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EIGENVALUES AND ENTROPIES UNDER THE HARMONIC-RICCI FLOW

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In this paper, the author discusses the eigenvalues and entropies under the harmonic-Ricci flow, which is the Ricci flow coupled with the harmonic map flow. We give an alternative proof of results for compact steady and expanding harmonic-Ricci breathers. In the second part, we derive some monotonicity formulas for eigenvalues of the Laplacian under the harmonic-Ricci flow. Finally, we obtain the first variation of the shrinker and expanding entropies of the harmonic-Ricci flow.

1. Introduction

Since the successful application of the Ricci flow to topological and geometric problems, several analogous flows have been studied, including the harmonic-Ricci flow [List 2006; Müller 2012], connection Ricci flow [Streets 2008], Ricci–Yang–Mills flow [Streets 2007; 2010; Young 2008], and renormalization group flows [He et al. 2008; Li 2012; Oliynyk et al. 2006; Streets 2009]. In this article, we study the

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eigenvalue problems of the harmonic-Ricci flow, which is the following coupled system:

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 4\,du(t)\otimes du(t), \tag{1-1}$$

$$\frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t). \tag{1-2}$$

For convenience, we introduce a new symmetric 2-tensor $\mathcal{S}_{g(t),u(t)}$ whose components S_{ij} are defined by

$$S_{ij} := R_{ij} - 2\partial_i u \partial_j u.$$

Its trace is $S_{g(t),u(t)} := g^{ij} S_{ij} = R_{g(t)} - 2|\nabla_{g(t)} u(t)|^2_{g(t)}$.

Suppose that M is a compact Riemannian manifold. For any Riemannian metric g and any smooth functions u, f, we have a number of functionals:

$$\mathcal{F}(g,u,f) = \int_M (R_g + |\nabla_g f|^2_g - 2|\nabla_g u|^2_g) e^{-f}\, dV_g,$$

$$\mathcal{E}(g,u,f) = \int_M (R_g - 2|\nabla_g u|^2_g) e^{-f}\, dV_g,$$

$$\mathcal{F}_k(g,u,f) = \int_M (kR_g + |\nabla_g f|^2_g - 2k|\nabla_g u|^2_g) e^{-f}\, dV_g.$$

List [2006] and Müller [2012] showed that, as in the case of Perelman's $\mathcal{F}$-functional, under the evolution equation

$$\begin{aligned}
&\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 4\,du(t)\otimes du(t),\\
&\frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t),\\
&\frac{\partial}{\partial t} f(t) = -\Delta_{g(t)} f(t) - R_{g(t)} + |\nabla_{g(t)} f(t)|^2_{g(t)} + 2|\nabla_{g(t)} u(t)|^2_{g(t)},
\end{aligned} \tag{1-3}$$

the evolution equation for the $\mathcal{F}$-functional is

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(g(t),u(t),f(t)) &= 2\int_M |\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t)|^2_{g(t)} e^{-f(t)}\, dV_{g(t)}\\
&\quad + 4\int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)} e^{-f(t)}\, dV_{g(t)},
\end{aligned} \tag{1-4}$$

which is nonnegative. Based on (1-4), we derive the following.

Theorem 1.1. *Under the evolution equation* (1-3), *one has*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(g(t),u(t),f(t))\\
&\qquad = 2\int_M |\mathcal{S}_{g(t),u(t)}|^2_{g(t)} e^{-f(t)}\, dV_{g(t)} + 4\int_M |\Delta_{g(t)} u(t)|^2_{g(t)} e^{-f(t)}\, dV_{g(t)},
\end{aligned} \tag{1-5}$$

and

$$\begin{aligned}(1\text{-}6)\quad &\frac{d}{dt}\mathscr{F}_k(g(t), u(t), f(t))\\ &= 2(k-1)\int_M |\mathcal{S}_{g(t),u(t)}|^2_{g(t)} e^{-f(t)}\, dV_{g(t)} + 2\int_M |\mathcal{S}_{g(t),u(t)}\\ &\quad + \nabla^2_{g(t)} f(t)|^2_{g(t)} e^{-f(t)}\, dV_{g(t)} + 4(k-1)\int_M |\Delta_{g(t)} u(t)|^2_{g(t)} e^{-f(t)}\, dV_{g(t)}\\ &\quad + 4\int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)} e^{-f(t)}\, dV_{g(t)}.\end{aligned}$$

As a corollary we give a new proof of the following result.

Corollary 1.2. *There is no compact steady harmonic-Ricci breather unless the manifold $(M, g(t))$ is Ricci-flat and $u(t)$ is a constant.*

To deal with the expanding harmonic-Ricci breather, we need the functionals

$$\mathscr{L}_+(g, u, \tau, f) = \tau^2 \int_M \Big(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_g u|^2_g\Big) e^{-f}\, dV_g,$$

$$\mathscr{L}_{+,k}(g, u, \tau, f) = \tau^2 \int_M \Big(k\Big(R_g + \frac{n}{2\tau}\Big) + \Delta_g f - 2k|\nabla_g u|^2_g\Big) e^{-f}\, dV_g.$$

Under the evolution equation

$$\begin{aligned}&\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 4\, du(t)\otimes du(t),\\ &\frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t),\\ &\frac{\partial}{\partial t} f(t) = -\Delta_{g(t)} f(t) + |\nabla_{g(t)} f(t)|^2_{g(t)} - R_{g(t)} + 2|\nabla_{g(t)} u(t)|^2_{g(t)},\\ &\frac{d}{dt}\tau(t) = 1,\end{aligned}$$

we have:

Theorem 1.3. *Under the evolution equation, one has*

$$\begin{aligned}(1\text{-}7)\quad &\frac{d}{dt}\mathscr{L}_+(g(t), u(t), \tau(t), f(t))\\ &= 2\tau(t)^2 \int_M \Big|\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t) + \frac{1}{2\tau(t)} g(t)\Big|^2_{g(t)} e^{-f(t)}\, dV_{g(t)}\\ &\quad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)} e^{-f(t)}\, dV_{g(t)}\end{aligned}$$

and

$$\begin{aligned}(1\text{-}8)\quad &\frac{d}{dt}\mathscr{L}_{+,k}(g(t),u(t),\tau(t),f(t))\\ &= 2\tau(t)^2\int_M |\mathcal{S}_{g(t),u(t)}+\nabla^2_{g(t)}f(t)+\frac{1}{2\tau(t)}g(t)|^2_{g(t)}e^{-f(t)}dV_{g(t)}\\ &\quad +2(k-1)\tau(t)^2\int_M |\mathcal{S}_{g(t),u(t)}+\frac{1}{2\tau(t)}g(t)|^2_{g(t)}e^{-f(t)}dV_{g(t)}\\ &\quad +4\tau(t)^2\int_M |\Delta_{g(t)}u(t)-\langle du(t),df(t)\rangle_{g(t)}|^2_{g(t)}e^{-f(t)}dV_{g(t)}\\ &\quad +4(k-1)\tau(t)^2\int_M |\Delta_{g(t)}u(t)|^2_{g(t)}e^{-f(t)}dV_{g(t)}.\end{aligned}$$

As a corollary, we obtain a new proof of the following.

Corollary 1.4. *There is no expanding harmonic-Ricci breather on compact Riemannian manifolds unless the manifold M is an Einstein manifold and $u(t)$ a constant.*

The second part of this paper focuses on the eigenvalue of the Laplacian operator under the harmonic-Ricci flow.

Theorem 1.5. *If $(g(t),u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $\lambda(t)$ denotes the eigenvalue of the Laplacian $\Delta_{g(t)}$ with eigenfunction $f(t)$,*

$$\begin{aligned}(1\text{-}9)\quad &\frac{d}{dt}\lambda(t)\cdot\int_M f(t)^2\,dV_{g(t)}\\ &= \lambda(t)\int_M S_{g(t),u(t)}f(t)^2\,dV_{g(t)}-\int_M S_{g(t),u(t)}|\nabla_{g(t)}f|^2_{g(t)}\,dV_{g(t)}\\ &\quad +2\int_M \langle \mathcal{S}_{g(t),u(t)},df(t)\otimes df(t)\rangle_{g(t)}\,dV_{g(t)}.\end{aligned}$$

Equation (1-9) is a general formula to describe the evolution of $\lambda(t)$ under the harmonic-Ricci flow. Under a curvature assumption, we can derive some monotonicity formulas for the eigenvalue $\lambda(t)$. Set

$$S_{\min}(0) := \min_{x\in M} S_{g(0),u(0)}(x), \tag{1-10}$$

the minimum of $S_{g(t),u(t)}$ over M at the time 0.

Theorem 1.6. *Let $(g(t),u(t))_{t\in[0,T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Suppose that $\mathcal{S}_{g(t),u(t)}-\alpha S_{g(t),u(t)}g(t)\geq 0$ along the harmonic-Ricci flow for some $\alpha\geq\frac{1}{2}$.*

(1) *If* $S_{\min}(0) \geq 0$, $\lambda(t)$ *is nondecreasing along the harmonic-Ricci flow for any* $t \in [0, T]$.

(2) *If* $S_{\min}(0) > 0$, *the quantity*

$$\left(1 - \frac{2}{n} S_{\min}(0)\, t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq n/(2S_{\min}(0))$.

(3) *If* $S_{\min}(0) < 0$, *the quantity*

$$\left(1 - \frac{2}{n} S_{\min}(0)\, t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

Corollary 1.7. *Let* $(g(t), u(t))_{t\in[0,T]}$ *be a solution of the harmonic-Ricci flow on a compact Riemannian surface* Σ *and let* $\lambda(t)$ *denote the eigenvalue of the Laplacian* $\Delta_{g(t)}$.

(1) *Suppose that* $\mathrm{Ric}_{g(t)} \leq \epsilon\, du(t) \otimes du(t)$ *where*

$$\epsilon \leq 4\frac{1-\alpha}{1-2\alpha}, \quad \alpha > \tfrac{1}{2}.$$

(i) *If* $S_{\min}(0) \geq 0$, $\lambda(t)$ *is nondecreasing along the harmonic-Ricci flow for any* $t \in [0, T]$.

(ii) *If* $S_{\min}(0) > 0$, *the quantity*

$$(1 - S_{\min}(0)\, t)^{2\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) *If* $S_{\min}(0) < 0$, *the quantity*

$$(1 - S_{\min}(0)\, t)^{2\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(2) *Suppose that*

$$|\nabla_{g(t)} u(t)|^2_{g(t)}\, g(t) \geq 2\, du(t) \otimes du(t).$$

(i) *If* $S_{\min}(0) \geq 0$, $\lambda(t)$ *is nondecreasing along the harmonic-Ricci flow for any* $t \in [0, T]$.

(ii) *If* $S_{\min}(0) > 0$, *the quantity*

$$(1 - S_{\min}(0)\, t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) *If* $S_{\min}(0) < 0$, *the quantity*

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

When we restrict to the Ricci flow, we obtain:

Corollary 1.8. *Let* $(g(t))_{t\in[0,T]}$ *be a solution of the Ricci flow on a compact Riemannian surface* Σ *and let* $\lambda(t)$ *denote the eigenvalue of the Laplacian* $\Delta_{g(t)}$.

(1) *If* $R_{\min}(0) \geq 0$, $\lambda(t)$ *is nondecreasing along the Ricci flow for any* $t \in [0, T]$.

(2) *If* $R_{\min}(0) > 0$, *the quantity* $(1 - R_{\min}(0)t)\lambda(t)$ *is nondecreasing along the Ricci flow for* $T \leq 1/R_{\min}(0)$.

(3) *If* $R_{\min}(0) < 0$, *the quantity* $(1 - R_{\min}(0)t)\lambda(t)$ *is nondecreasing along the Ricci flow for any* $t \in [0, T]$.

Remark 1.9. Let $(g(t))_{t\in[0,T]}$ be a solution of the Ricci flow on a compact Riemannian surface Σ with nonnegative scalar curvature and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Then $\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T]$.

Since

$$\mu(g, u) := \inf\left\{\mathscr{F}(g, u, f) \,\middle|\, f \in C^\infty(M),\ \int_M e^{-f}\, dV_g = 1\right\} \tag{1-11}$$

is the smallest eigenvalue of the operator $\Delta_{g,u} := -4\Delta_g + R_g - 2|\nabla_g u|_g^2$, we can consider the evolution equation for this eigenvalue under the harmonic-Ricci flow. To the operator $\Delta_{g,u}$ we associate a functional

$$\lambda_{g,u}(f) := \int_M f\,\Delta_{g,u} f\, dV_g. \tag{1-12}$$

When f is an eigenfunction of the operator $\Delta_{g,u}$ with the eigenvalue λ and normalized by $\int_X f^2\, dV_g = 1$, we obtain $\lambda_{g,u}(f) = \lambda$. Hence it suffices to study the evolution equation for $(d/dy)\lambda_{g,u}(f)$ under the harmonic-Ricci flow.

Theorem 1.10. *Suppose that* $(g(t), u(t))$ *is a solution of the harmonic-Ricci flow on a compact Riemannian manifold* M *and* $f(t)$ *is an eigenfunction of* $\Delta_{g(t),u(t)}$, *that is,* $\Delta_{g(t),u(t)} f(t) = \lambda(t) f(t)$ (*where* $\lambda(t)$ *is only a function of time* t), *with the normalized condition* $\int_M f(t)^2\, dV_{g(t)} = 1$. *Then we have*

$$\begin{aligned}\frac{d}{dt}\lambda(t) = \frac{d}{dt}\lambda_{g,u}(f(t)) = &\int_M 2\langle \mathscr{S}_{g(t),u(t)}, df(t)\otimes df(t)\rangle_{g(t)}\, dV_{g(t)} \\ &+ \int_M f(t)^2\big(|\mathscr{S}_{g(t),u(t)}|^2_{g(t)} + 2|\Delta_{g(t)}u(t)|^2_{g(t)}\big)\, dV_{g(t)}.\end{aligned} \tag{1-13}$$

List [2006] proved the nonnegativity of the operator $\mathcal{S}_{g(t),u(t)}$ is preserved by the harmonic-Ricci flow. Hence we get the following.

Corollary 1.11. *If* $\mathrm{Ric}_{g(0)} - 2du(0)\otimes du(0) \geq 0$, *the eigenvalues of the operator* $\Delta_{g(t),u(t)}$ *are nondecreasing under the harmonic-Ricci flow.*

Remark 1.12. If we choose $u(t) \equiv 0$, we obtain X. Cao's result [2007].

There is another expression for $d\lambda(t)/dt$.

Theorem 1.13. *Suppose that* $(g(t), u(t))$ *is a solution of the harmonic-Ricci flow on a compact Riemannian manifold* M *and* $f(t)$ *is an eigenfunction of* $\Delta_{g(t),u(t)}$, *that is,* $\Delta_{g(t),u(t)} f(t) = \lambda(t) f(t)$ *(where* $\lambda(t)$ *is only a function of time* t*), with the normalized condition* $\int_M f(t)^2\, dV_{g(t)} = 1$. *Then we have*

$$
\begin{aligned}
(1\text{-}14)\quad \frac{d}{dt}\lambda(t) = \frac{d}{dt}\lambda_{g,u}(f(t)) &= \frac{1}{2}\int_M |\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)}\varphi(t)|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)} \\
&\quad + \frac{1}{4}\int_M |\mathcal{S}_{g(t),u(t)}|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)} + \int_M |\langle du(t), d\varphi(t)\rangle_{g(t)}|^2 e^{-\varphi(t)}\, dV_{g(t)} \\
&\quad + 2\int_M |\nabla^2_{g(t)} u(t)|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)} - \int_M \Delta_{g(t)}(|\nabla_{g(t)} u(t)|^2_{g(t)}) e^{-\varphi(t)}\, dV_{g(t)} \\
&\quad + \frac{1}{4}\int_M |\mathcal{S}_{g(t),u(t)} + 4du(t)\otimes du(t)|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)},
\end{aligned}
$$

where $f(t)^2 = e^{-\varphi(t)}$.

Remark 1.14. When $u \equiv 0$, (1-14) reduces to J. Li's formula [2007].

Suppose that M is a compact manifold of dimension n. For any Riemannian metric g, any smooth functions u, f, and any positive number τ, we define

$$
(1\text{-}15)\qquad \mathcal{W}_\pm(g, u, f, \tau) := \int_M [\tau(S_g + |\nabla_g f|^2_g) \mp f \pm n] \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g.
$$

Set

$$
\mu_\pm(g, u, \tau) := \inf\left\{ \mathcal{W}_\pm(g, u, f, \tau) \,\middle|\, f \in C^\infty(M),\ \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g = 1 \right\},
$$

$$
\nu_-(g, u) := \inf\{\mu_-(g, u, \tau) \mid \tau > 0\}, \qquad \nu_+(g, u) := \sup\{\mu_+(g, u, \tau) \mid \tau > 0\}.
$$

The first variation of $\nu_\pm(g(s), u(s))$ is the following.

Theorem 1.15. *Suppose that* (M, g) *is a compact Riemannian manifold and* u *a smooth function on* M. *Let* h *be any symmetric covariant* 2*-tensor on* M *and set* $g(s) := g + sh$. *Let* v *be any smooth function on* M *and* $u(s) := u + sv$. *If*

$\nu_\pm(g(s),u(s)) = \mathcal{W}_\pm(g(s),u(s),f_\pm(s),\tau_\pm(s))$ for some smooth functions $f_\pm(s)$ with $\int_M e^{-f_\pm(s)}\,dV_g/(4\pi\tau_\pm(s))^{n/2} = 1$ and constants $\tau_\pm(s)>0$,

$$\frac{d}{ds}\Big|_{s=0}\nu_\pm(g(s),u(s)) = 4\tau_\pm\int_M v(\Delta_g u - \langle du, df_\pm\rangle_g)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g - \tau_\pm\int_M\left(\langle h,\mathcal{S}_{g,u}\rangle_g + \langle h,\nabla^2_g f\rangle_g \pm \frac{\operatorname{tr}_g h}{2\tau_\pm}\right)\frac{e^{-f_\pm}\,dV_g}{(4\pi\tau_\pm)^{n/2}}, \tag{1-16}$$

where $f_\pm := f_\pm(0)$ and $\tau_\pm := \tau_\pm(0)$. In particular, the critical points of $\nu_\pm(\cdot,\cdot)$ satisfy

$$\mathcal{S}_{g,u} + \nabla^2_g f \pm \frac{1}{2\tau_\pm}g = 0, \qquad \Delta_g u = \langle du, df_\pm\rangle_g.$$

Consequently, if $\mathcal{W}_\pm(g,u,f,\tau)$ and $\nu_\pm(g,u)$ achieve their extremum, (M,g) is a gradient expanding and shrinker harmonic-Ricci soliton according to the sign.

Corollary 1.16. *Suppose that (M,g) is a compact Riemannian manifold and u a smooth function on M. Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If $\nu_\pm(g(s),u(s)) = \mathcal{W}_\pm(g(s),u(s),f_\pm(s),\tau_\pm(s))$ for some smooth function $f_\pm(s)$ with $\int_M e^{-f_\pm(s)}\,dV/(4\pi\tau_\pm(s))^{n/2} = 1$ and a constant $\tau_\pm(s)>0$, and (g,u) is a critical point of $\nu_\pm(\cdot,\cdot)$, then*

$$\mathrm{Ric}_g = \mp\frac{1}{2\tau_\pm}g, \qquad f_\pm \equiv \text{constant}, \qquad u \equiv \text{constant}.$$

Thus, if $\mathcal{W}_\pm(g,u,\cdot,\cdot)$ achieve their minimum and (g,u) is a critical point of $\nu_\pm(\cdot,\cdot)$, (M,g) is an Einstein manifold and u is a constant function.

Remark 1.17. In the situation of Corollary 1.16, by normalization, we my choose $f_\pm = n/2$ and $u = 0$.

2. Notation and commuting identities

Let M be a compact Riemannian manifold of dimension n. For any vector bundle E over M, we denote by $\Gamma(M,E)$ the space of smooth sections of E. Set

$$\bigodot^2(M) := \{v = (v_{ij}) \in \Gamma(M, T^*M\otimes T^*M) \mid v_{ij} = v_{ji}\},$$
$$\bigodot^2_+(M) := \{g = (g_{ij}) \in \bigodot^2(M) \mid g_{ij} > 0\}.$$

Thus $\bigodot^2(M)$ is the space of all symmetric covariant 2-tensors on M, while $\bigodot^2_+(M)$ is the space of all Riemannian metrics on M. The space of all smooth functions on M is denoted by $C^\infty(M)$.

For a given Riemannian metric $g \in \bigodot^2_+(M)$, the corresponding Levi-Civita connection $\Gamma_g = (\Gamma^k_{ij})$ is given by

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{il} - \partial_\ell g_{ij}) \tag{2-1}$$

where $\partial_i := \partial/\partial x^i$ for a local coordinate system $\{x^1, \dots, x^n\}$. The Riemann tensor $\mathrm{Rm}_g = (R^k_{ijl})$ is determined by

$$R^k_{ij\ell} = \partial_i \Gamma^k_{j\ell} - \partial_j \Gamma^k_{i\ell} + \Gamma^k_{ip}\Gamma^p_{j\ell} - \Gamma^k_{jp}\Gamma^p_{i\ell}. \tag{2-2}$$

The Ricci curvature $\mathrm{Ric}_g = (R_{ij})$ is

$$R_{ij} = g^{k\ell} R^\ell_{kij}. \tag{2-3}$$

The scalar curvature R_g of the metric g now is given by

$$R_g = g^{ij} R_{ij}. \tag{2-4}$$

For any tensor $A = (A^{k_1\cdots k_q}_{j_1\cdots j_p})$ the covariant derivative of A is

$$\nabla_i A^{k_1\cdots k_q}_{j_1\cdots j_p} = \partial_i A^{k_1\cdots k_q}_{j_1\cdots j_p} - \sum_{r=1}^{p} \Gamma^m_{ij_r} A^{k_1\cdots k_q}_{j_1\cdots m\cdots j_p} + \sum_{s=1}^{q} \Gamma^{k_s}_{im} A^{k_1\cdots m\cdots k_q}_{j_1\cdots j_p}.$$

Next we recall the Ricci identity:

$$\nabla_i \nabla_j A^{\ell_1\cdots\ell_q}_{k_1\cdots k_p} - \nabla_j \nabla_i A^{\ell_1\cdots\ell_q}_{k_1\cdots k_p} = \sum_{r=1}^{q} R^{l_r}_{ijm} A^{\ell_1\cdots m\cdots\ell_q}_{k_1\cdots k_p} - \sum_{s=1}^{p} R^m_{ijk_s} A^{\ell_1\cdots\ell_q}_{k_1\cdots m\cdots k_p}.$$

In particular, for any smooth function $f \in C^\infty(M)$, we have

$$\nabla_i \nabla_j f = \nabla_j \nabla_i f.$$

The Bianchi identities are

$$0 = R_{ijk\ell} + R_{iklj} + R_{i\ell jk}, \tag{2-5}$$

$$0 = \nabla_q R_{ijk\ell} + \nabla_i R_{jqk\ell} + \nabla_j R_{qik\ell}, \tag{2-6}$$

and the contracted Bianchi identities are

$$0 = 2\nabla^j R_{ij} - \nabla_i R_g, \tag{2-7}$$

$$0 = \nabla_i R_{jk} - \nabla_j R_{ik} + \nabla^\ell R_{\ell kij}. \tag{2-8}$$

3. Harmonic-Ricci flow and the evolution equations

Motivated by the static Einstein vacuum equation, List [2006] introduced the harmonic-Ricci flow (originally called the Ricci flow coupled with the harmonic map flow). This flow is similar to the Ricci flow and is given by the coupled system

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 4\, du(t) \otimes du(t), \tag{3-1}$$

$$\frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t) \tag{3-2}$$

for a family of Riemannian metrics $g(t)$ and a family of smooth functions $u(t)$. Locally, we have

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + 4\partial_i u \cdot \partial_j u, \qquad \frac{\partial}{\partial t} u = \Delta_{g(t)} u(t). \tag{3-3}$$

Introduce a new symmetric tensor field $\mathcal{S}_{g(t),u(t)} = (S_{ij}) \in \bigodot^2(M)$,

$$S_{ij} := R_{ij} - 2\partial_i u \cdot \partial_j u. \tag{3-4}$$

Then its trace $S_{g(t),u(t)}$ is equal to

$$S_{g(t),u(t)} = g^{ij} S_{ij} = R_{g(t)} - 2|\nabla_{g(t)} u(t)|^2_{g(t)}. \tag{3-5}$$

The evolution equation for $R_{g(t)}$ is

$$\begin{aligned}\frac{\partial}{\partial t} R_{g(t)} = \Delta_{g(t)} R_{g(t)} + 2|\,\mathrm{Ric}_{g(t)}\,|^2_{g(t)} + 4|\Delta_{g(t)} u(t)|^2_{g(t)} \\ - 4|\nabla^2_{g(t)} u(t)|^2_{g(t)} - 8\langle \mathrm{Ric}_{g(t)}, du(t) \otimes du(t)\rangle_{g(t)}.\end{aligned} \tag{3-6}$$

Also, we have the evolution equation for $|\nabla_{g(t)} u|^2_{g(t)}$,

$$\frac{\partial}{\partial t} |\nabla_{g(t)} u(t)|^2_{g(t)} = \Delta_{g(t)} |\nabla_{g(t)} u(t)|^2_{g(t)} - 2|\nabla^2_{g(t)} u(t)|^2_{g(t)} - 4|\nabla_{g(t)} u(t)|^4_{g(t)}, \tag{3-7}$$

and the evolution equation for $S_{g(t),u(t)}$,

$$\frac{\partial}{\partial t} S_{g(t),u(t)} = \Delta_{g(t)} S_{g(t),u(t)} + 2|\mathcal{S}_{g(t),u(t)}|^2_{g(t)} + 4|\Delta_{g(t)} u(t)|^2_{g(t)}. \tag{3-8}$$

4. Entropies for harmonic-Ricci flow

Motivated by Perelman's entropy, List [2006] introduced a similar functional for the harmonic-Ricci flow:

$$\bigodot^2_+(M) \times C^\infty(M) \times C^\infty(M) \to \mathbb{R}, \quad (g, u, f) \mapsto \mathscr{F}(g, u, f)$$

where

$$\mathcal{F}(g,u,f) := \int_M (R_g + |\nabla_g f|_g^2 - 2|\nabla_g u|_g^2) e^{-f}\, dV_g. \tag{4-1}$$

He also showed that if $(g(t), u(t), f(t))$ satisfies the system

$$\begin{aligned}
\frac{\partial}{\partial t} g(t) &= -2\,\mathrm{Ric}_{g(t)} + 4\, du(t) \otimes du(t) - 2\nabla^2_{g(t)} f(t),\\
\frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)},\\
\frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) - R_{g(t)} + 2|\nabla_{g(t)} u(t)|^2_{g(t)},
\end{aligned} \tag{4-2}$$

the evolution of the entropy is given by

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}(g(t),u(t),f(t))\\
&\quad = 2\int_M \Big(|\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t)|^2_{g(t)}\\
&\qquad\qquad + 2|\Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)} \Big) e^{-f(t)}\, dV_{g(t)}\\
&\quad \geq 0.
\end{aligned} \tag{4-3}$$

Remark 4.1. The system (4-2) is equivalent to

$$\begin{aligned}
\frac{\partial}{\partial t} g(t) &= -2\,\mathrm{Ric}_{g(t)} + 4\, du(t) \otimes du(t),\\
\frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t),\\
\frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) - R_{g(t)} + |\nabla_{g(t)} f(t)|^2_{g(t)} + 2|\nabla_{g(t)} u(t)|^2_{g(t)}.
\end{aligned} \tag{4-4}$$

The same evolution of the entropy holds for system (4-4).

In particular, the entropy is nondecreasing and the equality holds if and only if $(g(t), u(t), f(t))$ satisfies

$$\begin{aligned}
\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t) &= 0,\\
\Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)} &= 0.
\end{aligned} \tag{4-5}$$

Definition 4.2. The $\mathcal{E}$-functional is defined as

$$\bigodot^2_+(M) \times C^\infty(M) \times C^\infty(M) \to \mathbb{R}, \quad (g,u,f) \mapsto \mathcal{E}(g,u,f),$$

where

$$\mathcal{E}(g,u,f) := \int_M (R_g - 2|\nabla_g u|_g^2) e^{-f}\, dV_g. \tag{4-6}$$

Proposition 4.3. *Under the evolution equation* (4-4), *one has*

$$\frac{d}{dt}\mathscr{E}(g(t),u(t),f(t)) = 2\int_M |\mathscr{S}_{g(t),u(t)}|^2_{g(t)} e^{-f(t)}\,dV_{g(t)} + 4\int_M |\Delta_{g(t)}u(t)|^2_{g(t)} e^{-f(t)}\,dV_{g(t)}. \tag{4-7}$$

Proof. Since $S_{g(t),u(t)} = R_{g(t)} - 2|\nabla_{g(t)}u(t)|^2_{g(t)}$ and

$$\frac{\partial}{\partial t}S_{g(t),u(t)} = \Delta_{g(t)}S_{g(t),u(t)} + 2|\mathscr{S}_{g(t),u(t)}|^2_{g(t)} + 4|\Delta_{g(t)}u(t)|^2_{g(t)},$$
$$\frac{\partial}{\partial t}dV_{g(t)} = -S_{g(t),u(t)}\,dV_{g(t)},$$

we have

$$\begin{aligned}
&\frac{d}{dt}\mathscr{E}(g(t),u(t),f(t)) \\
&= \int_M \Big(\frac{\partial}{\partial t}S_{g(t),u(t)}\Big)e^{-f(t)}\,dV_{g(t)} + \int_M S_{g(t),u(t)}\frac{\partial}{\partial t}(e^{-f(t)}\,dV_{g(t)}) \\
&= \int_M (\Delta_{g(t)}S_{g(t),u(t)} + 2|\mathscr{S}_{g(t),u(t)}|^2_{g(t)} + 4|\Delta_{g(t)}u(t)|^2_{g(t)})e^{-f(t)}\,dV_{g(t)} \\
&\qquad + \int_M S_{g(t),u(t)}\Big(-\frac{\partial}{\partial t}f(t) - S_{g(t),u(t)}\Big)e^{-f(t)}\,dV_{g(t)} \\
&= 2\int_M |\mathscr{S}_{g(t),u(t)}|^2_{g(t)}e^{-f(t)}\,dV_{g(t)} + 4\int_M |\Delta_{g(t)}u(t)|^2_{g(t)}e^{-f(t)}\,dV_{g(t)} \\
&\quad - \int_M S_{g(t),u(t)}\Big(\Delta_{g(t)}f(t) - |\nabla_{g(t)}f(t)|^2_{g(t)} + \frac{\partial}{\partial t}f(t) + S_{g(t),u(t)}\Big)e^{-f(t)}\,dV_{g(t)},
\end{aligned}$$

which implies (4-7). □

Definition 4.4. For any $k \geq 1$ we define

$$\mathscr{F}_k(g,u,f) := \int_M (kR_g + |\nabla_g f|^2_g - 2k|\nabla_g u|^2_g)e^{-f}\,dV_g. \tag{4-8}$$

Using the definition, it is easy to show that

$$\mathscr{F}_k(g,u,f) = (k-1)\mathscr{E}(g,u,f) + \mathscr{F}(g,u,f). \tag{4-9}$$

When $k = 1$, this is the $\mathscr{F}$-functional.

Theorem 4.5. *Under the evolution equation (4-4), one has*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_k(g(t),u(t),f(t)) &= 2(k-1)\int_M |\mathcal{S}_{g(t),u(t)}|^2_{g(t)} e^{-f(t)}\,dV_{g(t)} + 2\int_M |\mathcal{S}_{g(t),u(t)} \\ &\quad + \nabla^2_{g(t)} f(t)|^2_{g(t)} e^{-f(t)}\,dV_{g(t)} + 4(k-1)\int_M |\Delta_{g(t)}u(t)|^2_{g(t)} e^{-f(t)}\,dV_{g(t)} \\ &\quad + 4\int_M |\Delta_{g(t)}u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)} e^{-f(t)}\,dV_{g(t)}.\end{aligned} \tag{4-10}$$

Furthermore, the monotonicity is strict unless $g(t)$ is Ricci-flat, $u(t)$ is constant, and $f(t)$ is constant.

Proof. It immediately follows from (4-3) and (4-7). □

Set

$$\mu_k(g,u) := \inf\left\{\mathcal{F}_k(g,u,f)\,\middle|\, f \in C^\infty(M),\ \int_M e^{-f}\,dV_g = 1\right\}. \tag{4-11}$$

Then $\mu_k(g,u)$ is the lowest eigenvalue of $-4\Delta_g + k(R_g - 2|\nabla_g u|^2_g)$.

5. Compact steady harmonic-Ricci breathers

In this section we give an alternative proof on some results on compact steady harmonic-Ricci breathers that were proved in [List 2006; Müller 2012].

Definition 5.1. A solution $(g(t), u(t))$ of the harmonic-Ricci flow (1-1)–(1-2) is called a *harmonic-Ricci breather* if there exist $t_1 < t_2$, a diffeomorphism $\psi : M \to M$, and a constant $\alpha > 0$ such that

$$g(t_2) = \alpha\psi^* g(t_1), \quad u(t_2) = \psi^* u(t_1).$$

The cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, correspond to *shrinking*, *steady*, and *expanding harmonic-Ricci breathers*.

Theorem 5.2. *If $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M, the lowest eigenvalue $\mu_k(g(t), u(t))$ of the operator $-4\Delta_{g(t)} + k(R_{g(t)} - 2|\nabla_{g(t)} u(t)|^2_{g(t)})$ is nondecreasing under the harmonic-Ricci flow. The monotonicity is strict unless $g(t)$ is Ricci-flat and $u(t)$ is constant.*

Proof. The proof is similar to that given in [Li 2007]. For any $t_1 < t_2$, suppose that

$$\mu_k(g(t_2), u(t_2)) = \mathcal{F}_k(g(t_2), u(t_2), f_k(t_2))$$

for some smooth function $f_k(x)$. Being an initial value, $f_k(x) = f_k(x, t_2)$ for some smooth function $f_k(x,t)$ satisfying the evolution equation (4-4). The monotonicity formula (4-10) implies $\mu_k(g(t_2),u(t_2)) \ge \mathscr{F}_k(g(t_1),u(t_1), f_k(t_1)) \ge \mu_k(g(t_1),u(t_1))$. This completes the proof. □

Corollary 5.3. *On a compact Riemannian manifold, the lowest eigenvalues of* $-\Delta_{g(t)} + (1/2)(R_{g(t)} - 2|\nabla_{g(t)}u(t)|^2_{g(t)})$ *are nondecreasing under the harmonic-Ricci flow.*

Proof. Since $\mu_2(g(t), u(t))/4$ is the lowest eigenvalue of this operator, the result immediately follows from Theorem 5.2. □

Corollary 5.4. *There is no compact steady harmonic-Ricci breather unless the manifold* $(M, g(t))$ *is Ricci-flat and u is a constant.*

Proof. If $(g(t), u(t))$ is a steady harmonic-Ricci breather, then, for $t_1 < t_2$ given in the definition, we have

$$\mu_k(g(t_1), u(t_1)) = \mu_k(g(t_2), u(t_2)).$$

Hence, using Theorem 5.2, for any $t \in [t_1, t_2]$, we must have

$$\frac{d}{dt}\mu_k(g(t), u(t)) \equiv 0.$$

Thus $(M, g(t))$ is Ricci-flat and $u(t)$ is constant. □

6. Compact expanding harmonic-Ricci breathers

Inspired by [Li 2007], we define a new functional

$$\bigodot^2_+(M) \times C^\infty(M) \times C^\infty(\mathbb{R}) \times C^\infty(M) \to \mathbb{R}, \quad (g, u, \tau, f) \mapsto \mathscr{W}_+(g, u, \tau, f),$$

where $(\tau = \tau(t), t \in \mathbb{R})$.

$$\mathscr{W}_+(g, u, \tau, f) := \tau^2 \int_M \Big(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_g u|^2_g\Big) e^{-f}\, dV_g. \tag{6-1}$$

Similarly, we define a family of functionals

$$\mathscr{W}_{+,k}(g, u, \tau, f) := \tau^2 \int_M \Big(k\Big(R_g + \frac{n}{2\tau}\Big) + \Delta_g f - 2k|\nabla_g u|^2_g\Big) e^{-f}\, dV_g. \tag{6-2}$$

It's clear that $\mathscr{W}_{+,1}(g, u, \tau, f) = \mathscr{W}_+(g, u, \tau, f)$.

Lemma 6.1. *One has*

$$\mathscr{W}_+(g, u, \tau, f) = \tau^2 \mathscr{F}(g, u, f) + \frac{n}{2}\tau \int_M e^{-f}\, dV_g,$$

$$\mathscr{W}_{+,k}(g, u, \tau, f) = \tau^2 \mathscr{F}_k(g, u, f) + \frac{kn}{2}\tau \int_M e^{-f}\, dV_g,$$

$$\mathscr{W}_{+,k}(g, u, \tau, f) = \mathscr{W}_+(g, u, \tau, f) + (k-1)\Big(\tau^2 \mathscr{E}(g, u, f) + \frac{n}{2}\tau \int_M e^{-f}\, dV_g\Big).$$

Proof. Since $\Delta(e^{-f}) = (-\Delta f + |\nabla f|^2)e^{-f}$, it follows that

$$\begin{aligned}\mathcal{W}_+(g, u, \tau, f) - \tau^2\mathcal{F}(g, u, f) &= \frac{n}{2}\tau\int_M e^{-f}\,dV_g + \tau^2\int_M (\Delta_g f - |\nabla_g f|_g^2)e^{-f}\,dV_g \\ &= \frac{n}{2}\tau\int_M e^{-f}\,dV_g + \tau^2\int_M \Delta_g(e^{-f})\,dV_g = \frac{n}{2}\tau\int_M e^{-f}\,dV_g.\end{aligned}$$

We can similarly prove the remaining two relations. □

Theorem 6.2. *Under the coupled system*

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2\,\mathrm{Ric}_{g(t)} + 4\,du(t)\otimes du(t) - 2\nabla^2_{g(t)}f(t),\\ \frac{\partial}{\partial t}u(t) &= \Delta_{g(t)}u(t) - \langle du(t), df(t)\rangle_{g(t)},\\ \frac{\partial}{\partial t}f(t) &= -\Delta_{g(t)}f(t) - R_{g(t)} + 2|\nabla_{g(t)}u(t)|^2_{g(t)},\\ \frac{d}{dt}\tau(t) &= 1,\end{aligned}$$

the first variation formula for $\mathcal{W}_+(g(t), u(t), \tau(t), f(t))$ *is*

$$\begin{aligned}\text{(6-3)}\quad &\frac{d}{dt}\mathcal{W}_+(g(t), u(t), \tau(t), f(t)) \\ &= 2\tau(t)^2\int_M \Big|\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)}f(t) + \frac{1}{2\tau(t)}g(t)\Big|^2_{g(t)}e^{-f(t)}\,dV_{g(t)} \\ &\quad + 4\tau(t)^2\int_M |\Delta_{g(t)}u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)}e^{-f(t)}\,dV_{g(t)},\end{aligned}$$

and the first variation formula for $\mathcal{W}_{+,k}(g(t), u(t), \tau(t), f(t))$ *is*

$$\begin{aligned}\text{(6-4)}\quad &\frac{d}{dt}\mathcal{W}_{+,k}(g(t), u(t), \tau(t), f(t)) \\ &= 2\tau(t)^2\int_M \Big|\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)}f(t) + \frac{1}{2\tau(t)}g(t)\Big|^2_{g(t)}e^{-f(t)}\,dV_{g(t)} \\ &\quad + 2(k-1)\tau(t)^2\int_M \Big|\mathcal{S}_{g(t),u(t)} + \frac{1}{2\tau(t)}g(t)\Big|^2_{g(t)}e^{-f(t)}\,dV_{g(t)} \\ &\quad + 4\tau(t)^2\int_M |\Delta_{g(t)}u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2_{g(t)}e^{-f(t)}\,dV_{g(t)} \\ &\quad + 4(k-1)\tau(t)^2\int_M |\Delta_{g(t)}u(t)|^2_{g(t)}e^{-f(t)}\,dV_{g(t)}.\end{aligned}$$

Proof. Under this coupled system, we first observe that

$$\text{(6-5)}\qquad \frac{d}{dt}\left(\int_M e^{-f(t)}\,dV_{g(t)}\right) = 0.$$

In fact, from $\frac{\partial}{\partial t}\,dV_{g(t)} = -S_{g(t),u(t)} - \Delta_{g(t)} f(t)\,dV_{g(t)}$ we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\int_M e^{-f(t)} dV_{g(t)}\Big) &= \int_M \Big(-\frac{\partial}{\partial t} f(t)\cdot dV_{g(t)} + \frac{\partial}{\partial t} dV_{g(t)}\Big) e^{-f(t)} \\ &= \int_M [\Delta_{g(t)} f(t) + S_{g(t),u(t)} - S_{g(t),u(t)} - \Delta_{g(t)} f(t)] e^{-f(t)} dV_{g(t)} \\ &= 0.\end{aligned}$$

Lemma 6.1 and the identity (6-5) imply

$$\begin{aligned}&\frac{d}{dt}\mathcal{W}_+(g(t), u(t), \tau(t), f(t)) \\ &\quad = \tau(t)^2 \frac{d}{dt}\mathcal{F}(g(t), u(t), f(t)) + 2\tau(t)\mathcal{F}(g(t), u(t), f(t)) + \frac{n}{2}\int_M e^{-f(t)}\,dV_{g(t)} \\ &\quad = 2\tau(t)^2 \int_M |\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t)|^2_{g(t)} e^{-f(t)}\,dV_{g(t)} \\ &\qquad\qquad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t)\rangle_{g(t)}|^2 e^{-f(t)}\,dV_{g(t)} \\ &\qquad + 2\tau(t)\int_M (S_{g(t),u(t)} + |\nabla_{g(t)} f(t)|^2_{g(t)}) e^{-f(t)}\,dV_{g(t)} + \frac{n}{2}\int_M e^{-f(t)}\,dV_{g(t)},\end{aligned}$$

which is (6-3). Using Lemma 6.1 and the same method, we can prove (6-4). □

Remark 6.3. Under the coupled system

$$\begin{aligned}\frac{\partial}{\partial t} g(t) &= -2\,\mathrm{Ric}_{g(t)} + 4\,du(t)\otimes du(t), \\ \frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t), \\ \frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) + |\nabla_{g(t)} f(t)|^2_{g(t)} - R_{g(t)} + 2|\nabla_{g(t)} u(t)|^2_{g(t)}, \\ \frac{d}{dt}\tau(t) &= 1,\end{aligned}$$

the same formulas (6-3) and (6-4) hold for $\mathcal{W}_+$ and $\mathcal{W}_{+,k}$.

Define

$$\mu_+(g, u, \tau) := \inf\Big\{\mathcal{W}_+(g, u, \tau, f)\ \Big|\ f \in C^\infty(M),\ \int_M e^{-f}\,dV_g = 1\Big\}. \tag{6-6}$$

Lemma 6.4. *For any* $\alpha > 0$, *one has*

$$\mu_+(\alpha g, u, \alpha\tau) = \alpha\mu_+(g, u, \tau). \tag{6-7}$$

Proof. Set $\bar{g} := \alpha g$; then $R_{\bar{g}} = \alpha^{-1} R_g$, $\Delta_{\bar{g}} f = \alpha^{-1}\Delta_g f$, and $|\nabla_{\bar{g}} u|^2_{\bar{g}} = \alpha^{-1}|\nabla_{g(t)} u|^2_g$. Hence

$$\begin{aligned}\mathcal{W}_+(\bar{g}, u, \alpha\tau, f) &= \alpha^2\tau^2 \int_M \Big(R_{\bar{g}} + \frac{n}{2\alpha\tau} + \Delta_{\bar{g}} f - 2|\nabla_{\bar{g}} u|^2_{\bar{g}}\Big) e^{-f} dV_{\bar{g}} \\ &= \alpha\tau^2 \int_M \Big(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_{g(t)} u|^2_g\Big) \alpha^{n/2} e^{-f} dV_g.\end{aligned}$$

Since $f \mapsto f - (n/2)\ln\alpha$ is one-to-one and onto, by taking the infimum, we derive $\mu_+(\alpha g, u, \alpha\tau) = \alpha\mu_+(g, u, \tau)$. □

Definition 6.5. A solution $(g(t), u(t))$ of the harmonic-Ricci flow is called a *harmonic-Ricci soliton* if there exists a one-parameter family of diffeomorphisms $\psi_t : M \to M$, satisfying $\psi_0 = \mathrm{id}_M$, and a positive scaling function $\alpha(t)$ such that

$$g(t) = \alpha(t)\psi_t^* g(0), \quad u(t) = \psi_t^* u(0).$$

The cases $(\partial/\partial t)\alpha(t) = \dot{\alpha} < 0$, $\dot{\alpha} = 0$, and $\dot{\alpha} > 0$ correspond to *shrinking*, *steady*, and *expanding harmonic-Ricci solitons*, respectively. If the diffeomorphisms ψ_t are generated by a (possibly time-dependent) vector field $X(t)$ that is the gradient of some function $f(t)$ on M, the soliton is called a *gradient harmonic-Ricci soliton* and f is called the *potential of the harmonic-Ricci soliton*.

Müller [2012] showed that if $(g(t), u(t))$ is a gradient harmonic-Ricci soliton with potential f,

$$\begin{aligned}0 &= \mathrm{Ric}_{g(t)} - 2du(t) \otimes du(t) + \nabla^2_{g(t)} f(t) + cg(t), \\ 0 &= \Delta_{g(t)} u(t) - \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t)\rangle_{g(t)}\end{aligned}$$

for some constant c.

Corollary 6.6. *There is no expanding breather on compact Riemannian manifolds other than expanding gradient harmonic-Ricci solitons.*

Proof. The proof is similar to that given in [Li 2007]. Suppose there is an expanding breather on a compact Riemannian manifold M. Then, by definition, we have

$$g(t_2) = \alpha\Phi^* g(t_1), \quad u(t_2) = \Phi^* u(t_1)$$

for some $t_1 < t_2$, where Φ be a diffeomorphism and the constant $\alpha > 1$. Let $f_+(x)$ be a smooth function where $\mathcal{W}_+(g(t_2), u(t_2), \tau(t_2), f(t_2))$ attains its minimum. Then there exists a smooth function $f_+(x, t) : M \times [t_1, t_2] \to \mathbb{R}$ with initial value $f_+(x, t_2) = f_+(x)$ that satisfies the coupled system in Remark 6.3. Define a linear function

$$\tau : [t_1, t_2] \to (0, +\infty), \quad \tau(t_2) = T + t_2$$

where T is a constant. By the monotonicity formula, we have

$$\begin{aligned}\mu_+(g(t_2), u(t_2), \tau(t_2)) &= \mathcal{W}_+(g(t_2), u(t_2), \tau(t_2), f_+(t_2))\\ &\geq \mathcal{W}_+(g(t_1), u(t_1), \tau(t_1), f_+(t_1))\\ &\geq \mu_+(g(t_1), u(t_1), \tau(t_1)).\end{aligned}$$

Lemma 6.4 and the diffeomorphic invariant property of the functionals shows

$$\mu_+(g(t_1), u(t_1), \tau(t_1)) \leq \alpha\mu_+(g(t_1), u(t_1), \tau(t_1)),$$

which yields

$$\mu_+(g(t_1), u(t_1), \tau(t_1)) \geq 0,$$

since $\alpha > 1$.

If we impose an additional condition $\tau(t_2) = \alpha\tau(t_1)$ and $\tau(t_1) = T + t_1$, we have

$$\tau(t) = \frac{\alpha(t - t_1) - (t - t_2)}{\alpha - 1}, \quad T = \frac{t_2 - \alpha t_1}{\alpha - 1}.$$

Then

$$\frac{\tau(t_2)^{n/2}}{V_{g(t_2)}} = \frac{[\alpha(t_2 - t_1)/(\alpha - 1)]^{n/2}}{\alpha^{n/2} V_{g(t_1)}} = \frac{\tau(t_1)^{n/2}}{V_{g(t_1)}}.$$

The mean value theorem tells us that there exists a time $\bar{t} \in [t_1, t_2]$ with

$$\begin{aligned}0 &= \frac{d}{dt}\Big|_{t=\bar{t}} \log \frac{\tau(t)^{n/2}}{V_{g(t)}}\\ &= \frac{V_{g(\bar{t})}}{\tau(\bar{t})^{n/2}} \cdot \frac{(n/2)\tau(\bar{t})^{n/2-1} V_{g(\bar{t})} - \tau(\bar{t})^{n/2}(d/dt)|_{t=\bar{t}} V_{g(t)}}{V_{g(\bar{t})}^2}\\ &= \frac{n}{2\tau(\bar{t})} - \frac{1}{V_{g(\bar{t})}} \frac{\partial}{\partial t}\Big|_{t=\bar{t}} V_{g(\bar{t})}.\end{aligned}$$

From the evolution equation for the volume element $dV_{g(t)}$, we have

$$\frac{d}{dt} V_{g(t)} = \int_M \frac{\partial}{\partial t}\, dV_{g(t)} = \int_M (-S_{g(t),u(t)} - \Delta_{g(t)} f(t))\, dV_{g(t)} = -\int_M S_{g(t),u(t)}\, dV_{g(t)}.$$

Putting these together yields

$$0 = \frac{n}{2\tau(\bar{t})} + \frac{1}{V_{g(\bar{t})}} \int_M S_{g(\bar{t}),u(\bar{t})}\, dV_{g(\bar{t})} = \frac{1}{V_{g(\bar{t})}} \int_M \Big(S_{g(\bar{t}),u(\bar{t})} + \frac{n}{2\tau(\bar{t})}\Big)\, dV_{g(\bar{t})}.$$

If we set $\bar{f} = \log V_{g(\bar{t})}$,

$$0 = \mathcal{W}_+(g(\bar{t}), u(\bar{t}), \tau(\bar{t}), \bar{f}) \geq \mu_+(g(\bar{t}), u(\bar{t}), \tau(\bar{t})).$$

By the monotonicity of μ_+ we obtain

$$0 \le \mu_+(g(t_1), u(t_1), \tau(t_1)) \le \mu_+(g(\bar{t}), u(\bar{t}), \tau(\bar{t})) \le 0$$

Hence $\mu_+(g(t_1), u(t_1), \tau(t_1)) = \mu_+(g(t_2), u(t_2), \tau(t_2)) = 0$ and $\mathcal{W}_+ = 0$ on the interval $[t_1, t_2]$. This indicates that the first variation of $\mathcal{W}_+$ must vanish. So the expanding breather is a gradient soliton, that is,

$$\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t) + \frac{1}{2\tau(t)} g(t) = 0.$$

Moreover, in this case $\Delta_{g(t)} u(t) = \langle du(t), df(t)\rangle_{g(t)}$. □

Because of (6-7), we define

$$\text{(6-8)}\quad \mu_{+,k}(g, u, \tau) := \inf\left\{ \mathcal{W}_{+,k}(g, u, \tau, f) \;\middle|\; f \in C^{+\infty}(M),\ \int_M e^{-f} dV_g = 1 \right\}.$$

Due to Lemma 6.4, we still have

$$\text{(6-9)}\qquad \mu_{+,k}(\alpha g, u, \alpha\tau) = \alpha \mu_{+,k}(g, u, \tau).$$

Corollary 6.7. *If $(g(t), u(t))$ is an expanding harmonic-Ricci breather on compact Riemannian manifolds, M is an Einstein manifold and $u(t)$ is constant.*

Proof. Using the same method as in Corollary 6.6 and $\mu_{+,k}$, we can show that the first variation of $\mathcal{W}_{+,k}$ must vanish. Hence, from (6-4), one has

$$\begin{aligned} \mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)} f(t) + \frac{1}{2\tau(t)} g(t) &= 0, \\ \mathcal{S}_{g(t),u(t)} + \frac{1}{2\tau(t)} g(t) &= 0, \\ \Delta_{g(t)} u(t) &= \langle du(t), df(t)\rangle_{g(t)}, \\ \Delta_{g(t)} u(t) &= 0. \end{aligned}$$

The above four equations can be reduced to the coupled equation

$$\mathcal{S}_{g(t),u(t)} + \frac{1}{2\tau(t)} g(t) = 0 = \Delta_{g(t)} u(t),$$

which indicates that $u(t)$ is a constant and $\mathrm{Ric}_{g(t)} = -(1/(2\tau(t)))g(t)$. □

7. Eigenvalues of the Laplacian under the harmonic-Ricci flow

In this section we consider the eigenvalues of the Laplacian $\Delta_{g(t)}$ under the harmonic-Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 4\,du(t) \otimes du(t), \tag{7-1}$$

$$\frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t). \tag{7-2}$$

Suppose that $\lambda(t)$, which is a function of time t only, is an eigenvalue of the Laplacian $\Delta_{g(t)}$ with an eigenfunction $f(t) = f(x, t)$, that is,

$$-\Delta_{g(t)} f(t) = \lambda(t) f(t). \tag{7-3}$$

Taking the derivative with respect to t, we get

$$-\Big(\frac{\partial}{\partial t}\Delta_{g(t)}\Big) f(t) - \Delta_{g(t)}\Big(\frac{\partial}{\partial t} f(t)\Big) = \Big(\frac{d}{dt}\lambda(t)\Big) f(t) + \lambda(t)\frac{\partial}{\partial t} f(t).$$

Integrating the above equation with f yields

$$\begin{aligned} -\int_M f(t)\Big(\frac{\partial}{\partial t}\Delta_{g(t)}\Big) f(t)\, dV_{g(t)} - \int_M f(t)\Delta_{g(t)}\Big(\frac{\partial}{\partial t} f(t)\Big)\, dV_{g(t)} \\ = \frac{d}{dt}\lambda(t)\cdot\int_M f(t)^2\, dV_{g(t)} + \lambda(t)\int_M f(t)\frac{\partial}{\partial t} f(t)\, dV_{g(t)}. \end{aligned}$$

Since

$$\begin{aligned} -\int_M f(t)\Delta\Big(\frac{\partial}{\partial t} f(t)\Big)\, dV_{g(t)} &= -\int_M \Delta_{g(t)} f(t)\cdot\frac{\partial}{\partial t} f(t)\, dV_{g(t)} \\ &= \lambda(t)\int_M f(t)\frac{\partial}{\partial t} f(t)\, dV_{g(t)}, \end{aligned}$$

it follows that

$$\frac{d}{dt}\lambda(t)\cdot\int_M f(t)^2\, dV_{g(t)} = -\int_M f(t)\Big(\frac{\partial}{\partial t}\Delta_{g(t)}\Big) f(t)\, dV_{g(t)}. \tag{7-4}$$

If we set $v_{ij} = -2R_{ij} + 4\partial_i u\partial_j u$,

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \tfrac{1}{2} g^{k\ell}(\partial_i v_{\ell j} + \partial_j v_{il} - \partial_\ell v_{ij}).$$

We temporarily omit all subscripts t. Multiplying with g^{ij} on both sides, we obtain

$$\begin{aligned} g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij} &= \tfrac{1}{2} g^{kl}(2\nabla^i v_{li} - \nabla_l(g^{ij} v_{ij})) = g^{kl}\nabla^i v_{il} + \nabla^k S \\ &= g^{kl}\nabla^i(-2R_{il} + 4\nabla_i u\nabla_l u) + \nabla^k(R - 2|\nabla u|^2) \end{aligned}$$

$$
\begin{aligned}
&= -\nabla^k R + 4\Delta u \cdot \nabla^k u + 4\nabla_i u \cdot \nabla^i \nabla^k u + \nabla^k R - 4\nabla^k \nabla^i u \cdot \nabla_i u \\
&= 4\Delta u \cdot \nabla^k u.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial t}(\Delta f) &= \frac{\partial}{\partial t}(g^{ij}\nabla_i \nabla_j f) \\
&= \Big(\frac{\partial}{\partial t} g^{ij}\Big)\nabla_i \nabla_j f + g^{ij}\Big[\partial_i \partial_j \frac{\partial f}{\partial t} - \Big(\frac{\partial}{\partial t}\Gamma^k_{ij}\Big)\partial_k f - \Gamma^k_{ij}\partial_k \frac{\partial f}{\partial t}\Big] \\
&= \Big(\frac{\partial}{\partial t} g^{ij}\Big)\nabla_i \nabla_j f + \Delta_{g(t)}\Big(\frac{\partial}{\partial t} f\Big) - g^{ij}\Big(\frac{\partial}{\partial t}\Gamma^k_{ij}\Big)\nabla_k f \\
&= (2R_{ij} - 4\nabla_i u \nabla_j u)\nabla^i \nabla^j f - 4\Delta u \cdot \nabla^k u \nabla_k f + \Delta_{g(t)}\Big(\frac{\partial}{\partial t} f\Big).
\end{aligned}
$$

Plugging this into (7-4), we derive

$$
\begin{aligned}
&\frac{d}{dt}\lambda(t) \cdot \int_M f(t)^2\, dV_{g(t)} \\
&\quad = -2\int_M R_{ij}\nabla^i \nabla^j f\, dV + 4\int_M f\nabla^i u \nabla^j u \nabla_i \nabla_j f\, dV + 4\int_M f\Delta u \cdot \nabla^k u \nabla_k f\, dV.
\end{aligned}
$$

The first term can be rewritten as

$$
\begin{aligned}
-2\int_M f R_{ij}\nabla^i \nabla^j f\, dV &= \int_M \nabla^i (2f R_{ij})\nabla^j f\, dV \\
&= 2\int_M (\nabla^i f \cdot R_{ij} + f \cdot \nabla^i R_{ij})\nabla^j f\, dV \\
&= 2\int_M R_{ij}\nabla^i f \nabla^j f\, dV + \int_M f \nabla_j R \nabla^j f\, dV \\
&= 2\int_M R_{ij}\nabla^i f \nabla^j f\, dV - \int_M R \nabla_j (f\nabla^j f)\, dV \\
&= \lambda \int_R f^2\, dV - \int_M R|\nabla f|^2\, dV + 2\int_M R_{ij}\nabla^i f \nabla^j f\, dV.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Big(\frac{d}{dt}\lambda(t)\Big)\int_M f(t)^2\, dV_{g(t)} \\
&= \lambda(t)\int_M R_{g(t)} f(t)^2\, dV_{g(t)} + 2\int_M R_{ij}\nabla^i f \nabla^j f\, dV - \int_M R_{g(t)}|\nabla_{g(t)} f(t)|^2_{g(t)}\, dV_{g(t)} \\
&\qquad + 4\int_M f(\nabla^i u \nabla^j u \nabla_i \nabla_j f + \Delta u \nabla^k u \nabla_k f)\, dV.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\int_M f\nabla^i u\nabla^j u\nabla_i\nabla_j f\,dV &= -\int_M \nabla_i(f\nabla^i u\nabla^j u)\nabla_j f\,dV\\ &= -\int_M (\nabla_i f\nabla^i u\nabla^j u + f\Delta u\nabla^j u + f\nabla^i u\nabla_i\nabla^j u)\nabla_j f\,dV\\ &= -\int_M f\Delta u\langle\nabla u,\nabla f\rangle\,dV - \int_M \nabla^i u\nabla^j u\nabla_i f\nabla_j f\,dV - \int_M f\nabla^i u\nabla^j f\nabla_i\nabla_j u\,dV\end{aligned}$$

and therefore

$$\begin{aligned}\frac{d}{dt}\lambda(t)\int_M f(t)^2 dV_{g(t)} = \lambda(t)\int_M R_{g(t)} f(t)^2 dV_{g(t)} - 4\int_M f\nabla^i u\nabla^j f\nabla_i\nabla_j u\,dV\\ + 2\int_M S_{ij}\nabla^i f\nabla_j f\,dV - \int_M R_{g(t)}|\nabla_{g(t)} f(t)|^2_{g(t)}\,dV_{g(t)}.\end{aligned}$$

The last term here can be simplified as follows:

$$\begin{aligned}-\int_M f\nabla^i u\nabla^j f\nabla_i\nabla_j u\,dV &= \int_M \nabla^j(f\nabla_i u\nabla_j f)\nabla^i u\,dV\\ &= \int_M (\nabla^j f\nabla_i u\nabla_j f + f\nabla^j\nabla_i u\nabla_j f + f\nabla_i u\Delta f)\nabla^i u\,dV\\ &= \int_M |\nabla u|^2|\nabla f|^2\,dV + \int_M f\Delta f|\nabla u|^2\,dV + \int_M f\nabla^i u\nabla^j f\nabla_i\nabla_j u\,dV.\end{aligned}$$

Consequently,

$$-2\int_M f\nabla^i u\nabla^j f\nabla_i\nabla_j u\,dV = \int_M |\nabla u|^2|\nabla f|^2\,dV - \lambda\int_M f^2|\nabla u|^2\,dV.$$

Therefore we derive the following.

Theorem 7.1. *If $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $\lambda(t)$ denotes the eigenvalue of the Laplacian $\Delta_{g(t)}$, then*

$$\begin{aligned}\frac{d}{dt}\lambda(t)\cdot\int_M f(t)^2\,dV_{g(t)} &= \lambda(t)\int_M S_{g(t),u(t)} f(t)^2\,dV_{g(t)} - \int_M S_{g(t),u(t)}|\nabla_{g(t)} f(t)|^2_{g(t)}\,dV_{g(t)}\\ &\quad + 2\int_M \langle \mathcal{S}_{g(t),u(t)}, df(t)\otimes df(t)\rangle\,dV_{g(t)}.\end{aligned} \tag{7-5}$$

We set

$$S_{\min}(0) := \min_{x\in M} S(x,0). \tag{7-6}$$

Theorem 7.2. *Let $(g(t), u(t))_{t\in[0,T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Suppose that $\mathcal{S}_{g(t),u(t)} - \alpha S_{g(t),u(t)} g(t) \geq 0$ along the harmonic-Ricci flow for some $\alpha \geq \frac{1}{2}$.*

(1) *If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.*

(2) *If $S_{\min}(0) > 0$, the quantity*

$$\left(1 - \frac{2}{n} S_{\min}(0)t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq n/(2S_{\min}(0))$.

(3) *If $S_{\min}(0) < 0$, the quantity*

$$\left(1 - \frac{2}{n} S_{\min}(0)t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

Proof. By Theorem 7.1, we have

$$\frac{d}{dt}\lambda(t) \geq \frac{\int_M S_{g(t),u(t)} f(t)^2 \, dV_{g(t)}}{\int_M f(t)^2 \, dV_{g(t)}} \lambda(t) + (2\alpha - 1) \frac{\int_M S_{g(t),u(t)} |\nabla_{g(t)} f(t)|^2_{g(t)}}{\int_M f(t)^2 \, dV_{g(t)}}.$$

By definition we have $-f(t)\Delta_{g(t)} = \lambda(t) f(t)$. Integrating both sides yields that $\lambda(t) \geq 0$. Since

$$\frac{\partial}{\partial t} S_{g(t),u(t)} = \Delta_{g(t)} S_{g(t),u(t)} + 2|\mathcal{S}_{g(t),u(t)}|^2_{g(t)} + 4|\Delta_{g(t)} u(t)|^2_{g(t)}$$

and $|\mathcal{S}_{g(t),u(t)}|^2 \geq (1/n) S^2_{g(t),u(t)}$, it follows that

$$\frac{\partial}{\partial t} S_{g(t),u(t)} \geq \Delta_{g(t)} S_{g(t),u(t)} + \frac{2}{n} S^2_{g(t),u(t)}.$$

The corresponding ODE

$$\frac{d}{dt} a(t) = \frac{2}{n} a(t)^2, \quad a(t) = S_{\min}(0)$$

has the solution

$$a(t) = \frac{S_{\min}(0)}{1 - (2/n) S_{\min}(0)t}.$$

Then the maximum principle implies $S_{g(t),u(t)} \geq a(t)$ and hence, using the assumption that $2\alpha - 1 \geq 0$,

$$\frac{d}{dt}\lambda(t) \geq a(t)\lambda(t) + (2\alpha - 1)a(t)\frac{\int_M |\nabla_{g(t)} f(t)|^2_{g(t)}\, dV_{g(t)}}{\int_M f(t)^2\, dV_{g(t)}}.$$

By integration by parts, we note that

$$\int_M |\nabla f|^2\, dV = -\int_M f \cdot \Delta f\, dV = \lambda \int_M f^2\, dV,$$

which shows that

$$\frac{d}{dt}\lambda(t) \geq a(t)\lambda(t) + (2\alpha - 1)a(t)\lambda = 2\alpha a(t)\lambda(t)$$

and

$$\frac{d}{dt}\left(\lambda(t) \cdot \exp\left(-2\alpha \int_0^t a(\tau)\, d\tau\right)\right) \geq 0.$$

This inequality clearly implies the desired result. If $S_{\min}(0) \geq 0$, by the nonnegativity of $\mathcal{S}_{g(t)}$ preserved along the harmonic-Ricci flow, we conclude that $d\lambda(t)/dt \geq 0$. □

Corollary 7.3. *Let $(g(t), u(t))_{t\in[0,T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian surface Σ and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$.*

(1) *Suppose that* $\mathrm{Ric}_{g(t)} \leq \epsilon du(t) \otimes du(t)$ *where*

$$\epsilon \leq 4\frac{1-\alpha}{1-2\alpha}, \quad \alpha > \tfrac{1}{2}. \tag{7-7}$$

(i) *If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.*

(ii) *If $S_{\min}(0) > 0$, the quantity*

$$(1 - S_{\min}(0)\, t)^{2\alpha}\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) *If $S_{\min}(0) < 0$, the quantity*

$$(1 - S_{\min}(0)\, t)^{2\alpha}\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(2) *Suppose that*

$$|\nabla_{g(t)} u(t)|^2_{g(t)}\, g(t) \geq 2du(t) \otimes du(t). \tag{7-8}$$

(i) *If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.*

(ii) *If* $S_{\min}(0) > 0$, *the quantity*

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) *If* $S_{\min}(0) < 0$, *the quantity*

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

Proof. As above, we always omit subscripts t. In the surface case, we have $R_{ij} = \frac{1}{2} R g_{ij}$. Then

$$\begin{aligned} T_{ij} := S_{ij} - \alpha S g_{ij} &= \frac{R}{2} g_{ij} - 2\nabla_i u \nabla_j u - \alpha(R - 2|\nabla u|^2) g_{ij} \\ &= \left(\tfrac{1}{2} - \alpha\right) R g_{ij} - 2\nabla_i u \nabla_j u + 2\alpha |\nabla u|^2 g_{ij}. \end{aligned}$$

For any vector $V = (V^i)$, we calculate

$$\begin{aligned} T_{ij} V^i V^j &= \left(\tfrac{1}{2} - \alpha\right) R |V|^2 - 2(\nabla_i u V^i)^2 + 2\alpha |\nabla u|^2 |V|^2 \\ &\geq \left(\tfrac{1}{2} - \alpha\right) R |V|^2 - 2|\nabla u|^2 |V|^2 + 2\alpha |\nabla u|^2 |V|^2. \end{aligned}$$

If $R_{ij} \leq \epsilon \nabla_i u \nabla_j u$, then $T_{ij} V^i V^j = [(\frac{1}{2} - \alpha)\epsilon - 2 + 2\alpha]\, |\nabla u|^2 |V|^2 \geq 0$.

For the second case, we note that

$$\begin{aligned} T_{ij} V^i V^j &= R_{ij} V^i V^j - 2\nabla_i u V^i \nabla_j u V^j - \frac{R}{2} |V|^2 + |\nabla u|^2 |V|^2 \\ &\geq R_{ij} V^i V^j - |\nabla u|^2 |V|^2 - \frac{R}{2} |V|^2 + |\nabla u|^2 |V|^2 = 0. \end{aligned}$$

Hence the corresponding results follow by Theorem 7.2. □

When we consider the Ricci flow, we have the following two results derived from Corollary 7.3.

Corollary 7.4. *Let* $(g(t))_{t \in [0,T]}$ *be a solution of the Ricci flow on a compact Riemannian surface* Σ *and let* $\lambda(t)$ *denote the eigenvalue of the Laplacian* $\Delta_{g(t)}$.

(1) *If* $R_{\min}(0) \geq 0$, $\lambda(t)$ *is nondecreasing along the Ricci flow for any* $t \in [0, T]$.

(2) *If* $R_{\min}(0) > 0$, *the quantity* $(1 - R_{\min}(0)t)\lambda(t)$ *is nondecreasing along the Ricci flow for* $T \leq 1/R_{\min}(0)$.

(3) *If* $R_{\min}(0) < 0$, *the quantity* $(1 - R_{\min}(0)t)\lambda(t)$ *is nondecreasing along the Ricci flow for any* $t \in [0, T]$.

Remark 7.5. Let $(g(t))_{t\in[0,T]}$ be a solution of the Ricci flow on a compact Riemannian surface Σ with nonnegative scalar curvature and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Then $\lambda(t)$ is nondecreasing along the Ricci flow for $t\in[0,T]$.

8. Eigenvalues of the Laplacian-type under the harmonic-Ricci flow

Recall that

$$\mu(g,u)=\mu_1(g,u)=\inf\left\{\mathscr{F}(g,u,f)\;\Big|\;\int_M e^{-f}\,dV_g=1\right\}. \tag{8-1}$$

We showed that $\mu(g,u)$ is the smallest eigenvalue of the operator

$$-4\Delta_g+R_g-2|\nabla_g u|_g^2.$$

Inspired by [Cao 2007; 2008], we define a Laplacian-type operators associated with quantities g,u,c:

$$\Delta_{g,u,c}:=-\Delta_g+c(R_g-2|\nabla_g u|_g^2), \tag{8-2}$$

$$\Delta_{g,u}:=\Delta_{g,u,\frac{1}{2}}=-\Delta_g+\tfrac{1}{2}(R_g-2|\nabla_g u|_g^2). \tag{8-3}$$

Then $\mu(g,u)$ is the smallest eigenvalue of the operator $4\Delta_{g,u,1/4}$.

To the operator $\Delta_{g,u}$ we associate the functional

$$C^\infty(M)\to\mathbb{R},\quad f\mapsto\lambda_{g,u}(f):=\int_M f\,\Delta_{g,u}f\,dV_g. \tag{8-4}$$

When f is an eigenfunction of the operator $\Delta_{g,u}$ with the eigenvalue λ, that is, $\Delta_{g,u}f=\lambda f$ and is normalized by $\int_M f^2\,dV_g=1$, we obtain $\lambda_{g,u}(f)=\lambda$. The next lemma will deal with the evolution equation for $\lambda(f(t))$, where $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$ and the couple $(g(t),u(t))$ satisfies the harmonic-Ricci flow. Set

$$v_{ij}:=-2S_{ij}=-2R_{ij}+4\partial_i u\cdot\partial_j u,\quad v:=g^{ij}v_{ij}. \tag{8-5}$$

The symmetric tensor field thus obtained is denoted by $\mathscr{V}_{g(t),u(t)}=(v_{ij})$.

Lemma 8.1. *Suppose that $(g(t),u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$, that is, $\Delta_{g(t),u(t)}f(t)=\lambda(t)f(t)$ (where $\lambda(t)$ is only a function of time t only), with the normalized condition*

$$\int_M f(t)^2\,dV_{g(t)}=1.$$

Then we have

$$\frac{d}{dt}\lambda_{g(t),u(t)}(f(t) \\ = \int_M f(t)\big(\nabla^i v_{ik} - \tfrac{1}{2}\nabla_k v\big)\nabla^k f(t)\,dV_{g(t)} - \int_M f^2(t)\frac{\partial}{\partial t}|\nabla_{g(t)}u(t)|^2_{g(t)}\,dV_{g(t)} \\ + \int_M \bigg(\langle \mathcal{V}_{g(t),u(t)}, \nabla^2_{g(t)} f(t)\rangle_{g(t)} + \frac{1}{2}\Big(\frac{\partial}{\partial t}R_{g(t)}\Big) f(t)\bigg) f(t)\,dV_{g(t)}. \tag{8-6}$$

Before proving the lemma, we recall a formula that is an immediate consequence of the evolution equation:

$$\frac{\partial}{\partial t}(\Delta_{g(t)} f) \\ = -g^{ip}g^{jq}v_{pq}\nabla_i\nabla_j f - g^{ij}g^{k\ell}\nabla_i v_{j\ell}\nabla_k f + \tfrac{1}{2}\langle\nabla_{g(t)}v_{g(t)}, \nabla_{g(t)} f(t)\rangle_{g(t)} \tag{8-7}$$

where the metric $g(t)$ evolves by $\partial g_{ij}/\partial t = v_{ij}$.

Proof. Using (8-7) and integration by parts, we get

$$\begin{aligned}
&\frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) \\
&= \frac{\partial}{\partial t}\int_M \bigg(-\Delta_{g(t)} f(t) + \bigg(\frac{R_{g(t)}}{2} - |\nabla_{g(t)}u(t)|^2_{g(t)}\bigg) f(t)\bigg) f(t)\,dV_{g(t)} \\
&= \int_M \big(g^{ip}g^{jq}v_{pq}\nabla_i\nabla_j f + g^{ij}g^{kl}\nabla_i v_{jl}\nabla_k f - \tfrac{1}{2}\langle\nabla_{g(t)}v_{g(t)}, \nabla_{g(t)} f(t)\rangle_{g(t)}\big) f(t)\,dV_{g(t)} \\
&\quad + \int_M \bigg(-\Delta_{g(t)}\big(\frac{\partial}{\partial t} f(t)\big) + \bigg(\frac{R_{g(t)}}{2} - |\nabla_{g(t)}u(t)|^2_{g(t)}\bigg)\frac{\partial}{\partial t} f(t) \\
&\quad + \bigg(\frac{\partial}{\partial t}(\tfrac{1}{2}R_{g(t)}) - \frac{\partial}{\partial t}(|\nabla_{g(t)}u(t)|^2_{g(t)})\bigg) f(t)\bigg) f(t)\,dV_{g(t)} \\
&\qquad + \int_M \bigg(-\Delta_{g(t)} f(t) + \bigg(\frac{R_{g(t)}}{2} - |\nabla_{g(t)}u|^2_{g(t)}\bigg) f(t)\bigg)\frac{\partial}{\partial t}(f(t)\,dV_{g(t)}) \\
&= \int_M \bigg(g^{ip}g^{jq}v_{pq}\nabla_i\nabla_j f + \frac{1}{2}\Big(\frac{\partial}{\partial t}R_{g(t)}\Big) f(t)\bigg) f(t)\,dV_{g(t)} \\
&\quad + \int_M (g^{ij}g^{kl}\nabla_i v_{jl}\nabla_k f - \tfrac{1}{2}g^{kl}\nabla_l v\nabla_k f) f(t)\,dV_{g(t)} \\
&\quad + \int_M \Delta_{g(t),u(t)} f(t)\bigg(\frac{\partial}{\partial t} f(t)\,dV_{g(t)} + \frac{\partial}{\partial t}(f(t)\,dV_{g(t)})\bigg) \\
&\qquad - \int_M \frac{\partial}{\partial t}(|\nabla_{g(t)}u(t)|^2_{g(t)}) f(t)^2\,dV_{g(t)}.
\end{aligned}$$

Since $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$, it follows that

$$\int_M \Delta_{g(t),u(t)} f(t)\bigg(\frac{\partial}{\partial t} f(t)\,dV_{g(t)} + \frac{\partial}{\partial t}(f(t)\,dV_{g(t)})\bigg) = \lambda(t)\frac{\partial}{\partial t}\int_M f(t)^2\,dV_{g(t)} = 0$$

by the normalization condition. This completes the proof. □

Using (3-6), we find that the first term in the right side of (8-6) can be written as

$$\begin{aligned}
&\int_M \Big(v_{ij}\nabla^i\nabla^j f+\frac{1}{2}\Big(\frac{\partial}{\partial t}R_{g(t)}\Big)f(t)\Big)f(t)\,dV_{g(t)}\\
&=\int_M \big(-2f(t)\langle \mathrm{Ric}_{g(t)},\nabla^2_{g(t)}f(t)\rangle_{g(t)}+4f(t)\langle du(t)\otimes du(t),\nabla^2_{g(t)}f(t)\rangle_{g(t)}\big)\,dV_{g(t)}\\
&\qquad+\int_M \big((\tfrac{1}{2}\Delta_{g(t)}R_{g(t)}+|\mathrm{Ric}_{g(t)}|^2_{g(t)})f(t)^2+2f(t)^2|\Delta_{g(t)}u(t)|^2_{g(t)}\\
&\qquad\qquad-2f(t)^2|\nabla^2_{g(t)}u(t)|^2_{g(t)}-4f(t)^2\langle \mathrm{Ric}_{g(t)},du(t)\otimes du(t)\rangle_{g(t)}\big)\,dV_{g(t)}\\
&=\int_M \big(-2f(t)\langle \mathrm{Ric}_{g(t)},\nabla^2_{g(t)}f(t)\rangle_{g(t)}+(\tfrac{1}{2}\Delta_{g(t)}R_{g(t)}+|\mathrm{Ric}_{g(t)}|^2_{g(t)})f(t)^2\big)\,dV_{g(t)}\\
&\qquad+\int_M \big(4f(t)\langle du\otimes du,\nabla^2_{g(t)}f(t)\rangle_{g(t)}-4f^2\langle du(t)\otimes du(t),\mathrm{Ric}_{g(t)}\rangle_{g(t)}\\
&\qquad\qquad+2f(t)^2|\Delta_{g(t)}u(t)|^2_{g(t)}-2f(t)^2|\nabla^2_{g(t)}u(t)|^2_{g(t)}\big)\,dV_{g(t)}
\end{aligned}$$

For the second term in (8-6), using the contracted Bianchi identities, one has

$$\begin{aligned}
&\int_M (g^{ij}\nabla_i v_{jk}-\tfrac{1}{2}\nabla_k v)\nabla^k f\cdot f(t)\,dV_{g(t)}\\
&\qquad=\int_M \big(g^{ij}\nabla_i(-2R_{jk}+4\partial_j u\partial_k u)\\
&\qquad\qquad-\tfrac{1}{2}\nabla_k(-2R_{g(t)}+4|\nabla_{g(t)}u(t)|^2_{g(t)})\big)\nabla^k f\cdot f(t)\,dV_{g(t)}\\
&\qquad=\int_M 4f(t)\Delta_{g(t)}u(t)\langle\nabla_{g(t)}u(t),\,\nabla_{g(t)}f(t)\rangle_{g(t)}\,dV_{g(t)}\\
&\qquad\qquad+\int_M (4g^{ij}\nabla_j u\cdot\nabla_i\nabla_k u-2\nabla_k|\nabla_{g(t)}u(t)|^2_{g(t)})\nabla^k f\cdot f(t)\,dV_{g(t)}\\
&\qquad=\int_M 4f(t)\Delta_{g(t)}u(t)\langle\nabla_{g(t)}u(t),\,\nabla_{g(t)}f(t)\rangle_{g(t)}\,dV_{g(t)}
\end{aligned}$$

where in the last step we use the identity $\nabla_k|\nabla u|^2=2g^{pq}\nabla_k\nabla_p u\cdot\nabla_q u$. Therefore

$$\begin{aligned}
&\frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) \qquad (8\text{-}8)\\
&=\int_M \big(-2f(t)\langle \mathrm{Ric}_{g(t)},\nabla^2_{g(t)}f(t)\rangle_{g(t)}+(\tfrac{1}{2}\Delta_{g(t)}R_{g(t)}+|\mathrm{Ric}_{g(t)}|^2_{g(t)})f(t)^2\big)\,dV_{g(t)}\\
&\qquad+4f(t)\int_M \big(\langle du(t)\otimes du(t),\,\nabla^2_{g(t)}f(t)\rangle_{g(t)}-f(t)\langle du(t)\otimes du(t),\mathrm{Ric}_{g(t)}\rangle_{g(t)}\\
&\qquad\qquad+2f(t)^2|\Delta_{g(u)}u(t)|^2_{g(t)}-2f(t)^2|\nabla^2_{g(t)}u(t)|^2_{g(t)}\\
&\qquad\qquad\qquad+4f(t)\Delta_{g(t)}u(t)\langle\nabla_{g(t)}u(t),\,\nabla_{g(t)}f(t)\rangle_{g(t)}\big)\,dV_{g(t)}\\
&\qquad-\int_M (\Delta_{g(t)}|\nabla_{g(t)}u(t)|^2_{g(t)}-2|\nabla^2_{g(t)}u(t)|^2_{g(t)}-4|\nabla_{g(t)}u(t)|^4_{g(t)})f(t)^2\,dV_{g(t)}.
\end{aligned}$$

The above evolution equation can be simplified as follows.

Theorem 8.2. *Suppose* $(g(t), u(t))$ *is a solution of the harmonic-Ricci flow on a compact Riemannian manifold* M *and* $f(t)$ *is an eigenfunction of* $\Delta_{g(t),u(t)}$, *that is*, $\Delta_{g(t),u(t)} f(t) = \lambda(t) f(t)$ (*where* $\lambda(t)$ *is only a function of time* t *only*), *with the normalized condition* $\int_M f(t)^2 dV_{g(t)} = 1$. *Then we have*

$$\frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) = \int_M 2\langle \mathcal{S}_{g(t)}, df(t)\otimes df(t)\rangle_{g(t)}\, dV_{g(t)} + \int_M f(t)^2\big(|\mathcal{S}_{g(t)}|^2_{g(t)} + 2|\Delta_{g(t)}u(t)|^2_{g(t)}\big)\, dV_{g(t)}. \tag{8-9}$$

Proof. Calculate

$$\begin{aligned}
&\int_M 4f(t)\Delta_{g(t)}u(t)\langle\nabla_{g(t)}u(t), \nabla_{g(t)}f(t)\rangle_{g(t)}\, dV_{g(t)} \\
&\quad= -4\int_M \nabla_i u[\nabla^i f\cdot\langle\nabla u, \nabla f\rangle + f(\nabla^i\langle\nabla u, \nabla f\rangle)]\, dV \\
&\quad= -4\int_M |\langle\nabla u, \nabla f\rangle|^2\, dV_g - 4\int_M f\nabla_i u(\langle\nabla^i\nabla u, \nabla f\rangle + \langle\nabla u, \nabla^i\nabla f)\, dV.
\end{aligned}$$

By the same method, we have

$$\begin{aligned}
\int_M -\Delta_{g(t)}|\nabla_{g(t)}u(t)|^2_{g(t)} f(t)^2\, dV_{g(t)}
&= -\int_M |\nabla u|^2(2f\Delta f + 2|\nabla f|^2)\, dV \\
&= -2\int_M |\nabla f|^2|\nabla u|^2\, dV - 2\int_M f\Delta f|\nabla u|^2\, dV.
\end{aligned}$$

However,

$$\begin{aligned}
\int_M f\Delta f|\nabla u|^2\, dV &= \int_M -\nabla_i f\cdot\nabla^i(f|\nabla u|^2)\, dV \\
&= -\int_M \nabla_i f(\nabla^i f|\nabla u|^2 + f\nabla^i|\nabla u|^2)\, dV \\
&= -\int_M |\nabla u|^2|\nabla f|^2\, dV - \int_M f\nabla_i f\cdot\nabla^i|\nabla u|^2\, dV.
\end{aligned}$$

Therefore we arrive at

$$\begin{aligned}
\int_M -\Delta_{g(t)}|\nabla_{g(t)}u(t)|^2_{g(t)} f(t)^2\, dV_{g(t)}
&= 2\int_M f\nabla_i f\cdot\nabla^i|\nabla u|^2\, dV \\
&= 4\int_M f(t)\langle du(t)\otimes df(t), \nabla^2_{g(t)}u(t)\rangle_{g(t)}\, dV_{g(t)}.
\end{aligned}$$

Using the contracted Bianchi identities, we may simplify the term $\int_M \frac{1}{2} f^2 \Delta R\, dV$ as follows:

$$\begin{aligned}
&\int_N \frac{f(t)^2}{2} \Delta_{g(t)} R_{g(t)}\, dV_{g(t)} \\
&= -\frac{1}{2}\int_M \nabla_i R \cdot \nabla^i (f^2)\, dV \\
&= -\int_M \nabla_i R \cdot f \nabla^i f\, dV = -2\int_M \nabla^k R_{ki} \cdot f\nabla^i f\, dV \\
&= 2\int_M R_{ki} \nabla^k (f \nabla^j f)\, dV = 2\int_M R_{ki} (\nabla^k f \cdot \nabla^j f + f\nabla^k \nabla^j f)\, dV \\
&= 2\int_M \langle \mathrm{Ric}_{g(t)}, df(t) \otimes df(t)\rangle_{g(t)}\, dV_{g(t)} + 2\int_M f(t) \langle \mathrm{Ric}_{g(t)}, \nabla^2_{g(t)} f(t)\rangle_{g(t)}\, dV_{g(t)}.
\end{aligned}$$

Hence (8-8) becomes

$$\begin{aligned}
\frac{d}{dt}&\lambda_{g(t),u(t)}(f(t)) \\
&= \int_M \bigl(2\langle \mathrm{Ric}_{g(t)}, df(t)\otimes df(t)\rangle_{g(t)} + |\,\mathrm{Ric}_{g(t)}\,|^2_{g(t)} f(t)^2\bigr)\, dV_{g(t)} \\
&\qquad + \int_M \bigl(2|\Delta_{g(t)} u(t)|^2_{g(t)} + 4|\nabla_{g(t)} u(t)|^4_{g(t)}\bigr) f(t)^2\, dV_{g(t)} \\
&\qquad - \int_M 4 f(t)^2 \langle du(t)\otimes du(t), \mathrm{Ric}_{g(t)}\rangle_{g(t)}\, dV_{g(t)} \\
&\qquad - \int_M 4|\langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t)\rangle_{g(t)}|^2\, dV_{g(t)} \\
&= \int_M 2\langle \mathcal{S}_{g(t)}, df(t)\otimes df(t)\rangle_{g(t)}\, dV_{g(t)} \\
&\qquad\quad + \int_M f(t)^2 \bigl(|\,\mathrm{Ric}_{g(t)} - 2du(t)\otimes du(t)|^2_{g(t)} + 2|\Delta_{g(t)} u(t)|^2_{g(t)}\bigr)\, dV_{g(t)}
\end{aligned}$$

where, by definition, $S_{ij} = R_{ij} - 2\partial_i u \partial_j u$. □

List [2006] proved that the nonnegativity of the operator $\mathcal{S}_{g(t)}$ is preserved by the harmonic-Ricci flow. Hence we get the following.

Corollary 8.3. *If* $\mathrm{Ric}_{g(0)} - 2du(0)\otimes du(0) \ge 0$, *the eigenvalues of the operator* $\Delta_{g(t),u(t)}$ *are nondecreasing under the harmonic-Ricci flow.*

Remark 8.4. If we choose $u(t) \equiv 0$, we obtain Cao's result [2007].

9. Another formula for $\frac{d}{dt}\lambda(f(t))$

In this section we give another formula for $\frac{d}{dt}\lambda(f(t))$ using a method similar to that in [Li 2007]. Recall the formula

$$\frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) = \int_M 2\langle \mathcal{S}_{g(t),u(t)}, df(t)\otimes df(t)\rangle_{g(t)}\, dV_{g(t)} + \int_M f(t)^2\big(|\mathcal{S}_{g(t),u(t)}|^2_{g(t)} + 2|\Delta_{g(t)}u(t)|^2_{g(t)}\big)\, dV_{g(t)}.$$

Consider the function φ determined by $f^2(t) = e^{-\varphi(t)}$. Then we have

$$df = \frac{-e^{\varphi}d\varphi}{2f},\qquad \frac{\nabla f}{f} = -\frac{\nabla\varphi}{2},\qquad \frac{\Delta f}{f} = -\tfrac{1}{2}\Delta\varphi + \tfrac{1}{4}|\nabla\varphi|^2.$$

Hence

$$2\frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) = \int_M \langle \mathcal{S}_{g(t),u(t)}, d\varphi(t)\otimes d\varphi(t)\rangle_{g(t)} e^{-\varphi(t)}\, dV_{g(t)} + 2\int_M \big(|\mathcal{S}_{g(t),u(t)}|^2_{g(t)} + 2|\Delta_{g(t)}u(t)|^2_{g(t)}\big)e^{-\varphi}\, dV_{g(t)}.$$

Using integration by parts and contracted Bianchi identities yields

$$\begin{aligned}
&\int_M \langle \mathcal{S}_{g(t),u(t)}, d\varphi(t)\otimes d\varphi(t)\rangle_{g(t)} e^{-\varphi(t)}\, dV_{g(t)}\\
&= \int_M S_{ij}\nabla^i\varphi\nabla^j\varphi e^{-\varphi}\, dV = -\int_M S_{ij}\nabla^j\varphi\nabla^i(e^{-\varphi})\, dV\\
&= \int_M e^{-\varphi}\nabla^i(S_{ij}\nabla^j\varphi)\, dV\\
&= \int_M \nabla^i S_{ij}\cdot\nabla^j\varphi\cdot e^{-\varphi}\, dV + \int_M S_{ij}\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\, dV\\
&= \int_M \nabla^i R_{ij}\cdot\nabla^j\varphi\cdot e^{-\varphi}\, dV_g + \int_M S_{ij}\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\, dV + \int_M \nabla^i(-2\nabla_i u\nabla_j u)\nabla^j\varphi\cdot e^{-\varphi}\, dV_g\\
&= \frac{1}{2}\int_M R\Delta(e^{-\varphi})\, dV + \int_M S_{ij}\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\, dV - 2\int_M (\nabla^i u\nabla_j u)\nabla^i\nabla^j(e^{-\varphi})\, dV.
\end{aligned}$$

Thus

$$\int_M S_{ij}\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\, dV = \int_M S_{ij}\nabla^i\varphi\nabla^j\varphi e^{-\varphi}\, dV - \frac{1}{2}\int_M R\Delta(e^{-\varphi})\, dV + 2\int_M (\nabla^i u\nabla_j u)\nabla^i\nabla^j(e^{-\varphi}).$$

On the other hand, one gets

$$\begin{aligned}\int_M |\nabla^2_{g(t)}\varphi(t)|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)} &= \int_M \nabla_i\nabla_j\varphi\nabla^i\nabla_j\varphi\cdot e^{-\varphi}\, dV \\ &= -\int_M \nabla_j\varphi\cdot\nabla_i\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\, dV - \int_M \nabla_j\varphi\cdot\nabla^i\nabla^j\varphi\cdot\nabla_i(e^{-\varphi})\, dV \\ &= -\int_M \nabla_j\varphi\cdot\nabla_i\nabla^j\nabla^i\varphi\cdot e^{-\varphi}\, dV - \int_M \nabla_j\varphi\cdot\nabla^i\nabla^j\varphi\cdot\nabla_i(e^{-\varphi})\, dV.\end{aligned}$$

Since

$$\begin{aligned}\int_M \nabla_j\varphi\cdot\nabla^i\nabla^j\varphi\cdot\nabla_i(e^{-\varphi})\, dV &= -\int_M \nabla^i(\nabla_j\varphi\cdot\nabla_i(e^{-\varphi}))\nabla^j\varphi\, dV \\ &= -\int_M \nabla^j\varphi\cdot\nabla^i\nabla_j\varphi\cdot\nabla_i(e^{-\varphi})\, dV - \int_M |\nabla\varphi|^2\Delta(e^{-\varphi})\, dV,\end{aligned}$$

which implies

$$\int_M \nabla_j\varphi\cdot\nabla^i\nabla^j\varphi\cdot\nabla_i(e^{-\varphi})dV = -\frac{1}{2}\int_M |\nabla\varphi|^2\Delta(e^{-\varphi})\, dV,$$

it follows that

$$\int_M |\nabla^2\varphi|^2 e^{-\varphi}\, dV = -\int_M \nabla_j\varphi\cdot\nabla_i\nabla^j\nabla^i\varphi\cdot e^{-\varphi}\, dV + \frac{1}{2}\int_M |\nabla\varphi|^2\Delta(e^{-\varphi})\, dV.$$

By the Ricci identity the term $\nabla^i\nabla^j\nabla^i\varphi$ equals

$$\begin{aligned}\nabla_i\nabla^j\nabla^i\varphi &= g^{jk}g^{il}\nabla_i\nabla_k\nabla_l\varphi = g^{jk}g^{il}(\nabla_k\nabla_i\nabla_l\varphi - R^p_{ikl}\nabla_p\varphi) \\ &= \nabla^j\nabla_i\nabla^i\varphi - g^{jk}g^{il}R_{iklp}\nabla^p\varphi \\ &= \nabla^j\Delta\varphi + g^{jk}g^{il}R_{ikpl}\nabla^p\varphi = \nabla^j\Delta\varphi + g^{jk}R_{kp}\nabla^p\varphi.\end{aligned}$$

Hence

$$\begin{aligned}&-\int_M \nabla_j\varphi\cdot\nabla_i\nabla^j\nabla^i\varphi\cdot e^{-\varphi}\, dV \\ &\quad= -\int_M \nabla_i\varphi\cdot\nabla^j\Delta\varphi\cdot e^{-\varphi}\, dV - \int_M R_{kp}\nabla^k\varphi\cdot\nabla^p\varphi e^{-\varphi}\, dV \\ &\quad= \int_M \nabla^j\Delta\varphi\cdot\nabla_j(e^{-\varphi}) + \int_M R_{kp}\nabla^k\varphi\cdot\nabla^p(e^{-\varphi})\, dV \\ &\quad= -\int_M \Delta\varphi\cdot\Delta(e^{-\varphi}) - \int_M e^{-\varphi}(\nabla^p R_{kp}\cdot\nabla^k\varphi + R_{kp}\nabla^p\nabla^k\varphi) \\ &\quad= -\int_M \Delta(e^{-\varphi})\cdot\Delta\varphi\, dV + \frac{1}{2}\int_M \nabla_k R\cdot\nabla^k(e^{-\varphi})\, dV - \int_M e^{-\varphi}R_{kp}\nabla^k\nabla^p\varphi\, dV \\ &\quad= -\int_M \Delta(e^{-\varphi})(\Delta\varphi + \tfrac{1}{2}R) - \int_M R_{kp}\nabla^k\nabla^p\varphi\cdot e^{-\varphi}\, dV.\end{aligned}$$

Putting those formulas together, we obtain

$$\int_M 2S_{ij}\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\,dV+\int_M|\nabla^2\varphi|^2e^{-\varphi}\,dV$$
$$=\int_M S_{ij}\nabla^i\nabla_j\varphi\cdot e^{-\varphi}\,dV+\int_M(-2\nabla_iu\nabla_ju)\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\,dV$$
$$-\int_M\Delta(e^{-\varphi})\Big(\Delta\varphi+\frac{R}{2}-\tfrac{1}{2}|\nabla\varphi|^2\Big)\,dV$$
$$=\int_M S_{ij}\nabla^i\varphi\nabla^j\varphi\cdot e^{-\varphi}\,dV-\int_M\Delta(e^{-\varphi})(\Delta\varphi+R-\tfrac{1}{2}|\nabla\varphi|^2)\,dV$$
$$+2\int_M(\nabla_iu\nabla_ju\cdot\nabla^i\nabla^j(e^{-\varphi})-\nabla_iu\nabla_ju\cdot\nabla^i\nabla^j\varphi\cdot e^{-\varphi})\,dV.$$

Since f is an eigenfunction of λ, it induces

$$\lambda=-\frac{\Delta f}{f}+\frac{R}{2}-|\nabla u|^2=\tfrac{1}{2}\Delta\varphi-\tfrac{1}{4}|\nabla\varphi|^2+\frac{R}{2}-|\nabla u|^2,$$

and therefore

$$\int_M 2S_{ij}\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\,dV+\int_M|\nabla^2\varphi|^2e^{-\varphi}\,dV$$
$$=\int_M S_{ij}\nabla^i\varphi\nabla^j\varphi\cdot e^{-\varphi}\,dV-2\int_M\Delta(|\nabla u|^2)\cdot e^{-\varphi}\,dV$$
$$+2\int_M\nabla_iu\nabla_j(\nabla^i\nabla^j(e^{-\varphi})-\nabla^i\nabla^j\varphi\cdot e^{-\varphi})\,dV.$$

Plugging this into the expression of $\frac{d}{dt}\lambda(f(t))$ yields

$$2\frac{d}{dt}\lambda_{g(t),u(t)}(f(t))$$
$$=\int_M S_{ij}\nabla^i\varphi\nabla^j\varphi\cdot e^{-\varphi}\,dV+\int_M|\mathcal{S}|^2e^{-\varphi}\,dV+\int_M|\mathcal{S}|^2e^{-\varphi}\,dV+4\int_M|\Delta u|^2e^{-\varphi}\,dV$$
$$=\int_M|\mathcal{S}_{g(t),u(t)}+\nabla^2_{g(t)}\varphi(t)|^2_{g(t)}e^{-\varphi(t)}\,dV_{g(t)}+\int_M|\mathcal{S}_{g(t),u(t)}|^2_{g(t)}e^{-\varphi(t)}\,dV_{g(t)}$$
$$+4\int_M|\Delta_{g(t)}u(t)|^2_{g(t)}e^{-\varphi(t)}\,dV_{g(t)}+2\int_M\Delta_{g(t)}|\nabla_{g(t)}u(t)|^2_{g(t)}e^{-\varphi(t)}\,dV_{g(t)}$$
$$+\,2\int_M\nabla_iu\nabla_ju\big(-\nabla^i\nabla^j(e^{-\varphi})+\nabla^i\nabla^j\varphi\cdot e^{-\varphi}\big)\,dV$$

Now define

$$I:=\int_M(\nabla_iu\nabla_ju\cdot\nabla^i\nabla^j\varphi)e^{-\varphi}dV=-\int_M\nabla^i(\nabla_iu\nabla_ju\cdot e^{-\varphi})\nabla^j\varphi dV$$
$$=-\int_M\nabla^j\varphi(\Delta u\cdot\nabla_ju\cdot e^{-\varphi}+\nabla_iu\nabla^i\nabla_ju\cdot e^{-\varphi}-\nabla_iu\nabla_ju\nabla^i\varphi\cdot e^{-\varphi})dV$$
$$=-\int_M\nabla_ju\nabla^j\varphi\Delta u\cdot e^{-\varphi}dV-\int_M\nabla_iu\nabla^j\varphi\nabla^i\nabla_ju\cdot e^{-\varphi}dV+\int_M|\langle du,d\varphi\rangle|^2e^{-\varphi}dV,$$

$$\begin{aligned} II := \int_M \nabla_i u \nabla_j u \nabla^i \nabla^j (e^{-\varphi})\, dV &= \int_M \nabla^i \nabla^j (\nabla_i u \nabla_j u) e^{-\varphi}\, dV \\ &= \int_M \nabla^i (\nabla^j \nabla_i u \cdot \nabla_j u + \nabla_i u \Delta u) e^{-\varphi}\, dV \\ &= \int_M (\Delta \nabla^i u \cdot \nabla_i u + \nabla^i \Delta u \cdot \nabla_i u + |\nabla^2 u|^2 + |\Delta u|^2) e^{-\varphi}\, dV, \end{aligned}$$

$$\begin{aligned} III := \int_M \Delta(|\nabla u|^2) e^{-\varphi} dV &= 2\int_M \nabla^i (\nabla_i \nabla_j u \cdot \nabla^j u) e^{-\varphi} dV \\ &= 2\int_M (\Delta \nabla_j u \cdot \nabla^j u + |\nabla^2 u|^2) e^{-\varphi} dV. \end{aligned}$$

If we set

$$B := 2(III + I - II),$$

then

$$\begin{aligned} \frac{B}{2} = \int_M \big(\Delta \nabla_i u \cdot \nabla^i u - \nabla_i \Delta u \cdot \nabla^i u + |\nabla^2 u|^2 - |\Delta u|^2 + |\langle du, d\varphi\rangle|^2 \\ - \nabla_i u \cdot \nabla^i \varphi \cdot \Delta u - \nabla_i u \cdot \nabla^j \varphi \cdot \nabla^i \nabla_j u\big) e^{-\varphi}\, dV \\ = \int_M (R_{ij} \nabla^i u \nabla^j u + |\nabla^2 u|^2 - |\Delta u|^2 + |\langle du, d\varphi\rangle|^2 \\ - \nabla_i u \cdot \nabla^i \varphi \cdot \Delta u - \nabla_i u \cdot \nabla^j \varphi \cdot \nabla^i \nabla_j u) e^{-\varphi}\, dV. \end{aligned}$$

On the other hand,

$$\begin{aligned} -\int_M \nabla_i u \cdot \nabla^i \varphi \cdot \Delta u \cdot e^{-\varphi}\, dV &= \int_M (\nabla_i u \cdot \Delta u) \nabla^i (e^{-\varphi})\, dV \\ &= -\int_M \nabla^i (\nabla_i u \cdot \Delta u) e^{-\varphi}\, dV \\ &= \int_M (-|\Delta u|^2 - \nabla_i u \cdot \nabla^i \Delta u) e^{-\varphi}\, dV \end{aligned}$$

and

$$\begin{aligned} -\int_M \nabla_i u \nabla^j \varphi \nabla^i \nabla_j u \cdot e^{-\varphi}\, dV &= \int_M \nabla_i u \nabla^i \nabla_j u \nabla^j (e^{-\varphi})\, dV \\ &= -\int_M \nabla^j (\nabla_i u \nabla^i \nabla_j u) e^{-\varphi}\, dV \\ &= \int_M (-|\nabla^2 u|^2 - \nabla_i u \Delta \nabla^i u) e^{-\varphi}\, dV. \end{aligned}$$

Therefore

$$\frac{B}{2} = \int_M \big(-2|\Delta u|^2 + |\langle du, d\varphi\rangle|^2 - 2\langle \nabla u, \nabla \Delta u\rangle\big) e^{-\varphi}\, dV. \tag{9-1}$$

By definition,

$$\Delta(|\nabla u|^2) = \Delta(\nabla^i u \cdot \nabla_i u) = 2\nabla^i u \cdot \Delta\nabla_i u + 2|\nabla^2 u|^2.$$

So

$$\begin{aligned}\Delta|\nabla u|^2 &= 2|\nabla^2 u|^2 + 2(\nabla_i \Delta u + R_{ij}\nabla^j u)\nabla^i u\\ &= 2|\nabla^2 u|^2 + 2R_{ij}\nabla^i u \cdot \nabla^j u + 2\langle \nabla u, \nabla \Delta u\rangle.\end{aligned}$$

Plugging this into (9-1) yields

$$\frac{B}{2} = \int_M \big(-2|\Delta u|^2 + |\langle du, d\varphi\rangle|^2 + 2|\nabla^2 u|^2 - \Delta|\nabla u|^2 + 2R_{ij}\nabla^i u\nabla^j u\big)e^{-\varphi}\, dV.$$

Since

$$\begin{aligned}2R_{ij}\nabla^i u\nabla^j u &= 2(S_{ij} + 2\nabla_i u\nabla_j u)\nabla^i u\nabla^j u\\ &= 2S_{ij}\nabla^i u\nabla^j u + 4|\nabla u|^4\\ &= \tfrac{1}{4}|\mathcal{S} + 4\, du\otimes du|^2 - \tfrac{1}{4}|\mathcal{S}|^2,\end{aligned}$$

it follows that

$$\begin{aligned}\frac{B}{2} &= III + I - II\\ &= \int_M \big(|\langle du, d\varphi\rangle|^2 - 2|\Delta u|^2 - \tfrac{1}{4}|\mathcal{S}|^2 + 2|\nabla^2 u|^2 + \tfrac{1}{4}|\mathcal{S} + 4\, du\otimes du|^2\big)e^{-\varphi}\, dV - III.\end{aligned}$$

Hence

$$B = \int_M \big(-4|\Delta u|^2 + 2|\langle du, d\varphi\rangle|^2 - \tfrac{1}{2}|\mathcal{S}|^2 + 4|\nabla^2 u|^2 + \tfrac{1}{2}|\mathcal{S} + 4\, du\otimes du|^2\big)e^{-\varphi}\, dV - 2III.$$

Theorem 9.1. *Suppose that $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$, that is, $\Delta_{g(t),u(t)} f(t) = \lambda(t) f(t)$ (where $\lambda(t)$ is only a function of time t), with the normalized condition $\int_M f(t)^2\, dV_{g(t)} = 1$. Then we have*

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= \frac{d}{dt}\lambda_{g(t),u(t)}(f(t))\\ &= \tfrac{1}{2}\int_M |\mathcal{S}_{g(t),u(t)} + \nabla^2_{g(t)}\varphi(t)|^2_{g(t)} e^{-\varphi(t)} dV_{g(t)} + \frac{1}{4}\int_M |\mathcal{S}_{g(t),u(t)}|^2_{g(t)} e^{-\varphi(t)} dV_{g(t)}\\ &\qquad + \int_M |\langle du(t), d\varphi(t)\rangle_{g(t)}|^2 e^{-\varphi(t)}\, dV_{g(t)} + 2\int_M |\nabla^2_{g(t)} u(t)|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)}\\ &\qquad + \frac{1}{4}\int_M |\mathcal{S}_{g(t),u(t)} + 4\, du(t)\otimes du(t)|^2_{g(t)} e^{-\varphi(t)}\, dV_{g(t)}\\ &\qquad - \int_M \Delta_{g(t)}(|\nabla_{g(t)} u(t)|^2_{g(t)}) e^{-\varphi(t)}\, dV_{g(t)}.\end{aligned}$$

Remark 9.2. When $u \equiv 0$, this equation reduces to Li's formula [2007].

10. The first variation of expander and shrinker entropies

Suppose that M is a closed manifold of dimension n. We define

$$\mathcal{W}_\pm : \odot^2_+(M) \times C^\infty(M) \times C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}, \quad (g, u, f, \tau) \mapsto \mathcal{W}_\pm(g, u, f, \tau)$$

where

$$\mathcal{W}_\pm(g, u, f, \tau) := \int_M \big(\tau(S_{g,u} + |\nabla_g f|^2_g) \mp f \pm n\big) \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g. \tag{10-1}$$

Set

$$\mu_\pm(g, u, \tau) := \inf\left\{\mathcal{W}_\pm(g, u, f, \tau) \;\middle|\; f \in C^\infty(M),\ \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g = 1\right\},$$
$$\nu_\pm(g, u) := \sup\{\mu_\pm(g, u, \tau) \mid \tau > 0\}.$$

Lemma 10.1. *Suppose $\nu_\pm(g, u) = \mathcal{W}_\pm(g, u, f_\pm, \tau_\pm)$ for some functions $f_\pm$ and constants $\tau_\pm$ satisfying*

$$\int_M \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g = 1, \quad \tau_\pm > 0.$$

Then we must have

$$\tau_\pm(-2\Delta_g f_\pm + |\nabla_g f_\pm|^2_g - S_{g,u}) \pm f_\pm \mp n + \nu_\pm(g, u) = 0,$$
$$\int_M \frac{f_\pm e^{-f_\pm}}{(4\pi\tau)^{n/2}} dV_g = \frac{n}{2} \mp \nu_\pm(g, u).$$

Proof. Since g and u are fixed, we consider the corresponding Lagrangian multiplier function

$$\mathfrak{L}_\pm(f, \tau; \lambda) := \mathcal{W}_\pm(g, u, f, \tau) - \lambda\left(\int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g - 1\right).$$

Then the variation of $\mathfrak{L}_\pm$ in f direction is

$$\delta_f \mathfrak{L}_\pm(f, \tau; \lambda) = \int_M \big(2\tau \nabla^i f \nabla_i(\delta f) \mp \delta f + \lambda \delta f\big) \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g$$
$$- \int_M \big(\tau(S_{g,u} + |\nabla_g f|^2_g) \mp f \pm n\big)\delta f \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g.$$

By the divergence theorem, we calculate

$$\int_M \nabla^i f \cdot \nabla_i(\delta f) \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g = -\int_M \nabla_i\Big(\nabla^i f \frac{e^{-f}}{(4\pi\tau)^{n/2}}\Big)\delta f\, dV_g$$
$$= -\int_M (\Delta_g f - |\nabla_g f|^2_g)\delta f \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV_g.$$

Hence

$$\delta_f \mathfrak{L}_\pm(f,\tau;\lambda)=\int_M \big(\tau(-2\Delta_g f+|\nabla_g f|_g^2-S_{g,u})\pm f\mp n\mp 1+\lambda\big)\delta f\,\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,dV.$$

This implies that

$$\tau_\pm(-2\Delta_g f_\pm+|\nabla_g f_\pm|_g^2-S_{g,u})\pm f_\pm\mp n\mp 1+\lambda_\pm=0.$$

Since $f_\pm$ satisfies the normalized condition, it follows that

$$0=\lambda_\pm\mp 1+\int_M \big(\tau_\pm(-2\Delta_g f_\pm+|\nabla_g f_\pm|_g^2-S_{g,u})\pm f_\pm\mp n\big)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g.$$

From the identity

$$\int_M \Delta_g f\,\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,dV_g=\int_M |\nabla_g f|_g^2\,\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,dV_g$$

and the definition (10-1), we obtain

$$\nu_\pm(g,u)=\mathscr{W}_\pm(g,u,f_\pm,\tau_\pm)=\lambda_\pm\mp 1,$$

and, consequently,

$$\tau_\pm(-2\Delta_g f_\pm+|\nabla_g f_\pm|_g^2-S_{g,u})\pm f_\pm\mp n+\nu_\pm(g,u)=0.$$

The variation of $\mathfrak{L}_\pm$ with respect to τ indicates that

$$\begin{aligned}\delta_\tau \mathfrak{L}_\pm(f,\tau;\lambda)&=\int_M \delta\tau(S_{g,u}+|\nabla_g f|_g^2)\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,dV_g-\lambda\int_M\Big(-\frac{n}{2}\frac{\delta\tau}{\tau}\Big)\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,dV_g\\&\quad+\int_M\Big(-\frac{n}{2}\frac{\delta\tau}{\tau}\Big)\big(\tau(S_{g,u}+|\nabla_g f|_g^2)\mp f\pm n\big)\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,dV_g\\&=\int_M\delta\tau\Big(\Big(1-\frac{n}{2}\Big)(S_{g,u}+|\nabla_g f|_g^2)+\frac{n}{2\tau}(\lambda\pm f\mp n)\Big)\frac{e^{-f}\,dV_g}{(4\pi\tau)^{n/2}}.\end{aligned}$$

Using the first proved equation, we have

$$\begin{aligned}0&=\int_M\Big((\nu_\pm(g,u)\pm f_\pm\mp n)\Big(1-\frac{n}{2}\Big)+\frac{n}{2}(\nu_\pm(g,u)\pm f_\pm\mp n\pm 1)\Big)\frac{e^{-f_\pm}\,dV_g}{(4\pi\tau_\pm)^{n/2}}\\&=\int_M\Big(\nu_\pm\pm f_\pm\mp\frac{n}{2}\Big)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g\end{aligned}$$

and therefore we obtain the second one. □

For a symmetric 2-tensor $h=(h_{ij})\in\bigodot^2(M)$, we set

$$g(s):=g+sh$$

Then the variation of $g(s)$ is

$$\frac{\partial}{\partial s}\Big|_{s=0} R_{g(s)} = -h^{ij}R_{ij} + \nabla^i\nabla^j h_{ij} - \Delta_g(\mathrm{tr}_g h). \tag{10-2}$$

Theorem 10.2. *Suppose that (M, g) is a compact Riemannian manifold and u a smooth function on M. Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If*

$$\nu_\pm(g(s), u(s)) = \mathcal{W}_\pm(g(s), u(s), f_\pm(s), \tau_\pm(s))$$

for some smooth functions $f_\pm(s)$ with

$$\int_M e^{-f_\pm(s)}\, dV/(4\pi\tau_\pm(s))^{n/2} = 1$$

and constants $\tau_\pm(s) > 0$,

$$\frac{d}{ds}\Big|_{s=0}\nu_\pm(g(s), u(s)) = -\tau_\pm\int_M\Big(\langle h, \mathcal{S}_{g,u}\rangle_g + \langle h, \nabla_g^2 f\rangle_g \pm \frac{1}{2\tau_\pm}\mathrm{tr}_g h\Big)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g$$
$$+ 4\tau_\pm\int_M v(\Delta_g u - \langle du, df_\pm\rangle_g)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g,$$

where $f_\pm := f_\pm(0)$ and $\tau_\pm := \tau_\pm(0)$. In particular, the critical points of $\nu_\pm(\,\cdot\,,\cdot\,)$ satisfy

$$\mathcal{S}_{g,u} + \nabla_g^2 f \pm \frac{1}{2\tau_\pm} g = 0, \qquad \Delta_g u = \langle du, df_\pm\rangle_g.$$

Consequently, if $\mathcal{W}_\pm(g, u, f, \tau)$ and $\nu_\pm(g, u)$ achieve their minimums, (M, g) is a gradient expanding and shrinker harmonic-Ricci soliton according to the sign.

Proof. By definition, one has

$$\begin{aligned}
&\frac{d}{ds}\nu_\pm(g(s), u(s)) \\
&\quad= \frac{d}{ds}\mathcal{W}_\pm(g(s), u(s), f_\pm(s), \tau_\pm(s)) \\
&\quad= \int_M\Big(\frac{\partial}{\partial s}\tau_\pm(s)(S_{g(s),u(s)} + |\nabla_{g(s)} f_\pm(s)|^2_{g(s)})\Big)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\, dV_{g(s)} \\
&\quad\quad+ \int_M\Big(\tau_\pm(s)\frac{\partial}{\partial s}(S_{g(s),u(s)} + |\nabla_{g(s)} f_\pm(s)|^2_{g(s)}) \mp \frac{\partial}{\partial s} f_\pm(s)\Big)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\, dV_{g(s)} \\
&\quad\quad+ \int_M\big(\tau_\pm(s)(S_{g(s),u(s)} + |\nabla_{g(s)} f_\pm(s)|^2_{g(s)}) \mp f_\pm(s) \pm n\big) \\
&\quad\quad\quad\cdot\frac{\partial}{\partial s}\Big(\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\, dV_{g(s)}\Big).
\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial}{\partial s}S_{g(s),u(s)} &= \frac{\partial}{\partial s}R_{g(s)} - 2\frac{\partial}{\partial s}|\nabla_{g(s)}u(s)|^2_{g(s)}\\ &= \frac{\partial}{\partial s}R_{g(s)} - 2\left(\frac{\partial}{\partial s}g^{ij}\right)\nabla_i u\nabla_j u - 4g^{ij}\frac{\partial}{\partial s}\nabla_i u\cdot\nabla_j u\\ &= \frac{\partial}{\partial s}R_{g(s)} - 2(-g^{ip}g^{jq}h_{pq})\nabla_i u\nabla_j u - 4g^{ij}\nabla_i\left(\frac{\partial}{\partial s}u\right)\nabla_j u\\ &= \frac{\partial}{\partial s}R_{g(s)} + 2h_{pq}\nabla^p u\nabla^q u - 4\nabla_i\left(\frac{\partial}{\partial t}u\right)\nabla^i u\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial s}\left(\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_{g(s)}\right)\\ &=\left(-\frac{\partial}{\partial s}f_\pm(s)-\frac{n}{2\tau_\pm(s)}\frac{\partial}{\partial s}\tau_\pm(s)\right)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_{g(s)}+\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\frac{\partial}{\partial s}\,dV_{g(s)}\\ &=\left(-\frac{\partial}{\partial s}f_\pm(s)-\frac{n}{2\tau_\pm(s)}\frac{\partial}{\partial s}\tau_\pm(s)+\tfrac{1}{2}\operatorname{tr}_g h\right)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_{g(s)},\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{d}{ds}\nu_\pm(g(s),u(s))\\ &\quad=\int_M \frac{\partial}{\partial s}\tau_\pm(s)(S_{g(s),u(s)}+|\nabla_{g(s)}f_\pm(s)|^2_{g(s)})\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_{g(s)}\\ &\qquad+\int_M\Bigg(\tau_\pm(s)\bigg(\frac{\partial}{\partial s}R_{g(s)}+2h_{pq}\nabla^p u\nabla^q u-4\nabla_i\left(\frac{\partial}{\partial s}u\right)\nabla^i u\\ &\qquad-h_{pq}\nabla^p f\nabla^q f+2\nabla_i\left(\frac{\partial}{\partial s}f\right)\nabla^i f\bigg)\mp\frac{\partial}{\partial s}f_\pm(s)\Bigg)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_{g(s)}\\ &\qquad+\int_M\left(-\frac{\partial}{\partial s}f_\pm(s)-\frac{n}{2\tau_\pm(s)}\frac{\partial}{\partial s}\tau_\pm(s)+\tfrac{1}{2}\operatorname{tr}_g h\right)\\ &\qquad\quad\cdot\left(\tau_\pm(s)(S_{g(s),u(s)}+|\nabla_{g(s)}f_\pm(s)|^2_{g(s)})\mp f_\pm(s)\pm n\right)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_{g(s)}.\end{aligned}$$

From the equalities

$$\begin{aligned}\int_M \Delta_g\operatorname{tr}_g h\cdot e^{-f}\,dV_g &= \int_M \operatorname{tr}_g h\cdot\Delta_g(e^{-f})\,dV_g\\ &= \int_M \operatorname{tr}_g h(-\Delta_g f+|\nabla_g, f|^2_g)e^{-f}\,dV_g,\end{aligned}$$

$$\int_M \nabla^i\nabla^j h_{ij}\cdot e^{-f}\,dV_g = \int_M h_{ij}\nabla^i\nabla^j(e^{-f})\,dV$$
$$= \int_M h_{ij}(-\nabla^i\nabla^j f + \nabla^i f\nabla^j f)e^{-f}\,dV_g,$$
$$\int_M \nabla_i\left(\frac{\partial}{\partial s}f\right)\nabla^i f\,e^{-f}\,dV_g = \int_M -\frac{\partial}{\partial s}f(\Delta_g f - |\nabla_g f|_g^2)e^{-f}\,dV_g,$$
$$\int_M \Delta_g(e^{-f})\,dV_g = \int_M(-\Delta_g f + |\nabla_g f|_g^2)e^{-f}\,dV_g,$$

and Lemma 10.1, we obtain

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} & \nu_\pm(g(s),u(s))\\ &= \int_M \frac{\partial}{\partial s}\Big|_{s=0}\tau_\pm(s)(S_{g,u}+|\nabla_g f|_g^2)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g\\ &\quad + \int_M\bigg(\tau_\pm\bigg(-h^{ij}R_{ij}+\nabla^i\nabla_j h_{ij}-\Delta_g(\mathrm{tr}_g h)+2h_{pq}\nabla^p u\nabla^q u\\ &\qquad -4\nabla_i v\nabla^i u - h_{pq}\nabla^p f\nabla^q f + 2\nabla_i\left(\frac{\partial}{\partial s}\Big|_{s=0} f(s)\right)\nabla^i f\bigg)\mp\frac{\partial}{\partial s}\Big|_{s=0} f(s)\bigg)\\ &\qquad\cdot\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g + \int_M\left(-\frac{\partial}{\partial s}\Big|_{s=0} f_\pm(s) - \frac{n}{2\tau_\pm}(s)\frac{\partial}{\partial s}\Big|_{s=0}\tau_\pm(s)+\tfrac{1}{2}\,\mathrm{tr}_g h\right)\\ &\qquad\qquad\cdot\big(\tau_\pm(S_{g,u}+|\nabla_g f_\pm|_g^2)\mp f_\pm \pm n\big)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g.\end{aligned}$$

If we denote by B the last term and by A the remaining terms,

$$\begin{aligned}A = \int_M\bigg(&\frac{\partial}{\partial s}\Big|_{s=0}\tau_\pm(s)(|\nabla_g f_\pm|_g^2 + S_{g,u})\\ &- \tau_\pm(h^{ij}\nabla_i\nabla_j f_\pm + h^{ij}S_{ij} + 4\nabla_i v\cdot\nabla^i u)\mp\frac{\partial}{\partial s}f_\pm\bigg)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g\\ &+\int_M \tau_\pm(\Delta_g f_\pm - |\nabla_g f_\pm|_g^2)\left(\mathrm{tr}_g h - 2\frac{\partial}{\partial s}\Big|_{s=0} f(s)\right)\frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\,dV_g.\end{aligned}$$

The normalized condition

$$1 = \int_M \frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_g$$

implies

$$0 = \int_M\left(-\frac{\partial}{\partial s}\Big|_{s=0} f_\pm(s) - \frac{n}{2\tau_\pm}\frac{\partial}{\partial s}\Big|_{s=0}\tau_\pm(s)+\tfrac{1}{2}\,\mathrm{tr}_g h\right)\frac{e^{-f_\pm(s)}}{(4\pi\tau_\pm(s))^{n/2}}\,dV_g.$$

From Lemma 10.1, we conclude that

$$\tau_\pm S_{g,u} - \tau_\pm(|\nabla_g f_\pm|_g^2 - 2\Delta_g f_\pm) = \pm f_\pm \mp n + \nu_\pm(g,u).$$

Therefore

$$\tau_\pm(S_{g,u} + |\nabla_g f_\pm|_g^2) \mp f_\pm \pm n = 2\tau_\pm(|\nabla_g f_\pm|_g^2 - \Delta_g f_\pm) + \nu_\pm(g,u).$$

Plugging this into the definition of B yields

$$\begin{aligned}
B &= \int_M \left(-\frac{\partial}{\partial s}\bigg|_{s=0} f_\pm(s) - \frac{n}{2\tau_\pm}\frac{\partial}{\partial s}\bigg|_{s=0} \tau_\pm(s) + \tfrac{1}{2}\operatorname{tr}_g h \right) \\
&\qquad\qquad \cdot \big(2\tau_\pm(|\nabla_g f_\pm|_g^2 - \Delta_g f_\pm) + \nu_\pm(g,u)\big) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g \\
&= \int_M \left(-\frac{\partial}{\partial s}\bigg|_{s=0} f_\pm(s) - \frac{n}{2\tau_\pm}\frac{\partial}{\partial s}\bigg|_{s=0} \tau_\pm(s) + \tfrac{1}{2}\operatorname{tr}_g h \right) \\
&\qquad\qquad \cdot \big(2\tau_\pm(|\nabla_g f_\pm|_g^2 - \Delta_g f_\pm)\big) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g \\
&= \int_M \left(-\frac{\partial}{\partial s}\bigg|_{s=0} f_\pm(s) + \tfrac{1}{2}\operatorname{tr}_g h \right) 2\tau_\pm(|\nabla_g f_\pm|_g^2 - \Delta_g f_\pm) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g,
\end{aligned}$$

where we use the fact that

$$\int_M \Delta_g(e^{-f}) dV_g = 0.$$

Hence B cancels with the last term in A. Therefore the above variation equals

$$\begin{aligned}
&\frac{d}{ds}\bigg|_{s=0} \nu_\pm(g(s), u(s)) \\
&\quad = \int_M \left(\frac{\partial}{\partial s}\bigg|_{s=0} \tau_\pm(s) \left(|\nabla_g f_\pm|_g^2 + S_{g,u} \pm \frac{n}{2\tau_\pm} \right) - \tau_\pm \Big(h^{ij}\nabla_i\nabla_j f + h^{ij} S_{ij} \right. \\
&\qquad\qquad \left. \pm \frac{1}{2\tau_\pm} \operatorname{tr}_g h + 4v(\langle du, df\rangle - \Delta_g u) \Big) \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g.
\end{aligned}$$

To prove the theorem, it is sufficient to show that

$$\int_M \left(|\nabla_g f_\pm|_g^2 + S_{g,u} \pm \frac{n}{2\tau_\pm} \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV = 0.$$

Since M is compact, we have

$$0 = \int_M \Delta_g(e^{-f_\pm}) = \int_M (-\Delta_g f_\pm + |\nabla_g f_\pm|_g^2) e^{-f_\pm}\, dV.$$

Hence

$$\int_M \left(|\nabla_g f_\pm|^2 + S_{g,u} \pm \frac{n}{2\tau_\pm} \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV$$
$$= \int_M \left(2\Delta_g f_\pm - |\nabla_g f|_g^2 + S_{g,u} \pm \frac{n}{2\sigma_\pm} \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV.$$

Lemma 10.1 now indicates

$$\int_M \left(|\nabla_g f_\pm|^2 + S_{g,u} \pm \frac{n}{2\tau_\pm} \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV$$
$$= \int_M \left(\frac{\pm f_\pm \mp n + \nu_\pm(g,u)}{\tau_\pm} \pm \frac{n}{2} \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV$$
$$= \int_M \frac{1}{\tau_\pm} \left(\pm f_\pm \mp \frac{n}{2} + \nu_\pm(g,u) \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV$$
$$= \frac{1}{\tau_\pm} \left(\pm \frac{n}{2} - \nu_\pm(g,u) \mp \frac{n}{2} + \nu_\pm(g,u) \right) = 0.$$

The sign $+$ corresponds to the gradient expanding soliton and the sign $-$ to the gradient shrinker soliton. □

Corollary 10.3. *Suppose that (M, g) is a compact Riemannian manifold and u is a smooth function on M. Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If $\nu_\pm(g(s), u(s)) = \mathcal{W}_\pm(g(s), u(s), f_\pm(s), \tau_\pm(s))$ for some smooth function $f_\pm(s)$ with $\int_M e^{-f_\pm(s)}\, dV/(4\pi\tau_\pm(s))^{n/2} = 1$ and a constant $\tau_\pm(s) > 0$, and (g, u) is a critical point of $\nu_\pm(\cdot, \cdot)$, then*

$$\mathcal{S}_{g,u} = \mp \frac{1}{2\tau_\pm} g, \qquad f_\pm \equiv \text{constant}.$$

Thus, if $\mathcal{W}_\pm(g, u, \cdot, \cdot)$ achieve their minimum and (g, u) is a critical point of $\nu_\pm(\cdot, \cdot)$, (M, g, u) satisfies the static Einstein vacuum equation.

Proof. According to Lemma 10.1 and Theorem 10.2, we have

$$\tau_\pm(-2\Delta_g f_\pm + |\nabla_g f_\pm|_g^2 - S_{g,u}) \pm f_\pm \mp n$$
$$= -\nu_\pm = -\int_M \left(\tau_\pm(S_{g,u} + |\nabla_g f|_g^2) \mp f_\pm \pm n \right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g,$$

and hence

$$\begin{aligned} 2\Delta_g f_\pm - |\nabla_g f_\pm|_g^2 + S_{g,u} &= \int_M (S_{g,u} + |\nabla_g f_\pm|_g^2) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g \\ &= \int_M (S_{g,u} + \Delta_g f_\pm) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}}\, dV_g \\ &= \mp \frac{n}{2\tau_\pm} = S_{g,u} + \Delta_g f_\pm. \end{aligned}$$

From this we get $\Delta_g f_\pm = |\nabla_g f_\pm|_g^2$. After integrating on both sides, the functions $f_\pm$ must be constant, which implies $\mathcal{S}_g \pm (1/(2\tau_\pm))g = 0$. □

Remark 10.4. In the situation of Corollary 10.3, by normalization, we my choose $f_\pm = n/2$.

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