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Calculus of Variations

A priori estimates for Donaldson's equation over compact Hermitian manifolds

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Abstract In this paper we prove a priori estimates for Donaldson equation's

$$\omega\wedge(\chi+\sqrt{-1}\partial\bar{\partial}\varphi)^{n-1}=e^F(\chi+\sqrt{-1}\partial\bar{\partial}\varphi)^n,$$

over a compact complex manifold X of complex dimension n, where ω and χ are arbitrary Hermitian metrics. Our estimates answer a question of Tosatti-Weinkove (Asian J. Math. 14:19–40, 2010).

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1 Introduction

1.1 Donaldson's equation over compact Kähler manifolds

Let (X,ω) be a compact Kähler manifold of the complex dimension n, and χ another Kähler metric on X. In [3], Donaldson considered the following interesting equation

$$\omega\wedge\eta^{n-1}=c\eta^n,\quad [\eta]=[\chi], \tag{1.1}$$

where c is a constant, depending only on the Kähler classes of $[\chi]$ and $[\omega]$, given by

$$c=\frac{\int_X\omega\wedge\chi^{n-1}}{\int_X\chi^n}. \tag{1.2}$$

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He noted that a necessary condition for Eq. (1.1) is

$$nc\chi - \omega > 0, \tag{1.3}$$

and then conjectured that the condition (1.3) is also sufficient. For $n = 2$, Chen [1] observed that in this case the Eq. (1.1) reduces to a complex Monge-Ampère equation completely solved by Yau on his celebrated work on Calabi's conjecture [24].

1.2 J-flow and Donaldson's equation

To better understand the Eq. (1.1), Donaldson [3] and Chen [1] independently discovered the J-flow whose critical point gives the Eq. (1.1), and Chen showed that such flow always exists for all time. Using the J-flow, Chen [2] proved that if $n = 2$ and the holomorphic bisectional curvature of ω is nonnegative then the J-flow converges to a critical metric. Later, the curvature assumption was removed by Weinkove [22] and hence gave an alternative proof of Donaldson's conjecture on Kähler surfaces. For higher dimensional case, Weinkove [23] solved Donaldson's conjecture on a slightly strong condition

$$nc\chi - (n-1)\omega > 0 \tag{1.4}$$

using the J-flow. For more detailed discussions and related works, we refer to [4–7,15,16].

1.3 Donaldson's equation over compact Hermitian manifolds

Recently, the complex Monge-Ampère equation over compact Hermitian manifolds was solved Tosatti and Weinkove [17,18]. Other interesting estimates can be found in [19,25,26]. A parabolic proof was late given by Gill [8] by considering a parabolic complex Monge-Ampère equation. Other parabolic flows over compact Hermitian manifolds were considered in [14,19–21], where they obtained lots of interesting results parallel to those in Kähler case. By Tosatti-Weinkove's work, the author considers Donaldson's equation over compact Hermitian manifolds.

Let (X, ω) be a compact Hermitian manifold of the complex dimension n and χ another Hermitian metric on X. We denote by $\mathcal{H}_\chi$ the set of all real-valued smooth functions φ on X such that $\chi_\varphi := \chi + \sqrt{-1}\partial\overline{\partial}\varphi > 0$. Locally we have

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge dz^{\bar{j}}, \quad \chi = \sqrt{-1}\chi_{i\bar{j}}dz^i \wedge dz^{\bar{j}}. \tag{1.5}$$

For any real positive $(1,1)$-form $\alpha := \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$ and real $(1,1)$-form $\beta := \sqrt{-1}\beta_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$ we set

$$\mathrm{tr}_\alpha\beta := \alpha^{i\bar{j}}\beta_{i\bar{j}}. \tag{1.6}$$

We consider Donaldson's equation

$$\omega \wedge \chi_\varphi^{n-1} = e^F \cdot \chi_\varphi^n, \quad \varphi \in \mathcal{H}_\varphi \tag{1.7}$$

on X, where F is a given smooth function on X.

The main result of this paper is the following a priori estimates.

Theorem 1.1 *Let (X, ω) be a compact Hermitian manifold of the complex dimension n and χ another Hermitian metric. Let φ be a smooth solution of Donaldson's equation (1.7). Assume that*

$$\chi - \frac{n-1}{ne^F}\omega > 0. \tag{1.8}$$

Then

(1) *there exist uniform constant* $A > 0$ *and* $C > 0$*, depending only on* X, ω, χ*, and* F*, such that*

$$\mathrm{tr}_{\omega}\chi_{\varphi} \leq C \cdot e^{A(\varphi - \inf_X \varphi)}; \tag{1.9}$$

(2) *there exists a uniform constant* $C > 0$*, depending only on* X, ω, χ*, and* F*, such that*

$$||\varphi||_{C^0} \leq C; \tag{1.10}$$

(3) *there are uniform* C^{∞} *a priori estimates on* φ *depending only on* X, ω, χ*, and* F*.*

Meanwhile, Guan, Li and Sun [9,11–13] considered a priori estimates for Donaldson's equation over compact Hermitian manifolds under very general structure conditions rather than the condition (1.8).

Remark 1.2 As remarked in [17] (see page 22, line 27–28), to prove the zeroth estimate in Theorem 1.1 it suffices to show the second order estimate on φ. Our result gives an affirmative answer to the question in [17] (see page 22, line 28–30). Using the same argument in [17] (page 33), we can get a C^{α} estimate on φ for some $\alpha \in (0, 1)$. Differentiating (1.7) and applying the standard local elliptic estimates imply uniform C^{∞} estimates on φ.

There are some natural questions about the Eq. (1.7). Is condition (1.8) sufficient to product a solution to (1.7)? When ω and χ both are Kähler, it has been proved in [2,22,23] that this condition is sufficient. The second question is to consider a parabolic flow over compact Hermitian manifolds like the J-flow. Can we prove the long time existence and convergence of such a flow? Song and Weinkove [16] gave a necessary and sufficient condition for existence of solutions to the Donaldson's equation over compact Kähler manifolds (and also for convergence of the J-flow over compact Kähler manifolds). The last question then is whether we can find an analogous of above Song-Weinkove's condition. Those questions will be answered later.

Remark 1.3 Here and henceforth, when we say a "uniform constant" it should be understood to be a constant that depends only on X, ω, χ, and F. We will often write C or C' for such a constant, where the value of C or C' may differ from line to line. For the relation $P \leq CQ$ for a uniform constant C in the above sense, we write it as $P \lesssim Q$. $\mathrm{Re}(P)$ means the real part of P.

2 The second order estimates

2.1 Basic facts and notions

Let (X, ω) be a complex Hermitian manifold of the complex dimension n and χ another Hermitian metric on X. For a solution φ of Donaldson's equation (1.7), we denote by

$$\chi' := \chi + \sqrt{-1}\partial\overline{\partial}\varphi = \sqrt{-1}(\chi_{i\bar{j}} + \varphi_{i\bar{j}})dz^i \wedge dz^{\bar{j}}. \tag{2.1}$$

Also, we set $\chi'_{i\bar{j}} := \chi_{i\bar{j}} + \varphi_{i\bar{j}}$. Then we observe that

$$\mathrm{tr}_{\chi'}\omega = n\frac{\omega \wedge (\chi')^{n-1}}{(\chi')^n} = ne^F. \tag{2.2}$$

Consequently, $\mathrm{tr}_{\chi'}\omega$ is uniformly bounded away from zero and infinity. Let Δ_ω denote the Laplacian operator of the Chern connection associated to the Hermitian metric ω, and similarly for Δ_χ. Note that

$$\mathrm{tr}_\omega\chi' = g^{i\bar{j}}(\chi_{i\bar{j}} + \varphi_{i\bar{j}}) = \mathrm{tr}_\omega\chi + \Delta_\omega\varphi. \tag{2.3}$$

Remark 2.1 $\mathrm{tr}_\omega\chi'$ and $\mathrm{tr}_{\chi'}\omega$ are uniformly bounded from below away from zero. More precisely,

$$\mathrm{tr}_\omega\chi' \geq \frac{n}{e^F}, \quad \mathrm{tr}_{\chi'}\omega = ne^F. \tag{2.4}$$

The second assertion follows from (2.2), while the first inequality is obtained as follows. We choose a normal coordinate system so that

$$g_{i\bar{j}} = \delta_{ij}, \quad \chi'_{i\bar{j}} = \lambda'_i\delta_{ij}$$

for some $\lambda'_i, \ldots, \lambda'_n > 0$. Donaldson's equation then yields

$$ne^F = \sum_{1\leq i\leq n}\frac{1}{\lambda'_i}.$$

An elementary inequality shows that

$$\mathrm{tr}_\omega\chi' = \sum_{1\leq i\leq n}\lambda'_i \geq \frac{n^2}{\sum_{1\leq i\leq n}\frac{1}{\lambda'_i}} = \frac{n^2}{ne^F} = \frac{n}{e^F}.$$

We will frequently use the following

Lemma 2.2 (Guan-Li [10]) *At any point $p \in X$ there exists a holomorphic coordinates system centered at p such that, at p,*

$$g_{i\bar{j}} = \delta_{ij}, \quad \partial_j g_{i\bar{i}} = 0 \tag{2.5}$$

for all i and j. Furthermore, we can assume that $\chi'_{i\bar{j}}$ is diagonal.

Let $\widetilde{\Delta}$ denote the Laplacian operator associated to the Hermitian metric $h_{i\bar{j}}$ whose inverse matrix is given by

$$h^{i\bar{j}} := \chi'^{i\bar{\ell}}\chi'^{k\bar{j}}g_{k\bar{\ell}}; \tag{2.6}$$

and $\widetilde{\nabla}$ the associated covariant derivatives.

The basic idea to obtain the second order estimate, following from the method of Yau [24], is to consider the quantity

$$Q := \log(\mathrm{tr}_\omega\chi') - A\varphi \tag{2.7}$$

for some suitable constant A. Our first step is to estimate the term $\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi')$.

Definition 2.3 For convenience, we say that a term E is of **type I** if

$$|E|_\omega \lesssim 1, \tag{2.8}$$

and is of **type II** if

$$|E|_\omega \lesssim \mathrm{tr}_\omega\chi'. \tag{2.9}$$

It is east to see that any uniform constant is of type I and any type I term is of type II. We will use E_1 and E_2 to denote a type I and type II term, respectively.

2.2 The estimate for $\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi')$

Direct computation shows

$$\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi') = \frac{\widetilde{\Delta}\mathrm{tr}_\omega\chi'}{\mathrm{tr}_\omega\chi'} - \frac{|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2}{(\mathrm{tr}_\omega\chi')^2}. \tag{2.10}$$

By the definition, we have

$$\begin{aligned}
\widetilde{\Delta}\mathrm{tr}_\omega\chi' &= h^{i\bar{j}}\partial_i\partial_{\bar{j}}(g^{k\bar{\ell}}\chi'_{k\bar{\ell}}) \\
&= h^{i\bar{j}}\partial_i\left(-g^{k\bar{b}}g^{a\bar{\ell}}\partial_{\bar{j}}g_{a\bar{b}}\cdot\chi'_{k\bar{\ell}} + g^{k\bar{\ell}}\partial_{\bar{j}}\chi'_{k\bar{\ell}}\right) \\
&= h^{i\bar{j}}\Big[g^{k\bar{\ell}}\partial_i\partial_{\bar{j}}\chi'_{k\bar{\ell}} - g^{k\bar{b}}g^{a\bar{\ell}}\partial_i g_{a\bar{b}}\cdot\partial_{\bar{j}}\chi'_{k\bar{\ell}} - g^{k\bar{b}}g^{a\bar{\ell}}\partial_{\bar{j}}g_{a\bar{b}}\cdot\partial_i\chi'_{k\bar{\ell}} \\
&\quad - \Big(-g^{k\bar{q}}g^{p\bar{b}}\partial_i g_{p\bar{q}}\cdot g^{a\bar{\ell}}\partial_{\bar{j}}g_{a\bar{b}} - g^{k\bar{b}}g^{a\bar{q}}g^{p\bar{\ell}}\partial_i g_{p\bar{q}}\cdot\partial_{\bar{j}}g_{a\bar{b}} \\
&\quad + g^{k\bar{b}}g^{a\bar{\ell}}\partial_i\partial_{\bar{j}}g_{a\bar{b}}\Big)\chi'_{k\bar{\ell}}\Big].
\end{aligned}$$

Using the local coordinates in Lemma 2.2, we deduce that

$$\begin{aligned}
\widetilde{\Delta}\mathrm{tr}_\omega\chi' &= \sum_{1\le i,j\le n} h^{i\bar{i}}\partial_i\partial_{\bar{i}}\chi'_{k\bar{k}} - \sum_{1\le i,k,\ell\le n} h^{i\bar{i}}\partial_i g_{\ell\bar{k}}\cdot\partial_{\bar{i}}\chi'_{k\bar{\ell}} \\
&\quad - \sum_{1\le i,k,\ell\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{\ell\bar{k}}\cdot\partial_i\chi'_{k\bar{\ell}} + \sum_{1\le i,k,p\le n} h^{i\bar{i}}\partial_i g_{p\bar{k}}\cdot\partial_{\bar{i}}g_{k\bar{p}}\cdot\chi'_{k\bar{k}} \\
&\quad + \sum_{1\le i,k,q\le n} h^{i\bar{i}}\partial_i g_{k\bar{q}}\cdot\partial_{\bar{i}}g_{q\bar{k}}\cdot\chi'_{k\bar{k}} - \sum_{1\le i,k\le n} h^{i\bar{i}}\partial_i\partial_{\bar{i}}g_{k\bar{k}}\cdot\chi'_{k\bar{k}} \\
&= \sum_{1\le i,k\le n} h^{i\bar{i}}\partial_i\partial_{\bar{i}}\chi'_{k\bar{k}} - 2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\cdot\partial_i\chi'_{k\bar{j}}\right) + E_1,
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
E_1 &= \sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_i g_{j\bar{k}}\cdot\partial_{\bar{i}}g_{k\bar{j}}\cdot\chi'_{k\bar{k}} + \sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_i g_{k\bar{j}}\cdot\partial_{\bar{i}}g_{j\bar{k}}\cdot\chi'_{k\bar{k}} \\
&\quad - \sum_{1\le i,k\le n} h^{i\bar{i}}\partial_i\partial_{\bar{i}}g_{k\bar{k}}\cdot\chi'_{k\bar{k}}.
\end{aligned}$$

Since under the above mentioned local coordinates $\chi'_{i\bar{i}} = \lambda'_i\delta_{ij}$, it follows that $h^{i\bar{i}} = (\chi'^{i\bar{i}})^2 = 1/\lambda_i'^2$; hence $h^{i\bar{i}} \le e^{2F}$ using Remark 2.1. Therefore we see that E_1 is of type II, i.e.,

$$|E_1|_\omega \lesssim \mathrm{tr}_\omega\chi'. \tag{2.12}$$

The first term on the right hand side of (2.11) can be computed as follows: From Donaldson's equation (1.7), we obtain

$$ne^F = \mathrm{tr}_{\chi'}\omega = \chi'^{i\bar{j}}g_{i\bar{j}}$$

and, after taking the derivative with respect to $z^{\bar{\ell}}$,

$$n\partial_{\bar{\ell}}F\cdot e^F = -\chi'^{i\bar{b}}\chi'^{a\bar{j}}\partial_{\bar{\ell}}\chi'_{a\bar{b}}\cdot g_{i\bar{j}} + \chi'^{i\bar{j}}\partial_{\bar{\ell}}g_{i\bar{j}}.$$

Differentiating above equation again with respect to z^k yields

$$\begin{aligned} n\partial_k\partial_{\bar\ell}F\cdot e^F + n\partial_{\bar\ell}F\partial_k F\cdot e^F &= -\chi'^{i\bar b}\chi'^{a\bar j}g_{i\bar j}\partial_k\partial_{\bar\ell}\chi'_{a\bar b} - \chi'^{i\bar b}\chi'^{a\bar j}\partial_{\bar\ell}\chi'_{a\bar b}\partial_k g_{i\bar j} \\ &- \left(-\chi'^{i\bar q}\chi'^{p\bar b}\partial_k\chi'_{p\bar q}\cdot\chi'^{a\bar j}g_{i\bar j} - \chi'^{i\bar b}\chi'^{a\bar q}\chi'^{p\bar j}\partial_k\chi'_{p\bar q}\cdot g_{i\bar j}\right)\partial_{\bar\ell}\chi'_{a\bar b} \\ &- \chi'^{i\bar b}\chi'^{a\bar j}\partial_k\chi'_{a\bar b}\cdot\partial_{\bar\ell}g_{i\bar j} + \chi'^{i\bar j}\partial_k\partial_{\bar\ell}g_{i\bar j} \\ &= -\chi'^{i\bar b}\chi'^{a\bar j}g_{i\bar j}\partial_k\partial_{\bar\ell}\chi'_{a\bar b} - \chi'^{i\bar b}\chi'^{a\bar j}\partial_k\chi'_{a\bar b}\cdot\partial_{\bar\ell}g_{i\bar j} + \chi'^{i\bar j}\partial_k\partial_{\bar\ell}g_{i\bar j} \\ &- \left(-\chi'^{i\bar q}\chi'^{p\bar b}\partial_k\chi'_{p\bar q}\cdot\chi'^{a\bar j}g_{i\bar j} - \chi'^{i\bar b}\chi'^{a\bar q}\partial_k\chi'_{p\bar q}\cdot g_{i\bar j} + \chi'^{i\bar b}\chi'^{a\bar j}\partial_k g_{i\bar j}\right)\partial_{\bar\ell}\chi'_{a\bar b}. \end{aligned}$$

Multiplying above by $g^{k\bar\ell}$ on both sides implies

$$\begin{aligned} \left(\Delta_\omega F + |\nabla F|^2_\omega\right) ne^F &= -\sum_{1\le i,j,k,\ell\le n}\left(h^{i\bar j}g^{k\bar\ell}\partial_k\partial_{\bar\ell}\chi'_{i\bar j} - \chi'^{i\bar j}g^{k\bar\ell}\partial_k\partial_{\bar\ell}g_{i\bar j}\right) \\ &- \sum_{1\le i,j,k,\ell,a,b\le n}\chi'^{i\bar b}\chi'^{a\bar j}g^{k\bar\ell}\partial_k\chi'_{a\bar b}\cdot\partial_{\bar\ell}g_{i\bar j} + \sum_{1\le i,j,k,\ell,p,q\le n}h^{i\bar q}\chi'^{p\bar j}g^{k\bar\ell}\partial_k\chi'_{p\bar q}\cdot\partial_{\bar\ell}\chi'_{i\bar j} \\ &+ \sum_{1\le i,j,k,\ell,p,q\le n}h^{p\bar j}\chi'^{i\bar q}g^{k\bar\ell}\partial_k\chi'_{p\bar q}\cdot\partial_{\bar\ell}\chi'_{i\bar j} - \sum_{1\le i,j,k,\ell,a,b\le n}\chi'^{i\bar b}\chi'^{a\bar j}g^{k\bar\ell}\partial_{\bar k}g_{i\bar j}\cdot\partial_{\bar\ell}\chi'_{a\bar b}. \end{aligned}$$

Using the local coordinates (2.5) we arrive at

$$\begin{aligned} &\left(\Delta_\omega F + |\nabla F|^2_\omega\right) ne^F \\ &= -\sum_{1\le i,k\le n}h^{i\bar i}\partial_k\partial_{\bar k}\chi'_{i\bar i} + \sum_{1\le i,k\le n}\chi'^{i\bar i}\partial_k\partial_{\bar k}g_{i\bar i} + \sum_{1\le i,j,k\le n}h^{i\bar i}\chi'^{j\bar j}\partial_k\chi'_{j\bar i}\cdot\partial_{\bar k}\chi'_{i\bar j} \\ &\quad + \sum_{1\le i,j,k\le n}h^{i\bar i}\chi'^{j\bar j}\partial_k\chi'_{i\bar j}\cdot\partial_{\bar k}\chi'_{j\bar i} - 2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar i}\chi'^{j\bar j}\partial_k g_{i\bar j}\cdot\partial_{\bar k}\chi'_{j\bar i}\right). \end{aligned}$$

Equivalently,

$$\begin{aligned} \sum_{1\le i,k\le n}h^{i\bar i}\partial_k\partial_{\bar k}\chi'_{i\bar i} = &\sum_{1\le i,j,k\le n}h^{i\bar i}\chi'^{j\bar j}\partial_k\chi'_{j\bar i}\partial_{\bar k}\chi'_{i\bar j} + \sum_{1\le i,j,k\le n}h^{i\bar i}\chi'^{j\bar j}\partial_k\chi'_{i\bar j}\partial_{\bar k}\chi'_{j\bar i} \\ &- 2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar i}\chi'^{j\bar j}\partial_k g_{i\bar j}\cdot\partial_{\bar k}\chi'_{j\bar i}\right) \\ &+ \sum_{1\le i,k\le n}\chi'^{i\bar i}\partial_k\partial_{\bar k}g_{i\bar i} - \left(\Delta_\omega F + |\nabla F|^2_\omega\right) ne^F. \end{aligned} \tag{2.13}$$

Since

$$\begin{aligned} \partial_k\partial_{\bar k}\chi'_{i\bar i} &= \partial_k\partial_{\bar k}\left(\chi_{i\bar i} + \varphi_{i\bar i}\right) \\ &= \partial_k\partial_{\bar k}\chi_{i\bar i} + \partial_k\partial_{\bar k}\varphi_{i\bar i} \\ &= \partial_k\partial_{\bar k}\chi_{i\bar i} + \partial_i\partial_{\bar i}\varphi_{k\bar k} \\ &= \partial_k\partial_{\bar k}\chi_{i\bar i} + \partial_i\partial_{\bar i}\left(\chi'_{k\bar k} - \chi_{k\bar k}\right) \\ &= \partial_i\partial_{\bar i}\chi'_{k\bar k} + \left(\partial_k\partial_{\bar k}\chi_{i\bar i} - \partial_i\partial_{\bar i}\chi_{k\bar k}\right), \end{aligned}$$

we conclude that

$$\sum_{1\le i,k\le n} h^{i\bar{i}}\partial_i\partial_{\bar{i}}\chi'_{k\bar{k}} = \sum_{1\le i,k\le n} h^{i\bar{i}}\partial_k\partial_{\bar{k}}\chi'_{i\bar{i}} + \sum_{1\le i,k\le n} h^{i\bar{i}}\left(\partial_i\partial_{\bar{i}}\chi_{k\bar{k}} - \partial_k\partial_{\bar{k}}\chi_{i\bar{i}}\right). \quad (2.14)$$

Combining (2.13) and (2.14) yields

$$\begin{aligned}\sum_{1\le i,k\le n} h^{i\bar{i}}\partial_i\partial_{\bar{i}}\chi'_{k\bar{k}} = &\sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\cdot\partial_{\bar{k}}\chi'_{i\bar{j}}\\ &+\sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar{i}}\chi'^{j\bar{j}}\partial_k g_{j\bar{i}}\cdot\partial_{\bar{k}}\chi'_{i\bar{j}}\right)+E_2, \quad (2.15)\end{aligned}$$

where

$$E_2 = \sum_{1\le i,k\le n}\chi'^{i\bar{i}}\partial_k\partial_{\bar{k}}g_{i\bar{i}} + \sum_{1\le i,k\le n} h^{i\bar{i}}\left(\partial_i\partial_{\bar{i}}\chi_{k\bar{k}} - \partial_k\partial_{\bar{k}}\chi_{i\bar{i}}\right) - \left(\Delta_\omega F + |\nabla F|^2_\omega\right)ne^F.$$

By the same reason that $\chi'^{i\bar{i}} \le e^F$ and $h^{i\bar{i}} \le e^{2F}$, we observe that E_2 is of type I and

$$|E_2|_\omega \lesssim 1. \quad (2.16)$$

From (2.11) and (2.15), we get

$$\begin{aligned}\widetilde{\Delta}\mathrm{tr}_\omega\chi' = &\sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} + \sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar{i}}\chi'^{j\bar{j}}\partial_k g_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}\right)\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i\chi'_{k\bar{j}}\right)+E_1+E_2\\ = &\sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} + \sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar{i}}\chi'^{j\bar{j}}\partial_k g_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}\right)\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i\chi'_{k\bar{j}}\right)+E_2,\end{aligned}$$

since any type I term is also of type II.

2.3 The estimate for $\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi')$, continued: ω is Kähler

In the case that ω is Kähler, we in addition have $\partial_k g_{ij} = 0$ for any i, j, k in Lemma 2.2, and we deduce from the above equation that

$$\widetilde{\Delta}\mathrm{tr}_\omega\chi' = \sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} + \sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + E_2. \quad (2.17)$$

It remains to control the term $|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2/(\mathrm{tr}_\omega\chi')^2$. Notice that

$$\partial_i\left(\mathrm{tr}_\omega\chi'\right)=\partial_i\left(g^{k\bar{\ell}}\chi'_{k\bar{\ell}}\right)=g^{k\bar{\ell}}\partial_i\chi'_{k\bar{\ell}}=\sum_{1\le k\le n}\partial_i\chi'_{k\bar{k}}.$$

As in [17], we first give an inequality for $|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2/\mathrm{tr}_\omega\chi'$ and then we control the term $\mathrm{Re}\left(\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}(\partial_i\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}})\right)$. From

$$\begin{aligned}\frac{|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2}{\mathrm{tr}_\omega\chi'}&=\sum_{1\le i,j,k\le n}\frac{h^{i\bar{i}}\partial_i\chi'_{j\bar{j}}\partial_{\bar{i}}\chi'_{k\bar{k}}}{\mathrm{tr}_\omega\chi'}=\sum_{1\le j,k,i\le n}\frac{\sqrt{h^{i\bar{i}}}\partial_i\chi'_{j\bar{j}}\sqrt{h^{i\bar{i}}}\partial_{\bar{i}}\chi'_{k\bar{k}}}{\mathrm{tr}_\omega\chi'}\\
&\le\frac{1}{\mathrm{tr}_\omega\chi'}\sum_{1\le j,k\le n}\left(\sum_{1\le i\le n}h^{i\bar{i}}|\partial_i\chi'_{j\bar{j}}|^2\right)^{1/2}\left(\sum_{1\le i\le n}h^{i\bar{i}}|\partial_i\chi'_{k\bar{k}}|^2\right)^{1/2}\\
&=\frac{1}{\mathrm{tr}_\omega\chi'}\left[\sum_{1\le j\le n}\left(\sum_{1\le i\le n}h^{i\bar{i}}|\partial_i\chi'_{j\bar{j}}|^2\right)^{1/2}\right]^2\\
&=\frac{1}{\mathrm{tr}_\omega\chi'}\left[\sum_{1\le j\le n}\sqrt{\chi'_{j\bar{j}}}\left(\sum_{1\le i\le n}h^{i\bar{i}}\chi'^{j\bar{j}}|\partial_i\chi'_{j\bar{j}}|^2\right)^{1/2}\right]^2\\
&\le\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}|\partial_i\chi'_{j\bar{j}}|^2=\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\partial_{\bar{i}}\chi'_{j\bar{j}}.\end{aligned}$$

From

$$\begin{aligned}\partial_i\chi'_{j\bar{j}}=\partial_i\left(\chi_{j\bar{j}}+\varphi_{j\bar{j}}\right)&=\partial_i\chi_{j\bar{j}}+\partial_j\varphi_{i\bar{j}}=\partial_i\chi_{j\bar{j}}-\partial_j\chi_{i\bar{j}}+\partial_j\chi'_{i\bar{j}},\\
\partial_{\bar{i}}\chi'_{j\bar{j}}=\partial_{\bar{i}}\left(\chi_{j\bar{j}}+\varphi_{j\bar{j}}\right)&=\partial_{\bar{i}}\chi_{j\bar{j}}+\partial_{\bar{j}}\varphi_{j\bar{i}}=\partial_{\bar{i}}\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}+\partial_{\bar{j}}\chi'_{j\bar{i}},\end{aligned}$$

it follows that

$$\begin{aligned}\frac{|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2}{\mathrm{tr}_\omega\chi'}&\le\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\left(\partial_j\chi'_{i\bar{j}}+\partial_i\chi_{j\bar{j}}-\partial_j\chi_{i\bar{j}}\right)\left(\partial_{\bar{j}}\chi'_{j\bar{i}}+\partial_{\bar{i}}\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}\right)\\
&=\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\partial_{\bar{j}}\chi'_{j\bar{i}}+\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\left|\partial_i\chi_{j\bar{j}}-\partial_j\chi_{i\bar{j}}\right|^2\\
&\quad+2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\left(\partial_{\bar{i}}\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}\right)\right].\end{aligned}\tag{2.18}$$

Note that

$$\partial_j\chi'_{i\bar{j}}=\partial_j\left(\chi_{i\bar{j}}+\varphi_{i\bar{j}}\right)=\partial_j\chi_{i\bar{j}}+\partial_i\varphi_{j\bar{j}}=\partial_j\chi_{i\bar{j}}-\partial_i\chi_{j\bar{j}}+\partial_i\chi'_{j\bar{j}}.\tag{2.19}$$

Substituting (2.19) into (2.18) we obtain

$$\begin{aligned}\frac{|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2}{\mathrm{tr}_\omega\chi'} &\le \sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\partial_{\bar{j}}\chi'_{j\bar{i}} - \sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\left|\partial_i\chi_{j\bar{j}} - \partial_j\chi_{i\bar{j}}\right|^2 \\ &\quad + 2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\left(\partial_{\bar{i}}\chi_{j\bar{j}} - \partial_{\bar{j}}\chi_{j\bar{i}}\right)\right] \\ &\le \sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\partial_{\bar{j}}\chi'_{j\bar{i}} + 2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\left(\partial_{\bar{i}}\chi_{j\bar{j}} - \partial_{\bar{j}}\chi_{j\bar{i}}\right)\right].\end{aligned} \tag{2.20}$$

Lemma 2.4 *If ω is Kähler, then $\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi') \gtrsim -1$.*

Proof Calculate, since $h^{j\bar{j}} = \chi'^{j\bar{j}}\chi'^{j\bar{j}}$,

$$\begin{aligned}&\left|2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\left(\partial_{\bar{i}}\chi_{j\bar{j}} - \partial_{\bar{j}}\chi_{j\bar{i}}\right)\right]\right| \\ &\quad = \left|2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n} \sqrt{h^{j\bar{j}}}\sqrt{\chi'^{j\bar{j}}}\partial_i\chi'_{j\bar{j}}\cdot\sqrt{\chi'_{j\bar{j}}}h^{i\bar{i}}\left(\partial_{\bar{i}}\chi_{j\bar{j}} - \partial_{\bar{j}}\chi_{j\bar{i}}\right)\right]\right| \\ &\quad \le \sum_{1\le i,j\le n} h^{j\bar{j}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\partial_{\bar{i}}\chi'_{j\bar{j}} + \sum_{1\le i,j\le n}\chi'_{j\bar{j}}(h^{i\bar{i}})^2|\partial_{\bar{i}}\chi_{j\bar{j}} - \partial_{\bar{j}}\chi_{j\bar{i}}|^2 \\ &\quad \le \sum_{1\le i,j,k\le n} h^{k\bar{k}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{k}}\partial_{\bar{i}}\chi'_{k\bar{j}} + E_2 \\ &\quad = \sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} + E_2,\end{aligned} \tag{2.21}$$

where E_2 is a term of type II:

$$E_2 = \sum_{1\le i,j\le n}\chi'_{j\bar{j}}(h^{i\bar{i}})^2|\partial_{\bar{i}}\chi_{j\bar{j}} - \partial_{\bar{j}}\chi_{j\bar{i}}|^2.$$

From (2.10), (2.17), (2.20), and (2.21), we have

$$\begin{aligned}\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi') &\ge \frac{1}{\mathrm{tr}_\omega\chi'}\left[\sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + E_2 - \sum_{1\le i,j\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\partial_{\bar{j}}\chi'_{j\bar{i}}\right] \\ &= \frac{1}{\mathrm{tr}_\omega\chi'}\left(\sum_{1\le i\le n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + E_2\right) \\ &= \frac{1}{\mathrm{tr}_\omega\chi'}\left(\sum_{1\le i\le n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\left|\partial_k\chi'_{i\bar{j}}\right|^2 + E_2\right) \\ &\ge \frac{E_2}{\mathrm{tr}_\omega\chi'}.\end{aligned} \tag{2.22}$$

By the definition of type II terms, there exists a positive universal constant C satisfying $|E_2|_\omega \le C \cdot \mathrm{tr}_\omega \chi'$. Therefore

$$\widetilde{\Delta}\log(\mathrm{tr}_\omega \chi') \gtrsim -1.$$

Thus we complete the proof of the lemma. □

Theorem 2.5 *Let (X, ω) be a compact Kähler manifold of complex dimension n, and χ a Hermitian metric. Let φ be a smooth solution of Donaldson's equation*

$$\omega \wedge \chi_\varphi^{n-1} = e^F \chi_\varphi^n$$

where F is a smooth function on X. Assume that

$$\chi - \frac{n-1}{ne^F}\omega > 0.$$

Then there are uniform constants $A > 0$ and $C > 0$, depending only on X, ω, χ, and F, such that

$$\mathrm{tr}_\omega \chi_\varphi \le C \cdot e^{A(\varphi - \inf_X \varphi)}.$$

Proof Use the local coordinates in Lemma 2.2. The proof is similar to that in [22,23]. By the definition, one has

$$\widetilde{\Delta}\varphi = h^{k\bar{k}}\varphi_{k\bar{k}} = (\chi'^{k\bar{k}})^2(\chi'_{k\bar{k}} - \chi_{k\bar{k}}) = \sum_{1\le k\le n} \chi'^{k\bar{k}} - \mathrm{tr}_h \chi = \mathrm{tr}_{\chi'}\omega - \mathrm{tr}_h \chi.$$

Lemma 2.4 and (2.7) imply that

$$\begin{aligned}\widetilde{\Delta} Q = \widetilde{\Delta}\left[\log\left(\mathrm{tr}_\omega \chi'\right) - A\varphi\right] &\ge -C - A\left(\mathrm{tr}_{\chi'}\omega - \mathrm{tr}_h \chi\right)\\ &\ge -C - A\sum_{1\le i\le n}\chi'^{i\bar{i}} + A\sum_{1\le i\le n}\chi'^{i\bar{i}}\chi'^{i\bar{i}}\chi_{i\bar{i}}.\end{aligned}$$

Since φ is a solution of Donaldson's equation, it follows that $\mathrm{tr}_{\chi'}\omega = ne^F$ by (2.7) and hence, for any given positive uniform constants A and B (we will chose those constants later),

$$\widetilde{\Delta} Q \ge \left(Bne^F - C\right) - (A+B)\sum_{1\le i\le n}\chi'^{i\bar{i}} + A\sum_{1\le i\le n}\chi'^{i\bar{i}}\chi'^{i\bar{i}}\chi_{i\bar{i}}.$$

By the assumption we have $\chi \ge \frac{n-1}{ne^F}(1+\epsilon)\omega$ for some suitable number ϵ such that $0 < \epsilon < \frac{1}{n-1}$. Let $p \in X$ be a point where Q achieves its maximum; so $\widetilde{\Delta} Q \le 0$. At this point, we conclude that

$$\begin{aligned}0 &\ge \left(Bne^F - C\right) - (A+B)\sum_{1\le i\le n}\chi'^{i\bar{i}} + A\sum_{1\le i\le n}\chi'^{i\bar{i}}\chi'^{i\bar{i}}\chi_{i\bar{i}}\\ &\ge \left(Bne^F - C\right) - (A+B)\sum_{1\le i\le n}\chi'^{i\bar{i}} + A\frac{n-1}{ne^F}(1+\epsilon)\sum_{1\le i\le n}\chi'^{i\bar{i}}\chi'^{i\bar{i}}.\end{aligned}$$

We denote by λ'_i the eigenvalues of χ' at point p such that $\lambda'_1 \le \cdots \le \lambda'_n$. Hence

$$0 \ge \left(Bne^F - C\right) - (A+B)\sum_{1\le i\le n}\frac{1}{\lambda'_i} + A\frac{n-1}{ne^F}(1+\epsilon)\sum_{1\le i\le n}\frac{1}{\lambda'^2_i}.$$

In order to obtain the upper bound for λ'_i we need the following □

Lemma 2.6 *Let $\lambda_1, \ldots, \lambda_n$ be a sequence of positive numbers. Suppose*

$$0 \geq 1 - \alpha \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} + \beta \sum_{1 \leq i \leq n} \frac{1}{\lambda_i^2}$$

for some $\alpha, \beta > 0$ and $n \geq 2$. If

$$\frac{4}{n} \leq \frac{\alpha^2}{\beta} < \frac{4}{n-1} \tag{2.23}$$

holds, then

$$\lambda_i \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}} \tag{2.24}$$

for each i.

Proof Note that $\alpha - \sqrt{n\alpha^2 - 4\beta} > 0$ by (2.23). Since

$$1 + \sum_{1 \leq i \leq n} \left(\frac{\alpha}{2\sqrt{\beta}} - \frac{\sqrt{\beta}}{\lambda_i} \right)^2 \leq \frac{n\alpha^2}{4\beta}$$

it implies that

$$\sum_{1 \leq i \leq n} \left(\frac{\alpha}{2\sqrt{\beta}} - \frac{\sqrt{\beta}}{\lambda_i} \right)^2 \leq \frac{n\alpha^2 - 4\beta}{4\beta}.$$

The right hand side of the above inequality is nonnegative by (2.23). Consequently,

$$\frac{\alpha}{2\sqrt{\beta}} - \frac{\sqrt{\beta}}{\lambda_i} \leq \sqrt{\frac{n\alpha^2 - 4\beta}{4\beta}}$$

and then

$$\frac{\alpha - \sqrt{n\alpha^2 - 4\beta}}{2\sqrt{\beta}} \leq \frac{\sqrt{\beta}}{\lambda_i}.$$

Hence we obtain (2.24). □

To apply Lemma 2.6, we assume

$$Bne^F > C, \tag{2.25}$$

and set

$$\alpha \doteqdot \frac{A + B}{Bne^F - C}, \quad \beta \doteqdot \frac{A\frac{n-1}{ne^F}(1 + \epsilon)}{Bne^F - C}. \tag{2.26}$$

In the following we will find the explicit formulas for A and B in terms of C such that the assumption (2.25) and the condition (2.23) are both satisfied.

We choose a real number η satisfying

$$0 \leq \eta < 1. \tag{2.27}$$

Set

$$\frac{\alpha^2}{\beta} = \frac{4}{n - \eta}, \tag{2.28}$$

where α and β are given in (2.26). If (2.28) was valid, then the condition (2.23) is true. Equations (2.26) and (2.28) imply

$$(A+B)^2 = \frac{4}{n-\eta}(1+\epsilon)\left(Bne^F - C\right)\frac{n-1}{ne^F}A$$

so that

$$A^2 + B^2 + 2\left[1 - \frac{2(1+\epsilon)(n-1)}{n-\eta}\right]AB + \frac{4(1+\epsilon)(n-1)C}{(n-\eta)ne^F}A = 0.$$

The above relation can be rewritten as

$$\left[A + \left(1 - \frac{2(1+\epsilon)(n-1)}{n-\eta}\right)B\right]^2 = \left[\left(1 - \frac{2(1+\epsilon)(n-1)}{n-\eta}\right)^2 - 1\right]B^2 - \frac{4(1+\epsilon)(n-1)C}{(n-\eta)ne^F}A.$$

Taking

$$A = \left(-1 + \frac{2(1+\epsilon)(n-1)}{n-\eta}\right)B \tag{2.29}$$

we have $A > B$ and

$$B = \frac{\frac{4(1+\epsilon)(n-1)C}{(n-\eta)ne^F}\left(-1 + \frac{2(1+\epsilon)(n-1)}{n-\eta}\right)}{\left(1 - \frac{2(1+\epsilon)(n-1)}{n-\eta}\right)^2 - 1} = \frac{C}{ne^F}\cdot\frac{-(n-\eta)+2(1+\epsilon)(n-1)}{-(n-\eta)+(1+\epsilon)(n-1)}, \tag{2.30}$$

assuming

$$(1+\epsilon) > \frac{n-\eta}{n-1}. \tag{2.31}$$

From (2.30) and (2.31) we see that

$$\frac{Bne^F}{C} = \frac{-(n-\eta)+2(1+\epsilon)(n-1)}{-(n-\eta)+(1+\epsilon)(n-1)} > 1.$$

From the assumption $0 < \epsilon < \frac{1}{n-1}$ we have $0 < n-(n-1)(1+\epsilon) < 1$ and then such a η always exists. Hence Lemma 2.6 yields

$$\lambda_i' \le \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}}$$

where α and β are determined by (2.26), (2.29), and (2.30). Since $\mathrm{tr}_\omega\chi' = \sum_{i=1}^n \lambda_i'$, it follows that, at $p \in X$, $\mathrm{tr}_\omega\chi' \le C$ for some uniform constant C and, for any point $q \in X$,

$$Q(q) \le Q(p) = \log(\mathrm{tr}_\omega\chi')(p) - A\varphi(p) \le C - A\cdot\inf{}_X\varphi.$$

Equivalently, $\log(\mathrm{tr}_\omega\chi') \le C + A(\varphi - \inf_X\varphi)$.

2.4 The estimate for $\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi')$, continued: general case

Now we consider the general case that both ω and χ may not be Kähler. Using Lemma 2.2 we have

$$\begin{aligned}\widetilde{\Delta}\mathrm{tr}_\omega\chi' = &\sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} + \sum_{1\le i,j,k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + E_2\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar{i}}\chi'^{j\bar{j}}\partial_k g_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}\right)\\ &-2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i\chi'_{k\bar{j}}\right).\end{aligned}\tag{2.32}$$

As in [17] we deal with the last two terms by using the local coordinates in Lemma 2.2. Starting from the last term, we calculate

$$\begin{aligned}\sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_i\chi'_{k\bar{j}}\partial_{\bar{i}}g_{j\bar{k}} &= \sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i\left(\chi_{k\bar{j}}+\varphi_{k\bar{j}}\right)\\ &= \sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\left(\partial_i\chi_{k\bar{j}}+\partial_k\varphi_{i\bar{j}}\right)\\ &= \sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\left(\partial_i\chi_{k\bar{j}}+\partial_k\chi'_{i\bar{j}}-\partial_k\chi_{i\bar{j}}\right)\\ &= \sum_{i=1}^{n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_k\chi'_{i\bar{j}}+E_1,\end{aligned}\tag{2.33}$$

where E_1 is a term of type I and is given by

$$E_1 = \sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\left(\partial_i\chi_{k\bar{j}}-\partial_k\chi_{i\bar{j}}\right).\tag{2.34}$$

Taking the real part of (2.33) gives

$$\begin{aligned}&\left|2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n} h^{i\bar{i}}\partial_i\chi'_{k\bar{j}}\partial_{\bar{i}}g_{j\bar{k}}\right)\right|\\ &=\left|2\cdot\mathrm{Re}\left(\sum_{1\le i\le n}\sum_{1\le j\ne k\le n}\sqrt{h^{i\bar{i}}}\sqrt{\chi'^{j\bar{j}}}\partial_k\chi'_{i\bar{j}}\cdot\sqrt{h^{i\bar{i}}}\sqrt{\chi'_{j\bar{j}}}\partial_{\bar{i}}g_{j\bar{k}}\right)\right|+E_1\\ &\le\sum_{1\le i\le n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}+\sum_{i=1}^{n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\chi'_{j\bar{j}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i g_{k\bar{j}}+E_1\\ &\le\sum_{1\le i\le n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}+E_2,\end{aligned}\tag{2.35}$$

since $\sum_{i=1}^{n}\sum_{1\le j\ne k\le n} h^{i\bar{i}}\chi'_{j\bar{j}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i g_{k\bar{j}}$ is of type II. Similarly we have

$$\begin{aligned}&\left|2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\chi'^{i\bar{i}}\chi'^{j\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}\partial_k g_{i\bar{j}}\right)\right|\\&=\left|2\cdot\mathrm{Re}\left(\sum_{1\le i,j,k\le n}\sqrt{h^{j\bar{j}}}\sqrt{\chi'^{i\bar{i}}}\partial_{\bar{k}}\chi'_{j\bar{i}}\cdot\sqrt{\chi'^{i\bar{i}}}\partial_k g_{i\bar{j}}\right)\right|\\&\le\frac{1}{2}\sum_{1\le i,j,k\le n}h^{j\bar{j}}\chi'^{i\bar{i}}\partial_{\bar{k}}\chi'_{j\bar{i}}\partial_k\chi'_{i\bar{j}}+E_1=\frac{1}{2}\sum_{1\le i,j,k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_{\bar{k}}\chi'_{i\bar{j}}\partial_k\chi'_{j\bar{i}}+E_1,\end{aligned}\tag{2.36}$$

where

$$E_1=2\sum_{1\le i,j,k\le n}\chi'^{i\bar{i}}\partial_k g_{i\bar{j}}\partial_{\bar{k}}g_{j\bar{i}}\tag{2.37}$$

is a term of type I.

From (2.32), (2.35), and (2.36), we conclude that

$$\begin{aligned}\widetilde{\Delta}\mathrm{tr}_\omega\chi'\ge&\sum_{1\le i,j,k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}+\sum_{1\le i,j,k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}\\&-\sum_{i=1}^{n}\sum_{1\le j\ne k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}}-\frac{1}{2}\sum_{1\le i,j,k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}+E_2\\=&\frac{1}{2}\sum_{1\le i,j,k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}+\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\partial_{\bar{j}}\chi'_{j\bar{i}}+E_2.\end{aligned}\tag{2.38}$$

It remains to control the term $|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2/(\mathrm{tr}_\omega\chi')^2$. As in (2.20) one has

$$\begin{aligned}\frac{|\widetilde{\nabla}\mathrm{tr}_\omega\chi'|_h^2}{\mathrm{tr}_\omega\chi'}\le&\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_j\chi'_{i\bar{j}}\partial_{\bar{j}}\chi'_{j\bar{i}}\\&+2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\left(\partial_{\bar{i}}\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}\right)\right].\end{aligned}\tag{2.39}$$

Lemma 2.7 *One has* $\widetilde{\Delta}\log(\mathrm{tr}_\omega\chi')\gtrsim-1$.

Proof As in the proof of Lemma 2.4 we have

$$\begin{aligned}&\left|2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n}h^{i\bar{i}}\chi'^{i\bar{i}}\partial_i\chi'_{j\bar{j}}\left(\partial_{\bar{i}}\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}\right)\right]\right|\\&=\left|2\cdot\mathrm{Re}\left[\sum_{1\le i,j\le n}\sqrt{h^{j\bar{j}}}\sqrt{\chi'^{j\bar{j}}}\partial_i\chi'_{j\bar{j}}\cdot\sqrt{\chi'_{j\bar{j}}}h^{i\bar{i}}\left(\partial_i\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}\right)\right]\right|\\&\le\frac{1}{2}\sum_{1\le i,j\le n}h^{j\bar{j}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{j}}\partial_{\bar{i}}\chi'_{j\bar{j}}+2\sum_{1\le i,j\le n}\chi'_{j\bar{j}}(h^{i\bar{i}})^2\left|\partial_{\bar{i}}\chi_{j\bar{j}}-\partial_{\bar{j}}\chi_{j\bar{i}}\right|^2\\&\le\frac{1}{2}\sum_{1\le i,j,k\le n}h^{k\bar{k}}\chi'^{j\bar{j}}\partial_i\chi'_{j\bar{k}}\partial_{\bar{i}}\chi'_{k\bar{j}}+E_2=\frac{1}{2}\sum_{1\le i,j,k\le n}h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}}+E_2,\end{aligned}\tag{2.40}$$

where E_2 is a term of type II and given by

$$E_2 = 2 \sum_{1 \le i,j \le n} \chi'_{j\bar{j}} (h^{i\bar{i}})^2 \left| \partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right|^2.$$

Combining (2.40) with (2.10), (2.38), and (2.39), we arrive at

$$\begin{aligned} \widetilde{\Delta} \log(\mathrm{tr}_\omega \chi') \ge \frac{1}{\mathrm{tr}_\omega \chi'} \Bigg[& \frac{1}{2} \sum_{1 \le i,j,k \le n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \le i,j \le n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} \\ & - \sum_{1 \le i,j \le n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} - \frac{1}{2} \sum_{1 \le i,j,k \le n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + E_2 \Bigg] = \frac{E_2}{\mathrm{tr}_\omega \chi'}. \end{aligned}$$

By the definition of type II terms, there exists a positive uniform constant C satisfying $|E_2|_\omega \le C \cdot \mathrm{tr}_\omega \chi'$. Therefore

$$\widetilde{\Delta} \log(\mathrm{tr}_\omega \chi') \ge -C.$$

This complete the proof. □

By using the similar method as in the proof of Theorem 2.5, we have

Theorem 2.8 *Let (X, ω) be a compact Hermitian manifold of the complex dimension n, and χ another Hermitian metric. Let φ be a smooth solution of Donaldson's equation*

$$\omega \wedge \chi_\varphi^{n-1} = e^F \chi_\varphi^n,$$

where F is a smooth function on X. Assume that

$$\chi - \frac{n-1}{ne^F} \omega > 0.$$

Then there are uniform constants $A > 0$ and $C > 0$, depending only on X, ω, χ, and F, such that

$$\mathrm{tr}_\omega \chi_\varphi \le C \cdot e^{A(\varphi - \inf_X \varphi)}.$$

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