An Iterative Approach to Nonconvex QCQP with Applications in Signal Processing

Ahmad Gharanjik*‡, Bhavani Shankar‡, Mojtaba Soltanalian†, and Björn Ottersten*‡
* Signal Processing Department, KTH Royal Institute of Technology, ‡ SnT Centre, University of Luxembourg
†Department of Electrical and Computer Engineering, University of Illinois at Chicago

***Abstract*—This paper introduces a new iterative approach to solve or to approximate the solutions of the nonconvex quadratically constrained quadratic programs (QCQP). First, this constrained problem is transformed to an unconstrained problem using a specialized penalty-based method. A tight upper-bound for the alternative unconstrained objective is introduced. Then an efficient minimization approach to the alternative unconstrained objective is proposed and further studied. The proposed approach involves power iterations and minimization of a convex scalar function in each iteration, which are computationally fast. The important design problem of multigroup multicast beamforming is formulated as a nonconvex QCQP and solved using the proposed method.**

I. Introduction

Nonconvex QCQP is an important class of optimization problems that can be formulated as,

$$\mathcal{P}: \min_{\mathbf{x}\in\mathbb{C}^N} \quad \mathbf{x}^H\mathbf{A}_0\mathbf{x}$$
$$\text{s. t.} \quad \mathbf{x}^H\mathbf{A}_i\mathbf{x} \le c_i, \quad \forall i \in [M], \tag{1}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ are Hermitian matrices for all $i \in [M]$, M denotes the number of quadratic constraints, and $c_i \in \mathbb{R}$ (Please see the footnote for the notations)[1]. Herein, we are interested in *a subclass of nonconvex* QCQP problems with *convex* objective and *nonconvex* constraints, $\mathbf{A}_0$ is a positive definite (PD) matrix and $\mathbf{A}_i$ are Hermitian matrices with at least one negative eigenvalue [1]. This class of nonconvex QCQP problems captures many problems that are of interest to the signal processing and communications community such as beamforming design [2]–[4], radar optimal code design [5]–[8], multiple-input multiple-output (MIMO) and multiuser estimation and detection [9], as well as phase retrieval [10], [11]. The application is also extended to other domains such as portfolio risk management in financial engineering [12]. Nonconvex QCQP is known to be an NP-hard problem, i.e. at least as hard as NP-complete problems which are particularly deemed by optimization community to be difficult [13]. Due to its wide area of application, the nonconvex QCQP problem has been studied extensively in the optimization and signal processing literature. The NP-hardness of the problem has motivated the search for various efficient approaches to solve $\mathcal{P}$ including those based on the *semidefinite relaxation* (SDR) [9], [14], the *reformulation linearization technique* (RLT) [15], [16], and the *successive convex approximation* (SCA) [17]–[19]. Recently a variant of SCA known as *feasible point pursuit-successive convex approximation (FPP-SCA)* has also been proposed in [20]. To the best of our knowledge, SDR is yet the most prominent and widely used technique employed for tackling nonconvex QCQP. Note that $\mathcal{P}$ in (1) includes several class of widely known optimization problems including *binary quadratic programming* (BQP) (where $\mathbf{A}_i = \mathbf{e}_i\mathbf{e}_i^T$, $\mathbf{x} \in \mathbb{R}^N$) and *unimodular quadratic programming* (UQP) (where $\mathbf{A}_i = \mathbf{e}_i\mathbf{e}_i^T$, $\mathbf{x} \in \mathbb{C}^N$) [21].

In this paper, we propose a new iterative method to solve the nonconvex QCQP problem. We convert the nonconvex QCQP problem to an unconstrained problem in Section II. The reformulated optimization problem is then decomposed to subproblems which can be solved either analytically or using extremely efficient optimization tools, discussed in Section III. In the following, we show that the important signal processing problem of multigroup multicast beamforming can be formulated as a nonconvex QCQP that requires solving $\mathcal{P}$. The proposed formulation serves as a cornerstone to our numerical example in Section IV.

A. Application to Multigroup Multicast Beamforming

Consider the general *multigroup multicast beamforming* problem [3] for a downlink channel, with a n_{Tx}-antenna transmitter and K single-antenna users assigned to $G \le K$ multicast groups. We denote the subset of user indices in the k^{th} group by $\mathcal{G}_k$ for any $k \in [G]$. Let $\mathbf{h}_i \in \mathbb{C}^{n_{\mathrm{Tx}}}$ denote the channel between the transmit antennas and the i^{th} user. Also let $\mathbf{w}_k \in \mathbb{C}^{n_{\mathrm{Tx}}}$ denote the beamforming vector corresponding to the k^{th} group, $k \in [G]$, multicast group of users. The beamformed vector to k^{th} group takes the form $\mathbf{w}_k s_k$ with $\mathbf{E}[|s_k|^2] = 1$ where s_k is the symbol to be transmitted. The beamforming vectors are to be designed in order to enhance the network performance. In particular, the SINR value for any user $i \in \mathcal{G}_k$ (and any $k \in [G]$) is given by [3],

$$\mathbf{SINR}_i = \frac{\mathbf{w}_k^H\mathbf{R}_i\mathbf{w}_k}{\left(\sum_{j\in[G]\setminus\{k\}} \mathbf{w}_j^H\mathbf{R}_i\mathbf{w}_j\right) + \sigma_i^2}, \tag{2}$$

where $\mathbf{R}_i = \mathbf{E}\{\mathbf{h}_i\mathbf{h}_i^H\}$ is the covariance matrix of the i^{th} channel, σ_i^2 denotes the variance of the zero-mean additive white Gaussian noise (AWGN).

[1] $\mathbf{x}(k)$ is the k^{th} entry of the vector $\mathbf{x}$, $\|\mathbf{x}\|$ is the l_2-norm of $\mathbf{x}$, $\mathbf{X}^H$ is the complex conjugate of $\mathbf{X}$, $\mathbf{X}^T$ is the transpose of $\mathbf{X}$, and $\mathbf{Tr}(\mathbf{X})$ the trace of $\mathbf{X}$. $\mathbf{vec}(\mathbf{X})$ is the vector obtained by column-wise stacking of $\mathbf{X}$. $\|\mathbf{X}\|_F$ is the Frobenius norm of a matrix $\mathbf{X}$, $\otimes$ is the Kronecker product and $\mathbf{diag}(\mathbf{x})$ is the diagonal matrix formed by elements of $\mathbf{x}$. [M] denotes the set $\{1, 2, \cdots, M\}$, and $\mathbf{e}_i$ is the the i^{th} standard basis vector.

Consequently, the problem of minimizing total transmit power subject to constraints on user SINR performance in the network can be formulated as [2], [3],

$$\begin{aligned} \min_{\{\mathbf{w}_k\}_{k=1}^G}. \quad & \sum_{k=1}^{G} \|\mathbf{w}_k\|_2^2 \\ \text{s. t.} \quad & \mathbf{SINR}_i \geq \gamma_i \ , i \in [K] \end{aligned} \tag{3}$$

Note that by a specific reformulation, the SINR metric in (2) can be rewritten as a quadratic criterion. To see this, define the *stacked* beamforming vector $\mathbf{w} \in \mathbb{C}^N$ (with $N = n_{\text{Tx}}G$), $\widehat{\mathbf{R}}_i$ and $\widetilde{\mathbf{R}}_i$ as,

$$\mathbf{w} \triangleq \mathbf{vec}([\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_G]), \tag{4}$$

$$\widehat{\mathbf{R}}_i \triangleq \mathbf{diag}\left(\mathbf{e}_j\right) \otimes \mathbf{R}_i, \qquad \forall\, i \in [K], i \in \mathcal{G}_j \tag{5}$$

$$\widetilde{\mathbf{R}}_i \triangleq \left(\mathbf{I}_G - \mathbf{diag}\left(\mathbf{e}_j\right)\right) \otimes \mathbf{R}_i, \qquad \forall\, i \in [K], i \in \mathcal{G}_j \tag{6}$$

in which $\{\widehat{\mathbf{R}}_i\}$ and $\{\widetilde{\mathbf{R}}_i\}$ are PSD matrices. It can be easily verified that

$$\mathbf{SINR}_i = \frac{\mathbf{w}^H \widehat{\mathbf{R}}_i \mathbf{w}}{\mathbf{w}^H \widetilde{\mathbf{R}}_i \mathbf{w} + \sigma_i^2}, \quad \forall\, i \in [K]. \tag{7}$$

As a result, the SINR constraint in (3) can be rewritten as,

$$\mathbf{w}^H \widehat{\mathbf{R}}_i \mathbf{w} - \gamma_i \mathbf{w}^H \widetilde{\mathbf{R}}_i \mathbf{w} \geq \gamma_i \sigma_i^2, \tag{8}$$

or equivalently as $\mathbf{w}^H \mathbf{R}_i \mathbf{w} \geq 1$, where $\mathbf{R}_i$ is given by,

$$\mathbf{R}_i = \left(\widehat{\mathbf{R}}_i - \gamma_i \widetilde{\mathbf{R}}_i\right) / \left(\gamma_i \sigma_i^2\right). \tag{9}$$

The beamforming design problem for minimizing total transmit power with SINR constraint can thus be formulated as,

$$\min_{\mathbf{w}}. \quad \|\mathbf{w}\|^2 , \qquad \text{s. t.} \quad \mathbf{w}^H \mathbf{R}_i \mathbf{w} \geq 1, \quad \forall i \in [K]. \tag{10}$$

Note that this formulation may also be used to solve physical-layer multicasting and traditional multiuser transmit beamforming problems; see [4] and [2] for details.

II. Problem Reformulation

We begin our reformulation by rewriting $\mathcal{P}$ in an equivalent form. We can assume, without loss of generality, that $c_i \neq 0$; otherwise $\mathcal{P}$ will have a trivial solution of $\mathbf{x} = 0$ or it will be infeasible. Since $\mathbf{A}_0$ is a PD matrix, using the change of parameters by $\mathbf{A}_i \leftarrow \left(\mathbf{A}_0^{-\frac{1}{2}} \mathbf{A}_i \mathbf{A}_0^{-\frac{1}{2}}\right)/c_i$ and $\mathbf{x} \leftarrow \mathbf{A}_0^{\frac{1}{2}} \mathbf{x}$, the nonconvex QCQP of interest may be recast as,

$$\begin{aligned} \mathcal{P}_1 : \min_{\mathbf{x} \in \mathbb{C}^N}. \quad & \|\mathbf{x}\|^2 \\ \text{s. t.} \quad & \mathbf{x}^H \mathbf{A}_i \mathbf{x} \lhd_i 1, \quad \forall i \in [M], \end{aligned} \tag{11}$$

with $\mathbf{A}_i$ being Hermitian matrices. Here "$\lhd_i$" can represent any of "$\geq$", "$\leq$" or "$=$" for each i.

Now, let us define $\mathbf{x} = \sqrt{p}\,\mathbf{u}$, where $p \in \mathbb{R}^+$ and $\mathbf{u} \in \mathbb{C}^N$ is a unit norm vector. Then, (11) can be written as,

$$\begin{aligned} \mathcal{P}_1 : \min_{\mathbf{u},p}. \quad & p \\ \text{s. t.} \quad & \mathbf{u}^H \mathbf{A}_i \mathbf{u} \lhd_i \frac{1}{p}, \quad \forall i \in [M], \\ & \|\mathbf{u}\|^2 = 1. \end{aligned} \tag{12}$$

By introducing slack variables $\{t_i\}$, we transform all inequality constraints to equality constraints, viz.

$$\mathbf{u}^H \mathbf{A}_i \mathbf{u} + t_i = \frac{1}{p}, \quad \forall i \in [M], \tag{13}$$

where $t_i \in \mathbb{R}$. Therefore, $\mathcal{P}_1$ can be reformulated as,

$$\begin{aligned} \mathcal{P}_2 : \min_{\mathbf{u},p,\{t_i\}}. \quad & p \\ \text{s. t.} \quad & \mathbf{u}^H \left(\mathbf{A}_i + t_i \mathbf{I}\right) \mathbf{u} = \frac{1}{p}, \quad \forall i \in [M], \\ & \|\mathbf{u}\|^2 = 1. \end{aligned} \tag{14}$$

Any Hermitian matrix can be decomposed as a difference of two PSD matrices simply by partitioning the matrix into parts comprising only non-positive and non-negative eigenvalues. In particular, we consider,

$$\mathbf{A}_i = \mathbf{A}_i^+ - \mathbf{A}_i^-, \quad \mathbf{A}_i^+,\, \mathbf{A}_i^- \succeq 0,\, \forall i \in [M]. \tag{15}$$

We can also decompose t_i as $t_i = t_i^+ - t_i^-, \forall i \in [M]$ where

$$t_i^+ = \begin{cases} t_i & \text{if} \quad t_i > 0 \\ 0 & \text{if} \quad t_i \leq 0 \end{cases}, \quad t_i^- = \begin{cases} 0 & \text{if} \quad t_i > 0 \\ |t_i| & \text{if} \quad t_i \leq 0 \end{cases}. \tag{16}$$

Consequently, the constraint in $\mathcal{P}_2$ can be written as,

$$\mathbf{u}^H \left(\mathbf{A}_i^+ + t_i^+ \mathbf{I}\right) \mathbf{u} = \mathbf{u}^H \left(\mathbf{A}_i^- + (1/p + t_i^-)\mathbf{I}\right) \mathbf{u} \tag{17}$$

For notational simplicity, we define $\mathbf{C}_i = \mathbf{A}_i^- + (1/p + t_i^-)\mathbf{I}$ and $\mathbf{B}_i = \mathbf{A}_i^+ + t_i^+ \mathbf{I}$, where both matrices are PSD. Note that (17) holds if and only if $\|\mathbf{B}_i^{\frac{1}{2}} \mathbf{u}\| = \|\mathbf{C}_i^{\frac{1}{2}} \mathbf{u}\|$. In particular, the left-hand side of (17) is close to the right-hand side of (17) if and only if $\|\mathbf{B}_i^{\frac{1}{2}} \mathbf{u}\|$ is *close to* $\|\mathbf{C}_i^{\frac{1}{2}} \mathbf{u}\|$. Therefore, one can consider the following optimization problem as a penalized reformulation of $\mathcal{P}_2$

$$\begin{aligned} \mathcal{P}_3 : \min_{\mathbf{u},p,\{t_i\}}. \quad & p + \eta \sum_{i=1}^{M} \left(\|\mathbf{B}_i^{\frac{1}{2}} \mathbf{u}\| - \|\mathbf{C}_i^{\frac{1}{2}} \mathbf{u}\|\right)^2 \\ \text{s. t.} \quad & \|\mathbf{u}\|^2 = 1, \quad t_i \in \mathbb{R}, \quad \forall i \in [M], \end{aligned} \tag{18}$$

in which $\eta > 0$ determines the weight of the penalty-term added to the original objective of $\mathcal{P}_2$; and where $\mathcal{P}_3$ and $\mathcal{P}_2$ coincide as $\eta \to +\infty$. Note that optimizing $\mathcal{P}_3$ with respect to (w. r. t.) $\mathbf{u}$ may require rewriting $\mathcal{P}_3$ as a quartic objective in $\mathbf{u}$. To avoid this, we introduce another alternative objective:

$$\begin{aligned} \mathcal{P}_4 : \min_{\mathbf{u},p,\{t_i\},\{\mathbf{Q}_i\}}. \quad & p + \eta \sum_{i=1}^{M} \|\mathbf{B}_i^{\frac{1}{2}} \mathbf{u} - \mathbf{Q}_i \mathbf{C}_i^{\frac{1}{2}} \mathbf{u}\|^2 \\ \text{s. t.} \quad & \|\mathbf{Q}_i\|_F \leq 1, \quad t_i \in \mathbb{R} \quad \forall i \in [M], \\ & \|\mathbf{u}\|^2 = 1. \end{aligned} \tag{19}$$

In contrast to $\mathcal{P}_3$, the optimization problem $\mathcal{P}_4$ w. r. t. $\mathbf{u}$ can be easily cast as a problem of finding the largest eigenvalue of a PSD matrix—more on this later. To establish the equivalence of $\mathcal{P}_3$ and $\mathcal{P}_4$, observe that the minimizer $\mathbf{Q}_i$ of $\mathcal{P}_4$ should be a matrix with Frobenius norm less than or equal to 1 that satisfies the following condition,

$$\mathbf{Q}_i \mathbf{C}_i^{\frac{1}{2}} \mathbf{u} = \left(\|\mathbf{C}_i^{\frac{1}{2}} \mathbf{u}\| / \|\mathbf{B}_i^{\frac{1}{2}} \mathbf{u}\|\right) \mathbf{B}_i^{\frac{1}{2}} \mathbf{u}. \tag{20}$$

In this case, it will be straightforward to verify that,

$$\sum_{i=1}^{M} \|\mathbf{B}_i^{\frac{1}{2}}\mathbf{u} - \mathbf{Q}_i\mathbf{C}_i^{\frac{1}{2}}\mathbf{u}\|^2 = \sum_{i=1}^{M} \left(\|\mathbf{B}_i^{\frac{1}{2}}\mathbf{u}\|_2 - \|\mathbf{C}_i^{\frac{1}{2}}\mathbf{u}\|_2\right)^2. \quad (21)$$

In Section III, we present an analytical approach for the derivation of $\{\mathbf{Q}_i\}$.

III. Proposed Optimization Framework

We now propose an efficient iterative optimization framework based on a separate optimization of the objective of $\mathcal{P}_4$ and $\mathcal{P}_3$ over its partitions of variables $\mathbf{u}$, $\{\mathbf{Q}_i\}$, p, and $\{t_i\}$, at each iteration where the iterations can be initiated from any arbitrary setting.

A. Optimization w. r. t. $\mathbf{u}$

Consider p, $\{\mathbf{Q}_i\}$ and $\{t_i\}$ are fixed, then one can optimize $\mathcal{P}_4$ w. r. t. $\mathbf{u}$ via minimizing the criterion:

$$\sum_{i=1}^{K} \|\mathbf{B}_i^{\frac{1}{2}}\mathbf{u} - \mathbf{Q}_i\mathbf{C}_i^{\frac{1}{2}}\mathbf{u}\|^2 = \mathbf{u}^H\mathbf{R}\mathbf{u} \quad (22)$$

where $\mathbf{R} = \sum_{i=1}^{K}\left\{(\mathbf{B}_i + \mathbf{C}_i) - (\mathbf{B}_i^{\frac{1}{2}}\mathbf{Q}_i\mathbf{C}_i^{\frac{1}{2}} + \mathbf{C}_i^{\frac{1}{2}}\mathbf{Q}_i^H\mathbf{B}_i^{\frac{1}{2}})\right\}$. Minimizing $\mathbf{u}^H\mathbf{R}\mathbf{u}$ is equivalent to maximizing $\mathbf{u}^H(-\mathbf{R})\mathbf{u}$. In general, matrix $-\mathbf{R}$ is not PSD. However by diagonal loading (DL), one can make it PSD. Let us define diagonally loaded PD matrix $\widehat{\mathbf{R}} \triangleq -\mathbf{R} + \mu\mathbf{I}$ with $\mu > 0$ being larger than the minimum eigenvalue of $-\mathbf{R}$. Due to the fact that $\|\mathbf{u}\|^2 = 1$, DL will not change the solution of the optimization problem since it only adds a constant to the objective function: $\mathbf{u}^H\widehat{\mathbf{R}}\mathbf{u} = -\mathbf{u}^H\mathbf{R}\mathbf{u} + \mu$ in which μ is constant. Consequently, one can minimize (or decrease monotonically) the criterion in (22) by maximizing (or increasing monotonically) the objective of the following optimization problem:

$$\max_{\|\mathbf{u}\|^2=1} \quad \mathbf{u}^H\widehat{\mathbf{R}}\mathbf{u}\ . \quad (23)$$

Problem (23) is very well-known in that its solution is given by the unit-norm eigenvector corresponding to the largest eigenvalue of $\widehat{\mathbf{R}}$, which can be found efficiently using the power method iterations [22].

B. Tightening the Upper-Bound: Optimization w. r. t. $\{\mathbf{Q}_i\}$

Let $\mathbf{w}_i = \mathbf{B}_i^{\frac{1}{2}}\mathbf{u}$ and $\mathbf{v}_i = \mathbf{C}_i^{\frac{1}{2}}\mathbf{u}$ for notational simplicity. Then, the penalty term in $\mathcal{P}_3$ can be rewritten as $\sum_{i=1}^{M}(\|\mathbf{w}_i\| - \|\mathbf{v}_i\|)^2$. Using the following Lemma, we provide an upper-bound for this penalty term.

Lemma 1. *For any* $\mathbf{w}_i \in \mathbb{C}^N$, $\mathbf{v}_i \in \mathbb{C}^N$ *and* $\mathbf{Q}_i \in \mathbb{C}^{N\times N}$ *with* $\|\mathbf{Q}_i\|_F \leq 1$, *we have* $\|\mathbf{w}_i\| - \|\mathbf{v}_i\| \leq \|\mathbf{w}_i - \mathbf{Q}_i\mathbf{v}_i\|$.

Due to space limitation, the proof of the Lemma 1 is not included in the paper. Considering above lemma, it is straightforward to verify that

$$\sum_{i=1}^{M}(\|\mathbf{w}_i\| - \|\mathbf{v}_i\|)^2 \leq \sum_{i=1}^{M}\|\mathbf{w}_i - \mathbf{Q}_i\mathbf{v}_i\|^2. \quad (24)$$

As mentioned in Section II, optimal $\{\mathbf{Q}_i\}$ should satisfy (24) with equality. Hence, given $\mathbf{u}$, p and $\{t_i\}$, we must have

$$\|\mathbf{w}_i\| - \|\mathbf{v}_i\| = \min_{\|\mathbf{Q}_i\|_F\leq 1}\|\mathbf{w}_i - \mathbf{Q}_i\mathbf{v}_i\| \quad (25)$$

Now, the question to be addressed is finding optimal $\{\mathbf{Q}_i\}$. The typical method to find $\mathbf{Q}_i$ is to solve the optimization problem stated in (25). Interestingly, we show that in fact it is not necessary to numerically tackle such an optimization problem to find optimal $\{\mathbf{Q}_i\}$. Recall the optimality condition of $\mathbf{Q}_i$ in (20), which may be written as,

$$\mathbf{Q}_i\mathbf{v}_i = \Big(\|\mathbf{v}_i\|/\|\mathbf{w}_i\|\Big)\mathbf{w}_i\ . \quad (26)$$

Note that (26) can be recast as,

$$\mathbf{Q}_i\mathbf{v}_i = \frac{\mathbf{w}_i\|\mathbf{v}_i\|^2}{\|\mathbf{w}_i\|\|\mathbf{v}_i\|} = \frac{\mathbf{w}_i\mathbf{v}_i^H}{\|\mathbf{w}_i\|\|\mathbf{v}_i\|}\mathbf{v}_i\ . \quad (27)$$

Thus, the optimal $\mathbf{Q}_i = \mathbf{Q}_i^\star$ of $\mathcal{P}_4$ is immediately given by

$$\mathbf{Q}_i^\star = \left(\mathbf{w}_i\mathbf{v}_i^H\right)/\left(\|\mathbf{w}_i\|\|\mathbf{v}_i\|\right)\ . \quad (28)$$

It is straightforward to verify that $\mathbf{Q}_i^\star$ of (28) satisfies (25) and $\|\mathbf{Q}_i^\star\|_F = 1$. Note that given $\mathbf{u}$, p and $\{t_i\}$, calculation of $\mathbf{Q}_i^\star$ is not demanding from a computational point of view.

C. Optimization w. r. t. p

Now, assume that $\mathbf{u}$, $\{\mathbf{Q}_i\}$ and $\{t_i\}$ are given. Considering $\mathcal{P}_3$, the minimization w. r. t. p can be handled by the following optimization problem:

$$\min_{p} \quad p + \eta\sum_{i=1}^{M}\left(\alpha_i - \|\mathbf{C}_i^{\frac{1}{2}}\mathbf{u}\|\right)^2, \quad (29)$$

where $\alpha_i = \|\mathbf{B}_i^{\frac{1}{2}}\mathbf{u}\|$ is given for $i \in [M]$. We recall form (17) that $\mathbf{C}_i = \mathbf{A}_i^- + (1/p + t_i^-)\mathbf{I}$ is a function of p. Since $\mathbf{A}_i^-$ is a PSD matrix, it may be characterized by its eigen-value decomposition $\mathbf{A}_i^- = \mathbf{V}_i\mathbf{\Lambda}_i\mathbf{V}_i^H$ where $\mathbf{V}_i$ is a unitary matrix and Λ_i is a diagonal matrix formed from the eigenvalues of $\mathbf{A}_i^-$. As a result, $\mathbf{C}_i^{\frac{1}{2}}$ can be written as,

$$\mathbf{C}_i^{\frac{1}{2}} = \mathbf{V}_i\left(\mathbf{\Lambda}_i + (\frac{1}{p} + t_i^-)\mathbf{I}\right)^{\frac{1}{2}}\mathbf{V}_i^H. \quad (30)$$

Since multiplication with a unitary matrix does not change the ℓ_2-norm, we have that

$$\|\mathbf{C}_i^{\frac{1}{2}}\mathbf{u}\| = \left\|\left(\mathbf{\Lambda}_i + (\frac{1}{p} + t_i^-)\mathbf{I}\right)^{\frac{1}{2}}\mathbf{V}_i^H\mathbf{u}\right\| \quad (31)$$

$$= \left(\sum_{k=1}^{M}|\mathbf{a}_i(k)|^2\left(\boldsymbol{\lambda}_i(k) + \frac{1}{p} + t_i^-\right)\right)^{\frac{1}{2}} = \left(b_i + \frac{1}{p} + t_i^-\right)^{\frac{1}{2}},$$

where $\mathbf{a}_i = \mathbf{V}_i^H\mathbf{u}$, $\boldsymbol{\lambda}_i$ is a vector formed from diagonal elements of $\mathbf{\Lambda}_i$ $(\boldsymbol{\lambda}_i = \mathbf{diag}(\mathbf{\Lambda}_i))$ or equivalently from the eigenvalues of $\mathbf{A}_i^-$, and $b_i = \sum_{k=1}^{M}|\mathbf{a}_i(k)|^2\boldsymbol{\lambda}_i(k)$. In (32), we have used the fact that $\sum_{k=1}^{M} |\mathbf{a}_i(k)|^2 = \|\mathbf{a}_i\|^2 = 1$. The objective function of (29) now can be expanded as

$$f(p) = \tag{32}$$
$$p + \eta \sum_{i=1}^{M} \left(\alpha_i^2 + b_i + \frac{1}{p} + t_i^- - 2\alpha_i \left(b_i + \frac{1}{p} + t_i^- \right)^{\frac{1}{2}} \right).$$

By looking over its second derivative of $f(p)$, one can readily observe that $f(p)$ is not convex. Instead, we consider a change of parameters by $g(q) = f(1/q)$. Let $q^\star$ denotes the optimal q that minimize $g(q)$. Clearly, one can conclude that $p^\star = 1/q^\star$ will minimize $f(p)$. Therefore, in order to solve (29) it is sufficient to solve

$$\min_{q>0}. \quad g(q) \tag{33}$$

Now, let us have a deeper look at $g(q)$. The second derivative of $g(q)$ is given by

$$g''(q) = \frac{2}{q^3} + \eta \sum_{i=1}^{M} \frac{\alpha_i}{2} \left(b_i + q + t_i^- \right)^{-\frac{3}{2}}. \tag{34}$$

Since q, α_i, b_i and η have positive values, we can conclude that $g''(q) > 0$. This means that $g(q)$ is a convex function and we can use numerical methods like gradient descent algorithm to find the global optimum $q^\star$. Corresponding optimal solution for (29) will be given by $p^\star = 1/q^\star$. Note that convexity of $g(q)$ can also be concluded from definition of $g(q)$, as a *sum of convex functions* over $q > 0$.

D. Optimization w. r. t. t_i

Assuming $\mathbf{u}$ and p are known, the values of $\{t_i\}$ minimizing $\mathcal{P}_3$ and $\mathcal{P}_4$ can be calculated by using (13), that implies

$$t_i = \frac{1}{p} - \mathbf{u}^H \mathbf{A}_i \mathbf{u}. \tag{35}$$

However, it should be noted that at the optimal point, following conditions need to be satisfied for all $i \in [M]$,

$$\begin{cases} t_i \geq 0 & \text{if } ``\lhd_i" = ``\leq" \\ t_i = 0 & \text{if } ``\lhd_i" = ``=" \\ t_i \leq 0 & \text{if } ``\lhd_i" = ``\geq" \end{cases}, \tag{36}$$

otherwise it means that constraint in (12) is not satisfied and optimization problem $\mathcal{P}_1$ is not feasible. When the constraints (36) is imposed, the *optimal feasible* solution in each iteration can be found by,

$$t_i = \begin{cases} t_i & \text{if (36) is satisfied} \\ 0 & \text{if (36) is not satisfied} \end{cases} \tag{37}$$

Let us denote the variables generate at iteration r of the optimization framework by $\boldsymbol{x}^r = (\mathbf{u}^r, p^r, t_i^r)$. It can be shown that any *limit point* of the sequence $\boldsymbol{x}^r$ is an *stationary point*. A proof is not provided herein due to lack of space.

IV. A Numerical Example

In this section, a brief numerical example is provided to investigate the performance of the proposed method. To this end, we consider a multigroup multicast beamforming scenario with $G = 3$, $n_{\text{Tx}} = 4$ and $K = 15$ single-antenna users. We assume $\gamma_i = 1$ for all users. The entries of the channel vectors $\mathbf{h}_i$ are drawn from an i. i. d. complex Gaussian distribution with zero mean, with a variance set to 10. The Gaussian noise components received at each user antenna are assumed to have unit variance, i.e. $\sigma_i^2 = 1$ for all $i \in [K]$. We stop the optimization iterations whenever the objective decrease becomes bounded by 10^{-5} or number of iterations goes beyond 1000. Figure 1 shows the transition of objective function of $\mathcal{P}_3$ (equivalent to $\mathcal{P}_4$), with $\eta = 10$ in different iterations. It also shows the values of p in different iterations. It can be observed that objective function is monotonically decreasing. The difference between p and the objective of $\mathcal{P}_3$ denotes the penalty term of $\mathcal{P}_3$. Since $\eta = 10$ the penalty term might not be exactly zero, therefore resulted SINR for users, $\hat{\gamma}_i$, might be slightly less than targeted γ_i. In this case, one can readily find the feasible beamforming vector $\mathbf{w}$ by simply scaling it. The results leading to Figure 1 was obtained in 2.5 seconds on a standard PC, while SDR followed by a randomization step (with 1000 realizations) took 3.5 seconds. Also our approach resulted in $p^\star = 1.22$ while SDR achieved $p^\star_{\text{SDR}} = 1.37$. Note that the lower bound for $p^\star$ achieved by SDR (corresponding to high-rank solution) was $p^\star_{\text{LB}} = 1.16$.

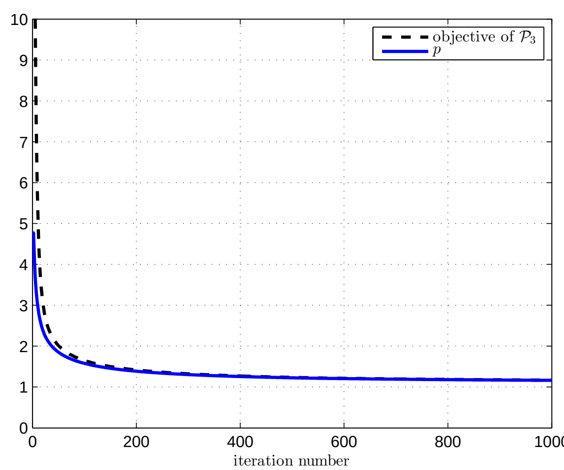

Fig. 1. Transition of the objective function $\mathcal{P}_3$ and the parameter p vs. iteration number. The weight of the penalty-term (η) is set to 10.

V. Conclusion

An iterative approach is proposed to tackle the nonconvex QCQPs. Each iteration of the proposed method requires solving a set of subproblems, which is accomplished by computationally efficient steps. The multigroup multicast beamforming problem is formulated as a nonconvex QCQP and solved using the proposed method. Numerical results showed the proposed approach is computationally efficient and produces quality results.

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