic volatility and leverage effect (generally negative correlation between stock returns and their volatilities), on the other hand the possibility of default. It extends the well known CEV model, initially introduced to correct the Black-Scholes model, which usually underprices or overprices European-style options.

As a particular case of (2.1), we present the model of the stock price independently of the model of the short rate in order to underlying their own particular characteristics singularly. We have:

$$S_t = \mathbb{1}_{\{\tau>t\}} e^{X_t}$$
$$dX_t = \left(r_t + b + (c - \frac{1}{2})\sigma^2 e^{2(\beta-1)X_t}\right) dt + \sigma e^{(\beta-1)X_t} dW_t^1$$
$$\tau = \inf\{t \geq 0 : \int_0^t (b + c\sigma^2 e^{2(\beta-1)X_s}) ds \geq \epsilon\},$$

so that the default intensity as a function of the underlying asset is defined by

$$\gamma(x) = b + c\sigma^2(t,x), \qquad \sigma(t,x) = \sigma e^{2(\beta-1)x}$$

with parameters $b \geq 0$, $c \geq 0$ and $\sigma > 0$ (here we consider a constant parameter version of the model; the general version allows for deterministic time-dependent parameters). Therefore, default intensity is an affine function of the instantaneous stock variance $\sigma^2(t,x)$ with non-negative coefficients (the greater the stock volatility, the greater the intensity). As in CEV, to get the stock volatility $\sigma(t,x)$ be a negative power of the stock price, the volatility elasticity parameter β need to be less than 1, i.e. $\beta < 1$ (note that for $\beta = 1$ we will find the B&S model).

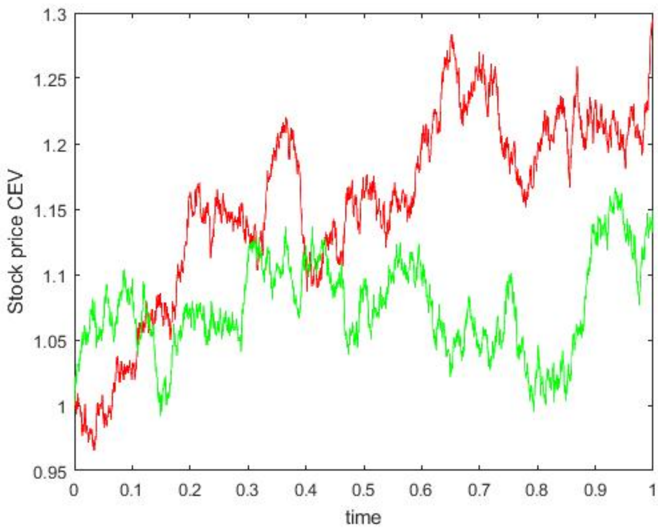


Figure 3.1: JDCEV process, Euler scheme simulation

We also assume, unlike Carr and Linetsky (2006) [6], that stock price process can only jump to default (by definition τ is the first jump of a doubly-stochastic

Poisson process with intensity $\gamma(t,x)$) and we will not consider the case where it can either diffuse to default reaching the bankruptcy level.

Let S be a defualtable bond that pays 1 at time $T > t$ if no default occurs prior to maturity (i.e., $S_T > 0$, $\tau > T$) and zero recovery otherwise. In view of (1.12) and (2.2), assuming zero interest rate, we have

$$V_t = \mathbb{E}[\mathbb{1}_{\{\tau>T\}}|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}u(t,X_t;T), \tag{3.1}$$

where

$$u(t,X_t;T) = \mathbb{E}[e^{-\int_t^T \gamma(s,X_s)ds}|X_t]. \tag{3.2}$$

Note that we can take the expectation with respect the process X_t (or equivalently w.r.t. S_t) instead of $\mathscr{F}_t = \sigma(X_u, u \le t)$ because of the Markov property of the Brownian motion that define X and the choice of zero interest rate.

Recalling (2.4), we have found that the function $u(t,x;T)$ is equally the price of a bond, the survival probability $Q(t,\tau > T)$ and the solution of the following Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})u(t,x) = 0, & t \in [0,T[, x \in \mathbb{R}^d \\ u(T,x) = 1, & x \in \mathbb{R}^d, \end{cases} \tag{3.3}$$

where $\mathcal{A} + \gamma(t,x)$ is the characteristic operator of X and

$$\mathcal{A} = \frac{1}{2}\sigma^2 e^{2(\beta-1)x}\partial_{xx} + \left(b + (c - \frac{1}{2})\sigma^2 e^{2(\beta-1)x}\right)\partial_x - \gamma(t,x).$$

The exact price $u(t,x;T)$ in this setting is computed explicitly in Mendoza-Arriaga et al. (2010) [17] as follows

$$\begin{aligned} u(t,x;T) = \sum_{n=0}^{\infty} e^{-(b+\omega n)(T-t)} \frac{\Gamma(1+c/|\beta|)\Gamma(n+1/(2|\beta|))}{\Gamma(\nu+1)\Gamma(1/(2|\beta|))n!} \\ \times A^{1/(2|\beta|)} e^x \exp\left(-Ae^{-2\beta x}\right) {}_1F_1(1-n+c/|\beta|;\nu+1;Ae^{-2\beta x}) \end{aligned} \tag{3.4}$$

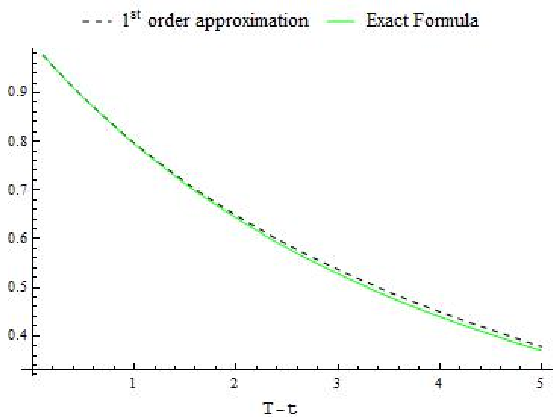
where ${}_1F_1$ is the Kummer confluent hypergeometric function, $\Gamma(x)$ is a Gamma function and

$$\nu = \frac{1+2c}{2|\beta|}$$
$$A = \frac{b}{\sigma^2|\beta|}$$
$$\omega = 2|\beta|b.$$

Now, in view of (3.4), we can apply the method presented in Section (2.3), and then compare the results with the exact formula for the survival probability. We use a *Mathematica* notebook to implement formula in (2.20) and find the explicit expression of the functions $u_n(t,x)$, choosing expansion as in Example (2.2), with $\bar{x} = X_t$. We stop our approximation method at the second order, i.e. $n = 2$. Here an example for the unidimensional case in which formulas can be steel written concisely:

$$u_0(t,x;T) = e^{-(b+c\sigma^2 e^{2(\beta-1)x})(T-t)}$$
$$u_1(t,x;T) = e^{-(b+c\sigma^2 e^{2(\beta-1)x})(T-t)}(-\sigma^2 bce^{2x(\beta-1)}(T-t)^2(\beta-1)$$
$$+\frac{1}{2}\sigma^4 ce^{4x(\beta-1)}(T-t)^2(\beta-1) - \sigma^4 c^2 e^{2x(\beta-1)}(T-t)^2(\beta-1))$$
$$u_2(t,x;T) = e^{-(b+c\sigma^2 e^{2(\beta-1)x})(T-t)}(-\sigma^4 ce^4 x(\beta-1)(T-t)^2(\beta-1)^2 - \frac{2}{3}\sigma^2 b^2 ce^{2x(\beta-1)}(T-t)^3(\beta-1)^2 \dots$$
$$\dots + \frac{1}{2}\sigma^8 c^4 e^{8x(\beta-1)}(T-t)^4(\beta-1)^2).$$

We choose as test parameters $\sigma = 0.3$, $\beta = \frac{2}{3}$, $b = 0.01$, $c = 2$, $S_0 = 1$, and we plot the bond prices as the maturity T varies from $T = t = 0$ to $T = 5$.



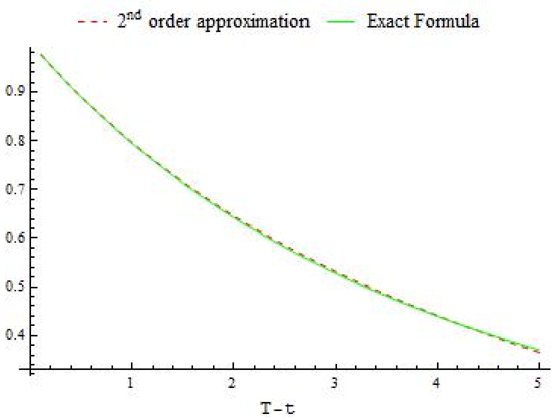


Figure 3.3: The dashed lines correspond to the approximations $u_0(t,x;t)$ and $u_1(t,x;T)$, while the solid line is the exact survival probability, computed by truncating equation (3.4) at $n = 70$.

Maturities	Exact price	2^{nd}order approx.	Rel.Error	Abs.Error
0.25	0.942314	0.942447	0.0140915%	0.0132787%
0.5	0.88907	0.889466	0.0445082%	0.039571%
0.75	0.839819	0.840611	0.0942323%	0.0791381%
1.	0.794206	0.795477	0.160051%	0.127113%
1.25	0.751908	0.7537	0.238274%	0.17916%
1.5	0.712636	0.714949	0.324687%	0.231384%
1.75	0.676126	0.678929	0.414534%	0.280277%
2.	0.642143	0.645369	0.502498%	0.322676%
2.25	0.610471	0.614028	0.5827%	0.355721%
2.5	0.580918	0.584686	0.648679%	0.376829%
2.75	0.553307	0.557144	0.693377%	0.383651%
3.	0.527482	0.531222	0.709107%	0.374041%
3.25	0.503297	0.506757	0.687517%	0.346025%
3.5	0.480623	0.483601	0.619542%	0.297766%
3.75	0.459341	0.461616	0.495348%	0.227534%
4.	0.439344	0.440681	0.304269%	0.133679%
4.25	0.420535	0.420681	0.0347332%	0.0146065%
4.5	0.402823	0.40151	−0.32581%	0.131244%
4.75	0.386128	0.383074	−0.790975%	0.305418%
5.	0.370376	0.365282	−1.37552%	0.50946%

Table 3.1: Relative and Absolute Errors between 2^{nd} order approximation method and the exact formula truncated at $n = 70$.

As expected, for short time, the approximation is good, and just at order 2 we reach a relative error level lower then 1%. We have seen the case of zero interest rate (the case of deterministic interest rate is analogous), however, when pricing a long-maturity option, the stochastic feature of interest rates has a stronger impact on the price. For this reason we need to let short rate process enter in the valuation and study general stochastic-local volatility model as in (2.1).

In the literature have been proposed numerous way on how to specify the dynamic of the short rate under the equivalent martingale measure, see for instance [4]. We will choose for our tests the Vasicek (1977) model, an endogenous term-structure model which allows to find analytical solutions for bonds and options prices. First, we give the notion of affine term structure and we study how to get solutions for these models.

3.2 Affine Term Structure

Definition 3.1. A model is said to possess an affine term structure if

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3.5}$$

where $P(t,T)$ is the value of the Zero Coupon Bond at time t and A, B are deterministic functions of time.

Since

$$P(t,T) = \mathbb{E}[e^{-\int_t^T r_s ds}|\mathscr{F}_t],$$

it's clear that the existence of an affine term structure for an interest-rate model is extremely useful from both an analytical and computational point of view. A first question is whether there is a relationship between the coefficients of the short rate and affinity in the above sense. Assume a generic risk-neutral dynamics for the short rate

$$dr(t) = b(t,r_t)dt + \sigma(t,r_t)dW_t. \tag{3.6}$$

The conditions on b and σ such that the resulting model presents an affine term structure is simply that b and σ^2 need to be affine functions, i.e.

$$\begin{aligned} b(t,x) &= \lambda(t)x + \eta(t) \\ \sigma^2(t,x) &= \gamma(t)x + \delta(t), \end{aligned} \tag{3.7}$$

where $\lambda, \eta, \gamma, \delta$ are deterministic time functions. The functions A and B can be obtained from these coefficients by solving the following differential equations:

$$\begin{cases} \partial_t B(t,T) + \lambda(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 = -1, \\ B(T,T) = 0, \end{cases} \tag{3.8}$$

$$\begin{cases} \partial_t [\log A(t,T)] - \eta(t)B(t,T) + \frac{1}{2}\delta(t)B(t,T)^2 = 0, \\ A(T,T) = 1. \end{cases} \tag{3.9}$$

The first equation is a Riccati differential equation that, in general, need to be solved numerically. Anyhow, in the case of Vasicek we have

$$\begin{aligned} \lambda(t) &= -k \\ \eta(t) &= k\theta \\ \gamma(t) &= 0 \\ \delta(t) &= \sigma^2, \end{aligned} \quad . \tag{3.10}$$

The equations are explicitly solvable for these particular models, obtaining the expressions for A and B.

In conclusion, we have seen that affinity in the coefficients guarantees affinity of the term structure. The converse is also true under a particular assumption. When the model has an affine term structure and time-homogeneous coefficients $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$, then these coefficients are necessarily affine functions of x of the form (3.7), for suitable constants $\lambda, \eta, \gamma, , \delta$.

3.2.1 Vasicek model

Vasicek model specifies that the instantaneous spot rate evolves following the stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t, \qquad r(0) = r_0, k, \theta, \sigma > 0. \tag{3.11}$$

The equation is linear and can be solved explicitly. Integrating equation (3.11) we have

$$\begin{aligned} r_t &= r_0 e^{-tk} + \theta e^{-tk}(e^{tk} - 1) + \int_0^t e^{-k(t-s)} \sigma dW_s \\ &= r_0 e^{-tk} + \theta(1 - e^{-tk}) + \sigma \int_0^t e^{-k(t-s)} dW_s. \end{aligned}$$

Moreover $r_t \sim N(m_t, V_t)$ is normally distributed with mean and variance given by

$$\begin{aligned} m_t &:= \mathbb{E}[r_t] = r_0 e^{-kt} + \theta(1 - e^{-kt}) \\ V_t &:= Var[r_t] = \frac{\sigma^2}{2k}(1 - e^{-2kt}). \end{aligned}$$

We can observe that, for any time t, the rate r_t can be negative with positive probability. However, this drawback is counterbalanced by the easier analytical tractability that is implied by a Gaussian density.

It results also that the short rate r_t is mean reverting, since the limit of the expected rate, for t going to infinity, is equal to θ. This means that the drift of the process is positive whenever the short rate is below θ, and negative otherwise, so as to be pushed to be closer on average to the level θ.

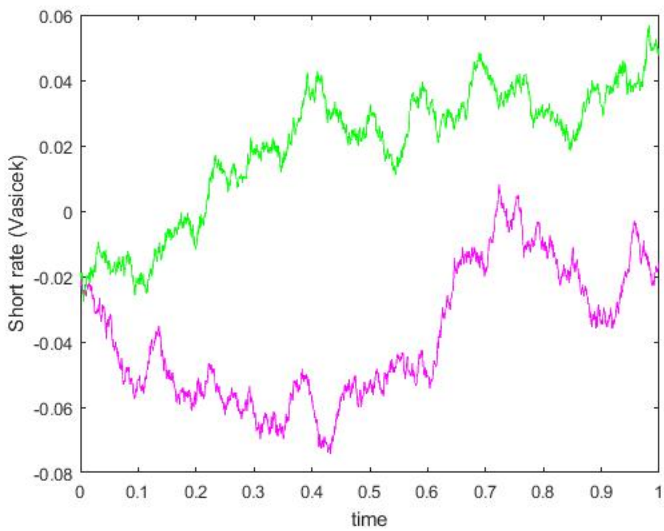


Figure 3.4: Vasicek short rate process, Euler scheme simulation

As mentioned before, Vasicek model has an affine term structure, with coefficients $\lambda(t) = -k, \eta(t) = k\theta, \gamma(t) = 0, \delta(t) = \sigma^2$. We can solve the differential equations

$$\begin{cases} \partial_t B(t,T) - kB(t,T) = -1, \\ B(T,T) = 0, \end{cases} \tag{3.12}$$

$$\begin{cases} \partial_t[\log A(t,T)] - k\theta B(t,T) + \frac{1}{2}\sigma^2 B(t,T)^2 = 0, \\ A(T,T) = 1. \end{cases} \tag{3.13}$$

The first equation is an ODE, whose solution is

$$B(t,T) = \frac{1}{k}(1 - e^{-k(T-t)}), \tag{3.14}$$

from the second equations, on the other hand, we get

$$A(t,T) = \exp\{(\theta - \frac{\sigma^2}{2k^2})(B(t,T) - T + t) - \frac{\sigma^2}{4k}B^2(t,T)\}. \tag{3.15}$$

Therefore the price of the zero coupon bond has an explicit and closed formula, that is

$$P(t,T) = \exp\{(\theta - \frac{\sigma^2}{2k^2})(B(t,T) - T + t) - \frac{\sigma^2}{4k}B^2(t,T) - \frac{1}{k}(1 - e^{-k(T-t)})r_t\}. \tag{3.16}$$

Having an exact formula for the ZCB price is a trivial result in order to calibrate the parameters of the model.

3.3 Testing the approximation method

In this section we compare the method presented in Section (2.3) with Monte Carlo method for the bi-dimensional default intensity model on one name with stochastic short interest, chosen as in Section (3.1) and (3.2.1). In particular we focus on the approximate price of a Zero Coupon Bound given by (2.1). In our case, the model is given by

$$
\begin{aligned}
S_t &= \mathbb{1}_{\{\tau>t\}} e^{X_t}, \\
dX_t &= \left(r_t + b + (c - \frac{1}{2})\sigma^2 e^{2(\beta-1)X_t}\right) dt + \sigma e^{(\beta-1)X_t} dW_t^1 \\
dr_t &= k(\theta - r_t)dt + \delta dW_t^2 \\
\rho dt &= d < W^1, W^2 >_t \\
\tau &= \inf\{t \geq 0 : \int_0^t \gamma(s, X_s) ds \geq \epsilon\}.
\end{aligned}
\tag{3.17}
$$

Therefore, with respect to the general model in (2.1) we have:

$$
\begin{aligned}
\mu(t,x,r) &= r + \gamma(t,x,r) - \frac{1}{2}\sigma^2(t,x,r) \\
\gamma(t,x) &= b + c\sigma^2 e^{2(\beta-1)x} \\
\sigma(t,x,r) &= \sigma e^{(\beta-1)x} \\
\alpha(t,x,r) &= k(\theta - r) \\
\beta(t,x,r) &= \delta.
\end{aligned}
$$

First of all, we simplify the expression of the SDE describing the interest rate. We apply Itô's formula to cancel the linear dependence in the drift term. We set $y_t = e^{kt} r_t$, then we obtain

$$
\begin{aligned}
dy_t &= k y_t dt + e^{kt} dr_t \\
&= k y_t dt + e^{kt}(k(\theta - r_t)dt + \delta dW_t^2) \\
&= k y_t dt + k e^{kt}\theta dt - k y_t dt + e^{kt}\delta dW_t^2 \\
&= k e^{kt}\theta dt + e^{kt}\delta dW_t^2
\end{aligned}
$$

We write the bi-dimensional stochastic process to the extent of finding the associated pricing partial differential operator. Hence, in terms of the instantaneous correlation coefficient we set, as usual,

$$
\begin{cases}
W_t^1 = \widehat{W}_t^1 \\
W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2}\, d\widehat{W}_t^2,
\end{cases}
\tag{3.18}
$$

where $\widehat{W}_t = (\widehat{W}_t^1, \widehat{W}_t^2)$ is the standard Brownian Motion. If we set $Y_t = (X_t, y_t)$, then we have:

$$dY_t = m(t, X_t, y_t)dt + C(t, X_t, y_t)d\widehat{W}_t, \tag{3.19}$$

where

$$m(t,x,y) = \begin{pmatrix} \mu(t,x,y) \\ ke^{kt}\theta \end{pmatrix}$$

and

$$C(t,x,y) = \begin{pmatrix} \sigma(t,x,y) & 0 \\ \delta e^{kt}\rho & \delta e^{kt}\sqrt{1+\rho^2} \end{pmatrix}.$$

If we set $C = C(t,x,y)$ we can find the operator of (3.19) by computing $C \cdot C^*$,

$$C \cdot C^* = \begin{pmatrix} \sigma^2(t,x,y) & e^{kt}\delta\rho\sigma(t,x,y) \\ e^{kt}\delta\rho\sigma(t,x,y) & (e^{kt}\delta)^2 \end{pmatrix}.$$

We write as in (2.11)

$$\mathcal{A} = \sum_{|\alpha| \leq 2} a_\alpha(t,x,y) D^\alpha_{x,y}, \tag{3.20}$$

where

$$a_{(2,0)}(t,x,y) = \frac{1}{2}\sigma^2(t,x,y), \quad a_{(1,0)}(t,x,y) = \mu(t,x,y), \quad a_{(1,1)}(t,x,y) = e^{kt}\delta\rho\sigma(t,x,y),$$
$$a_{(0,2)}(t,x,y) = (e^{kt}\delta)^2, \quad a_{(0,1)}(t,x,y) = ke^{kt}\theta, \quad a_{(0,0)}(t,x,y) = -\gamma(t,x,y) - e^{-kt}y.$$

Now, we can apply the approximation method to compute the price of a defaultable ZCB, i.e.

$$V_t = \mathbb{1}_{\{\tau > t\}} \mathbb{E}[e^{-\int_t^T (r_s + \gamma_s)ds} | \mathscr{F}_t]. \tag{3.21}$$

We implement formula in (2.20) finding the functions $u_n(t,x)$. We choose Taylor expansion for the coefficients as in Example (2.2), with time-dependent $(\bar{x}(t), \bar{y}(t))$ as follows

$$\bar{x}(t) = x, \qquad \bar{y}(t) = \bar{y} + \theta(e^{ks} - e^{kt}). \tag{3.22}$$

We stop our approximation method at the second order, i.e. $n = 2$.

In order to test the validity of our results, we also compute the ZCB price via Monte Carlo approximation. For this reason, we will give here below an overview of this method and we will explain how it can be applied in our case.

3.3.1 Monte Carlo Method

Monte Carlo simulation has become an essential tool in the pricing of derivative securities and in risk management. This method, in fact, allows to calculate the expected value of a random variable whose distribution is known. It is essentially based on the strong law of large numbers which ensures that this approximation converges to the correct mean value as the number of draws increases. Let us recall the statement of the theorem to clarify the notations.

Theorem 3.1. *Let* (X_n) *be a sequence of i.i.d. random variables with* $\mathbf{E}[X_1] < \infty$*. If* $\mathbf{E}[X_1] = \mathbf{E}[X] = \mu$ *and*

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \tag{3.23}$$

then

$$\lim_{n \to +\infty} M_n = \mathbb{E}[X] \qquad a.s. \tag{3.24}$$

This result means that, as long as we generate a large number of realizations in an independent way, almost surely we can use M_n to estimate $\mathbb{E}[X]$. From Markov's inequality we can study the error of the Monte Carlo method. In fact, setting $\sigma = var(X_1)$, for every $\epsilon > 0$ we have

$$\begin{aligned} P(|M_n - \mu| \geq \epsilon) &\leq \frac{var(M_n)}{\epsilon^2} \\ &= \frac{\frac{1}{n^2} var(X_1 + \cdots + X_n)}{\epsilon^2} = \frac{\frac{1}{n^2} var(X_1)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}. \end{aligned}$$

Therefore, if we set $p = \frac{\sigma^2}{n\epsilon^2}$ we find the following estimation for the error

$$P(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} = p, \tag{3.25}$$

where

- n is the number of samples
- ϵ is the maximum approximation error
- p is the probability that M_n is not in the confidence interval $[\mu - \epsilon, \mu + \epsilon]$.

We want to observe that, in order to get a good approximation, a great number of realizations are needed and using a calculator, it is almost impossible to generate completely independent random variables. However, these errors due to pseudo-random nature of the results, in general can be estimated if the generator works well.

Euler Scheme

As we have seen before, pricing derivative securities reduces to calculate an expected value (or equivalently to solve a partial differential equation). Therefore, numerical approximation of this expected value by Monte Carlo typically involves simulating paths of stochastic processes used to describe the evolution of the underlying asset prices, interest rates and when necessary model parameters. We present first the Euler Scheme for estimate simulations of stochastic differential equations, and then how it can be used in conjunction with the Monte Carlo. Let (X,r) be a stochastic process solution of the SDE

$$dX_t = \mu(t,X_t,r_t)dt + \sigma(t,X_t,r_t)dW_t^1$$
$$dr_t = \alpha(t,X_t,r_t)dt + \beta(t,X_t,R_t)dW_t^2$$
$$\rho(t,X_t,r_t) = \langle W_1,W_2\rangle_t$$

We recall that only the coefficient $\mu(t,X_t,r_t)$ depends on both X_t and r_t (by (3.17)), so that we can simplify our explanation.
We want to calculate the value of a defaultable zero coupon bond with zero recovery payoff given by (3.21).

We see that we need in theory all the value of the processes in the interval $[0,T]$. To get this result we first divide the time horizon $[0,T]$ into small increments of length h obtaining a time grid $0 = t_0 < t_1 < \ldots < t_n = T$. Then we set:

- n = number of increments
- s = number of realizations

- $X(i,j), r(i,j)$ = matrix of the processes where the rough-index represents the time and the column-index the realization
- $t(i)$ = i-th vector of n increments.

We illustrate the main steps of the method:

Step 1: First of all, we produce s independent realizations Z_i and $Z_i^{'}$ of the normal standard distribution $\mathcal{N}(0,1)$ to simulate the paths of the stochastic processes. Then, using an iterative formula we can determine all the value $r_{t_i}^{(j)}$ and $X_{t_i}^{(j)}$. Given that $W_{t_{i+1}}^1 - W_{t_i}^1 = \sqrt{t_{i+1} - t_i} \cdot Z_{t_i}$ and $W_{t_{i+1}}^2 - W_{t_i}^2 = \sqrt{t_{i+1} - t_i}(\rho Z_{t_i} + \sqrt{1-\rho^2})Z_{t_i}^{'}$, for our particular model, we have:

$$
\begin{aligned}
r_{t_{i+1}}^{(j)} &= r_{t_i}^{(j)} + k(\theta - r_{t_i}^{(j)})(t_{i+1} - t_i) + \delta(W_{t_{i+1}}^2 - W_{t_i}^2) \\
X_{t_{i+1}}^{(j)} &= X_{t_i}^{(j)} + (r_{t_i}^{(j)} + b + (c - \frac{1}{2})\sigma^2 e^{2(\beta-1)X_{t_i}^{(j)}})(t_{i+1} - t_i) + \sigma^2 e^{2(\beta-1)X_{t_i}^{(j)}}(W_{t_{i+1}}^1 - W^1 t_i)
\end{aligned}
$$

Step 2: Then we have to compute numerically integral of the form

$$I(j) \approx \int_0^T (r_s^{(j)} + \gamma_s(X_s^{(j)}))ds,$$

through Newton-Côtes type-methods for instance.

Step 3: Finally we have to compute the approximation of the price, that is, for every sample

$$f(j) = \exp(-I(j))$$

and by Monte Carlo the approximate price of the bond will be

$$\text{Bond}_s = \frac{1}{s}\sum_{j=1}^{s} f(j).$$

To reach a 99% Interval confidence we also need to compute the sample standard deviation

$$s_C = \sqrt{\frac{1}{s-1}\sum_{j=1}^{s}(\text{Bond}_s - f(j))^2}.$$

In this way we will have that with probability $p = 99\%$ the true value will be in the window

$$[\text{Bond}_s - 2.58\frac{s_C}{\sqrt{n}}, \text{Bond}_s + 2.58\frac{s_C}{\sqrt{n}}].$$

3.3.2 Numerical Results

Now, we are finally able to compute the price of the ZCB and compare the numerical results between the two methods. We remark that we could do the same analyses to price all European-style options with payoff of the form $h(X_T)$, simply manipulating expression in (2.16) for the approximation method, and multiplying the payoff function in the last step for the Monte Carlo.

We choose as test parameters

$$X_0 = 0, \quad b = 0.1, \quad c = 1, \quad \sigma = 0.2 \quad \beta = 0.5$$
$$y_0 = r_0 = 0.02, \quad \theta = 0.03, \quad \delta = 0.05, \quad \rho = -0.3,$$

then we use Monte Carlo Method respectively with 10000 and 100000 iterations to compute the value of the ZCB as the maturity T varies from 0 to 2. We choose small times in order to verify the convergence of the approximation method chosen as said before of order $n = 2$.

We can summarize the results in the following figures and tables:

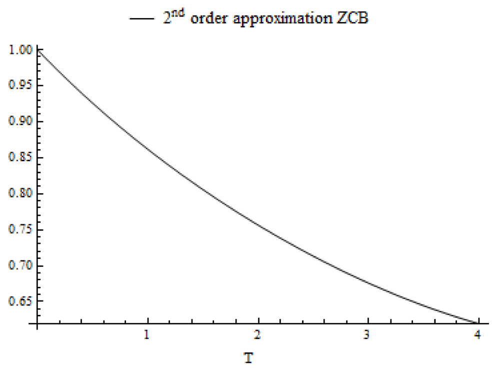
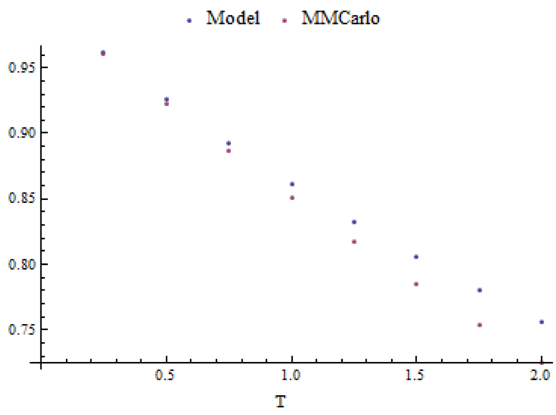
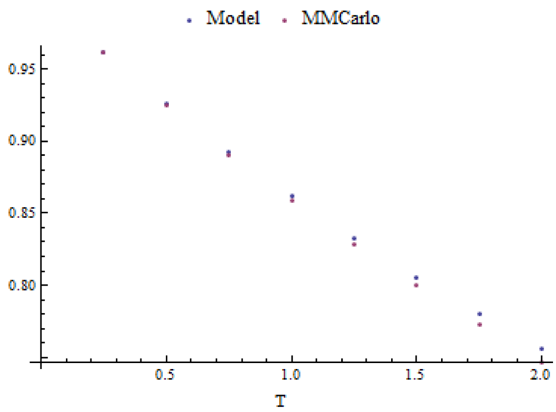


Figure 3.5: Zero coupon Bond prices using approximation method of order $n = 2$ from $T = 0$ to $T = 4$



(a) Comparison between the approximation method and Monte Carlo with 10000 iterations



(b) Comparison between the approximation method and Monte Carlo with 100000 iterations

Maturities	Model	MMCarlo	Confidence	Rel.Error	Abs.Error
0.25	0.961532	0.960707	0.00064±	0.0857206%	0.0824231%
0.5	0.925875	0.92269	0.00102±	0.343966%	0.318469%
0.75	0.8927	0.886449	0.0015±	0.700307%	0.625164%
1.	0.861737	0.851035	0.01784±	1.24189%	1.07018%
1.25	0.832769	0.817888	0.000345±	1.78685%	1.48803%
1.5	0.805625	0.785594	0.00213±	2.48634%	2.00306%
1.75	0.780172	0.754237	0.00289±	3.32431%	2.59353%
2.	0.756312	0.725145	0.001579±	4.12096%	3.11673%

Table 3.2: Relative and Absolute Errors between 2^{nd} order approximation method and the Monte Carlo with 10000 simulation.

Maturities	Model	MMCarlo	Confidence	Rel.Error	Abs.Error
0.25	0.961532	0.961226	0.00026±	0.0317971%	0.0305739%
0.5	0.925875	0.924967	0.00065±	0.0980272%	0.090761%
0.75	0.8927	0.89071	0.00107±	0.2229%	0.198983%
1.	0.861737	0.858664	0.0015±	0.356629%	0.30732%
1.25	0.832769	0.828102	0.001817±	0.560359%	0.46665%
1.5	0.805625	0.799659	0.00221±	0.740495%	0.596561%
1.75	0.780172	0.772398	0.00251±	0.996405%	0.777367%
2.	0.756312	0.746615	0.00282±	1.28217%	0.96972%

Table 3.3: Relative and Absolute Errors between 2^{nd} order approximation method and the Monte Carlo with 100000 simulation.

We can see that the approximate price obtained from the Monte Carlo method is near to the approximate price obtained from Taylor expansion. In particular, increasing the number of iterations, absolute and relative errors (with respect to the model price) decrease as T varies. However, as mentioned before, we want to stress that the series approach implemented via *Wolfram Mathematica* is better than the simulations in terms of computing time and that, despite their very long expressions, formulas do not require any special functions.

Chapter 4

Market Calibration

The purpose of this work is to find a method to calibrate the default intensity model with stochastic short rate on one name, chosen as in Section (3.1) and (3.2.1). We initially calibrate on the term structure the Vasicek model to the market default-free bond price, then by replacing the parameters, in order to calibrate the JDCEV model, we use two different approaches. First, we find an approximation for the default probabilities and we calibrate the model to the market data. Secondly, we find an explicit formula for the par CDS spread in our model and we approximate it using technique in Section (2.3). CDS are quoted through the rates (or "spreads") R in their premium legs that render them fair at inception. We calibrate our intensity model to credit data and lastly we find the model parameters matching the default probabilities implicit in CDS prices (by bootstrapping) to the default probabilities implied by the model itself.

The calibration of a financial model consists in finding those parameters such that the model prices fit the market ones. This is a well known example of inverse problem and in particular it is known as the inverse problem of mathematical finance. This problem turns out to be ill-posed: existence and uniqueness are not guaranteed, as well as continuous dependence of the solution to market data. Therefore, one usually transforms the problem in an optimization problem. In particular a constrained, non linear least squares minimization problem, setting the residuals as the square of the relative errors. Hence, if $x \in \mathbb{R}^n$ is the vector of the parameters, $y \in \mathbb{R}^m$ the market data vector and h is the function from our model which gives the price, we can write the problem as

$$\mathit{find}\ x \in \Omega\ s.t.\ \mathcal{F}(\widehat{x}) = \min_{x\in\Omega} \mathcal{F}(x), \tag{4.1}$$

where Ω is the feasible region for the parameters x and

$$\mathcal{F}(x) = \frac{1}{2}\sum_{j=1}^{m}(h(x) - y_j)^2. \tag{4.2}$$

We presents now the calibration of the Vasicek model on its term structure.

4.1 Vasicek Calibration

We have seen in Section (3.2.1) that Vasicek model possesses an affine term structure. This means that we have a closed formula for the price of a default-free zero coupon bond, as in (3.16). Through market default-free bond prices we can therefore calibrate the model, knowing the initial value at start date $r_0 = -0.002$. We present in following table the times to maturities, the market prices and the model prices computed after calibration together with absolute and relative errors. Maturities goes from 0 to 10 years with intervals of three months.

Time to Maturities	Mark. Bond	Model Bond	Rel. Errors
0.25	1.00069	1.00047	0.0223762%
0.5	1.00148	1.00086	0.0616003%
0.75	1.00235	1.00118	0.117062%
1.	1.00326	1.00143	0.182161%
1.25	1.00413	1.0016	0.252231%
1.5	1.00498	1.00169	0.327191%
1.75	1.00579	1.00171	0.405731%
2.	1.00654	1.00164	0.487196%
2.25	1.00724	1.00149	0.570973%
2.5	1.00784	1.00125	0.654102%
2.75	1.00831	1.00093	0.73217%
3.	1.00858	1.00052	0.799797%
3.25	1.00861	1.00002	0.852005%
3.5	1.00839	0.999421	0.889174%
3.75	1.00795	0.998732	0.914391%
4.	1.00734	0.997947	0.932285%
4.25	1.0066	0.997064	0.947447%
4.5	1.00572	0.99608	0.95893%
4.75	1.00468	0.994994	0.963705%
5.	1.00342	0.993805	0.958057%
5.25	1.00191	0.992509	0.938539%
5.5	1.00016	0.991105	0.905213%
5.75	0.998169	0.989591	0.859413%
6.	0.995961	0.987965	0.802879%
6.25	0.993551	0.986224	0.737418%
6.5	0.990945	0.984368	0.663735%
6.75	0.988148	0.982394	0.582295%
7.	0.985163	0.9803	0.493635%

Table 4.1: Relative and Absolute Errors between market and model price of default-free bond.

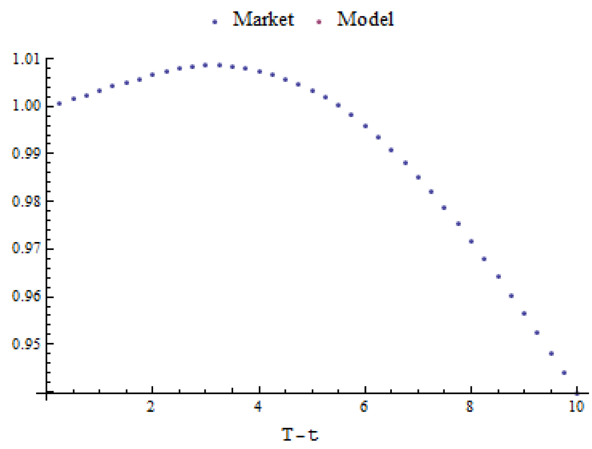


Figure 4.1: Graph of the market and model bond prices.

We implement a *Mathematica* notebook where, after defining the least-square problem, we use the function `NMinimize`, which applies different methods for non linear optimization (e.g. Differential Evolution or Nelder-Mead methods), with the constrains on the parameters given by the model, i.e. $k, \theta, \delta > 0$. We have:

$$k = 1.80726376335, \qquad \theta = 7.56533947221 \cdot 10^{-6}, \qquad \delta = 0.10788042275.$$

4.2 Credit Default Swaps

In this section we formally introduce CDS contracts and their payoff. A *Credit Default Swap* is a basic protection contract between two parties, called the *protection buyer* and the *protection seller*, typically designed to transfer to the protection seller the financial loss the protection buyer would suffer if a particular default event happened to a third part, called the *reference* or *credit entity.* It's important to note that in contemporary CDS contracts neither the protection seller nor protection buyer are obliged to have investments in the underlying reference credit.

CDS's are now actively traded and have become, even after the recent financial crisis, a sort of basic product of the single-name credit derivatives area (analogously to interest-rate swaps being a basic product in the interest-rate derivatives world). As a consequence, the need is no longer to have a model to be used to value CDS's, but rather to consider a model that can be calibrated to CDS's, i.e. to take CDS's as inputs, in order to price more complex credit derivatives.

The protection buyer pays the premium leg consisting of a regular premium (or coupon) payments (e.g. every three months) up to expiry of the CDS, which cease if a default occurs, whereas the protection seller agrees to make a single protection premium in case the pre-specified default event happens at time $\tau \in [0, T]$. The premium leg consists in the payment of a rate R at times $T_1, \dots, T_M$, ending payments in case of default. We assume the year fraction $T_i - T_{i-1}$ to be constant and we denote it by α. We model the protection leg as a random payment Z_τ at default time τ if this is before the expiry (end of protection) of the CDS (time T) and nothing otherwise.

Protection Seller	$\rightarrow$ protection Z_τ at default τ if $\tau \in [0, T]$ $\rightarrow$	Protection Buyer
	$\leftarrow$ rate R at $T_1, \dots, T_M$ or until default τ $\leftarrow$	

We use an intensity model with the technique presented in Chapter (1) and recalling (1.6) we define the Hazard process by

$$\Gamma_t = \Gamma(t, X_t) = \int_0^t \gamma(t, X_t) dt, \tag{4.3}$$

where X_t is the underlying log-price process as in (2.1); for simplicity, we write $\gamma(t, x) = \gamma_t$. Furthermore, we denote by $D(t, s) = e^{-\int_t^s r_u du}$ the stochastic discount factor. In this way, from the prospective of the buyer, the payoff of the CDS at time t is

$$CDS\Pi_t(R) = \mathbb{1}_{\{t<\tau\leq T\}} Z_\tau D(t, \tau) - \sum_{i=1}^{M} \alpha R \mathbb{1}_{\{\tau>T_i\}} D(t, T_i), \tag{4.4}$$

Therefore, the no arbitrage price is

$$CDS_t(R) = \mathbb{E}\left[\mathbb{1}_{\{t<\tau\leq T\}} Z_\tau D(t, \tau) - \sum_{i=1}^{M} \alpha R \mathbb{1}_{\{\tau>T_i\}} D(t, T_i) | \mathscr{G}_t\right]. \tag{4.5}$$

In order to find an explicit formula to compute the par CDS spread, and then calibrate the model to the market data, we assume the protection premium Z_τ to be constant, typically $Z_\tau = L = 1 - R_{\mathrm{EC}}$, where R_{EC} is called recovery. In fact, usually, we can think at the case when the buyer, let's call him "A", owns a corporate bond issued by the reference entity "C" and is waiting for the coupons and final notional payment from "C". If "C" defaults before the bond maturity, "A" does not receive such payments. "A" then goes to the protection seller "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that "A" would face in case "C" defaults.

We want to stress that the par CDS spread R is the value that makes the contract fair, i.e. such that the present value of the two exchanged flows is zero. We need this value because this is the way how market quotes CDS's: CDS are quoted via their fair R's.

In view of (1.3), (1.4) and (1.5) in Corollary (1.1), we have

$$
\begin{aligned}
CDS_t(R) &= L\,\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\int_t^T D(t,v)e^{\Gamma_t-\Gamma_v}d\Gamma_v|\mathscr{F}_t\right] - \alpha R\sum_{i=1}^{M}\mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T_i)e^{\Gamma_t-\Gamma_{T_i}}|\mathscr{F}_t]\\
&= D^{-1}(0,t)e^{\Gamma_t}\mathbb{1}_{\{\tau>t\}}\left(L\,\mathbb{E}\left[\int_t^T D(0,v)e^{-\Gamma_v}d\Gamma_v|\mathscr{F}_t\right] - \alpha R\sum_{i=1}^{M}\mathbb{E}[D(0,T_i)e^{-\Gamma_{T_i}}|\mathscr{F}_t]\right)\\
&= e^{\int_0^t(r_s+\gamma_s)ds}\mathbb{1}_{\{\tau>t\}}\left(L\,\mathbb{E}\left[\int_t^T e^{-\int_0^v(r_s+\gamma_s)ds}\gamma_v dv|\mathscr{F}_t\right] - \alpha R\sum_{i=1}^{M}\mathbb{E}[e^{-\int_0^{T_i}(r_s+\gamma_s)ds}|\mathscr{F}_t]\right)\\
&= e^{\int_0^t(r_s+\gamma_s)ds}\mathbb{1}_{\{\tau>t\}}\left(L\int_t^T \mathbb{E}\left[e^{-\int_0^v(r_s+\gamma_s)ds}\gamma_v|\mathscr{F}_t\right]dv - \alpha R\sum_{i=1}^{M}\mathbb{E}[e^{-\int_0^{T_i}(r_s+\gamma_s)ds}|\mathscr{F}_t]\right)\\
&= e^{\int_0^t(r_s+\gamma_s)ds}\mathbb{1}_{\{\tau>t\}}\left(L\int_t^T \bar{u}(t,x;r,v)dv - \alpha R\sum_{i=1}^{M}u(t,x;r,T_i)\right) \qquad (4.6)
\end{aligned}
$$

where $u(\cdot,x;r,T)$ and $\bar{u}(\cdot,x;r,v)$ can be computed by the approximation method as in Section (2.3). In particular $u(\cdot,x;r,T)$ is the price of the defaultable ZCB and $\bar{u}(\cdot,x;r,v)$ can be approximated solving (2.13) and (2.14), setting $\varphi(x)=\gamma_v(x)$.

Finally, by setting the CDS price equal to zero we can solve in R and find the par CDS spread at time t,

$$
R(t,T) = \frac{L\int_t^T \bar{u}(t,x;r,v)dv}{\alpha\sum_{i=1}^{M}u(t,x;r,T_i)} \qquad (4.7)
$$

Remark 4.1. Assuming zero correlation, i.e. $\rho = 0$, we have that the Brownian motions describing X_t and r_t are independent. In this case, we can rewrite (4.6) obtaining

$$\begin{aligned} CDS_t(R) &= e^{\int_0^t (r_s+\gamma_s)ds}\mathbb{1}_{\{\tau>t\}}\Bigg(L\int_t^T \mathbb{E}[e^{-\int_0^v r_s}|\mathscr{F}_t]\mathbb{E}[e^{-\int_0^v \gamma_s}\gamma_v|\mathscr{F}_t]dv \\ &\quad -\alpha R\sum_{i=1}^{M}\mathbb{1}_{\{\tau>t\}}\mathbb{E}[e^{-\int_0^{T_i} r_s}|\mathscr{F}_t]\mathbb{E}[e^{-\int_0^{T_i}\gamma_s}|\mathscr{F}_t]\Bigg) \\ &= e^{\int_0^t (r_s+\gamma_s)ds}\mathbb{1}_{\{\tau>t\}}\left(-L\int_t^T \mathbb{E}[e^{-\int_0^v r_s}|\mathscr{F}_t]\frac{\partial}{\partial v}\mathbb{E}[e^{-\int_0^v \gamma_s}|\mathscr{F}_t]dv - \dots\right). \end{aligned}$$

We have that $Q(t,T) = \mathbb{E}[e^{-\int_0^T \gamma_s}|\mathscr{F}_t]$ is the survival probability, hence setting $p(t,x;r,T) = \mathbb{E}[e^{-\int_0^T r_s}|\mathscr{F}_t]$ we obtain a formula for the par CDS spread with no correlation

$$R(t,T) = \frac{-L\int_t^T p(t,x;r,v)\frac{\partial}{\partial v}Q(t,v)dv}{\alpha\sum_{i=1}^{M} p(t,x;r,T_i)Q(t,T_i)}. \tag{4.8}$$

4.3 Calibration on CDS spreads

Now we are able to calibrate the JDCEV model to market CDS spreads, by replacing the parameters found for the Vasicek model. First, we compute the so called implicit survival probabilities by a bootstrapping formula obtained recursively from (4.8) as follows

$$\begin{aligned} Q(t,T_1) &= \frac{\alpha R(t,T_1)}{\alpha R(t,T_1)+L} \\ Q(t,T_i) &= \frac{-(\alpha R(t,T_i)+L)\sum_{j=1}^{i-1} D(t,T_j)Q(t,T_j) + L\sum_{j=1}^{i} D(t,T_j)Q(t,T_{j-1})}{\alpha R(t,T_i)D(t,T_i)+LD(t,T_i)}, \quad i \geq 2. \end{aligned}$$

We note that we need sets of data containing the maturities, the discount factors (i.e. the risk-free ZCB prices) and the CDS spreads.

The first approach is to approximate survival probabilities in our model through expansion method. We estimate

$$Q(t,T) = \mathbb{E}[e^{-\int_0^T \gamma_s}|\mathscr{F}_t] = \mathbb{E}[e^{-\int_0^T (b+c\sigma^2 e^{2(\beta-1)X_s})ds}|\mathscr{F}_t] = u(t,x;r,T). \tag{4.9}$$

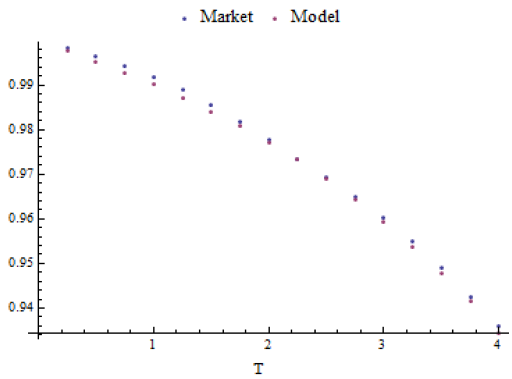
This can be easily done just by changing the $a_{(0,0)}(t,x,y)$ coefficient of the operator (3.20) of the model. As in the case of Vasicek calibration, we set a least square problem with the implicit survival probabilities, then we approximate $u(t,x;r,T)$ at order $n=2$ getting an explicit formula that allows to find the minimizing parameters. We present the numerical results in table (4.2) and in the figures below.

Setting the constrains for the parameters $\sigma > 0$, $b, c \geq 0$, $\beta < 1$ we get

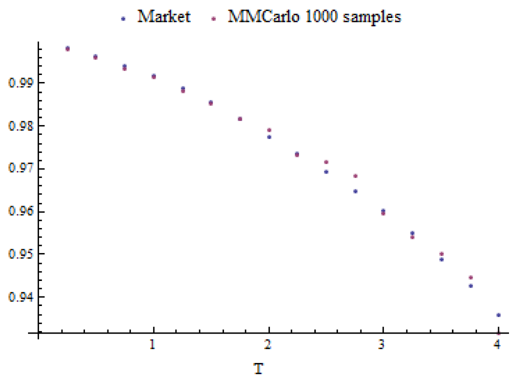
$$\sigma = 0.3567912680, \quad b = 0.00203245509886,$$
$$c = 0.056300595825, \quad \beta = 0.375063514676.$$

Time to Maturities	Mark. Prob	Model Prob	Rel.Error	Abs.Error
0.25	0.998208	0.997666	0.0542674%	0.0541701%
0.5	0.996258	0.995245	0.101657%	0.101277%
0.75	0.994152	0.992707	0.145386%	0.144535%
1.	0.991767	0.99002	0.176179%	0.174728%
1.25	0.988892	0.987153	0.175879%	0.173925%
1.5	0.985545	0.984075	0.149181%	0.147025%
1.75	0.981702	0.980753	0.0967183%	0.0949485%
2.	0.97759	0.977154	0.0446338%	0.0436335%
2.25	0.973459	0.973245	0.0219604%	0.0213775%
2.5	0.969267	0.968994	0.0281691%	0.0273033%
2.75	0.964892	0.964366	0.0544661%	0.0525539%
3.	0.96015	0.959329	0.0854945%	0.0820875%
3.25	0.954879	0.953848	0.107944%	0.103073%
3.5	0.949016	0.94789	0.118634%	0.112585%
3.75	0.942627	0.941421	0.127942%	0.120602%
4.	0.935855	0.934407	0.154722%	0.144797%

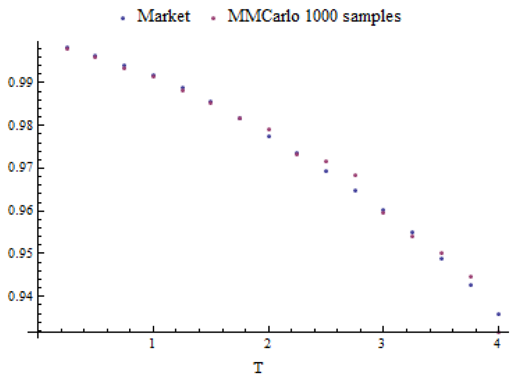
Table 4.2: Relative and Absolute Errors between implicit and model survival probabilities.



(a) Implicit and model survival probability.



(b) Implicit s.p. and Monte Carlo estimate



(c) Approximate s.p. and Monte Carlo estimate

Now, we calibrate the model to CDS spreads in view of (4.7). We have to apply the approximation method to the functions $u(\cdot,x;r,T)$ and $\bar{u}(\cdot,t;r,v)$ to find and explicit formula thanks to which we will find the optimal parameters. We have already calculate the function u in Subsection (3.3.2) where we have seen that it is the price of a defaultable zero coupon bond. Let us remind the expression of $\bar{u}$:

$$\bar{u}(t,x;r,v) = \mathbb{E}\left[e^{-\int_0^v (r_s+\gamma_s)ds}\gamma_v|\mathscr{F}_t\right]. \tag{4.10}$$

In this case, we could follow the approximation method solving (2.13) and (2.14), setting $\varphi(x) = \gamma_v(x)$. However, in this way, we would find an expression that takes very long to be computed. Hence, we rewrite (4.10) as follows. We only need to observe that

$$\begin{aligned}
&\partial_v \left(e^{-\int_0^v (\gamma_s + r_s) ds} \right) = -(\gamma_v - r_v) e^{-\int_0^v (\gamma_s + r_s) ds} \\
&\Longrightarrow \gamma_v e^{-\int_0^v (\gamma_s + r_s) ds} = -\partial_v \left(e^{-\int_0^v (\gamma_s + r_s) ds} \right) - r_v e^{-\int_0^v (\gamma_s + r_s) ds} \\
&\Longrightarrow \mathbb{E}\left[\gamma_v e^{-\int_0^v (\gamma_s + r_s) ds} | \mathscr{F}_t \right] = \mathbb{E}\left[-\partial_v \left(e^{-\int_0^v (\gamma_s + r_s) ds} \right) - r_v e^{-\int_0^v (\gamma_s + r_s) ds} | \mathscr{F}_t \right],
\end{aligned}$$

then,

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T (\gamma_v e^{-\int_0^v (\gamma_s + r_s) ds}) dv | \mathscr{F}_t \right] \\
=&\mathbb{E}\left[\int_t^T (-\partial_v \left(e^{-\int_0^v (\gamma_s + r_s) ds} \right) - r_v e^{-\int_0^v (\gamma_s + r_s) ds}) dv | \mathscr{F}_t \right] \\
=&-\mathbb{E}\left[\int_t^T \partial_v \left(e^{-\int_0^v (\gamma_s + r_s) ds} \right) dv | \mathscr{F}_t \right] - \mathbb{E}\left[\int_t^T r_v e^{-\int_0^v (\gamma_s + r_s) ds} dv | \mathscr{F}_t \right] \\
=&-\mathbb{E}\left[e^{-\int_t^T (\gamma_s + r_s) ds} | \mathscr{F}_t \right] - \int_t^T \mathbb{E}\left[r_v e^{-\int_0^v (\gamma_s + r_s) ds} | \mathscr{F}_t \right] dv \\
=&-u(t, x; r, T) - \int_t^T \widehat{u}(v, x; r, T) dv.
\end{aligned}$$

Therefore, we can rewrite (4.7) as

$$R(t, T) = L \frac{-u(t, x; r, T) - \int_t^T \widehat{u}(v, x; r, T) dv}{\alpha \sum_{i=1}^M u(t, x; r, T_i)}. \tag{4.11}$$

This formula is easier to compute since we know that the Vasicek is a deterministic-coefficients process with normal distribution (see (3.11)). Setting the constrains for the parameters $\sigma > 0$, $b, c \geq 0$, $\beta < 1$ we get

$$\begin{aligned}
&\sigma = 0.14680028182798, \quad b = 0.001098769809, \\
&c = 0.350607043483, \quad \beta = -1.762292575318, \quad \rho = -0.075584678937.
\end{aligned}$$

Therefore, having market data for CDS spreads, we are able in this case to find a minimizing value for the correlation parameter ρ. We summarize the results after replacing the parameters to the approximate formula for the CDS spread in table (4.3). The spreads estimate is not so accurate as before, due to the fact that we have approximated functions in (4.11) at order $n = 1$ to avoid an excessive computing time.

Time to Maturities	Market Spreads	Model Spreads	Abs.Error	Rel.Error
0.25	0.00430851	0.00470817	−0.0399663%	−9.27614%
0.5	0.0045025	0.00477789	−0.0275386%	−6.11628%
0.75	0.00469649	0.00485304	−0.0156547%	−3.33327%
1.	0.0049646	0.00497854	−0.00139449%	−0.280887%
1.25	0.00535106	0.0051716	0.0179464%	3.3538%
1.5	0.00582786	0.00543718	0.0390682%	6.7037%
1.75	0.00633408	0.0057749	0.0559179%	8.8281%
2.	0.0068	0.00618175	0.0618252%	9.09195%
2.25	0.00717167	0.00665286	0.0518818%	7.23426%
2.5	0.00748821	0.007181	0.0307215%	4.10264%
2.75	0.00779196	0.00775516	0.00367989%	0.472268%
3.	0.0081242	0.00835808	−0.0233878%	−2.87878%
3.25	0.00850947	0.00896287	−0.0453392%	−5.32809%
3.5	0.00894881	0.00952914	−0.0580332%	−6.48502%
3.75	0.00942144	0.0100008	−0.0579312%	−6.14886%
4.	0.0099	0.0103107	−0.0410651%	−4.14799%
4.25	0.0103596	0.010406	−0.00463797%	−0.447699%
4.5	0.010797	0.0103242	0.0472821%	4.37917%
4.75	0.0112111	0.0104064	0.0804699%	7.17772%
5.	0.0115909	0.0119536	−0.0362721%	−3.12936%

Table 4.3: Relative and Absolute Errors between market and model CDS Spreads with parameters from CDS calibration.

Finally, once calibrated our intensity model, we have to check whether the model parameters match the default probabilities implicit in CDS prices (by bootstrapping) to the default probabilities implied by the model itself. We substitute in the computation of the implicit and approximate survival probabilities parameters from CDS calibration. We can see in table (4.4) that we find low levels for the relative errors, mostly for small times, verifying the validity of the technique presented in Section (2.3) and its adaptability to more complex valuation contracts as credit default swaps.

Time to Maturities	Mark. Prob	Model Prob	Rel.Error	Abs.Error
0.25	0.998208	0.998093	0.0115415%	0.0115208%
0.5	0.996258	0.996097	0.0161915%	0.0161309%
0.75	0.994152	0.993935	0.0218449%	0.0217172%
1.	0.991767	0.991533	0.0236441%	0.0234494%
1.25	0.988892	0.988817	0.00759371%	0.00750936%
1.5	0.985545	0.985719	−0.0176153%	−0.0173607%
1.75	0.981702	0.982169	−0.0475276%	−0.0466579%
2.	0.97759	0.978101	−0.0522909%	−0.051119%
2.25	0.973459	0.973453	0.000648175%	0.000630972%
2.5	0.969267	0.968162	0.114055%	0.11055%
2.75	0.964892	0.962168	0.28229%	0.272379%
3.	0.96015	0.955416	0.493097%	0.473447%
3.25	0.954879	0.947848	0.736291%	0.703069%
3.5	0.949016	0.939413	1.01184%	0.960252%
3.75	0.942627	0.93006	1.33318%	1.25669%
4.	0.935855	0.919739	1.72205%	1.61159%

Table 4.4: Relative and Absolute Errors between implicit and model survival probabilities with parameters from CDS calibration.

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