Combinatorial Problems At The Interface Of Discrete And Convex Geometry

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Dedicated to my parents.

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Abstract

This thesis consists of three chapters. The first two chapters concern lattice points and convex sets. In the first chapter we consider convex lattice polygons with minimal perimeter. Let n be a positive integer and $\|\cdot\|$ any norm in $\mathbb{R}^2$. Denote by B the unit ball of $\|\cdot\|$ and $\mathcal{P}_{B,n}$ the class of convex lattice polygons with n vertices and least $\|\cdot\|$-perimeter. We prove that after suitable normalisation, all members of $\mathcal{P}_{B,n}$ tend to a fixed convex body, as $n \to \infty$.

In the second chapter we consider maximal convex lattice polygons inscribed in plane convex sets. Given a convex compact set $K \subset \mathbb{R}^2$ what is the largest n such that K contains a convex lattice n-gon? We answer this question asymptotically. It turns out that the maximal n is related to the largest affine perimeter that a convex set contained in K can have. This, in turn, gives a new characterisation of K_0, the convex set in K having maximal affine perimeter.

In the third chapter we study a combinatorial property of arbitrary finite subsets of $\mathbb{R}^d$. Let $X \subset \mathbb{R}^d$ be a finite set, coloured with $\lfloor \frac{d+3}{2} \rfloor$ colours. Then X contains a rainbow subset $Y \subset X$, such that any ball that contains Y contains a positive fraction of the points of X.

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Notation

$\mathrm{A}(\cdot)$	Maximal affine perimeter	43
$\mathrm{AP}(\cdot)$	Affine perimeter	40
$\mathcal{B}$	The set of Euclidean balls in $\mathbb{R}^d$	68
$\mathcal{B}_c$	The family of the complements of Euclidean balls in $\mathbb{R}^d$	79
$b(C)$	The barycentre of C	45
$C(\Delta)$	The convex lattice chain in T with set of edges $\mathbb{P} \cap \Delta$	52
$C(\Delta, \mathbb{Z}_t)$	The $\mathbb{Z}_t$ lattice chain that corresponds to $C(\Delta)$	55
C^*	The unique convex body with radius of curvature $R(u) = \frac{1}{3}\rho^3(u)$	45
$\mathcal{C}$	The set of convex bodies in $\mathbb{R}^2$	40
$\mathcal{C}(K)$	The set of all convex bodies contained in K	43
$\mathcal{C}_0$	The set of all $C \in \mathcal{C}$ with $b(C) = 0$	45
$D(T)$	The special parabola within a triangle $T = \mathrm{conv}\{p_0, p_1, p_2\}$ tangent to p_0p_1 and p_1p_2 at p_0 and p_2 respectively	41
$\mathrm{dist}(\cdot, \cdot)$	Hausdorff distance	39

$F(K)$	The unique subset K_0 of K	44
$\mathcal{F}$	A family of sets F expressed in the form $F = \{x \in \mathbb{R}^d : \sum_{i=1}^m \alpha_i f_i(x) \leq 0\}$, for some given family of functions $f_i : \mathbb{R}^d \to \mathbb{R}$	77
$G(C)$	The unique body C^*	45
$h_K(\cdot)$	The support function of K	19
K_0	The unique subset of K with maximal affine perimeter	44
$\mathcal{K}$	The set of equivalence classes in $\{F(K) : K \in \mathcal{C}\}$	46
$\mu(\cdot)$	Möbius function	33
$m(K, \mathbb{Z}_t)$	The maximal number of vertices of a convex $\mathbb{Z}_t$ lattice n-gon contained in a plane convex body K	38
$m(T)$	The maximal length of a convex lattice chain within T	47
$\mathrm{O}(\cdot)$	The big 'oh' notation	21
$\mathcal{P}_{B,n}$	The class of convex lattice n-gons with least perimeter with respect to B	18
$\mathbb{P}$	The set of primitive vectors	19

$\mathrm{per}(n)$	The perimeter of a minimiser $Q(n)$	18
$Q(n)$	A member of the class $\mathcal{P}_{B,n}$	18
$\rho_C(\cdot)$	The radial function of C	45
$w(\cdot)$	Lattice width	20
$\mathbb{Z}_t$	The lattice $\frac{1}{t}\mathbb{Z}^2$	38
$\zeta(\cdot)$	Riemann zeta function	33

Introduction

The topic of this dissertation is combinatorial properties of convex and discrete sets in Euclidean spaces. We shall be considering three separate problems whose solutions are presented in three chapters (see also [P1],[BP] and [P2]). In the first two chapters we deal with convex lattice polygons in the plane and in the third chapter we study a combinatorial property of arbitrary finite subsets of $\mathbb{R}^d$.

A *convex lattice polygon* is a convex polygon whose vertices all lie on the integer lattice $\mathbb{Z}^2$. Let $n \in \mathbb{N}$ be a positive integer and $\|\cdot\|$ be a norm on the plane. We consider the set of all convex lattice polygons with exactly n vertices, that is, convex lattice n-gons and for each fixed number of vertices n, we examine the n-gons that have the least perimeter with respect to a norm $\|\cdot\|$. In other words, we look for a polygon with edges $x_1, \dots, x_n$ for which $\sum \|x_i\|$ is as small as possible. We denote the class of minimal polygons by $\mathcal{P}_{B,n}$, where B is the unit ball of the norm.

In the first chapter we show that the polygons in $\mathcal{P}_{B,n}$ have a 'limit shape'. Namely, as $n \to \infty$, all members of $\mathcal{P}_{B,n}$, after suitable normalisation, converge to a fixed convex body. We also describe this limit shape explicitly in terms of B.

Inspired by a result of Arnold in [Ar80], Vershik was the first to pose the question of whether the members of some class of convex lattice polygons have a limit shape. Jarnik appears to have studied the concept of the limit shape in [Ja25], where he dealt with a strictly convex curve of length $\ell \to \infty$.

Limit shape theorems have been the subject of interest of many papers. The most remarkable results, due to Bárány, appear in [Bá95] (see also [Si] for a different viewpoint and [Ve94]) and [Bá97]. In [Bá95] it is proved that almost all convex $\frac{1}{n}$-lattice polygons contained in the square $[-1,1]^2$ are close to a fixed convex body. In [Bá97], this result is generalised to the case of convex $\frac{1}{n}$-lattice polygons contained in a convex body $K \subset \mathbb{R}^2$.

The class of convex lattice polygons with minimal perimeter was studied by Stojaković in [St01] and [St03], for the case of the ℓ_p-norm. In [St01] he obtains the expression for the ℓ_q-perimeter of the convex lattice polygons having the minimal ℓ_p-perimeter, with respect to the number of their vertices, where p, q are positive integers or ∞. In [St03] it is proved that as $n \to \infty$, the south-east arcs of the convex lattice polygons having minimal ℓ_p-perimeter converge, after a suitable scaling, to a curve which is described explicitly in parametric form. In the first chapter we deal with the case of a general norm. Our approach is different from that of Stojaković: in the previous work, the precise formula for the norm was essential. We avoid this by expressing the support function and the vertices of the limit shape as an integral over the unit ball of the norm.

In the second chapter, we deal with another limit shape theorem. Let $K \subset \mathbb{R}^2$ be a convex body, that is, a convex compact set with non-empty interior. Consider all convex lattice polygons that are contained in K and have maximal number of vertices. We determine the maximal number of vertices that a convex lattice polygon contained in K can have. To be precise, let $\mathbb{Z}_t = \frac{1}{t}\mathbb{Z}^2$ be a shrunken copy of the lattice $\mathbb{Z}^2$ (where we think of t as large) and define a convex $\mathbb{Z}_t$ n-gon to be a convex polygon with exactly n vertices, all belonging to the lattice $\mathbb{Z}_t$. Now, define the number $m(K, \mathbb{Z}_t)$ to be the maximal number of vertices n, such that there is a convex $\mathbb{Z}_t$ n-gon contained in K. In Chapter 2, we begin by determining the asymptotic behaviour of $m(K, \mathbb{Z}_t)$, as $t \to \infty$. Consider now the set of all convex bodies S that are contained in K. It is known from [Bá97] that there is a unique convex body $K_0 \subset K$ which has maximal affine perimeter which we denote $\mathrm{A}(K)$. We show that $m(K, \mathbb{Z}_t)$ and $\mathrm{A}(K)$ are related. Moreover, we prove that as $t \to \infty$, any maximiser for $m(K, \mathbb{Z}_t)$ converges to K_0. With the proof of this limit shape theorem, we derive a characterisation of the body K_0. We also obtain some side results related to K_0 and K.

The problem of estimating $m(K, \mathbb{Z}_t)$ goes back to Jarnik's paper [Ja25] in which he asked and answered the following question: what is the maximal number of lattice points which lie on a strictly convex curve of length ℓ, as ℓ tends to ∞? His estimate is $\frac{3}{\sqrt[3]{2\pi}} \cdot \ell^{2/3}(1 + o(1))$. From this bound he concluded that when the strictly convex curve is the circle of radius r, a convex polygon contained in this circle has at most $3\sqrt[3]{2\pi}r^{2/3}(1 + o(1))$ vertices. This follows from our estimate as well. Andrews in [An63] showed that a convex lattice polygon P has at most $c(\mathrm{Area}\, P)^{1/3}$ vertices, where $c > 0$ is a universal constant, and generally, a convex lattice polytope $P \subset \mathbb{R}^d$ with non-empty interior can have at most $c'(\mathrm{vol}\, P)^{(d-1)/(d+1)}$ vertices, where the constant $c' > 0$ depends on dimension only.

In the third chapter we consider a problem which is of a different nature from the first two, in the sense that we are now moving to a combinatorial property of discrete sets. Here we are interested in finite sets $X \subset \mathbb{R}^d$ in general position. Any d points in a d-dimensional space, define a hyperplane. "General position" means that no $d + 1$ points of X belong to the same hyperplane. Let us consider the planar case first. Let X be a finite set of points on the plane with no three points co-linear. Consider a colouring of the points of X with two colours, say red and blue. We prove that any such set contains a pair of points, one red and one blue, with the following property: any disc that contains this pair, contains many of the points of X. By many we mean a positive fraction of the total number of points. In general we prove that any finite set $X \subset \mathbb{R}^d$ in general position coloured with $\lfloor \frac{d+3}{2} \rfloor$ colours contains a rainbow subset, that is a subset which consists of exactly one point of each colour, with the property that any Euclidean ball that contains it, contains a positive fraction of the points of X. We show that the number of colours is best possible. We also provide an example where all points in any such ball (except for the points of the rainbow set itself) come from a single colour. In the final section of this chapter, we show that the result still holds if instead of Euclidean balls we consider more general families of subsets of $\mathbb{R}^d$.

This "points-balls" problem was introduced by Neumann-Lara and Urrutia in [NU88]. They proved a planar, non-coloured case, namely, that any finite set $X \subset \mathbb{R}^2$ contains two points such that any disc that contains them, will contain at least $\lceil (n-2)/60 \rceil$ points of X, where n is the number of points in X. This expression was improved by Hayward in [Ha89] to $\lceil \frac{5}{84}(n-2) \rceil$ and Hayward, Rappaport and Wenger in [HRW89] showed that if $\Pi(n)$ is the largest number, such that for every n-point set X on the plane, there exist two points $x, y \in X$ with this property, then $\lfloor n/27 \rfloor + 2 \leq \Pi(n) \leq \lceil n/4 \rceil + 1$. The planar case was later generalised by Bárány, Schmerl, Sidney and Urrutia in [BSSU89] to the case of $X \subset \mathbb{R}^d$, and Bárány and Larman in [BL90] generalised the d-dimensional result from the case of Euclidean balls, to the case of ellipsoids and more generally, quadrics in $\mathbb{R}^d$.

Our approach is in some ways similar to and in others different from those that appear in the previous articles. The planar result uses the facts that among any five points on the plane, there are four that form a convex quadrilateral Π and in any convex quadrilateral, one of the diagonals has the property that any disc that contains it, contains another vertex of Π. To extend this to d dimensions what is used is the Gale transform by which n points in $\mathbb{R}^d$ are mapped to $\mathbb{R}^{n-d-1}$. In the paper [BSSU89], $d+3$ points in $\mathbb{R}^d$ are mapped into the plane. In both cases, a simple counting argument which we also use, gives the main result. A somewhat different approach appears in [BL90]. They observed that the family of quadrics in $\mathbb{R}^d$ can be induced by halfspaces in $\mathbb{R}^m$, by using a family of $m = (d+1)(d+2)/2$

functions. Using these functions, a set $Z \subset \mathbb{R}^d$ of $(m+1)$ points is mapped into $\mathbb{R}^m$. With these observations, the final element in their argument is the trivial fact that Z is linearly dependent.

The coloured version, as it is often the case, requires a stronger tool. In our case it will be a beautiful result of Vrećica and Živaljević which is the following coloured Tverberg type Theorem from [VŽ94]. (For other results of this type, and especially for the case of $d+1$ colours, see [BL90] or [ŽV92].) Let $C_1, C_2, \dots, C_k \subset \mathbb{R}^d$ be disjoint finite sets (colours), each of cardinality $|C_i| = 2p-1$, where p is a prime that satisfies $p(d-k+1) \leq d$. Then there are p disjoint rainbow sets $A_1, \dots, A_p \subset \cup_{i=1}^k C_i$ whose convex hulls have a common point. We will apply this result for the case when $p=2$, which is a coloured Radon type Theorem. The proof of this result uses topological methods.

Chapter 1

The Limit Shape of Convex Lattice Polygons with Minimal Perimeter

1.1 Introduction and results

Let $n \in \mathbb{N}$ be a positive integer. A *convex lattice n-gon* is a convex polygon with n vertices all lying on the lattice $\mathbb{Z}^2$. We can define the perimeter of a polygon with respect to a norm $\|\cdot\|$ as the sum of the lengths of the edges, where the length is with respect to the specified norm. We shall consider the class of convex lattice n-gons having the least perimeter with respect to some given norm. This class was studied in [St01] and [St03]. In particular, in [St01] an expression for the ℓ_q-perimeter is given, for the convex lattice polygons having the minimal ℓ_p-perimeter, with respect to the number of their vertices, where p, q are positive integers or ∞. In [St03] it is proved that as $n \to \infty$, the south-east arc of the convex lattice polygons having minimal ℓ_p-perimeter tends, after a suitable scaling, to a certain curve which is explicitly described in parametric form.

In this paper, we consider the case of a general norm. Let $\|\cdot\|$ be any norm on

$\mathbb{R}^2$ and let B denote its unit ball. For n a positive integer $\mathcal{P}_{B,n}$ denotes the class of all convex lattice n-gons with least perimeter with respect to $\|\cdot\|$. We will call every member of $\mathcal{P}_{B,n}$ a *minimiser*. The perimeter of a minimiser $Q(n)$ will be denoted by per(n). As we will show in Remark 1.4.1, per(n) is increasing. Since the perimeter is invariant under lattice translations, we shall always consider minimisers that are centred at the origin, if they are centrally symmetric. In the case that n is odd, the minimisers cannot be centrally symmetric but we shall show that asymptotically their centres of mass can also be taken to be the origin.

In this chapter we are dealing with the following question: As the number of vertices n tends to ∞, do all sequences of minimisers, after suitable normalisation, converge to a fixed convex body? In other words, do the members of the class $\mathcal{P}_{B,n}$ have a limit shape?

Denote by $h_K(\cdot)$ the support function of a convex body K (the definition will be given in the next section). In Theorems 1.1.1 and 1.1.2, the unit vectors $u \in \mathbb{R}^2$ are vectors of length one with respect to the Euclidean norm. We prove the following limit shape theorem.

Theorem 1.1.1. *Let $u \in \mathbb{R}^2$ be a unit vector. If $Q(n) \in \mathcal{P}_{B,n}$ for all n,*

$$\lim_{n\to\infty} n^{-3/2} h_{Q(n)}(u) = \frac{1}{4}\frac{\pi}{\sqrt{6(\text{Area}\,B)^3}} \int_{x\in B} |x\cdot u|dx,$$

where the integration is taken over the unit ball B of the norm.

It is sometimes more 'convenient' to express a polygon as the convex hull of its vertices. The vertices of $Q(n)$ satisfy the following limit shape theorem.

Theorem 1.1.2. *Let $u \in \mathbb{R}^2$ be any unit vector and let $z(u)$ be the vertex of a minimiser $Q(n)$ which satisfies $u\cdot z(u) = \max\{u\cdot x,\ x \in Q(n)\}$. Then*

$$\lim_{n\to\infty} n^{-3/2} z(u) = \frac{1}{2}\frac{\pi}{\sqrt{6(\text{Area}\,B)^3}} \int_{\substack{x\in B\\ x\cdot u\geq 0}} x dx.$$

Both theorems say that the members of $\mathcal{P}_{B,n}$ have a 'limit shape'. We prove only Theorem 1.1.1. Theorem 1.1.2 can be proved using similar arguments. Note that the result of [St01] can be recovered from Theorem 1.1.2 when the integral is taken over the unit ball of the ℓ_p norm.

The proof is composed of four steps. We first obtain an expression for the perimeter of a minimiser, when the number of vertices n belongs to a certain subsequence $(n_k)_{k=0}^{\infty}$ of $\mathbb{N}$. We then extend the result to include any even number and then any odd number of vertices. We finally obtain the limit shape of a minimiser $Q(n)$, for $n \in \mathbb{N}$. The cases of n_k-gons and $2m$-gons are straightforward. Most of the difficulty arises in the case when n is odd.

It is worth mentioning that other limit shape theorems have also been obtained, for example in [Bá95], [Si] and [Ve94]. It is proved that, as $n \to \infty$ almost all convex $\frac{1}{n}\mathbb{Z}^2$ lattice polygons contained in the square $[-1, 1]^2$ are very close to a fixed convex body. The proofs of this result are essentially different from ours and are more involved: in our case, when the number of vertices belongs to $(n_k)_{k=1}^{\infty}$, the minimiser is unique. The size of the class of convex $\frac{1}{n}\mathbb{Z}^2$ lattice polygons contained in $[-1, 1]^2$ is asymptotically $\exp cn^{2/3}$.

1.2 Preliminaries

Let $\mathbb{P}$ be the set of primitive vectors $z \in \mathbb{Z}^2$, i.e. lattice points whose coordinates are co-prime. If P is a convex lattice n-gon we can regard the edges of P in the obvious way as a sequence of distinct integer vectors whose sum is 0. Our aim is to find n distinct, non-zero lattice vectors $x_1, \ldots, x_n$, which sum to 0 and with $\sum \|x_i\|$ as small as possible. In the case when n is even, we may restrict our attention to primitive vectors. In the case when n is odd this is not always possible. In Remark 1.2.3 at the end of this section, we discuss the case when n is odd.

Let $K \subset \mathbb{R}^2$ be a convex body, that is, a compact, convex set with non-empty interior and let $u \in \mathbb{R}^2$ be a unit vector. The *support function* $h_K(\cdot)$ of K is given by

$$h_K(u) = \max\{x \cdot u \ : \ x \in K\}.$$

A *zonotope* $Z \subset \mathbb{R}^2$ is a convex body which is the Minkowski sum of finitely many line segments,

$$Z = \sum_{i=1}^{m}[x_i, y_i] = \{\sum_{i=1}^{m} z_i, \ z_i \in [x_i, y_i], \ i = 1, \dots, m\}.$$

Let $t_1, t_2, \dots$ be the possible norms of primitive vectors, taken in increasing order and let n_k be the number of primitive vectors with norm at most t_k. Then n_k is the number of primitive vectors contained in $t_k B$ and there are no primitive vectors in the interior of $t_{k+1}B \backslash t_k B$. It is easy to see that since the set $\{p : p \in \mathbb{P} \cap t_k B\}$ consists of the shortest n_k integer vectors which sum to zero, for the case of the n_k-gon, the minimiser $Q(n_k)$ will be the zonotope

$$Q(n_k) = \sum_{p \in \mathbb{P} \cap t_k B} [0, p], \tag{1.1}$$

whose edges are the primitive vectors in $t_k B$. Note that in this case, the minimiser is unique (up to lattice-translations). For convenience only, we will be treating $Q(n_k)$ as the 0-symmetric zonotope

$$Q(n_k) = \frac{1}{2} \sum_{p \in \mathbb{P} \cap t_k B} \left[-\frac{p}{2}, \frac{p}{2}\right]$$

which is a translated copy of (1.1). This is not necessarily a lattice polygon but it has the same perimeter as (1.1).

We will need to know the size of $\mathbb{P} \cap tB$ and more generally, of sums of the form $\sum_{p \in \mathbb{P} \cap tB} f(p)$, for certain functions f. Let $K \subset \mathbb{R}^2$ be a convex body and denote by $w(K)$ the *lattice width* of K, that is

$$w(K) = \min_{w \in \mathbb{Z}^2 \backslash \{0\}} \max\{w \cdot (x - y) \ : \ x, y \in K\}. \tag{1.2}$$

It is well known that the density of $\mathbb{P}$ in $\mathbb{Z}^2$ is asymptotically $6/\pi^2$. Thus, one would expect that

$$|\mathbb{P} \cap K| \approx \frac{6}{\pi^2} \operatorname{Area}(K).$$

This is indeed the case when $w(K)$ is large enough. We state in Lemma 1.2.1 below a slightly more general result which we will need. The proof of Lemma 1.2.1 is given in Section 1.7.

Lemma 1.2.1. *Let $K \subset \mathbb{R}^2$ be a 0-symmetric convex body with $\mathrm{w} = w(K) > 3$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous function of order $\alpha \geq 0$, i.e. $f(\lambda x) = |\lambda|^\alpha f(x)$, $\lambda \in \mathbb{R}$. Assume that $|f(x)| \leq c$, for $x \in K$, (for some constant c) and that f varies by at most V on any unit square Q intersecting K. Then*

$$\left| \sum_{p \in \mathbb{P} \cap K} f(p) - \frac{6}{\pi^2} \int_{x \in K} f(x)dx \right| \leq \mathrm{Area}(K) \left(\frac{2}{3} V + \frac{42(c+V)\log \mathrm{w}}{\mathrm{w}} \right).$$

An immediate consequence of Lemma 1.2.1, for the case when $f \equiv 1$, is a well known estimate for $n_k = |\mathbb{P} \cap t_k B|$ (see for instance [Ja25] or [BT04]). In this case, $V = 0$ and the lattice width $\mathrm{w} = w(tB)$ is within constant times t. For convenience, we will be using Landau's big 'oh' notation, $\mathrm{O}(\cdot)$.

Lemma 1.2.2. *The number n_k of primitive vectors contained in $t_k B$, $t_k > 0$ is*

$$|\mathbb{P} \cap t_k B| = t_k^2 \left\{ \frac{6}{\pi^2} \mathrm{Area}\, B + \mathrm{O}\left(\frac{\log t_k}{t_k} \right) \right\}.$$

Remark 1.2.3.

In Section 1.4 we will see that a minimiser $Q(n)$, where n is even, with $n_k \leq n < n_{k+1}$, contains all n_k primitive vectors from $t_k B$ as edges and does not contain any edges longer than t_{k+1}. Thus, an integer vector $z \in \mathbb{Z}^2 \backslash \mathbb{P}$, with $\|z\| \leq t_{k+1}$ cannot appear as an edge of $Q(n)$, as it will contain a primitive vector p that has been used.

In Section 1.5 we will see that this is not necessarily true for a minimiser $Q(n)$, when n is odd. In this case, not all primitive vectors from $t_k B$ are necessarily used as edges of $Q(n)$ and there can be minimisers whose edges are not all primitive vectors. It is not hard to see that these can only come from the set of edges that appear 'alone', i.e. z is an edge $Q(n)$ but $-z$ is not. Indeed, if there is a pair of edges $\pm v$ from $\mathbb{Z}^2 \backslash \mathbb{P}$, then it contains a pair of edges $\pm p$ from $\mathbb{P}$, of shorter, primitive vectors that have not appeared as edges. Replacing $\pm v$ with $\pm p$ gives a polygon with smaller perimeter. Hence, the only possible non-primitive edges come from the set of edges that appear alone and in Section 1.5 we will prove that these are few and short.

1.3 The case $n = n_k$

1.3.1 The perimeter of $Q(n_k)$

We saw in the previous section that for the subsequence $\{n_k\}_{k=0}^{\infty}$ of $\mathbb{N}$, the minimiser is $Q(n_k) = \sum_{p \in \mathbb{P} \cap t_k B} \left[-\frac{p}{2}, \frac{p}{2}\right]$. Thus, the perimeter of $Q(n_k)$ is

$$\operatorname{per}(n_k) = \sum_{p \in \mathbb{P} \cap t_k B} \|p\|.$$

Applying Lemma 1.2.1, for $f(x) = \|x\|$, we get

Proposition 1.3.1. *The perimeter* $\operatorname{per}(n_k)$ *of* $Q(n_k)$ *is*

$$\operatorname{per}(n_k) = t_k^3 \left\{ \frac{6}{\pi^2} \int_{x \in B} \|x\| dx + \mathrm{O}\left(\frac{\log t_k}{t_k}\right) \right\}, \tag{1.3}$$

or in terms of the number of vertices,

$$\operatorname{per}(n_k) = n_k^{3/2} \left\{ \frac{\pi}{\sqrt{6(\operatorname{Area} B)^3}} \int_{x \in B} \|x\| dx + \mathrm{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right) \right\}. \tag{1.4}$$

Proof. The expression (1.3) follows immediately if we apply Lemma 1.2.1 for the case $f(x) = \|x\|$.

For (1.4) we have that $n_k = \mathrm{O}(t_k^2)$ and $t_k = \mathrm{O}(n_k^{1/2})$ from Lemma 1.2.2. Namely, from the expression for n_k, it is easy to see that

$$\left| t_k - \frac{\pi}{\sqrt{6 \operatorname{Area} B}} n_k^{1/2} \right| \leq c \log n_k,$$

so,

$$t_k = n_k^{1/2} \left\{ \frac{\pi}{\sqrt{6 \operatorname{Area} B}} + \mathrm{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right) \right\}.$$

Now, by replacing t_k in (1.3) we get

$$\begin{aligned}
\operatorname{per}(n_k) &= n_k^{3/2}\left(\frac{\pi}{\sqrt{6\,\text{Area}\,B}} + \mathrm{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right)\right)^3 \frac{6}{\pi^2}\int\limits_{x\in B}\|x\|dx + \mathrm{O}\left(n_k \log n_k\right) \\
&= n_k^{3/2}\frac{6}{\pi^2}\frac{\pi^3}{\sqrt{(6\,\text{Area}\,B)^3}}\int\limits_{x\in B}\|x\|dx\left(1 + \mathrm{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right)\right)^3 + \mathrm{O}\left(n_k \log n_k\right) \\
&= n_k^{3/2}\frac{\pi}{\sqrt{6(\text{Area}\,B)^3}}\int_{x\in B}\|x\|dx + \mathrm{O}\left(n_k \log n_k\right).
\end{aligned}$$

Thus,

$$\operatorname{per}(n_k) = n_k^{3/2}\left\{\frac{\pi}{\sqrt{6(\text{Area}\,B)^3}}\int_{x\in B}\|x\|dx + \mathrm{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right)\right\}.$$

□

1.3.2 The support function of $Q(n_k)$

As before, $h_{Q(n_k)}(u)$ denotes the support function of $Q(n_k)$, for $u \in \mathbb{R}^2$ a unit vector, $h_{Q(n_k)}(u) = \max\{x \cdot u : x \in Q(n_k)\}$. For a zonotope $Z = \sum_{i=1}^k S_i$, where $S_i = [-\alpha_i v_i, \alpha_i v_i]$, the support function is given by $h_Z(u) = \sum_{i=1}^k \alpha_i |u \cdot v_i|$ (see [Sch93]). Since $Q(n_k)$ is a 0-symmetric zonotope, we have the following

Proposition 1.3.2. *The support function of the minimiser $Q(n_k)$ is given by*

$$h_{Q(n_k)}(u) = \frac{t_k^3}{4}\left\{\frac{6}{\pi^2}\int_{x\in B}|x\cdot u|dx + \mathrm{O}\left(\frac{\log t_k}{t_k}\right)\right\},$$

or in terms of n_k,

$$h_{Q(n_k)}(u) = n_k^{3/2}\left\{\frac{\pi}{4\sqrt{6(\text{Area}\,B)^3}}\int_{x\in B}|x\cdot u|dx + \mathrm{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right)\right\},$$

for all $u \in \mathbb{R}^2$.

Proof. We have that

$$\begin{aligned}
h_{Q(n_k)}(u) &= \sum_{\substack{p\in\mathbb{P}\cap t_k B \\ p\cdot u\geq 0}}\left|\frac{p}{2}\cdot u\right| = \frac{1}{4}\sum_{p\in\mathbb{P}\cap t_k B}|p\cdot u| \\
&= \frac{t_k^3}{4}\left\{\frac{6}{\pi^2}\int_{x\in B}|x\cdot u|dx + \mathrm{O}\left(\frac{\log t_k}{t_k}\right)\right\}.
\end{aligned}$$

where again we applied Lemma 1.2.1, for $f(x) = |x \cdot u|$. □

Since we have determined the asymptotic behaviour of the support function of $Q(n_k)$ we have determined the asymptotic behaviour of $Q(n_k)$ itself, i.e.

$$Q(n_k) = \{x \in \mathbb{R}^2 : x \cdot u \leq h_{Q(n_k)}(u), u \in \mathbb{R}^2, \|u\|_2 = 1\}.$$

1.4 The case $n = 2m$

Our aim now is to reduce the case in which n is an even, positive integer $2m$ say, to the case already discussed. There is a $k \in \mathbb{N}$ such that $n_k \leq 2m < n_{k+1}$. We can construct a convex lattice $2m$-gon, $Q(2m)$, with minimal perimeter by adding to the set of edges of $Q(n_k)$ any disjoint set of $(2m - n_k)$ primitive vectors from $t_{k+1}B$ whose sum is zero. Since the shortest available vectors are used, the polygon will have minimal perimeter. As an even number of vertices is missing, we may obviously choose a union of pairs $\{q, -q\}$, $q \in t_{k+1}B$. These are not the only choices, but whatever minimiser we consider, all primitive vectors from $t_k B$ will be used and no edge will be longer than t_{k+1}. These facts, combined with a bound on $|n_{k+1} - n_k|$ will show that as long as the required number of primitive vectors sum to 0 and come from at most $t_{k+1}B$, the results obtained for perimeter (and the support function) can be extended to any even number. Clearly, in this case the minimiser $Q(2m)$ is not unique, since the choice of a set of $(2m - n_k)$ primitive vectors from at most $t_{k+1}B$ is not unique. Passing, as before, to the 0-symmetric zonotope, we have that

$$Q(2m) = \sum_{\substack{i=1 \\ p_i \in P \cap t_k B}}^{n_k} \left[-\frac{p_i}{2}, \frac{p_i}{2}\right] + \sum_{\substack{j=n_k+1 \\ q_j \in P \cap \mathrm{bd}(t_{k+1}B)}}^{2m} \left[-\frac{q_j}{2}, \frac{q_j}{2}\right].$$

Our problem is to show that, although there are many minimisers, they all have essentially the same shape when n is large.

1.4.1 The perimeter of $Q(2m)$

To extend the results of the case $n = n_k$ to any even number of vertices, we will need the following two remarks.

Remark 1.4.1. *Let n be any positive integer. Then,* $\operatorname{per}(n) \le \operatorname{per}(n+1)$.

Proof. Indeed, let $Q(n+1)$ be a convex lattice $(n+1)$-gon with perimeter $\operatorname{per}(n+1)$. If we remove any one of its vertices and take the convex hull of the remaining n, we get a convex lattice n-gon with perimeter say $\operatorname{per}'(n)$. Obviously, $\operatorname{per}'(n) \le \operatorname{per}(n+1)$. Since $\operatorname{per}(n)$ is the least perimeter of a convex lattice n-gon, we have $\operatorname{per}(n) \le \operatorname{per}'(n)$. Hence, $\operatorname{per}(n) \le \operatorname{per}(n+1)$. □

Note that equality in Remark 1.4.1 can only hold if the specified norm is not strictly convex.

Remark 1.4.2. $|n_{k+1} - n_k| = \mathrm{O}(t_{k+1})$.

Proof. From the definition of the sequence n_k it is clear that $|n_{k+1} - n_k|$ is the number of primitive vectors on the boundary of $t_{k+1}B$. Therefore,

$$|n_{k+1} - n_k| = |\mathbb{P} \cap \operatorname{bd}(t_{k+1}B)| \le |\mathbb{Z}^2 \cap \operatorname{bd}(t_{k+1}B)| \le ct_{k+1},$$

where the constant $c > 0$ depends on the perimeter of the unit ball B. So, $|n_{k+1} - n_k| = \mathrm{O}(t_{k+1})$. □

Proposition 1.4.3. *The perimeter of $Q(n)$, for $n = 2m$ is*

$$\operatorname{per}(n) = n^{3/2}\left\{\frac{\pi}{\sqrt{6(\operatorname{Area} B)^3}}\int_{x\in B}\|x\|dx + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\right\}.$$

Proof. From the previous remarks we have,

$$\begin{aligned}|\operatorname{per}(n_{k+1}) - \operatorname{per}(n)| &= |(\operatorname{per}(n_k) + |n_{k+1} - n_k|t_{k+1}) - (\operatorname{per}(n_k) + |n - n_k|t_{k+1})| \\ &\le |n_{k+1} - n|t_{k+1} \le |n_{k+1} - n_k|t_{k+1} \le ct_{k+1}^2.\end{aligned}$$

So, $\mathrm{per}(n) = \mathrm{per}(n_{k+1}) + \mathrm{O}(t_{k+1}^2)$. Therefore, for n even, the perimeter in terms of n is

$$\mathrm{per}(n) = n^{3/2}\left\{\frac{\pi}{\sqrt{6(\mathrm{Area}\,B)^3}}\int_{x\in B}\|x\|dx + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\right\}. \tag{1.5}$$

□

1.4.2 The support function of $Q(2m)$

Proposition 1.4.4. *The support function of any minimiser $Q(n)$, for $n = 2m$ is given by*

$$h_{Q(n)}(u) = n^{3/2}\left\{\frac{\pi}{4\sqrt{6(\mathrm{Area}\,B)^3}}\int_{x\in B}|x\cdot u|dx + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\right\},$$

for all unit vectors $u \in \mathbb{R}^2$.

Proof. The support function of $Q(n)$ is,

$$h_{Q(n)}(u) = \sum_{\substack{i=1\\ p_i\in\mathbb{P}\cap t_k B\\ p_i\cdot u\geq 0}}^{n_k}\left|\frac{p_i}{2}\cdot u\right| + \sum_{\substack{j=n_k+1\\ q_j\in\mathbb{P}\cap\mathrm{bd}(t_{k+1}B)\\ q_j\cdot u\geq 0}}^{2m}\left|\frac{q_j}{2}\cdot u\right|. \tag{1.6}$$

For the second sum, since any two norms on the plane are equivalent, we may assume that $c_1\|q_j\|_2 \leq \|q_j\| \leq c_2\|q_j\|_2$. Thus we have,

$$\sum_{\substack{j=n_k+1\\ q_j\in\mathbb{P}\cap\mathrm{bd}(t_{k+1}B)\\ q_j\cdot u\geq 0}}^{2m}\left|\frac{q_j}{2}\cdot u\right| \leq \frac{1}{4}\sum_{j=n_k+1}^{2m}\|q_j\|_2\cdot\|u\|_2 \leq \frac{1}{4c_1}|2m-n_k|t_{k+1} \leq cn_{k+1}.$$

So,

$$\sum_{\substack{j=n_k+1\\ q_j\in\mathbb{P}\cap\mathrm{bd}(t_{k+1}B)\\ q_j\cdot u\geq 0}}^{2m}\left|\frac{q_j}{2}\cdot u\right| = \mathrm{O}(n_{k+1}) = \mathrm{O}(n).$$

The first sum in (1.6) is the support function of $Q(n_k)$. So $h_{Q(n_k)}$ and $h_{Q(n)}$ differ by at most $\mathrm{O}(n)$. Therefore, for $n = 2m$

$$h_{Q(n)}(u) = n^{3/2}\left\{\frac{\pi}{4\sqrt{6(\mathrm{Area}\,B)^3}}\int_{x\in B}|x\cdot u|dx + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\right\}. \tag{1.7}$$

□

1.5 The case $n = 2m + 1$

Let $n = 2m + 1$ be a positive odd number. As before we choose $k \in \mathbb{N}$ to be such that $n_k < n < n_{k+1}$. Since n_k, n_{k+1} are even numbers, we have that $n_k \leq n - 1 < n < n + 1 \leq n_{k+1}$. No minimiser $Q(n)$ (as n is an odd number) can be centrally symmetric. There is no immediately obvious choice of a set of $2m + 1$ vectors which sum to zero to give $Q(n)$ as there is in the case of an even number. We saw that for an even number $2m$ with $n_k \leq 2m < n_{k+1}$, all edges are primitive vectors, all primitive vectors from $t_k B$ are used and no primitive vectors longer than t_{k+1} are used. In the present case it may be that none of these statements is necessarily true. Some of the edges that appear 'alone', that is, edges p for which $-p$ is not an edge may not be primitive vectors. It may be that not all of the n_k edges of $Q(n_k)$ are used and some edges may not belong to $t_{k+1}B$. Therefore we need to know an upper bound for the lengths of the edges used. We also need to estimate the number of edges of a minimiser $Q(n)$ that appear alone. We estimate these quantities in the next three lemmas.

Lemma 1.5.1. *The length of each edge of a minimiser $Q(n)$ is at most $2t_{k+1}$.*

Proof. Suppose there is one edge of $Q(n)$ longer than $2t_{k+1}$. Then, from the definition of the minimiser for an even number of vertices we have

$$\text{per}(2m + 2) = \text{per}(2m) + 2t_{k+1} < \text{per}(2m + 1).$$

So, $\text{per}(n + 1) < \text{per}(n)$. From Remark 1.4.1, this is impossible. □

Our aim is to find how many of these vectors do not come from $t_k B$. For this, we will first need an upper bound for $|J|$, the number of edges of a minimiser that appear alone.

Lemma 1.5.2. *The number of vectors not chosen in pairs is bounded above by $cn^{3/4}\sqrt{\log n}$, where c is a positive constant.*

Proof. Let $\{\pm u_i, i \in I\}$, $\{v_j, j \in J\}$ be the edges of a minimiser $Q(n)$. So there are $2|I|$ primitive vectors that appear in pairs and $|J|$ single integer vectors, all of length at most $2t_{k+1}$, which sum to zero, with $2|I| + |J| = n$. Consider the following sets of vectors:

$$U_1 = \{\pm u_i, i \in I\} \cup \{\pm v_j, j \in J\},\ U_2 = \{\pm u_i, i \in I\}.$$

The vectors in both sets sum to zero and the sum of their lengths is obviously $2\,\mathrm{per}(n)$. Using Remark 1.4.1 we have,

$$\begin{aligned} 2\,\mathrm{per}(n+1) \geq 2\,\mathrm{per}(n) &= \sum_{u \in U_1} \|u\| + \sum_{v \in U_2} \|v\| \\ &\geq \mathrm{per}(2(|I| + |J|)) + \mathrm{per}(2|I|) \\ &= \mathrm{per}(n + |J|) + \mathrm{per}(n - |J|). \end{aligned}$$

Thus we have

$$2\,\mathrm{per}(n+1) \geq \mathrm{per}(n - |J|) + \mathrm{per}(n + |J|). \tag{1.8}$$

Since in (1.8) the numbers $n+1, n-|J|, n+|J|$ are all even, we can use the expression (1.5) for the perimeter for an even number of vertices and obtain an upper bound for $|J|$. In (1.5), set $D = \frac{\pi}{\sqrt{6(\text{Area } B)^3}} \int_{x \in B} \|x\| dx$ and A for the constant implied by the O notation. We know that $|J| < n$, but we need a slight improvement of this. From (1.8), we have,

$$\begin{aligned} 2(n+1)^{3/2} D(1 + \frac{1}{D} E_1) &\geq \mathrm{per}(n - |J|) + \mathrm{per}(n + |J|) \\ &> \mathrm{per}(n + |J|) = (n + |J|)^{3/2} D(1 + \frac{1}{D} E_2), \end{aligned}$$

where E_1, E_2 are the error terms. For n sufficiently large and since $(5/3)^{3/2} > 2$, this gives,

$$\left(\frac{5}{3}n\right)^{3/2} D \geq (n + |J|)^{3/2} D$$

and hence $|J| < (5/3 - 1)n = \frac{2}{3}n$. Inequality (1.8) can be written

$$\begin{aligned} 2(n+1)^{3/2} \left\{ D + A \frac{\log(n+1)}{(n+1)^{1/2}} \right\} &\geq (n - |J|)^{3/2} \left\{ D - A \frac{\log(n - |J|)}{(n - |J|)^{1/2}} \right\} \\ &+ (n + |J|)^{3/2} \left\{ D - A \frac{\log(n + |J|)}{(n + |J|)^{1/2}} \right\} \end{aligned}$$

$$= \left\{(n - |J|)^{3/2} + (n + |J|)^{3/2}\right\} D$$
$$- A\{(n - |J|)\log(n - |J|) + (n + |J|)\log(n + |J|)\}. \qquad (1.9)$$

Using the fact that $0 < |J| < \frac{2}{3}n$ we get,

$$(n - |J|)\log(n - |J|) + (n + |J|)\log(n + |J|) < n \log n + \frac{5}{3}n \log \frac{5}{3}n$$
$$< 3n \log n.$$

Combining this with (1.9) we get,

$$2D\left(1 + \frac{1}{n}\right)^{3/2} + 2A\frac{\log n}{\sqrt{n}} \geq \left\{\left(1 - \frac{|J|}{n}\right)^{3/2} + \left(1 + \frac{|J|}{n}\right)^{3/2}\right\} D - 3A\frac{\log n}{\sqrt{n}}$$

and hence

$$\left\{\left(1 - \frac{|J|}{n}\right)^{3/2} + \left(1 + \frac{|J|}{n}\right)^{3/2}\right\} D \leq 2D\left(1 + \frac{1}{n}\right)^{3/2} + 5A\frac{\log n}{\sqrt{n}}.$$

Using the Taylor expansions of the functions $(1+x)^{3/2}$ and $(1-x)^{3/2}$, for $x \in (0,1)$ we get that

$$\left(1 - \frac{|J|}{n}\right)^{3/2} + \left(1 + \frac{|J|}{n}\right)^{3/2} \geq 2\left(1 + \frac{3}{8}\frac{|J|^2}{n^2}\right)$$

and $\left(1 + \frac{1}{n}\right)^{3/2} < 1 + \frac{2}{n} + \frac{1}{n^2} < 1 + \frac{3}{n}$. So

$$\left(1 + \frac{3}{8}\frac{|J|^2}{n^2}\right) D < D\left(1 + \frac{3}{n}\right) + \frac{5}{2}A\frac{\log n}{\sqrt{n}},$$

from which we finally get

$$|J| < cn^{3/4}\sqrt{\log n}. \qquad (1.10)$$

□

Write $E_n = \{\pm u_i,\ i \in I\} \cup \{v_j,\ j \in J\}$ for the set of edges of $Q(n)$ and $E_{n_{k+1}} = \mathbb{P} \cap t_{k+1}B$ for the set of edges of $Q(n_{k+1})$. In the next lemma we show that the size of the symmetric difference of E_n and $E_{n_{k+1}}$ is small.

Lemma 1.5.3. $|E_{n_{k+1}} \triangle E_n| \leq 2|J| + c'\sqrt{n_{k+1}} < 3cn^{3/4}\sqrt{\log n}.$

Proof. Write I as the disjoint union $I = I_{\text{ins}} \cup I_{\text{out}}$, where I_{ins} are the edges that appear in pairs and that come from inside $t_{k+1}B$ and I_{out} are the edges that appear in pairs and come from outside $t_{k+1}B$. We distinguish two cases.

Suppose $I_{\text{out}} \neq \varnothing$. This implies that for each pair of vectors $\{-p, p\}$ from t_{k+1}, either one of the vectors $-p$ and p or both, appear as edges of $Q(n)$. Indeed, if we suppose that neither $-p$ nor p has been used, then by replacing a pair $\pm p_i, i \in I_{\text{out}}$ with $\{-p, p\}$ we produce a polygon with smaller perimeter which is impossible. Thus only single vectors from inside $t_{k+1}B$ are missed. But since $t_{k+1}B$ is symmetric, the number of single vectors missed from $t_{k+1}B$ is equal to the number of single vectors used from $t_{k+1}B$. But the total number of single vectors that are used as edges of $Q(n)$ is $|J|$. So, at most $|J|$ vectors from $t_{k+1}B$ do not appear in E_n. Therefore $|E_{n_{k+1}} \backslash E_n| \leq |J|$. Moreover, this implies that at least $|n_{k+1} - |J||$ edges of $Q(n)$ come from inside $t_{k+1}B$ and therefore at most $|n - (n_{k+1} - |J|)|$ remain to be used from outside. So,

$$\begin{aligned} |E_n \backslash E_{n_{k+1}}| &\leq |n - (n_{k+1} - |J|)| \leq |n_{k+1} - n| + |J| < |n_{k+1} - n_k| + |J| \\ &\leq c' \sqrt{n_{k+1}} + |J|. \end{aligned}$$

Thus in this case,

$$|E_{n_{k+1}} \triangle E_n| \leq 2|J| + c' \sqrt{n_{k+1}} < 3cn^{3/4} \sqrt{\log n}.$$

Suppose $I_{\text{out}} = \varnothing$. Then no pair of edges of $Q(n)$ comes from outside $t_{k+1}B$. Therefore only the edges of $Q(n)$ that appear alone can come from outside $t_{k+1}B$, which implies that $|E_n \backslash E_{n_{k+1}}| \leq |J|$. Also, all $|I|$ pairs of edges of $Q(n)$ come from inside $t_{k+1}B$, so, at least $2|I|$ of the vectors of $t_{k+1}B$ appear as edges in $Q(n)$. Therefore at most $|n_{k+1} - 2|I||$ vectors are missed out, i.e.

$$\begin{aligned} |E_{n_{k+1}} \backslash E_n| &\leq |n_{k+1} - 2|I|| = |n_{k+1} - (n - |J|)| \leq |n_{k+1} - n_k| + |J| \\ &\leq c\sqrt{n_{k+1}} + |J|. \end{aligned}$$

Again, in this case,

$$|E_{n_{k+1}} \triangle E_n| \le 2|J| + c'\sqrt{n_{k+1}} < 3cn^{3/4}\sqrt{\log n}.$$

□

1.5.1 The perimeter of $Q(2m+1)$

We are finally in a position to compute the asymptotic behaviour of the perimeter for n odd. We have that $|E_{n_{k+1}} \triangle E_n| < 3cn^{3/4}\sqrt{\log n}$ and the length of each edge of a minimiser $Q(n)$ cannot exceed $2t_{k+1}$. So, for the perimeter of $Q(n)$ we have,

$$|\text{per}(n_{k+1}) - \text{per}(n)| \le \sum_{p \in E_{n_{k+1}} \triangle E_n} \|p\| \le |E_{n_{k+1}} \triangle E_n| 2t_{k+1} \le 6c'n^{5/4}\sqrt{\log n}.$$

Therefore,

$$\left| \text{per}(n) - n^{3/2} \frac{\pi}{\sqrt{6(\text{Area}\, B)^3}} \int_{x \in B} \|x\| dx \right| \le c'' n^{5/4} \sqrt{\log n},$$

and so,

$$\text{per}(n) = n^{3/2} \left\{ \frac{\pi}{\sqrt{6(\text{Area}\, B)^3}} \int_{x \in B} \|x\| dx + \mathrm{O}\left(\frac{\sqrt{\log n}}{n^{1/4}} \right) \right\}.$$

□

1.5.2 The support function of $Q(2m+1)$

Since a minimiser $Q(n)$ has an odd number of vertices, it is not centrally symmetric. We shall show however that asymptotically there is a natural translate of each minimiser, whose centre of mass is close to the origin and that these translates converge to the same (0-symmetric) limit shape as the minimisers for even n. Indeed, $Q(n)$ may be taken to be the Minkowski sum of the 0-symmetric zonotope $Z_1 = \sum_{i \in I} \pm u_i$ and the polygon $P_2 = \sum_{j \in J} q_j$. If we choose P_2 so that the edges $\{v_j, j = 1, \dots |J|\}$ are numbered in increasing order according to their slopes and place v_1 at the origin, then it is easy to see that P_2 is contained in a ball of diameter at most $|J| \cdot \sqrt{n} \leq cn^{5/4}\sqrt{\log n}$. Once we apply the appropriate normalisation $1/n^{3/2}$, what we get for the centre of gravity x_0 of a minimiser $Q(n)$ is $\|x_0\| \leq \frac{\sqrt{\log n}}{n^{1/4}}$. Therefore, for the support function of a minimiser $Q(n)$ for the case n odd we have,

$$\left| h_{Q(n)}(u) - n^{3/2}\frac{\pi}{4\sqrt{6(\operatorname{Area} B)^3}}\int_{x\in B}|x\cdot u|dx \right| \leq c'n\log n + c''n^{5/4}\sqrt{\log n}$$
$$\leq cn^{5/4}\sqrt{\log n}.$$

Therefore, for $n = 2m+1$ odd

$$h_{Q(n)}(u) = n^{3/2}\left\{\frac{\pi}{4\sqrt{6(\operatorname{Area} B)^3}}\int_{x\in B}|x\cdot u|dx + \mathrm{O}\left(\frac{\sqrt{\log n}}{n^{1/4}}\right)\right\}. \tag{1.11}$$

□

1.6 The limit shape Theorem

We can now prove Theorems 1.1.1 and 1.1.2. All minimisers, after the appropriate normalisation, converge to a fixed convex body, as the number of vertices n tends to ∞. We prove only Theorem 1.1.1; Theorem 1.1.2 can be shown similarly.

Proof of Theorem 1.1.1. From the expressions 1.7 and 1.11, for any $n \in \mathbb{N}$ we have,

$$\left| \frac{h_{Q(n)}(u)}{n^{3/2}} - \frac{\pi}{4\sqrt{6(\operatorname{Area} B)^3}}\int_{x\in B}|x\cdot u|dx \right| \leq c'\frac{\sqrt{\log n}}{n^{1/4}}.$$

Therefore,

$$\lim_{n\to\infty} n^{-3/2}h_{Q(n)}(u) = \frac{1}{4}\frac{\pi}{\sqrt{6(\operatorname{Area} B)^3}}\int_{x\in B}|x\cdot u|dx.$$

□

1.7 Proof of Lemma 1.2.1

In this final section we give a proof of the crucial Lemma 1.2.1 which we used throughout the preceding sections. We will use the Möbius function

$$\mu(d) = \begin{cases} 1, & \text{if } d = 1 \\ 0, & \text{if } p^2 | d, \text{ for some prime } p \\ (-1)^k, & \text{if } d = p_1 p_2 \cdots p_k, \text{ where } p_i\text{'s are distinct primes} \end{cases}$$

We mention here two equivalent forms of the Möbius inversion formula, which will be used in the proof of Lemma 1.2.1 below. From the second form we will only need the $s = 2$ case. For details see [HW79] or [Ap76].

$$\sum_{d|k} \mu(d) = \begin{cases} 1, & k = 1 \\ 0, & \text{otherwise} \end{cases}, \tag{1.12}$$

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)}, \ (s > 1) \tag{1.13}$$

where $\zeta(\cdot)$ denotes the Riemann zeta function, $\zeta(s) = \sum_{d=1}^{\infty} \frac{1}{d^s}$.

Proof. Let us be reminded that our aim is to prove that, the sum of the values of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, over all primitive vectors in a plane convex body K can be approximated by the integral of f over K times the density of $\mathbb{P}$ in $\mathbb{Z}^2$, provided that f is bounded on K and it does not vary much over a unit square that intersects K.

We may assume that K is in *standard position.* This means that the lattice width of K, $w(K) = \mathrm{w}$, is obtained for $w = (0, 1)$ in (1.2). Write $[-v/2, v/2]$ for the intersection of the x-axis with K. Once we have fixed K so that the lattice width occurs in the direction $(0, 1)$ we may further assume that after a suitable shear, the tangent to K at the point $(v/2, 0)$ has slope between 1 and ∞. Using the fact that the width of K in the directions $(1, 1)$ and $(1, 0)$ is at least w, we have that $2v \geq \mathrm{w}$. Denote by $\mathrm{P}(K)$ and $\mathrm{Area}(K)$ the perimeter and area of K

respectively. Under these assumptions, we get

$$\frac{v\,\mathrm{w}}{2} \leq \mathrm{Area}(K) \leq v\,\mathrm{w} \tag{1.14}$$

and

$$\mathrm{P}(K) \leq 2v + 4\,\mathrm{w} \leq 10v \leq 10\frac{\mathrm{Area}(K)}{\mathrm{w}}. \tag{1.15}$$

The method is standard: we use the Möbius inversion formula (1.12) and re-write $\sum_{p\in\mathbb{P}\cap K} f(p)$ as follows.

$$\begin{aligned}\sum_{p\in\mathbb{P}\cap K} f(p) &= \sum_{z=(k,l)\in\mathbb{Z}^2\cap K} f(z) \sum_{d|k,d|l} \mu(d) = \sum_{d=1}^{\infty} \mu(d) \sum_{\substack{z=(k,l)\in\mathbb{Z}^2\cap K\\ d|k,d|l}} f(z)\\ &= \sum_{d=1}^{\lfloor \mathrm{w}\rfloor} \mu(d) \sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K} f(dw),\end{aligned}$$

which gives,

$$\sum_{p\in\mathbb{P}\cap K} f(p) = \sum_{d=1}^{\lfloor \mathrm{w}\rfloor} \mu(d) d^{\alpha} \sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K} f(w), \tag{1.16}$$

by the homogeneity of f. As the sum is now over all lattice points in $\mathbb{Z}^2 \cap \frac{1}{d}K$, we shall approximate it with the integral over $\frac{1}{d}K$. In order to do so, let us write $Q(w)$ for a unit square intersecting $\frac{1}{d}K$ and centred at $w \in \mathbb{Z}^2$. We call w *inside* and write w_{ins}, if $Q(w_{ins})$ lies entirely in the interior $\mathrm{int}(\frac{1}{d}K)$ of $\frac{1}{d}K$, *boundary* and write w_{bd}, if $w_{bd} \in \mathrm{int}(\frac{1}{d}K)$ but $Q(w_{bd}) \cap \frac{1}{d}K^c \neq \varnothing$ and *outside*, w_{out}, if $w_{out} \notin \frac{1}{d}K$ but $Q(w_{out}) \cap \frac{1}{d}K \neq \varnothing$. In the next claim we compare the second sum on the right hand side of (1.16) with the integral of f over $\frac{1}{d}K$.

Claim 1.7.1.

$$\left|\sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K} f(w) - \frac{1}{d^{\alpha+2}}\int_{x\in K} f(x)dx\right| \leq V\frac{1}{d^{\alpha+2}}\,\mathrm{Area}(K) + (c+V)\frac{20v}{d^{\alpha+1}}.$$

Proof. Clearly, the number of lattice points in $\frac{1}{d}K$ is the number of w_{ins} and w_{bd}.

Using this, we can write for the integral of f over $\frac{1}{d}K$,

$$\begin{aligned}\int_{x\in\frac{1}{d}K} f(x)dx &= \sum_{w=w_{ins}}\int_{Q(w)} f(x)dx + \sum_{w=w_{bd}}\int_{Q(w)\cap\frac{1}{d}K} f(x)dx \\ &\quad + \sum_{w=w_{out}}\int_{Q(w)\cap\frac{1}{d}K} f(x)dx \\ &= \sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K}\int_{Q(w)} f(x)dx - \sum_{w=w_{bd}}\int_{Q(w)\cap\frac{1}{d}K^c} f(x)dx \\ &\quad + \sum_{w=w_{out}}\int_{Q(w)\cap\frac{1}{d}K} f(x)dx \qquad (1.17)\end{aligned}$$

For the first sum in (1.17) we have,

$$\left|\int_{Q(w)} f(x)dx - f(w)\right| \leq \int_{Q(w)} |f(x)-f(w)|dx \leq V,$$

from which we get,

$$\left|\sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K} f(w) - \sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K}\int_{Q(w)} f(x)dx\right| \leq V|\mathbb{Z}^2\cap\frac{1}{d}K| \qquad (1.18)$$

Now we need a bound on $|\mathbb{Z}^2\cap\frac{1}{d}K|$. This can be obtained as follows,

$$|\mathbb{Z}^2\cap\frac{1}{d}K| = \sum_{w\in w_{ins}\cup w_{bd}} 1 = \sum_{w\in w_{ins}\cup w_{bd}} \text{Area}(Q(w)) \leq \text{Area}\left(\frac{1}{d}K\right) + \frac{10v}{d},$$

where the last term comes from (1.15) and is the perimeter of the smallest box that contains $\frac{1}{d}K$. Hence,

$$|\mathbb{Z}^2\cap\frac{1}{d}K| \leq \frac{1}{d^2}\text{Area}(K) + \frac{10v}{d}. \qquad (1.19)$$

The second and third sums in (1.17) are at most $c\dfrac{20v}{d}$. Therefore, if we combine (1.17), (1.18) and (1.19) we get,

$$\left|\sum_{w\in\mathbb{Z}^2\cap\frac{1}{d}K} f(w) - \frac{1}{d^{\alpha+2}}\int_{x\in K} f(x)dx\right| \leq V\frac{1}{d^{\alpha+2}}\text{Area}(K) + (c+V)\frac{20v}{d^{\alpha+1}},$$

where we applied change of variables and the homogeneity of f. This completes the proof of the Claim. □

Combining Claim 1.7.1 with (1.16) we get,

$$\left|\sum_{p\in\mathbb{P}\cap K} f(p) - \sum_{d=1}^{\lfloor w\rfloor}\frac{\mu(d)}{d^2}\int_{x\in K} f(x)dx\right| \leq \sum_{d=1}^{\lfloor w\rfloor}\frac{\mu(d)}{d^2}V\,\text{Area}(K) + \sum_{d=1}^{\lfloor w\rfloor}\frac{\mu(d)}{d}(c+V)20v,$$

which yields,

$$\left|\sum_{p\in\mathbb{P}\cap K} f(p) - \sum_{d=1}^{\lfloor w\rfloor}\frac{\mu(d)}{d^2}\int_{K} f(x)dx\right| \leq V\,\text{Area}(K)\sum_{d=1}^{\lfloor w\rfloor}\frac{1}{d^2} + (c+V)20v\sum_{d=1}^{\lfloor w\rfloor}\frac{1}{d}, \tag{1.20}$$

where we used the fact that $|\mu(d)| \leq 1$. Now using the inequalities

$$\sum_{d=1}^{\lfloor w\rfloor}\frac{1}{d} \leq 1 + \log\lfloor w\rfloor \leq 1 + \log w,$$

and

$$\left|\sum_{d=1}^{\lfloor w\rfloor}\frac{\mu(d)}{d^2} - \frac{6}{\pi^2}\right| = \left|\sum_{d=\lfloor w\rfloor+1}^{\infty}\frac{\mu(d)}{d^2}\right| \leq \sum_{d=\lfloor w\rfloor+1}^{\infty}\frac{|\mu(d)|}{d^2} \leq \sum_{d=\lfloor w\rfloor+1}^{\infty}\frac{1}{d^2} < \frac{1}{\lfloor w\rfloor} < \frac{2}{w},$$

we get from (1.20)

$$\begin{aligned}\left|\sum_{p\in\mathbb{P}\cap K} f(p) - \frac{6}{\pi^2}\int_K f(x)dx\right| &\leq \frac{2}{w}\int_K f(x)dx + \frac{6}{\pi^2}V\,\text{Area}(K)\\ &\quad +20v(c+V)(1+\log w)\\ &\leq \frac{2c}{w}\,\text{Area}(K) + \frac{6}{\pi^2}V\,\text{Area}(K)\\ &\quad +20v(c+V)(1+\log w).\end{aligned}$$

From this, using (1.15) we get

$$\begin{aligned}\left|\sum_{p\in\mathbb{P}\cap K} f(p) - \frac{6}{\pi^2}\int_K f(x)dx\right| &\leq \text{Area}(K)\frac{2c}{w} + \frac{6}{\pi^2}V\,\text{Area}(K)\\ &\quad + \text{Area}(K)\frac{40(c+V)(1+\log w)}{w}.\end{aligned}$$

Now using the fact that $w > 3$ we get (finally!)

$$\left|\sum_{p\in\mathbb{P}\cap K} f(p) - \frac{6}{\pi^2}\int_K f(x)dx\right| \leq \text{Area}(K)\left(\frac{2}{3}V + \frac{42(c+V)\log w}{w}\right).$$

which is what we were supposed to prove. □

Remark 1.7.2.

It would be interesting to see whether the result of this chapter holds when B is not the unit ball of a norm. Suppose $B \subset \mathbb{R}^2$ is a convex body, with centre of gravity at the origin, but not necessarily 0-symmetric. Is there a limit shape for the convex lattice polygons with minimal perimeter with respect to their vertices, when the perimeter is taken with respect to B?

Chapter 2

On Maximal Convex Lattice Polygons Inscribed in a Plane Convex Set

2.1 Introduction and results

Assume $K \subset \mathbb{R}^2$ is a fixed convex body, that is, a compact, convex set with non-empty interior. Let $\mathbb{Z}^2$ denote the (usual) lattice of integer points and write $\mathbb{Z}_t = \frac{1}{t}\mathbb{Z}^2$: a shrunken copy of $\mathbb{Z}^2$, when t is large. A *convex* $\mathbb{Z}_t$ *lattice* n*-gon* is, by definition, a convex polygon with exactly n vertices each belonging to the lattice $\mathbb{Z}_t$. Define

$$m(K, \mathbb{Z}_t) = \max\{n : \text{ there is a convex } \mathbb{Z}_t \text{ lattice } n\text{-gon contained in } K\}.$$

In this chapter, we determine the asymptotic behaviour of $m(K, \mathbb{Z}_t)$, as $t \to \infty$. Let $\mathrm{A}(K)$ denote the supremum of the affine perimeters of all convex sets $S \subset K$. (Section 2.2 is devoted to the affine perimeter and its properties.) We now state the main result of this chapter.

Theorem 2.1.1. *Under the above conditions*

$$\lim_{t\to\infty} t^{-2/3} m(K, \mathbb{Z}_t) = \frac{3}{(2\pi)^{2/3}} \mathrm{A}(K).$$

Let $\mathrm{AP}(S)$ denote the affine perimeter of a convex set $S \subset \mathbb{R}^2$. It is shown in [Bá97] (see also Theorem 2.3.4 below) that there is a unique $K_0 \subset K$ with $\mathrm{AP}(K_0) = \mathrm{A}(K)$. This unique K_0 has the interesting "limit shape" property (see [Bá97]) that the overwhelming majority of the convex $\mathbb{Z}_t$ lattice polygons contained in K are very close to K_0, in the Hausdorff metric. This property applies to the case of maximal convex lattice polygons as well. Let $\mathrm{dist}(\cdot,\cdot)$ denote the Hausdorff distance.

Theorem 2.1.2. *For any maximiser Q_t in the definition of $m(K, \mathbb{Z}_t)$,*

$$\lim_{t\to\infty} \mathrm{dist}(Q_t, K_0) = 0.$$

The problem of estimating $m(K, \mathbb{Z}_t)$ has a long history. Jarník proved in [Ja25] that a strictly convex curve of length ℓ in the plane contains at most

$$\frac{3}{\sqrt[3]{2\pi}} \cdot \ell^{2/3}(1 + o(1))$$

lattice points and that this estimate is best possible. When the strictly convex curve is the circle of radius r, Jarník's estimate gives that a convex polygon contained in this circle has at most $3\sqrt[3]{2\pi} r^{2/3}(1 + o(1))$ vertices. The same bound follows from Theorem 2.1.1 as well.

Andrews [An63] showed that a convex lattice polygon P has at most $c(\mathrm{Area}\, P)^{1/3}$ vertices where $c > 0$ is a universal constant. The smallest known value of c is $(8\pi^2)^{1/3} < 5$ which follows from an inequality of Rényi and Sulanke [RS63] (see [Ra93]), but we will not be needing this fact. We will use Andrews' estimate when dealing with degenerate triangles T. In the $K, \mathbb{Z}_t$ setting, this implies that

$$m(K, \mathbb{Z}_t) \leq 20 t^{2/3} (\mathrm{Area}\, T)^{1/3}. \tag{2.1}$$

Remark 2.1.3.

The lattice points on the curve giving the extremum, form a convex lattice polygon, which is called Jarník's polygon. It is clear that its edges are "short" primitive vectors. This phenomenon will reappear in the proofs of Theorems 2.1.1 and 2.2.1.

Remark 2.1.4.

Actually, Andrews [An63] proved much more: namely, that a convex lattice polytope $P \subset \mathbb{R}^d$ with non-empty interior can have at most $c(\operatorname{vol} P)^{(d-1)/(d+1)}$ vertices where the constant $c > 0$ depends on dimension only.

2.2 Affine perimeter

In this section we collect some facts concerning the affine perimeter that will be used in the proofs.

Let $\mathcal{C}$ denote the set of convex bodies in $\mathbb{R}^2$, that is, compact convex sets with non-empty interior. Given $S \in \mathcal{C}$, choose a subdivision $x_1, \ldots, x_n, x_{n+1} = x_1$ of the boundary ∂S and lines ℓ_i, $i = 1, \ldots, n$ supporting S at x_i. Denote by y_i the intersection of ℓ_i and ℓ_{i+1} and by T_i the triangle $\operatorname{conv}\{x_i, y_i, x_{i+1}\}$ (and also its area). The affine perimeter $\operatorname{AP}(S)$ of $S \in \mathcal{C}$ is defined as

$$\operatorname{AP}(S) = 2 \lim \sum_{i=1}^{n} \sqrt[3]{T_i},$$

where the limit is taken over a sequence of subdivisions with $\max_{1\ldots,n} |x_{i+1} - x_i| \to 0$. The existence of the limit and its independence of the sequence chosen, follow from the fact, implied by the inequality in (2.4) below, that $\sum_{i=1}^{n} \sqrt[3]{T_i}$ decreases as the subdivision is refined. Therefore, the affine perimeter is the infimum,

$$\operatorname{AP}(S) = 2 \inf \sum_{i=1}^{n} \sqrt[3]{T_i}.$$

It is easy to see that (see also the property in (2.2.1) below) the affine perimeter is invariant under area preserving affine transformations. Note also that, by the definition, $\mathrm{AP}(P) = 0$, when P is a polygon.

The same definition applies for a compact convex curve Γ: a subdivision $x_1, \dots, x_{n+1}$ on Γ, together with the supporting lines at x_i, defines the triangles $T_1, \dots, T_n$, and $\mathrm{AP}(\Gamma)$ is the infimum of $2\sum_{i=1}^n \sqrt[3]{T_i}$.

Alternatively, given unit vectors $d_1, \dots, d_{n+1}$ (in clockwise order on the unit circle), there is a subdivision $x_1, \dots, x_{n+1}$ on Γ with tangent line ℓ_i at x_i which is orthogonal to d_i. The subdivision defines triangles $T_1, \dots, T_n$, and

$$\mathrm{AP}(\Gamma) = 2 \inf \sum_{i=1}^{n} \sqrt[3]{T_i}$$

where now the infimum is taken over all n and all choices of unit vectors $d_1, \dots, d_{n+1}$. Note that the triangles T_i are determined by Γ and $d_1, \dots, d_{n+1}$ uniquely (unless d_i is orthogonal to a segment contained in Γ in which case we can take the midpoint of this segment for x_i). We will call them the triangles induced by directions $d_1, \dots, d_{n+1}$ on Γ.

2.2.1 Properties of the map $\mathrm{AP} : \mathcal{C} \to \mathbb{R}$

We mention here some properties of the map $\mathrm{AP} : \mathcal{C} \to \mathbb{R}$ that will be used throughout the chapter.

(2.2.1) $\mathrm{AP}(LS) = (\det L)^{1/3}\, \mathrm{AP}(S)$, for $L : \mathbb{R}^2 \to \mathbb{R}^2$ linear.

(2.2.2) If the boundary of S is twice differentiable, then

$$\mathrm{AP}(S) = \int_{\partial S} \kappa^{1/3} ds = \int_0^{2\pi} r^{2/3} d\phi,$$

where κ is the curvature and r the radius of curvature at the boundary point with outer normal vector $u(\phi) = (\cos\phi, \sin\phi)$.

(2.2.3) Given a triangle $T = \mathrm{conv}\{p_0, p_1, p_2\}$, let $D = D(T)$ be the unique parabola which is tangent to p_0p_1 and p_1p_2 at p_0 and p_2 respectively. Among all convex curves connecting p_0 and p_2 within the triangle T, the arc of the parabola D is the unique one with maximal affine length, and $\mathrm{AP}(D) = 2\sqrt[3]{T}$. We call D the special parabola in T.

(2.2.4) Let T be the triangle as in (2.2.3) and let q_1, q_2 be points on the sides p_0p_1 and p_1p_2 respectively. Let p_3 be a point on q_1q_2 and write T_1 and T_2 for the triangles $\mathrm{conv}\{p_0, q_1, p_3\}$ and $\mathrm{conv}\{p_3, q_2, p_2\}$ respectively. Then we have, (see Figure 2.1)

$$\sqrt[3]{T} \geq \sqrt[3]{T_1} + \sqrt[3]{T_2}.$$

Moreover, equality holds if, and only if, q_1q_2 is tangent to the parabola D at the point p_3 (see [Bl23]).

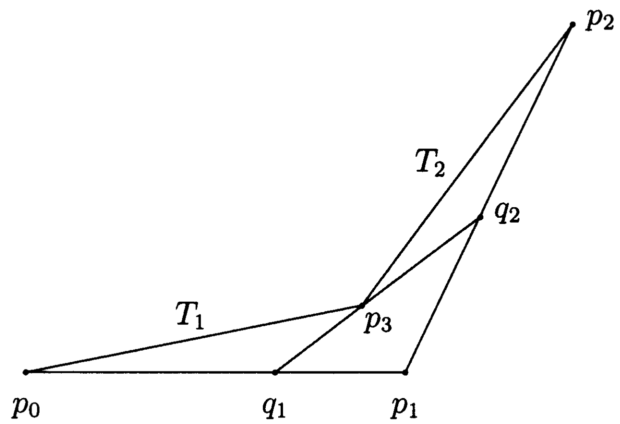

Figure 2.1: $\sqrt[3]{T} \geq \sqrt[3]{T_1} + \sqrt[3]{T_2}$

It is clear from the definition of the affine perimeter that, for a polygon K, $\mathrm{AP}(K) = 0$. This shows further that the map $\mathrm{AP} : \mathcal{C} \to R$ is not continuous ($\mathcal{C}$ is equipped with the Hausdorff metric). It is known however, that it is upper semi-continuous (see for instance [Lu91]).

The following theorem will be used for the proof of the main theorems. It is similar, in spirit, to a result of Vershik [Ve94]. Assume Γ is a compact convex curve in the plane. For $\varepsilon > 0$ we denote by $U_\varepsilon(\Gamma)$ the ε-neighbourhood of Γ.

Let $m(\Gamma, \varepsilon, \mathbb{Z}_t)$ denote the maximum number of vertices that a convex $\mathbb{Z}_t$ lattice curve lying in $U_\varepsilon(\Gamma)$ can have.

Theorem 2.2.1. *Under the above conditions*

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} t^{-2/3} m(\Gamma, \varepsilon, \mathbb{Z}_t) = \frac{3}{(2\pi)^{2/3}}\,\mathrm{AP}(\Gamma).$$

For the proof of Theorem 2.2.1 we will need the following fact which is a consequence of the upper semi-continuity of the affine perimeter.

Proposition 2.2.2. *For every compact convex curve Γ and for every $\eta > 0$ there exist $\varepsilon > 0$, an integer n, and unit vectors $d_1, \ldots, d_{n+1}$ such that for every compact convex curve $\Gamma' \subset U_\varepsilon(\Gamma)$ the triangles $T_1, \ldots, T_n$ induced by $d_1, \ldots, d_{n+1}$ on Γ' satisfy*

$$2\sum_{i=1}^{n} \sqrt[3]{T_i} \le \mathrm{AP}(\Gamma) + \eta.$$

Proof. Let Γ be a compact, convex curve and $\eta > 0$. Suppose the assertion is false. Then for every $\varepsilon > 0$, there is $\Gamma' \subset U_\varepsilon(\Gamma)$, such that $2\sum_{i=1}^{n} \sqrt[3]{T_i} > \mathrm{AP}(\Gamma)$. As this is true for any choice of unit vectors $d_1, \ldots, d_{n+1}$ and any n, we have that $\mathrm{AP}(\Gamma') > \mathrm{AP}(\Gamma) + \eta$, for any $\eta > 0$. This contradicts the upper semicontinuity of the functional AP. □

2.3 Maximal affine perimeter

In this section, we shall be interested in the subset of a convex body K, with maximal affine perimeter. Given $K \in \mathcal{C}$, let $\mathcal{C}(K)$ denote the set of all convex bodies contained in K, that is, $\mathcal{C}(K) = \{S \in \mathcal{C} : \ S \subset K\}$. Define the map $\mathrm{A} : \mathcal{C} \to \mathbb{R}$ by

$$\mathrm{A}(K) = \sup\{\mathrm{AP}(S),\ S \in \mathcal{C}(K)\}.$$

The following result comes from [Bá97].

Theorem 2.3.1. *For every $K \in \mathcal{C}$ there exists a unique $K_0 \in \mathcal{C}(K)$ such that $\mathrm{AP}(K_0) = \mathrm{A}(K)$.*

Proposition 2.3.2. *The function* $\mathrm{A} : \mathcal{C} \to \mathbb{R}$ *is continuous.*

We omit the simple proof.

Theorem 2.3.1 shows that there is a mapping $F : \mathcal{C} \to \mathcal{C}$, given by

$$F(K) = K_0.$$

The map F is affinely equivariant, that is, for a nondegenerate affine map $L : \mathbb{R}^2 \to \mathbb{R}^2$, we have that $F(LK) = LF(K)$.

Proposition 2.3.3. *The mapping* $F : \mathcal{C} \to \mathcal{C}$ *is continuous.*

Proof. Let $K_n, K \in \mathcal{C}$, such that $K_n \to K$. Choose a convergent subsequence of $(F(K_n))$ and let us denote by K^* its limit. From the uniqueness of K_0, since K^* is contained in K, it suffices to show that $\mathrm{AP}(K^*) = \mathrm{A}(K)$. For this, by the definitions of F and A, it is enough to show that $\mathrm{AP}(K^*) \geq \mathrm{AP}(F(K))$. Using the facts that AP is upper semi-continuous and A is continuous we get,

$$\mathrm{AP}(K^*) \geq \limsup \mathrm{AP}(F(K_n)) = \lim \mathrm{A}(K_n) = \mathrm{AP}(F(K)).$$

$\square$

2.3.1 Properties of K_0

The unique $F(K) = K_0$ has interesting properties. Clearly, $\partial K_0 \cap \partial K \neq \varnothing$, as otherwise a slightly enlarged copy of K_0 would be contained in K and have larger affine perimeter. Since $\partial K_0 \cap \partial K$ is closed, $\partial K_0 \backslash \partial K$ is the union of countably many arcs, called *free arcs.*

(2.3.1) Each free arc is an arc of a parabola whose tangents at the end points are tangent to K as well.

(2.3.2) The boundary of K_0 contains no line segment.

The last statement is made quantitative in [Bá99]. Assume that Area$(K) = 1$. Assume further that the ellipsoid of maximal area, E_0, inscribed in K_0 is a circle. This can be arranged by using a suitable area preserving affine transformation.

(2.3.3) Under these conditions the radius of curvature at each point on the boundary of K_0 is at most 240.

From the proofs of our main Theorems 2.1.1 and 2.1.2 we get a characterisation of K_0. For $C \in \mathcal{C}$, the *barycentre* (or *centre of gravity*) of C is defined by

$$b(C) = \frac{1}{\operatorname{Area} C} \int_{x \in C} x dx.$$

Define $\mathcal{C}_0$ as the collection of all $C \in \mathcal{C}$ with $b(C) = 0$. Fix $C \in \mathcal{C}_0$ and let $u \in S^1$ be a unit vector. The *radial function*, $\rho(u) = \rho_C(u)$ is, as usual, defined as

$$\rho_C(u) = \max\{t > 0 : tu \in C\}.$$

The condition $\int_C x dx = 0$ can be rewritten

$$\int_{S^1} \rho(u)^3 du = 0.$$

(Here du denotes vector integration on S^1.) By Minkowski's classical theorem (see [Sch93]), there is a unique (up to translation) convex body $C^* = G(C)$ whose radius of curvature at the boundary point with outer normal vector u, is exactly $R(u) = \frac{1}{3}\rho^3(u)$. The following characterisation theorem describes the sets $F(K)$ when $K \in \mathcal{C}$.

Theorem 2.3.4. *For each $K \in \mathcal{C}$, there is a unique $C \in \mathcal{C}_0$, such that K_0 is a translated copy of $G(C) = C^*$. Moreover, for every $C \in \mathcal{C}_0$ the set $G(C) = C^* \in \mathcal{C}$ satisfies $F(C^*) = C^*$.*

This theorem immediately implies the following result.

Corollary 2.3.5. *Assume $K \in \mathcal{C}$. Then $F(K) = K$ holds if, and only if, K has well-defined and continuous radius of curvature $R(u)$ (for each $u \in S^1$) and $\sqrt[3]{3R(u)}$ is the radial function of a convex set $C \in \mathcal{C}_0$.*

We say that two sets $K_1, K_2 \in \mathcal{C}$ are equivalent, if they are translates of each other. Write $\mathcal{K}$ for the set of equivalence classes in $\{F(K) : K \in \mathcal{C}\}$. The two theorems above show that the map $G : \mathcal{C}_0 \to \mathcal{K}$ is one-to-one. It can be shown that the map $G : \mathcal{C}_0 \to \mathcal{K}$ is continuous in both directions but we will not need this fact here.

Theorem 2.3.4 implies the following strengthening of (2.3.3).

Corollary 2.3.6. *For any $K \in \mathcal{C}$ there is a non-degenerate linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that the radius of curvature $R(u)$ of $F(L(K)) = (L(K))_0$ at any point of its boundary satisfies*

$$\frac{1}{3} \leq R(u) \leq \frac{8}{3}.$$

Remark 2.3.7.

Theorem 2.3.4 and Corollary 2.3.5 may extend to higher dimensions. Unfortunately, the uniqueness of the maximal affine surface area convex set contained in a fixed convex body in $\mathbb{R}^d$ for $d > 2$ is not known.

2.4 "Large" and "small" triangles

The key step in the proof of our theorems is a result about large triangles. Though the proofs may appear to be rather technical, the idea behind them is simple. Let us give here an informal description.

We are interested in the maximal convex $\mathbb{Z}_t$ lattice polygons inscribed in a convex body K, when t is large. This is the same as considering the maximal $\mathbb{Z}^2$ lattice polygons inscribed in the blown up copy tK of K. Theorems 2.1.1 and 2.1.2 show that any such maximiser is very close to the subset K_0 of K with maximal affine perimeter. As we saw earlier, the boundary of this body K_0 is the union of countably many parabolic arcs, whose tangents at the end points are tangent to K as well. These tangent lines to K (and K_0) will define our "large" triangles. We will be interested in finding the set of vectors that will build up the arc of Q_t within each such triangle T. We shall prove that each large triangle, naturally gives rise to a "small" triangle, Δ, so that the edges of the arc of a maximiser Q_t within T are the primitive vectors in Δ. These connections will become clearer in the next two subsections.

2.4.1 Large triangles

We start with a definition which is slightly more general than necessary.

Definition 2.4.1.

Let $T = \text{conv}\{p_0, p_1, p_2\}$ be a (non-degenerate) triangle in $\mathbb{R}^2$. A *convex lattice chain* within T (from the side $[p_0, p_1]$ to the side $[p_1, p_2]$) is a sequence of points $x_0, \dots, x_n$ such that

(i) the points $p_0, x_0, \dots, x_n, p_2$ are in convex position

(ii) $z_i = x_i - x_{i-1} \in \mathbb{Z}^2$, for each $i = 1, \dots, n$.

The *length* of this convex lattice chain is n. Define $m(T)$ as the *maximal length* that a convex lattice chain within T can have. For simplicity we denote the area of T by the same letter T.

Assume now that $a, b \in \mathbb{R}^2$ are two non-parallel vectors and t_1, t_2 are almost equal and large. Setting $p_1 - p_0 = t_1 a$ and $p_2 - p_1 = t_2 b$ gives the "large" triangle $T = \text{conv}\{p_0, p_1, p_2\}$.

Theorem 2.4.2. *Assume $t_1, t_2 \to \infty$ with $t_1/t_2 \to 1$. Then*

$$\lim m(T) \cdot T^{-1/3} = \frac{6}{(2\pi)^{2/3}}.$$

Clearly it suffices to show this when $t_1 = t_2 = t$ and $t \to \infty$. This will be done in Section 2.5.

We shall need this result in the $\mathbb{Z}_t$ setting as well. So, given a triangle T in the plane, we define $m^*(T, \mathbb{Z}_t)$ as the length of a maximal $\mathbb{Z}_t$ lattice chain from vertex p_0 to vertex p_2 within T. The previous theorem states that

$$\lim_{t\to\infty} t^{-2/3} m^*(T, \mathbb{Z}_t) = \frac{6\sqrt[3]{T}}{(2\pi)^{2/3}}.$$

Now let Q_t be a maximal $\mathbb{Z}_t$ lattice chain in T (from p_0 to p_2). The next theorem relates Q_t to the special parabola $D(T)$ defined in (2.2.3).

Theorem 2.4.3. *Under the above conditions*

$$\lim_{t\to\infty} \operatorname{dist}(Q_t, D(T)) = 0$$

The proof of this result which is given in Section 2.6 shows the close connection between maximal convex lattice chains and the inequality discussed in (2.2.4).

From the proof of Theorem 2.4.2 we will be able to give a simple construction of a convex $\mathbb{Z}_t$ lattice curve in the triangle T which is almost maximal and is very close to the parabolic arc $D(T)$. This construction will be used in the characterisation Theorem 2.3.4.

Remark 2.4.4.

It would be interesting to understand the behaviour of $m(T)$, for general triangles T, whose areas tend to ∞. Write $w(T)$ for the lattice width of the triangle T. If $w \in \mathbb{Z}^2$ is the direction in which the lattice width of T is attained, then the lattice points belonging to any translated copy of T are contained in $\lceil w(T) \rceil$ consecutive lattice lines. Each such line contains at most two vertices from a convex lattice chain. Thus,

$$m(T) \leq 2\lceil w(T) \rceil < 2w(T) + 2.$$

Hence, if $w(T)$ is much smaller than $T^{1/3}$, the asymptotic estimate

$$m(T) \approx \frac{6}{(2\pi)^{2/3}} \sqrt[3]{T}$$

of Theorem 2.4.2 does not hold.

2.4.2 Small triangles

Assume now that $u, v \in \mathbb{R}^2$ are non-parallel vectors. Define the triangle Δ as

$$\Delta = \text{conv}\{0, u, v\}.$$

Its area will also be denoted by Δ and its lattice width by $w(\Delta)$. Again we write $\mathbb{P}$ for the set of primitive vectors in $\mathbb{Z}^2$.

We will need the size of $\mathbb{P} \cap \Delta$ which, as estimated in Lemma 1.2.1 (or Lemma 1.2.2) is

$$\left| |\mathbb{P} \cap \Delta| - \frac{6}{\pi^2}\Delta \right| \leq 30\Delta \frac{\log w(\Delta)}{w(\Delta)}. \tag{2.2}$$

Let T be the "large" triangle of the previous subsection. In our application, $u = \lambda a$, and $v = \lambda b$ with $\lambda \approx t^{1/3}$. Thus $w(\Delta)$ is of order $t^{1/3}$ which is large and the triangle Δ is "small" compared to T.

Any given $x \in \Delta$ can be written uniquely as $x = \alpha(x)u + \beta(x)v$. Clearly, $\alpha(x) = x \cdot v^{\perp}/u \cdot v^{\perp}$, and $\int_{\Delta} \alpha(x)dx = \Delta/3$. We state the following result, which can be derived from Lemma 1.2.1.

Theorem 2.4.5. *Assume $w(\Delta)$ is large enough ($w(\Delta) \geq 6$). Then*

$$\left| \sum_{p \in \mathbb{P} \cap \Delta} \alpha(p) - \frac{6}{\pi^2} \int_{x \in \Delta} \alpha(x)dx \right| < 30\Delta \frac{\log w(\Delta)}{w(\Delta)}$$

and

$$\left| \sum_{p \in \mathbb{P} \cap \Delta} \beta(p) - \frac{6}{\pi^2} \int_{x \in \Delta} \beta(x)dx \right| < 30\Delta \frac{\log w(\Delta)}{w(\Delta)}.$$

Notice that the estimate is invariant under lattice preserving affine transformations.

We are now in the position to begin the proofs of the main results.

2.5 Proof of Theorem 2.4.2

We assume $t = t_1 = t_2$ and set $U = ta$, $V = tb$. We shall find an upper and a lower bound for the maximal length $m(T)$ of a chain in T. For $x \in \mathbb{R}^2$, there is a unique representation $x = \bar{\alpha}(x)U + \bar{\beta}(x)V$. We start with the upper bound.

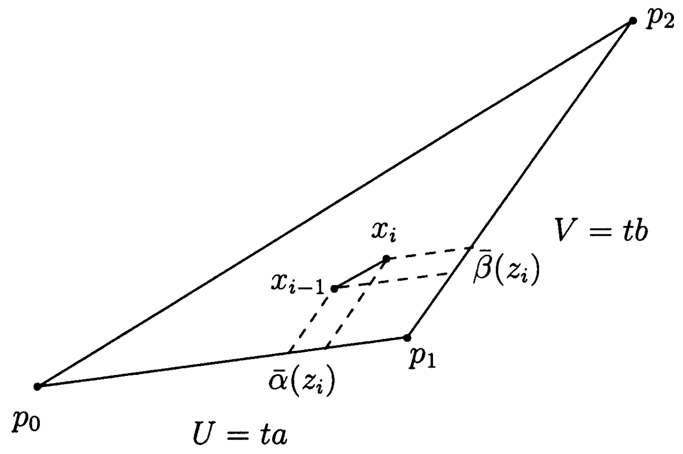

Figure 2.2: The unique representation of $z_i = \bar{\alpha}(z_i)U + \bar{\beta}(z_i)V$.

Let $x_0, \ldots, x_n$ be the sequence of vertices of a maximal lattice chain in T. So $m(T) = n$. The vectors $z_i = x_i - x_{i-1}$ all lie in $\mathbb{Z}^2$ and all belong to the cone $\text{pos}\{a, b\}$. Clearly, since $\bar{\alpha}(U) = \bar{\beta}(V) = 1$, the edges z_i must satisfy

$$\sum_{i=1}^{n} \bar{\alpha}(z_i) \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} \bar{\beta}(z_i) \leq 1, \tag{2.3}$$

as otherwise the lattice chain would extend beyond p_2. Define the norm (essentially an ℓ_1 norm) $\|\cdot\|$ as follows,

$$\|x\| = |\bar{\alpha}(x)| + |\bar{\beta}(x)|. \tag{2.4}$$

Since the z_i are non-parallel vectors from $\mathbb{Z}^2 \cap \text{pos}\{a, b\}$, we have

$$\sum_{i=1}^{n} \|z_i\| \geq \sum \|p\|, \tag{2.5}$$

where the second sum is taken over the shortest (in $\|\cdot\|$ norm) n primitive vectors in $\text{pos}\{a, b\}$. The set of these shortest n vectors from $\mathbb{P} \cap \text{pos}\{a, b\}$ is exactly $\mathbb{P} \cap \Delta$, where $\Delta = \text{conv}\{0, \lambda a, \lambda b\}$, for some suitable $\lambda > 0$.

The proof of the upper bound is based on identifying which $\lambda > 0$ will make the sum $\sum_{\mathbb{P} \cap \Delta} \|p\|$ almost equal to, but slightly larger than 2. Then, if it were such that $|\mathbb{P} \cap \Delta| \leq n$, according to (2.5) we would have

$$\sum_{i=1}^{n} \bar{\alpha}(z_i) + \sum_{i=1}^{n} \bar{\beta}(z_i) \geq \sum_{i=1}^{n} \|z_i\| \geq \sum_{p \in \mathbb{P} \cap \Delta} \|p\| > 2,$$

contradicting (2.3). So, for the λ which we shall identify, $|\mathbb{P} \cap \Delta| > m(T) = n$. Using this and the estimate (2.2) for $|\mathbb{P} \cap \Delta|$, we will derive the upper bound on $m(T)$.

The computation is as follows. Setting $u = \lambda a$, $v = \lambda b$,

$$\bar{\alpha}(x) = \frac{V^{\perp} \cdot x}{V^{\perp} \cdot U} = \frac{\lambda}{t} \frac{v^{\perp} \cdot x}{v^{\perp} \cdot u} = \frac{\lambda}{t} \alpha(x).$$

Write Δ_0 for the triangle $\text{conv}\{0, a, b\}$ (and its area). We have $\Delta = \lambda^2 \Delta_0$, $w(\Delta) = \lambda w(\Delta_0)$ and

$$\int_{\Delta} \alpha(x) dx = \int_{\Delta} \beta(x) dx = \frac{1}{3} \Delta = \frac{1}{3} \lambda^2 \Delta_0.$$

By Theorem 2.4.5

$$\begin{aligned} \sum_{p \in \mathbb{P} \cap \Delta} \bar{\alpha}(p) &= \frac{\lambda}{t} \sum_{p \in \mathbb{P} \cap \Delta} \alpha(p) \geq \frac{\lambda}{t} \left[\frac{6}{\pi^2} \frac{\lambda^2 \Delta_0}{3} - 30 \lambda^2 \Delta_0 \frac{\log w(\Delta)}{w(\Delta)} \right] \\ &= \frac{2 \lambda^3 \Delta_0}{\pi^2 t} \left(1 - 15 \pi^2 \frac{\log w(\Delta)}{w(\Delta)} \right). \end{aligned}$$

Now set

$$\lambda = \sqrt[3]{\frac{\pi^2 t}{2 \Delta_0} (1 + \delta)},$$

where $\delta > 0$ will be specified. Now $\lambda > \sqrt[3]{\frac{\pi^2 t}{2 \Delta_0}}$, so, for large enough t,

$$\frac{\log w(\Delta)}{w(\Delta)} = \frac{\log \lambda w(\Delta_0)}{\lambda w(\Delta_0)} \leq c_1 t^{-1/3} \log t$$

with a constant $c_1 > 0$ depending only on Δ_0. Choose $\delta = 30 \pi^2 c_1 t^{-1/3} \log t$. With this choice

$$\sum_{p \in \mathbb{P} \cap \Delta} \bar{\alpha}(p) \geq \left(1 + 30 \pi^2 c_1 t^{-1/3} \log t\right) \left(1 - 15 \pi^2 c_1 t^{-1/3} \log t\right) > 1$$

if t is large enough. Similarly $\sum_{p\in\mathbb{P}\cap\Delta}\bar{\beta}(p) > 1$ and so $\sum_{p\in\mathbb{P}\cap\Delta}\|p\| > 2$. Consequently,

$$\begin{aligned}
m(T) &\leq |\mathbb{P}\cap\Delta| < \frac{6}{\pi^2}\Delta + 30\Delta\frac{\log w(\Delta)}{w(\Delta)}\\
&= \frac{6}{\pi^2}\Delta\left(1+5\pi^2\frac{\log w(\Delta)}{w(\Delta)}\right)\\
&\leq \frac{6}{\pi^2}\left(\frac{\pi^2 t}{2\Delta_0}\right)^{2/3}\Delta_0(1+\delta)^{2/3}\left(1+5\pi^2c_1t^{-1/3}\log t\right)\\
&= \frac{6}{(2\pi)^{2/3}}T^{1/3}\left(1+O(t^{-1/3}\log t)\right). \qquad (2.6)
\end{aligned}$$

Note that the implied constant depends on Δ_0 only.

For the lower bound we work in a similar manner. We now want to find which $\lambda > 0$ makes the sums $\sum_{p\in\mathbb{P}\cap\Delta}\bar{\alpha}(p)$ and $\sum_{p\in\mathbb{P}\cap\Delta}\bar{\beta}(p)$ almost equal to, but slightly smaller than one. Then the primitive vectors in Δ can serve as edges of a convex lattice chain in T, so $m(T) \geq |\mathbb{P}\cap\Delta|$. For later reference we denote this convex lattice chain by $C(\Delta)$. The computation is similar to the previous one. We set

$$\lambda = \sqrt[3]{\frac{\pi^2 t}{2\Delta_0}(1-\delta)} \quad\text{and}\quad \delta = \min\left(\frac{1}{2}, 15\pi^2c_2t^{-1/3}\log t\right).$$

Thus $\lambda \geq \sqrt[3]{\frac{\pi^2 t}{4\Delta_0}}$ is of order $\sqrt[3]{t}$. Consequently $\frac{\log w(\Delta)}{w(\Delta)} \leq c_2t^{-1/3}\log t$ for some c_2 (depending only on Δ_0) for all large enough t. Then $\delta < 1/2$, if t is large enough and so

$$\delta = 15\pi^2c_2t^{-1/3}\log t.$$

We check that these choices give the desired inequalities $\sum\bar{\alpha}(p), \sum\bar{\beta}(p) \leq 1$.

$$\begin{aligned}
\sum_{p\in\mathbb{P}\cap\Delta}\bar{\alpha}(p) &= \frac{\lambda}{t}\sum_{p\in\mathbb{P}\cap\Delta}\alpha(p) \leq \frac{\lambda}{t}\left[\frac{6}{\pi^2}\frac{\Delta}{3} + 30\Delta\frac{\log w(\Delta)}{w(\Delta)}\right]\\
&= \frac{2\lambda^3\Delta_0}{\pi^2 t}\left(1+15\pi^2\frac{\log w(\Delta)}{w(\Delta)}\right)\\
&= (1-\delta)\left(1+15\pi^2c_2t^{-1/3}\log t\right) < 1,
\end{aligned}$$

and similarly for $\sum_{p\in\mathbb{P}\cap\Delta}\bar{\beta}(p) \le 1$. Hence,

$$\begin{aligned} m(T) &\ge |\mathbb{P}\cap\Delta| \ge \frac{6}{\pi^2}\Delta - 30\Delta\frac{\log w(\Delta)}{w(\Delta)} \\ &\ge \frac{6}{(2\pi)^{2/3}}T^{1/3}\left(1 - O(t^{-1/3}\log t)\right). \end{aligned} \tag{2.7}$$

Finally, combining (2.6) and (2.7) we get

$$\frac{6}{(2\pi)^{2/3}}T^{1/3}\left(1 - O(t^{-1/3}\log t)\right) \le m(T) \le \frac{6}{(2\pi)^{2/3}}T^{1/3}\left(1 + O(t^{-1/3}\log t)\right),$$

which completes the proof. □

Remark 2.5.1.

The preceding proof contains the construction of an almost maximal lattice chain in T, namely $C(\Delta)$. The edges of this chain are the vectors in $\mathbb{P}\cap\Delta$. Its length is

$$|\mathbb{P}\cap\Delta| = \frac{6}{(2\pi)^{2/3}}T^{1/3}\left(1 + O(t^{-1/3}\log t)\right).$$

The chain $C(\Delta) = (x_0, x_1, \dots, x_n)$ almost connects the two special vertices, p_0 an p_2, of T:

$$\sum_{p\in\mathbb{P}\cap\Delta} p = \left(\sum_{p\in\mathbb{P}\cap\Delta}\bar{\alpha}(p)\right)U + \left(\sum_{p\in\mathbb{P}\cap\Delta}\bar{\beta}(p)\right)V.$$

Here the coefficients of U and V are between $1 - O(t^{-1/3}\log t)$ and 1. So setting $x_0 - p_0 \approx \alpha_0 a$ and $p_2 - x_n \approx \beta_0 b$ we have α_0, $\beta_0 = O(t^{2/3}\log t)$.

2.6 Proof of Theorem 2.4.3

In this section we shall prove Theorem 2.4.3, namely that as $t \to \infty$, a maximal convex $\mathbb{Z}_t$ lattice chain Q_t converges, in the Hausdorff metric, to the special parabolic arc $D(T)$ within T. From the proof of Theorem 2.4.2, we obtained a construction of an almost maximal lattice chain $C(\Delta)$. Let us call this chain $C(\Delta, \mathbb{Z}_t)$, when we refer to it in the $\mathbb{Z}_t$ setting. We show in Claim 2.6.1 at the end of this section, that this chain also converges to the special parabolic arc $D(T)$.

Proof. Now $T = \text{conv}\{p_0, p_1, p_2\}$ is fixed and Q_t is a maximal $\mathbb{Z}_t$ lattice chain within T. Recall that $D = D(T)$ is the unique parabola within T that is tangent to p_0p_1 and p_1p_2 at the points p_0 and p_1 respectively and that D has maximal affine length among all convex curves in T.

Let ℓ be a line, with fixed direction d, supporting Q_t at a vertex p_3. Let $q \in D(T)$ be the point where the tangent to $D(T)$ goes in direction d. We shall prove that p_3 tends to q.

Assume ℓ intersects p_0p_1 at q_1 and p_1p_2 at q_2. Let T_1 and T_2 be the triangles with vertices p_0, q_1, p_3 and p_3, q_2, p_2, respectively (see Figure 2.1). (Of course $\ell, p_3, q_1, q_2, T_1, T_2$ all depend on t.) Clearly, since p_3 is on a maximal chain Q_t

$$m^*(T, \mathbb{Z}_t) = m^*(T_1, \mathbb{Z}_t) + m^*(T_2, \mathbb{Z}_t). \tag{2.8}$$

Choose convergent subsequences of q_1, p_3, q_2 as $t \to \infty$. Assuming on the one hand that none of the p_i and q_j coincide in the limit, we may apply the $\mathbb{Z}_t$ version of Theorem 2.4.2. This yields,

$$\begin{aligned} m^*(T_1, \mathbb{Z}_t) + m^*(T_2, \mathbb{Z}_t) &= \frac{6}{(2\pi)^{2/3}} \left(\sqrt[3]{T_1} + \sqrt[3]{T_2}\right) \left(1 + O\left(\frac{\log t}{\sqrt[3]{t}}\right)\right) \\ &\leq \frac{6}{(2\pi)^{2/3}} \sqrt[3]{T} \left(1 + O\left(\frac{\log t}{\sqrt[3]{t}}\right)\right) = m^*(T, \mathbb{Z}_t), \end{aligned}$$

which, combined with (2.8) gives,

$$\sqrt[3]{T} = \sqrt[3]{T_1} + \sqrt[3]{T_2}.$$

(Note that T_1 and T_2 denote the limiting triangles.) In view of the property of the affine perimeter discussed in (2.2.4), this equality is possible, if and only if, ℓ (in the limit) is tangent to the parabola $D(T)$ at the point q. Thus p_3 tends to q.

On the other hand, if one of the triangles, say T_1, becomes degenerate, then we may use Andrews' estimate (2.1). This gives, for the limiting triangles T_1 and T_2,

$$\frac{6}{(2\pi)^{2/3}} \sqrt[3]{T} \leq 20\sqrt[3]{T}_1 + \frac{6}{(2\pi)^{2/3}} \sqrt[3]{T}_2.$$

Since T_1 is degenerate, it has area 0, so the inequality is impossible since $T_2 < T$. This completes the proof. □

In the $\mathbb{Z}_t$ setting, the chain $C(\Delta)$ corresponds to the $\mathbb{Z}_t$ lattice chain which we shall denote by $C(\Delta, \mathbb{Z}_t)$ which is almost maximal and almost connects p_0 to p_2 within T. We show next that this chain, too, is very close to the parabola $D(T)$.

Claim 2.6.1. $\lim_{t\to\infty} \operatorname{dist}(C(\Delta, \mathbb{Z}_t), D(T)) = 0$

Proof. For this proof we work in the $\mathbb{Z}^2$ setting and divide by t in the end. Recall that Δ is the triangle $\operatorname{conv}\{0, \lambda a, \lambda b\}$.

Fix a vector d on the segment (a, b), that is, $d = (1 - s)a + sb$ with $s \in (0, 1)$ fixed, and let $\Delta(s) = \operatorname{pos}(a, d) \cap \Delta$. The tangent line to $C(\Delta)$ in direction d goes through the point $q \in C(\Delta)$ (see Figure 2.3.). Now

$$q - p_0 = (q - x_0) + (x_0 - p_0) = \sum_{p \in \mathbb{P} \cap \Delta(s)} p + (x_0 - p_0).$$

Here $x_0 - p_0$ is $O(t^{2/3} \log t)a$ from Remark 2.5.1. Clearly, the sum of the vectors in $\mathbb{P} \cap \Delta(s)$ can be estimated in the same way as for $\mathbb{P} \cap \Delta$ in Remark 2.5.1 and we get that

$$\sum_{\mathbb{P} \cap \Delta(s)} p = st(a + d)\left(1 + O(t^{-1/3} \log t)\right).$$

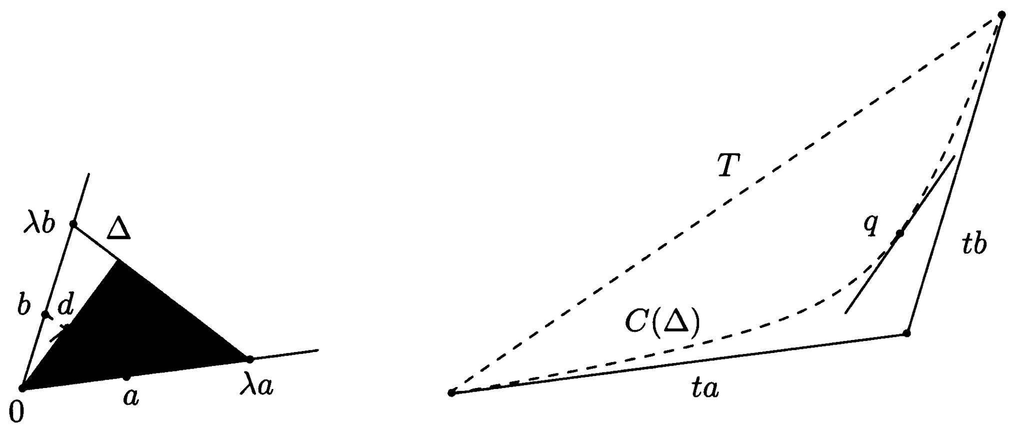

Figure 2.3: The definition of the region $\Delta(s)$.

The point $st(a+d)$ is on the parabola $D(T)$ and the tangent at that point has direction d as one can readily check. Thus $q - st(a+d)$ is at most $O(t^{2/3}\log t)$. Dividing by t we get the claim. □

Remark 2.6.2.

Since the primitive points are distributed evenly in a small triangle, instead of summing them, we could take the integral of the vector x in Δ. In this case the triangle does not have to be small. With the previous notation we get,

$$z(s) = \int_{\Delta(s)} x dx = \frac{1}{3}\Delta(s)(a+d) = \frac{1}{3}\Delta s((2-s)a + sb).$$

This is a curve, parametrised by s. It is very easy to see that this curve is exactly the special parabola inscribed in the triangle $\frac{1}{3}\Delta \operatorname{conv}\{0, a, a+b\}$. The tangent to this parabola at $z(s)$ is parallel to d.

Remark 2.6.3.

The moral is that the maximal $\mathbb{Z}_t$ lattice chain in T, the special chain $C(\Delta, \mathbb{Z}_t)$ and $D(T)$ are all very close to one another. Furthermore, the chain $C(\Delta, \mathbb{Z}_t)$ is almost explicitly described and the curve $z(s)$ can be computed from Δ. This is the main idea behind the proof of Theorem 2.3.4.

2.7 Proof of Theorem 2.2.1

In this section, we prove Theorem 2.2.1 which will be used for the proof of the main theorems given in Section 2.8. Recall that $U_\varepsilon(\Gamma)$ is the ε-neighbourhood of Γ and $m(\Gamma, \varepsilon, \mathbb{Z}_t)$ is the maximal number of vertices of a convex $\mathbb{Z}_t$ lattice curve contained in $U_\varepsilon(\Gamma)$.

Proof. The proof consists of two parts. First we show that the lim sup of $t^{-2/3}|\operatorname{vert} Q_t|$, over a sequence of convex $\mathbb{Z}_t$ lattice curves $Q_t \subset U_\varepsilon(\Gamma)$, can only be slightly larger than $3(2\pi)^{-2/3}\operatorname{AP}(\Gamma)$. Then we construct a sequence, P_t, of convex

$\mathbb{Z}_t$ lattice polygons lying in $U_\varepsilon(\Gamma)$ with almost as many as $3(2\pi)^{-2/3}t^{2/3}\,\mathrm{AP}(\Gamma)$ vertices.

Let η be a (small) positive number, and choose $\varepsilon > 0$ and unit vectors $d_1, \ldots, d_{k+1}$ for Γ and η according to Proposition 2.2.2. Let Q_t (with $t \to \infty$) be a sequence of convex $\mathbb{Z}_t$ lattice curves in $U_\varepsilon(\Gamma)$. We will show that this sequence contains a subsequence, to be denoted by Q_τ, such that

$$\limsup \tau^{-2/3} |\operatorname{vert} Q_\tau| \le \frac{3}{(2\pi)^{2/3}}(\mathrm{AP}(\Gamma) + 3\eta).$$

This will prove our first goal.

For each t, the directions d_i induce triangles $T_i(t)$, $i = 1, \ldots, k$ on Q_t, with vertices $z_i(t)$. Now we choose a convergent subsequence Q_τ such that $\lim_\tau z_i(\tau) = x_i$ for each i. Let $Q \subset U_\varepsilon(\Gamma)$ be the limit of the subsequence Q_τ. The triangle $T_i(\tau)$ tends to a triangle T_i on Q, for each $i = 1, \ldots, k$ (we include the possibility that some limiting triangle T_i is degenerate). From Proposition 2.2.2 we know that the triangles T_i, $i = 1, \ldots k$ on Q satisfy,

$$\mathrm{AP}(Q) \le 2\sum_{i=1}^{k} \sqrt[3]{T_i} \le \mathrm{AP}(\Gamma) + \eta. \tag{2.9}$$

Next we estimate $|\operatorname{vert} Q_\tau|$. We do this by using the estimate for $m^*(T_i(\tau), \mathbb{Z}_\tau)$ of (the $\mathbb{Z}_t$ version of) Theorem 2.4.2, for all $i = 1, \ldots, k$. When the limiting triangle T_i is non-degenerate, we get

$$m^*(T_i(\tau), \mathbb{Z}_\tau) \le \frac{6}{(2\pi)^{2/3}}\tau^{2/3}(\sqrt[3]{T_i(\tau)} + O(\tau^{-1/3}\log \tau)),$$

Since the constant implied by the 'big oh' term depends on the triangle T_i only, we may write this as

$$m^*(T_i(\tau), \mathbb{Z}_\tau) \le \frac{6}{(2\pi)^{2/3}}\tau^{2/3}\left(\sqrt[3]{T_i} + \frac{\eta}{k}\right),$$

for large enough τ and non-degenerate T_i. When a triangle T_i is degenerate, Andrews' estimate works once more. In this case we get,

$$m^*(T_i(\tau), \mathbb{Z}_\tau) \le 20\tau^{2/3}\sqrt[3]{T_i(\tau)} \le \tau^{2/3}\left(\sqrt[3]{T_i} + \frac{\eta}{k}\right),$$

again for large enough τ. Here the triangle in question is degenerate so $T_i = 0$. Thus for all i and large τ we have

$$m^*(T_i(\tau), \mathbb{Z}_\tau) \leq \frac{6}{(2\pi)^{2/3}}\tau^{2/3}\left(\sqrt[3]{T_i} + \frac{\eta}{k}\right),$$

So we have, again for large enough τ,

$$\begin{aligned} |\operatorname{vert} Q_\tau| &\leq \sum_{i=1}^{k} m^*(T_i(\tau), \mathbb{Z}_\tau) \leq \sum_{i=1}^{k} \frac{6}{(2\pi)^{2/3}}\tau^{2/3}\left(\sqrt[3]{T_i} + \frac{\eta}{k}\right) \\ &= \frac{3}{(2\pi)^{2/3}}\tau^{2/3} 2\sum_{i=1}^{k}\left(\sqrt[3]{T_i} + \frac{\eta}{k}\right) \leq \frac{3}{(2\pi)^{2/3}}\tau^{2/3}(\mathrm{AP}(\Gamma) + 3\eta), \end{aligned}$$

where for the last inequality we used (2.9).

Next comes the construction of the sequence P_t. Assume $\varepsilon > 0$ is small. We will find a sequence of $\mathbb{Z}_t$ lattice polygons P_t in $U_\varepsilon(\Gamma)$ such that the number of vertices of P_t is at least

$$|\operatorname{vert} P_t| \geq 3(2\pi)^{-2/3}t^{2/3}(\mathrm{AP}(\Gamma) - 2\varepsilon),$$

provided that t is large enough. The construction is as follows: choose points $x_i \in \Gamma$ and lines ℓ_i tangent to Γ at x_i, $i = 1, \ldots, k$ so that the induced triangles T_i all lie in $U_{\varepsilon/2}(\Gamma)$. We assume (for convenience rather than necessity) that the slopes of the lines ℓ_i are irrational. By the properties of the affine perimeter

$$2\sum_{i=1}^{k} \sqrt[3]{T_i} \geq \mathrm{AP}(\Gamma). \tag{2.10}$$

For each i there is a $\mathbb{Z}_t$ lattice square, of side length $1/t$, containing x_i. Move ℓ_i to a parallel position $\ell_i(t)$, that contains a vertex, $z_i(t)$, of this square. Here $z_i(t) \in \mathbb{Z}_t$ is chosen so that the whole square and Γ lie on the same side of $\ell_i(t)$. Replace each x_i by $z_i(t) \in \mathbb{Z}_t$ and each triangle T_i by the corresponding $T_i(t)$. Note that the $z_i(t)$ are in convex position.

Recall that the proof of the lower bound of Theorem 2.4.2 produced an almost maximal $\mathbb{Z}^2$ lattice chain $C(\Delta)$ in the triangle T, where the edges of $C(\Delta)$ are the vectors in $\mathbb{P} \cap \Delta$. In the $\mathbb{Z}_t$ setting, this gives an almost maximal $\mathbb{Z}_t$ lattice chain $C(\Delta_i(t), \mathbb{Z}_t)$ in $T_i(t)$. Now fix i and let $z_i(t), y_0, y_1, \ldots, y_n, z_{i+1}(t)$ be the vertices of $C(\Delta_i(t), \mathbb{Z}_t)$. Note that $y_j - y_{j-1}$ is in $\mathbb{Z}_t$ but y_j may not be.

We now show how this chain can be changed a little so that it is an almost maximal $\mathbb{Z}_t$ lattice chain within $T_i(t)$ with all of its vertices in $\mathbb{Z}_t$. For this, note first that $v = (y_0 - z_i(t)) + (z_{i+1}(t) - y_n)$ is in $\mathbb{Z}_t$. Also, the slope of v is between the slopes of two consecutive edges of the chain $C(\Delta_i(t), \mathbb{Z}_t)$, say the j-th and the $(j+1)$-st. Then the vectors

$$y_1 - y_0, y_2 - y_1, \ldots, y_j - y_{j-1}, v, y_{j+1} - y_j, \ldots, y_n - y_{n-1},$$

in this order, form the edges of a chain from $z_i(t)$ to $z_{i+1}(t)$ within $T_i(t)$ with all vertices in $\mathbb{Z}_t$. Let $C_i(t)$ denote this chain. Then $C_i(t)$ has at least as many edges as the chain $C(\Delta_i(t), \mathbb{Z}_t)$.

Since $T_i \subset U_{\varepsilon/2}(\Gamma)$, both $T_i(t)$ and $C_i(t)$ lie in $U_\varepsilon(\Gamma)$ if t is large enough. Thus the union of the $C_i(t)$ form a convex $\mathbb{Z}_t$ lattice curve P_t in $U_\varepsilon(\Gamma)$. The construction of P_t is complete. We shall now find a lower bound on the number of vertices of P_t.

The number of edges in $C_i(t)$ is at least as large as the number of edges in $C(\Delta_i(t), \mathbb{Z}_t)$ which is, by Theorem 2.4.2, at least

$$\frac{6}{(2\pi)^{2/3}} \sqrt[3]{T_i(t)} \left(1 - O(t^{-1/3} \log t)\right),$$

provided T_i is nondegenerate. Note that for such a T_i,

$$\sqrt[3]{T_i(t)} \left(1 - O(t^{-1/3} \log t)\right) \geq \sqrt[3]{T_i} - \frac{\varepsilon}{k}$$

if t is large enough. For degenerate T_i, we may use the trivial estimate saying that $C_i(t)$ has at least $t^{2/3} \sqrt[3]{T_i} = 0$ edges. So we have

$$\begin{aligned}
|\operatorname{vert} P_t| &\geq \sum_{i=1}^{k} \frac{6}{(2\pi)^{2/3}} t^{2/3} \left(\sqrt[3]{T_i} - \frac{\varepsilon}{k}\right) \\
&\geq \frac{3}{(2\pi)^{2/3}} t^{2/3} \sum_{i=1}^{k} 2 \left(\sqrt[3]{T_i} - \frac{\varepsilon}{k}\right) \\
&\geq \frac{3}{(2\pi)^{2/3}} t^{2/3} (\mathrm{AP}(\Gamma) - 2\varepsilon),
\end{aligned}$$

where the last inequality comes from (2.10). This completes the proof. □

Remark 2.7.1.

The chain $C_i(t)$ is made up of primitive vectors in $\Delta_i(t)$ plus possibly one more vector which we ignore as it is short. So the edges of P_t are essentially the primitive vectors in $\cup_{i=1}^k \Delta_i(t)$. In the proof of Theorem 2.3.4 we will need to show that the set $\cup_{i=1}^k \Delta_i(t)$ has nice properties.

2.8 Proof of Theorems 2.1.1 and 2.1.2

Using Theorem 2.2.1, we can now give a simple proof of both Theorems 2.1.1 and 2.1.2.

Proof. Let Q_t be any maximiser in the definition of $m(K, \mathbb{Z}_t)$ and choose a subsequence Q'_t. We show that Q'_t contains a further subsequence, to be denoted by Q_τ, with $\tau \to \infty$ such that

$$\lim_{\tau\to\infty} \tau^{-2/3} |\operatorname{vert} Q_\tau| = \frac{3}{(2\pi)^{2/3}} \operatorname{A}(K),$$

and

$$\lim_{\tau\to\infty} \operatorname{dist}(Q_\tau, K_0) = 0.$$

This will prove both theorems.

The proof is based on Theorem 2.2.1. Choose a convergent subsequence Q_τ from Q'_t, and let S be the limit of Q_τ. Clearly $S \subset K$ and $S \in \mathcal{C}$. By Theorem 2.2.1,

$$\lim_{\tau\to\infty} \tau^{-2/3} |\operatorname{vert} Q_\tau| = \frac{3}{(2\pi)^{2/3}} \operatorname{AP}(S).$$

By the definition of $\operatorname{A}(K)$, we have that $\operatorname{AP}(S) \le \operatorname{A}(K)$. Assume that $\operatorname{AP}(S) < \operatorname{A}(K)$. Then $\operatorname{AP}(S) + \eta < \operatorname{A}(K)$, for some positive η. A slightly shrunken homothetic copy of K_0, say K'_0, can be placed in K so that $U_\varepsilon(\partial K'_0) \subset K$ for some positive ε, and $\mathrm{AP}(S) + \eta/2 < \mathrm{AP}(K_0')$. Now Theorem 2.2.1 implies the existence of a convex $\mathbb{Z}_t$ lattice polygon P_τ in $U_\varepsilon(\partial K_0') \subset K$ with

$$|\operatorname{vert} P_\tau| > \frac{3}{(2\pi)^{2/3}}\tau^{2/3}(\mathrm{AP}(S) + \eta/4).$$

This contradicts the maximality of Q_τ. Hence $\mathrm{AP}(S) = \mathrm{A}(K)$ and $S = K_0$ follows from the uniqueness of K_0. □

2.9 Proof of Theorem 2.3.4

In this section we give the proof of Theorem 2.3.4, which characterises the body K_0.

Proof. We assume first that K is a convex polygon. Then K_0 is tangent to the edges $E_1, \dots, E_k$ of K at points $p_1, \dots, p_k$. We may assume that the boundary of K is exactly $\cup_{i=1}^k E_i$ (since we can delete the edges not touched by K_0). Denote by v_i the common vertex of the edges E_i and E_{i+1} and the outer angle at v_i by $\phi_i > 0$. Set $T_i = \operatorname{conv}\{p_i, v_i, p_{i+1}\}$. Then $\mathrm{A}(K) = 2\sum_{i=1}^k \sqrt[3]{T_i}$ is the solution of the maximisation problem

$$\max\left(2\sum_{i=1}^k \sqrt[3]{\frac{1}{2}x_i(e_{i+1} - x_{i+1})\sin\phi_i}\right),$$

where $e_i = \|E_i\|$ and $x_i = \|p_i - v_i\|$. (See Figure 2.4.)

The solution is unique (according to Theorem 2.3.1). So, taking derivatives, we get the necessary conditions for the extremum. As a simple computation shows, they are

$$x_i^{-2/3}(e_{i+1} - x_{i+1})^{1/3}(\sin\phi_i)^{1/3} = x_{i-1}^{1/3}(e_i - x_i)^{-2/3}(\sin\phi_{i-1})^{1/3},$$

for $i = 1, \dots, k$. These can be written as

$$\frac{x_i}{\sqrt[3]{T_i}} = \frac{e_i - x_i}{\sqrt[3]{T_{i-1}}}, \quad i = 1, \dots, k. \tag{2.11}$$

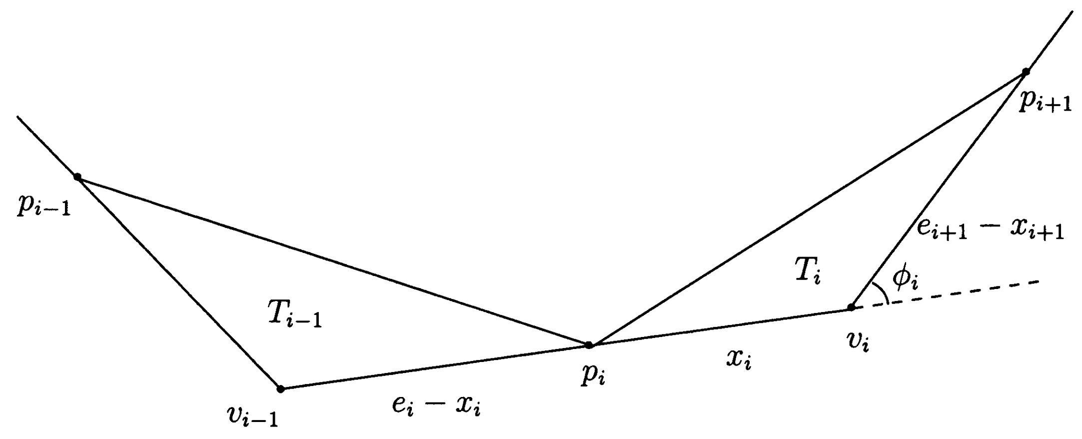

Figure 2.4: The triangles T_{i-1}, T_i.

Now, for each i, define the triangle

$$\Delta_i = \frac{1}{\sqrt[3]{T_i}} \operatorname{conv}\{0, v_i - p_i, p_{i+1} - v_i\}.$$

So T_i determines Δ_i. Conversely, Δ_i determines T_i uniquely (up to translation) in the following way. If $\Delta_i = \operatorname{conv}\{0, a_i, b_i\}$ then

$$\operatorname{conv}\{0, \Delta_i a_i, \Delta_i(a_i + b_i)\}$$

is a translated copy of T_i.

Each Δ_i has 0 as a vertex. Note that Δ_i and Δ_{i-1} share an edge; namely, the edge going in direction $v_i - v_{i-1}$. This edge has length $x_i/\sqrt[3]{T_i}$ if considered as an edge of the triangle Δ_i and length $(e_i - x_i)/\sqrt[3]{T_{i-1}}$ if considered as an edge of Δ_{i-1}. That they are equal follows from the necessary conditions (2.11) for the extremum.

Claim 2.9.1. $\cup_{i=1}^{k} \Delta_i$ *is a convex set.*

Proof. Our aim is two show that for two consecutive triangles Δ_{i-1}, Δ_i, the union $\Delta_{i-1} \cup \Delta_i$ forms a convex set. To simplify the notation we assume that $i = 2$ and $\Delta_1 = \operatorname{conv}\{0, a_1, b_1\}$, $\Delta_2 = \operatorname{conv}\{0, a_2, b_2\}$. Here, as we have seen, $b_1 = a_2$. We have to show that the angle, ψ say, at the common vertex $b_1 = a_2$ of $\Delta_1 \cup \Delta_2$ is less than π (see Figure 2.5).

If $\phi_1 + \phi_2 \geq \pi$, then $\psi < \pi$ follows immediately. So assume $\phi_1 + \phi_2 < \pi$. Then $\psi < \pi$, if and only if, $\Delta_1 + \Delta_2 > \Delta$ where $\Delta = \text{conv}\{0, a_1, b_2\}$.

As both the convexity of $\cup\Delta_i$ and the affine perimeter are invariant under linear transformations we may assume that $a_1 = (1, 0)$ and $b_2 = (0, 1)$, and write $b_1 = a_2 = (r\cos\alpha, r\sin\alpha)$ with $\alpha \in (0, \pi/2)$ and $r > 0$. With this notation $\Delta_1 = \frac{r}{2}\sin\alpha$, $\Delta_2 = \frac{r}{2}\cos\alpha$ so our aim is to show that

$$r(\sin\alpha + \cos\alpha) > 1. \tag{2.12}$$

Now T_1 and T_2 are determined by Δ_1 and Δ_2 (see Figure 2.6 for notations). In particular, the lengths of the edges of the triangles T_1 and T_2 are,

$$\|v_1 - p_1\| = \frac{1}{3}\Delta_1 = \frac{r}{6}\sin\alpha, \qquad \|p_2 - v_1\| = \frac{1}{3}r\Delta_1 = \frac{r^2}{6}\sin\alpha,$$
$$\|v_2 - p_2\| = \frac{1}{3}r\Delta_2 = \frac{r}{6}\cos\alpha, \qquad \|p_3 - v_2\| = \frac{1}{3}r\Delta_2 = \frac{r^2}{6}\cos\alpha.$$

Then,

$$A = \|v_2 - v_1\|\cos\alpha = (r^2/6)(\sin\alpha + \cos\alpha)\cos\alpha,$$
$$B = \|v_2 - v_1\|\sin\alpha = (r^2/6)(\sin\alpha + \cos\alpha)\sin\alpha.$$

The special parabolic arc within the triangle $T = \text{conv}\{p_1, p_0, p_3\}$ connecting p_1 and p_3 must intersect the segment $[v_1, v_2]$ as otherwise, replacing T_1 and T_2 by T

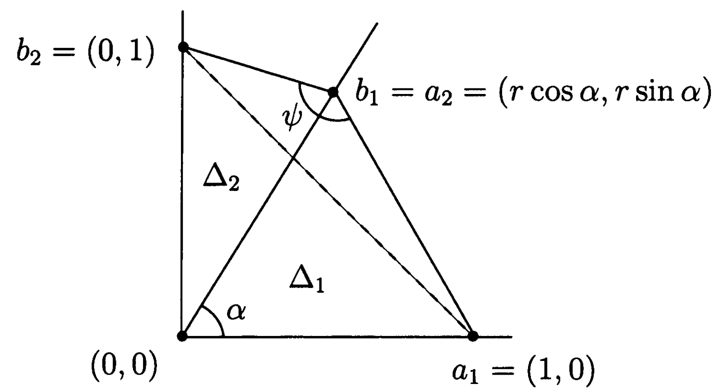

Figure 2.5: The triangles Δ_1, Δ_2.

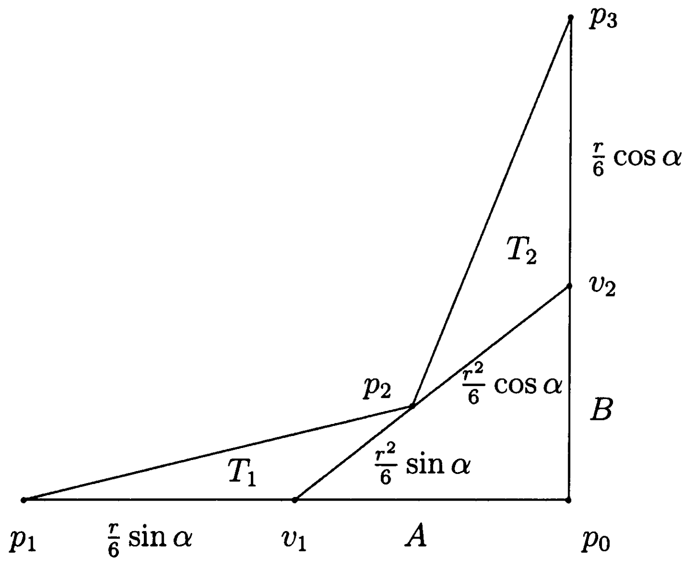

Figure 2.6: The corresponding triangles T_1, T_2.

would increase the affine perimeter of K_0. This happens, if and only if,

$$\frac{A}{A + \|p_1 - v_1\|} > \frac{B}{B + \|p_3 - v_2\|}.$$

This is equivalent to the statement $AB > \frac{r^2}{36}\sin\alpha\cos\alpha$, or

$$(r^4/36)(\sin\alpha + \cos\alpha)^2 \sin\alpha\cos\alpha > \frac{r^2}{36}\sin\alpha\cos\alpha.$$

This in turn states that $r^2(\sin\alpha + \cos\alpha)^2 > 1$, which implies (2.12). This completes the proof of Claim 2.9.1 □

We now show that the convex set $\cup_{i=1}^k \Delta_i$ has its barycentre at 0. Remark 2.6.2 shows that

$$\int_{\Delta_i} x dx = \frac{1}{3}\Delta_i(a_i + b_i) = p_{i+1} - p_i.$$

This immediately implies that

$$\int_{\cup_{i=1}^k \Delta_i} x dx = \sum_{i=1}^{k}(p_{i+1} - p_i) = 0.$$

We are almost finished with the proof for the case when K is a polygon. Define C to be a copy of $\cup_{i=1}^k \Delta_i$, rotated clockwise around the origin by $\pi/2$. Let u

be a unit vector, and let $z(u)$ be the unique point with outer normal u on the boundary of K_0. The radius of curvature, $R(u)$ of K_0 at $z(u)$ is by definition, the limit, as $v \to u$, of the length of the arc of ∂K_0 between $z(u)$ and $z(v)$, divided by the angle between $u, v \in S^1$. We may assume both directions u and v lie in the triangle $\Delta_i^\perp$. Define $\Delta(u, \imath \int_{\Delta(u,v)}$ ›e $\Delta(u,v) = \mathrm{pos}\{u,v\} \cap \Delta_i^\perp$. Then

$$\int_{\Delta(u,v)} x dx \approx \frac{1}{3}\rho_C(u)^3 \|u - v\| u,$$

and so $R(u) = \frac{1}{3}\rho_C(u)^3$. This proves the first half of Theorem 2.3.4 in the case when K is a polygon.

Now assume that $K \subset \mathbb{R}^2$ is arbitrary, and let P_n be a sequence of convex polygons tending to $F(K) = K_0$ with $K_0 \subset P_n$. Then $F(P_n) \to K_0$ as well. Also, by the previous argument, $F(P_n) = G(C_n)$ with a unique $C_n \in \mathcal{C}_0$ and $R_{F(P_n)}(u) = \rho_{C_n}^3(u)/3$. By the property (2.3.3) $R_{F(P_n)}(u)$ is bounded. Then one can choose a convergent subsequence from C_n tending to $C \in \mathcal{C}_0$. It is easy to see that not only the subsequence but the whole sequence (C_n) tends to C implying, in turn, that $\rho_{C_n} \to \rho_C$ and so $\lim R_{F(P_n)}(u) = \rho_C^3(u)/3$ for each u. It follows now that $R_{K_0}(u) = \rho_C^3(u)/3$.

The second half of the theorem is easy: $G(C) = C^* \in \mathcal{C}$ clearly. Choose a sufficiently dense set of directions $d_1, \ldots, d_{n+1}$ and consider the induced triangles, T_i, on C^*. The corresponding "small triangles" Δ_i are very close to $C^\perp \cap \mathrm{pos}\{d_i, d_{i+1}\}$ where $C^\perp$ is a copy of C rotated by $\pi/2$ (anticlockwise) since $R_{C^*}(u) = \rho_C^3(u)/3$. Then the rotated copy of $\cup \Delta_i$ is very close to C. □

2.10 Proof of Corollary 2.3.6

Proof. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a nondegenerate linear map. For $K \in \mathcal{C}$ we have $F(LK) = LF(K)$. Assume now that $C \in \mathcal{C}_0$ is the unique convex set whose existence is guaranteed by Theorem 2.3.4. We claim that LC is the convex set corresponding to LK. It suffices to check this when K is a convex polygon. The proof of Theorem 2.3.4 shows that $\cup_{i=1}^k \Delta_i$ is a convex polygon with each Δ_i a well defined triangle. Since the integral $\int_{\Delta_i(s)} x dx$, for $s \in (0,1)$ describes the special parabola in T_i, the integral $\int_{L\Delta_i(s)} x dx$ describes the special parabola in LT_i. As C is a rotated copy of $\cup_{i=1}^k \Delta_i$, LC is also a rotated copy of $\cup_{i=1}^k L\Delta_i$. This proves the claim.

Given $K \in \mathcal{C}$ and the corresponding $C \in \mathcal{C}_0$, choose a linear transformation $L\mathbb{R}^2 \to \mathbb{R}^2$ that carries C into *isotropic position*. This means (see [KLS95]) that $b(C) = 0$ and the matrix of inertia about 0, $\frac{1}{\text{Area}\, C} \int_{x \in C} x x^T dx$ is the identity matrix. Kannan, Lovász, and Simonovits [KLS95] prove that, with this positioning, C contains a circle $B(r)$ and is contained in a circle $B(R)$, both centred at the origin, with $R/r \leq 2$. So, we may take $r = 1$ and then $R \leq 2$. In this position, the radial function $\rho_{LC}(u)$ clearly satisfies

$$1 \leq \rho_{LC}(u) \leq 2.$$

Since the radius of curvature of $F(LK)$ is $R(u) = \frac{1}{3}\rho_{LC}^3(u)$, it follows that

$$\frac{1}{3} \leq R(u) \leq \frac{8}{3}.$$

□

Chapter 3

A Combinatorial Property of Points and Balls, A Coloured Version

3.1 Introduction and results

Any finite set X of points in the plane in general position contains two points such that any disc that contains them necessarily contains a positive fraction of the points of X. This statement was introduced and proved by Neumann-Lara and Urrutia in [NU88]. They showed that any such set contains two points such that any disc that contains them, will contain at least $\lceil (n-2)/60 \rceil$ points of X, where n is the number of points in X. Hayward, Rappaport and Wenger in [HRW89] showed that if $\Pi(n)$ is the largest number, such that for every n-point set X on the plane, there exist two points $x, y \in X$ with the property that any disc that contains them, contains $\Pi(n)$ of the points of X, then $\lfloor n/27 \rfloor + 2 \leq \Pi(n) \leq \lceil n/4 \rceil + 1$. This lower bound was significantly improved by Hayward in [Ha89] where he obtained the expression $\lceil \frac{5}{84}(n-2) \rceil$.

The planar case was later generalised by Bárány, Schmerl, Sidney and Urrutia in [BSSU89] to the case of $X \subset \mathbb{R}^d$. They proved that any n-point set $X \subset \mathbb{R}^d$ in general position contains a subset A with $|A| = \lfloor \frac{1}{2}(d+3) \rfloor$ such that any ball that contains it contains $c(d)|X|$ points, where the constant $c(d) > 0$ is at least $(k!(d-k-1)!)/d!$ and $k = \lfloor \frac{1}{2}(d+3) \rfloor$. They also showed that the bound on the number of points of A is best possible.

Bárány and Larman in [BL90] generalised this result from the case of Euclidean balls, to the case of ellipsoids and more generally, quadrics in $\mathbb{R}^d$. They show that there is a constant $c(d) > 0$ such that any n-point set $X \subset \mathbb{R}^d$ contains a subset Y, with $|Y| = \lfloor \frac{1}{4}d(d+3) \rfloor$, with the property that any quadric that contains Y contains $c(d)|X|$ points of X. In this case $c(d) \geq k^{-1}2^{-(k+1)}$, where $k = (d+1)(d+2)/2$. Again it is shown that the bound on the number of points in Y is best possible.

Here we consider the coloured case. Before stating our results, let us introduce some notation. Throughout the chapter, k will equal $\lfloor \frac{d+3}{2} \rfloor$. The *system of sets* $C_1, C_2, \ldots, C_k \subset \mathbb{R}^d$ will be k non-empty, disjoint, finite sets (also called colour classes, or colours) with $|C_i| = s$, $s \geq 3$ for all $i = 1, \ldots, k$. Their union $\cup_{i=1}^k C_i$ will be denoted by X and n will be the cardinality of X. We will call X a *coloured set*, or equivalently, a set partitioned into m colour classes C_i. A set $Y \subset X$ is a *transversal* for the system $C_1, \ldots, C_k$, if $|Y \cap C_i| = 1$, for all $i = 1, \ldots, k$. Let $\mathcal{B} \subset \mathcal{P}(\mathbb{R}^d)$ denote the set of Euclidean balls in $\mathbb{R}^d$. Note that for the coloured case, the assumption that the points are in general position is not necessary. Our main result is the following.

Theorem 3.1.1. *For any $d \geq 1$, there is a constant $c(d) > 0$ such that for any system $C_1, C_2, \ldots, C_k \subset \mathbb{R}^d$, there is a transversal $Y \subset X$, such that for any $B \in \mathcal{B}$ with $Y \subset B$, we have that $|B \cap X| \geq c(d)|X|$.*

The constant here is $c(d) = \frac{1}{2k3^k}$. If the sets $C_1, \ldots, C_k$ are thought of as colours, then Theorem 3.1.1 says that there is a rainbow set $Y \subset X$ such that any ball that contains it, contains a positive fraction of the points of X.

The main tool for proving Theorem 3.1.1 is Lemma 3.1.2 below, the proof of which uses the following beautiful result of Živaljević and Vrećica from [ŽV92].

Theorem 3.1 (Vrećica and Živaljević). *Let $C_1, C_2, \dots, C_k \subset \mathbb{R}^d$ be disjoint finite sets, with $|C_i| = 2p-1$, $p(d-k+1) \le d$ where p is a prime. Then there are $A_1, \dots, A_p \subset \cup_{i=1}^k C_i$ with $A_i \cap A_j = \varnothing$, for $i \neq j$, $|A_i \cap C_j| = 1$, for all i, j and $\bigcap_{i=1}^{p} \operatorname{conv} A_i \neq \varnothing$.*

We will apply the $p = 2$ case of this result in the proof of Lemma 3.1.2. Then k can be taken to be $\lfloor \frac{d+3}{2} \rfloor$.

Lemma 3.1.2. *Let $X \subset \mathbb{R}^d$ be a finite set, $X = C_1 \cup C_2 \cup \dots \cup C_k$, with $C_i \cap C_j = \varnothing$, for all $i \neq j$, where $k = \lfloor \frac{d+3}{2} \rfloor$ and $|C_i| = 3$. Then there is $Y \subset X$ with $|Y| = k$, $|Y \cap C_i| = 1$, for all $i = 1, \dots, k$, such that for any $B \in \mathcal{B}$ with $Y \subset B$, we have $B \cap (X \backslash Y) \neq \varnothing$.*

In the next two Theorems, the sets C_i, $i = 1, \dots, k$ can be finite or infinite. The following Theorem shows that the number of colour classes in Theorem 3.1.1 is best possible.

Theorem 3.1.3. *For any $d \ge 1$ there is a system $C_1, C_2, \dots, C_{k-1} \subset \mathbb{R}^d$, such that for any transversal $Y \subset X$, there is $B \in \mathcal{B}$ with $Y \subset B$ and $B \cap X = Y$.*

Though we know from Theorem 3.1.1 that we can always find a rainbow subset $Y \subset X$ such that any $B \in \mathcal{B}$ that contains it, will contain a positive fraction of the points of X, we cannot hope that this fraction will come from many of the colour classes. In fact, all points (except for the points of Y) might come from a single colour class.

Theorem 3.1.4. *For any $d \ge 1$ there is a system $C_1, C_2, \dots, C_k$, such that for any transversal $Y \subset X$, there is a ball $B \in \mathcal{B}$ such that $Y \subset B$, $|B \cap C_i| = 1$, for all $i \in \{1, \dots, k\} \backslash \{i_0\}$ and $|B \cap C_{i_0}| \ge c(d)|X|$, for some $i_0 \in \{1, \dots, k\}$.*

In Section 3.5, we show that the result of Theorem 3.1.1 also holds if instead of the family of Euclidean balls we consider more general families $\mathcal{F} \subset \mathcal{P}(\mathbb{R}^d)$, where $F \in \mathcal{F}$ is given by

$$F = \left\{ x \in \mathbb{R}^d \ : \ \sum_{i=1}^{m} \alpha_i f_i(x) \leq 0 \right\},$$

for $f_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, \dots, m$ and $\alpha_i \in \mathbb{R}$ not all zero.

Remark 3.1.5.

For the proof of our main Theorem, B. von Stengel (private communication) suggested that another possible approach could be along the lines of the proof of Lemma 1 of [BSSU89]. An attempt towards this direction showed that it might be possible to do so, for the case when $d = 2m$.

We have $C_1, \dots, C_k \subset \mathbb{R}^d$, where $k = \lfloor \frac{d+3}{2} \rfloor$, $X = \cup_{i=1}^{k} C_i$, and each C_i, $i = 1, \dots, k$ has cardinality $|C_i| = 3$. If $d = 2m$ even, then $k = m + 1$ and $n = |X| = 3(m+1)$. If we apply the Gale transform to the points of X, then we have $n = 3(m+1)$ points in $\mathbb{R}^{m+2}$. By the Centre Transversal Theorem (see [Ma]), there is a hyperplane ($(k-1)$-flat) such that both halfspaces contain at least $\frac{1}{(m+2)-k+2}|C_i|$ points from each colour. Thus, for the case when $d = 2m$, both halfspaces contain at least one point of each colour class.

Unfortunately, this method fails if $d = 2m + 1$, since in this case we can only get that both halfspaces contain at least $\frac{1}{(m+3)-k+2}|C_i| = \frac{1}{4}3$ points from each C_i.

3.2 Proof of Lemma 3.1.2 and Theorem 3.1.1

In this section, we prove Lemma 3.1.2 and Theorem 3.1.1. We start with the proof of Lemma 3.1.2. In the proof we use the notation x^2 and xy for the squared norm of x and the scalar product of x, y respectively. The proofs are similar to the ones given in [BSSU89].

Proof. By the result of Vrećica and Živaljević, for the case $p = 2$ we have $A_1, A_2 \subset X$, with $A_1 \cap A_2 = \varnothing$, $|A_i \cap C_j| = 1$, for all i, j and $\operatorname{conv} A_1 \cap \operatorname{conv} A_2 \neq \varnothing$. Suppose there is no subset Y of X that satisfies the statement of the Lemma. In particular, the subsets A_1, A_2 do not satisfy it. Then, there are $B_1, B_2 \in \mathcal{B}$ such that $A_1 \subset B_1$, $A_2 \subset B_2$ and $B_1 \cap X \backslash A_1 = \varnothing$ and $B_2 \cap X \backslash A_2 = \varnothing$. In particular, $B_1 \cap A_2 = \varnothing$ and $B_2 \cap A_1 = \varnothing$. Write $B_1 = \{x \in \mathbb{R}^d \ : \ (x - c_1)^2 \leq r_1^2\}$ and $B_2 = \{x \in \mathbb{R}^d \ : \ (x - c_2)^2 \leq r_2^2\}$, for some $c_1, c_2 \in \mathbb{R}^d$, $r_1, r_2 > 0$. Then, all of the following hold:

$$x^2 - 2c_1 x + c_1^2 \leq r_1^2 \quad \text{and} \quad x^2 - 2c_2 x + c_2^2 > r_2^2, \ \forall x \in A_1,$$

$$y^2 - 2c_2 y + c_2^2 \leq r_2^2 \quad \text{and} \quad y^2 - 2c_1 y + c_1^2 > r_1^2, \ \forall y \in A_2.$$

Hence,

$$2(c_2 - c_1)x < (r_1^2 - r_2^2) + (c_2^2 - c_1^2), \ \forall x \in A_1 \tag{3.1}$$

and

$$2(c_2 - c_1)y > (r_1^2 - r_2^2) + (c_2^2 - c_1^2), \ \forall y \in A_2. \tag{3.2}$$

Let $z \in \operatorname{conv} A_1 \cap \operatorname{conv} A_2$. Then

$$z = \sum_{x \in A_1} \lambda(x) x = \sum_{y \in A_2} \mu(y) y,$$

for some $0 \leq \lambda(x), \mu(y) \leq 1$, with $\sum_{x \in A_1} \lambda(x) = \sum_{y \in A_2} \mu(y) = 1$.

Multiply (3.1) by $\lambda(x) \geq 0$, $x \in A_1$ and sum over all $x \in A_1$ and (3.2) by $\mu(y) \geq 0$, $y \in A_2$ and sum over all $y \in A_2$. Then,

$$(r_1^2 - r_2^2) + (c_2^2 - c_1^2) \ < \ 2(c_2 - c_1)z < (r_1^2 - r_2^2) + (c_2^2 - c_1^2).$$

This means that if the assertion is false for both A_1, A_2, then a point in the intersection of their convex hulls, will lie in both open half spaces of the hyperplane defined by the intersection of the boundaries of B_1 and B_2, which is impossible. $\square$

Using Lemma 3.1.2 and a standard counting argument which was also used in [BSSU89], [Ha89] and [BL90], we can now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Since for every $Z \subset X$ with $|Z \cap C_i| = 3$ there is a $Y \subset Z$ which satisfies the properties of Lemma 3.1.2, there is $\mathcal{Z} \subset \mathcal{P}(X)$ and $Y \subset X$, with $|Y \cap C_i| = 1$, for $i = 1, \ldots, k$, such that Y is a subset of every $Z \in \mathcal{Z}$ and

$$|\mathcal{Z}| \geq \frac{\binom{s}{3}^k}{\binom{s}{1}^k} = \frac{(s-2)^k(s-1)^k}{6^k}, \tag{3.3}$$

where $|C_i| = s$, for all $i = 1, \ldots, k$, $\sum_{i=1}^k |C_i| = sk = |X|$ and $k = \lfloor \frac{d+3}{2} \rfloor$.

Let $B \in \mathcal{B}$ be such that $Y \subset B$ and write $|B \cap X| = m$. From Lemma 3.1.2, each $Z \in \mathcal{Z}$ contains a point of $B \backslash Y$. So, the number of ways to extend Y to Z, provided that $B \cap (Z \backslash Y) \neq \varnothing$ is at most

$$\binom{m-k}{1}\binom{s-1}{2}^{k-1}\binom{s-2}{1}. \tag{3.4}$$

The first term is the number of ways to choose a point of Z from the remaining $m-k$ points in B. The colour class to which this point belongs, will be extended by one point, which is the third term. The second term is the number of ways to extend the remaining $k-1$ colour classes by two points. The expression of (3.4) can be re written as

$$(m-k)\frac{(s-2)^{k-1}(s-1)^{k-1}}{2^{k-1}}(s-2) \tag{3.5}$$

This is an upper bound on $|\mathcal{Z}|$. From (3.3) and (3.5) we get

$$\frac{1}{2 \cdot 3^k}(s-1) \leq m-k,$$

or

$$\frac{sk-k}{2k3^k} \leq m-k.$$

Since $sk = |X|$ and $m = |B \cap X|$ we get $|B \cap X| \geq \frac{1}{2k3^k}|X| = c(d)|X|$. $\square$

3.3 Proof of Theorem 3.1.3

In this section, we prove Theorem 3.1.3: namely, that the number of colour classes in Theorem 3.1.1 is best possible.

Proof. This is very similar to Theorem 6 in [BSSU89]. Consider the moment curve $\gamma(t) = (t, t^2, \ldots, t^d)$. We first deal with the case when d is odd, $d = 2m - 1$. Let

$$\epsilon > t_1 > t_2 > \cdots > t_m > 0 \tag{3.6}$$

be positive real numbers and $\epsilon > 0$ to be specified later. We shall find constants $c_0, \ldots, c_{2d}$ so that the polynomials

$$p(t) = (t - c_1)^2 + (t^2 - c_2)^2 + \cdots + (t^{d-1} - c_{d-1})^2 + (t^d - c_d)^2 \tag{3.7}$$

and

$$q(t) = c_0 + [(t - t_1)(t - t_2) \cdots (t - t_m)]^2 r(t), \tag{3.8}$$

where

$$r(t) = c_{d+1} + c_{d+2}t + \cdots + c_{2d}t^{2m-2}, \tag{3.9}$$

are identically equal. This implies that when t takes the values $t_1, \ldots, t_m$ the points $\gamma(t_i)$, $i = 1, \ldots, m$ belong to the surface of the ball centred at $(c_1, \ldots, c_d)$ of radius $\sqrt{c_0}$. It is not hard to see that the c_i's can be uniquely determined in the order $c_{2d}, \ldots, c_0$ and are polynomial functions of $t_1, \ldots, t_m$ with constant terms given by,

$$c_i = \begin{cases} 0 + O(\epsilon), & i \text{ odd} \\ 1/2 + O(\epsilon), & i \in \{2, \ldots, d-1\},\ i \text{ even} \\ 1 + O(\epsilon), & i \in \{d+1, \ldots, 2d\},\ i \text{ even} \end{cases}$$

and finally

$$c_0 = \frac{m-1}{4} + O(\epsilon).$$

It is also easy to see that for these values of $c_{d+1},\dots,c_{2d}$, the polynomial term $r(t)$ is positive when $t>0$. Choose $\epsilon>0$ so small, so that c_0 is positive. Let $C_i = U_{\epsilon'}(\gamma(s_i)) = \{\gamma(t),\ t\in[s_i-\epsilon', s_i+\epsilon']\}$, $i=1,\dots,m$ to be sufficiently small neighbourhoods of $\gamma(s_i)$, where $\gamma(s_i)$ are points taken from the part of the moment curve with $\{\gamma(t),\ t\in(0,\epsilon)\}$ and choose $0<\epsilon'<\epsilon$ so small, so that the C_i's are disjoint. Set X to be $X=\cup_{i=1}^m C_i$ (or the union of finite subsets of C_i). Then, for any $Y\subset X$ with $|Y\cap C_i|=1$, for all $i=1,\dots m$, there is a ball $B\in\mathcal{B}$ centred at $(c_1,\dots,c_d)$ of radius $\sqrt{c_0}$, that contains Y, for which $B\cap X=Y$.

The case when d is even, $d=2m$ can be dealt with similarly. In this case, the constants c_i, $i=1,\dots,2d-1$ are not uniquely defined ($c_{d+1}\in\mathbb{R}$). In this case, we may choose $c_{d+1}=0+O(\epsilon)$. Then the c_i's are given by

$$c_i = \begin{cases} 0+O(\epsilon), & i \text{ odd},\ i\in\{1,\dots,d+1\}, \text{ or} \\ & i \text{ even},\ i\in\{d+2,\dots,2d-2\} \\ 1/2+O(\epsilon), & i \text{ even},\ i\in\{2,\dots,d\} \\ 1+O(\epsilon), & i \text{ odd},\ i\in\{d+3,\dots,2d-1\} \end{cases}$$

and finally

$$c_0 = \frac{m}{4}+O(\epsilon).$$

□

3.4 Proof of Theorem 3.1.4

We now proceed with the proof of Theorem 3.1.4. We shall give a construction of a finite set $X=\bigcup_{i=1}^k C_i$, with the property that for any choice of a transversal Y, there is a ball B that contains Y and such that all points in the intersection $B\cap X$, other than the points of Y, come from a single colour class. The construction involves the $(m-1)$-simplex, where $d=2m-1$. The proof for even dimensions is the same so we only prove the case when d is odd.

Proof. Here $d = 2m - 1$ and so $k = m + 1$. For the first m colour classes, we take

$$C_i = \{x \in \mathbb{R}^d,\ x = (u_i, xe_i) \in \mathbb{R}^{m-1} \times \mathbb{R}^m,\ |x| < \varepsilon\},$$

for $i = 1, \ldots, m$ and some $\varepsilon > 0$. Here u_i, $i = 1, \ldots, m$ are the vertices of an $(m-1)$-dimensional simplex, that is, the convex hull of m affinely independent points in the $m - 1$ dimensional space $\mathbb{R}^{m-1}$. We choose these vertices to be the following unit vectors.

$$\begin{aligned} u_1 &= (1, 0, 0, \ldots, 0) \\ u_2 &= -u_1 = (-1, 0, 0, \ldots, 0) \\ u_3 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ u_m &= (0, 0, 0, \ldots, 1) \end{aligned}$$

The set $\{e_i,\ i = 1, \ldots, m\}$ is the usual basis of $\mathbb{R}^m$. Therefore, for each $i = 1, \ldots, m$ the colour class C_i is defined as small segment in the direction of e_i, one for each of the m remaining dimensions. The final colour class will be a small segment at the centre of gravity of the simplex, $\left(0, \frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0\right)$.

We shall show that with this choice of the u_i's, for any $Y \subset \bigcup_{i=1}^{m+1} C_i$, transversal, $Y = \{y_i \in C_i :\ i = 1, \ldots, m+1\}$, there is a ball B, centred at $(u_0, v_0) \in \mathbb{R}^d$ of radius R (both to be specified), such that $B \cap C_i = \{y_i\}$, for $i = 1, \ldots, m$ and $C_{m+1} \subset B$.

Write $u_0 = (u_0(1), \ldots, u_0(m-1)) \in \mathbb{R}^{m-1}$ and $v_0 = (v_0(1), \ldots, v_0(m)) \in \mathbb{R}^m$. By definition of the colour classes, the points y_i, $i = 1, \ldots, m$, are of the form $y_i = (u_i, x_i e_i)$. The ball B with the desired properties must satisfy,

$$((u_0, v_0) - (u_i, x_i e_i)) \cdot (0, e_i) = 0 \tag{3.10}$$

$$\|u_0 - u_i\|^2 + \|v_0 - x_i e_i\|^2 = R^2, \tag{3.11}$$

for $i = 1, \ldots, m$. Equation (3.11) shows that each y_i lies on the surface of the ball and equation (3.10), that each colour class is tangent to the ball at the point y_i.

Note that here we use the same notation $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^{m-1}, \mathbb{R}^m$ and $\mathbb{R}^d$. From the m equations (3.10) we get $v_0(i) = x_i$, for all $i = 1, \ldots, m$. Using this, the m equations of (3.11) can be rewritten as,

$$\|u_0 - u_i\|^2 + \sum_{j \neq i} x_j^2 = R^2, \quad i = 1, \ldots, m, \tag{3.12}$$

or

$$\|u_0 - u_i\|^2 + \|v_0\|^2 = R^2 + x_i^2, \quad i = 1, \ldots, m. \tag{3.13}$$

The first two equations of (3.13) give

$$u_0 \cdot u_1 = \frac{x_2^2 - x_1^2}{4}. \tag{3.14}$$

Since $u_1 = (1, 0, \ldots, 0)$, this yields $u_0(1) = \frac{x_2^2 - x_1^2}{4}$.

From the first and (i)-th equation in (3.13), for $i = 3, \ldots, m$ we get

$$u_0 \cdot u_i = \frac{x_1^2 - x_i^2}{2} + u_0 \cdot u_1, \tag{3.15}$$

where we also used the fact that $\|u_i\| = 1$, for all $i = 1, \ldots, m$. From this, using (3.14) we get,

$$u_0 \cdot u_i = \frac{x_1^2 + x_2^2 - 2x_i^2}{4}, \quad i = 3, \ldots, m. \tag{3.16}$$

Since $u_0 \cdot u_i = u_0(i-1)$, for $i = 3, \ldots, m$ we obtain the remaining coordinates of u_0,

$$u_0(i-1) = \frac{x_1^2 + x_2^2 - 2x_i^2}{4}, \quad i = 3, \ldots, m. \tag{3.17}$$

To specify the radius R, we may use any one of the equations from (3.13). Using say the first, we get

$$R = \sqrt{\sum_{i=3}^{m} \frac{(x_1^2 + x_2^2 - 2x_i^2)^2}{16} + \sum_{i=2}^{m} x_i^2 + \frac{(x_2^2 - x_1^2)^2}{16} - \frac{x_2^2 - x_1^2}{2} + 1}. \tag{3.18}$$

Hence, for any choice of transversal $Y = \{y_i, i = 1, \ldots, m+1\}$, the ball B, centred at

$$(u_0, v_0) = \left(\frac{x_2^2 - x_1^2}{4}, \frac{x_1^2 + x_2^2 - 2x_3^2}{4}, \ldots, \frac{x_1^2 + x_2^2 - 2x_m^2}{4}, x_1, x_2, \ldots, x_m \right)$$

of radius R, satisfies (3.10) and (3.11). That is, it is tangent to the intervals (colour classes) C_i at the points y_i, for $i = 1, \dots, m$ and $C_{m+1} \subset B$. This completes the proof. □

3.5 General families

The result of Theorem 3.1.1 can be generalised from the family of Euclidean balls to families of subsets defined by a set of m functions. Given $\{f_i : \mathbb{R}^d \to \mathbb{R},\ i = 1, \dots, m\}$, let $\mathcal{F} \subset \mathcal{P}(\mathbb{R}^d)$ be the family of sets of the form

$$F = \left\{ x \in \mathbb{R}^d \ : \ \sum_{i=1}^{m} \alpha_i f_i(x) \leq 0 \right\}, \tag{3.19}$$

for some $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ not all zero. If $T : \mathbb{R}^d \to \mathbb{R}^m$ is the map defined by $T(x) = (f_1(x), \dots, f_m(x))$, then a typical set F is given by $T(x) \cdot a \leq 0$, for some $a = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, $a \neq 0$. Thus $\mathcal{F}$ is induced by the halfspaces in $\mathbb{R}^m$ whose bounding hyperplane passes through the origin. We will see in Remark 3.5.4 an example of such a family of $d+2$ functions which produce the Euclidean balls (and their complements).

In this case, Lemma 3.1.2 and Theorem 3.1.1 are formulated as follows.

Lemma 3.5.1. *Let $X \subset \mathbb{R}^d$, be a finite set, $X = X_1 \cup X_2 \cup \cdots \cup X_k$, with $X_i \cap X_j = \varnothing$, for all $i \neq j$, where $k = \lfloor \frac{m+3}{2} \rfloor$ and $|X_j| = 3$. Then there is a transversal $Y \subset X$ such that for any $F \in \mathcal{F}$ with $Y \subset F$, we have $F \cap (X \setminus Y) \neq \varnothing$.*

Theorem 3.5.2. *Let $\mathcal{F}$ be the family of sets in $\mathbb{R}^d$ generated by a family of m functions as in (3.19). Then there is a positive constant $c(m) > 0$ such that for any system $X_1, X_2, \dots, X_k$, in $\mathbb{R}^d$ with $k = \lfloor \frac{m+3}{2} \rfloor$ there is a transversal $Y \subset X$, with the property that for any $F \in \mathcal{F}$ with $Y \subset F$, it follows $|F \cap X| \geq c(m)|X|$.*

We only prove Lemma 3.5.1 as the counting argument for Theorem 3.5.2 is the same as for Theorem 3.1.1.

Proof. Consider the points $T(x) \in \mathbb{R}^m, x \in X$. These are in divided in k classes $C(X_j), j = 1, \ldots, k$ inherited from $\mathbb{R}^d$, with $|C(X_j)| = |X_j| = 3$, $j = 1, \ldots, k$, with $k = \lfloor \frac{m+3}{2} \rfloor$. From the Theorem of Vrećica and Živaljević, there are $A_1, A_2 \subset \mathbb{R}^m$, with $A_1 \cap A_2 = \varnothing$, $|A_i \cap C(X_j)| = 1$, for all i, j and $\operatorname{conv} A_1 \cap \operatorname{conv} A_2 \neq \varnothing$.

Write $A_1 = \{T(x_1), \ldots, T(x_k)\}$, $A_2 = \{T(y_1), \ldots, T(y_k)\}$, where $x_j, y_j \in X_j$, $j = 1, \ldots, k$. Let $Y_1 = \{x_1, \ldots, x_k\} \subset X$ and $Y_2 = \{y_1, \ldots, y_k\} \subset X$. We claim that either Y_1 or Y_2 is the set Y in the statement of the Lemma. That is, either for any $F \in \mathcal{F}$ that contains Y_1, we have $F \cap Y_2 \neq \varnothing$ (and therefore F contains a point of X), or, for any $F \in \mathcal{F}$ that contains Y_2, we have $F \cap Y_1 \neq \varnothing$. Indeed, suppose both assertions are false. Then there are $F_1, F_2 \in \mathcal{F}$,

$$F_1 = \left\{ x \in \mathbb{R}^d \; : \; \sum_{i=1}^{m} \alpha_i f_i(x) \leq 0 \right\}, \quad F_2 = \left\{ x \in \mathbb{R}^d \; : \; \sum_{i=1}^{m} \beta_i f_i(x) \leq 0 \right\},$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ not all zero, $\beta_1, \ldots, \beta_m \in \mathbb{R}$ not all zero, such that $Y_1 \subset F_1$, $Y_2 \cap F_1 = \varnothing$ and $Y_2 \subset F_2$, $Y_1 \cap F_2 = \varnothing$. This means that there are $a = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $b = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $a, b \neq 0$, such that all of the following hold:

$$a \cdot T(x_j) \leq 0, \; \forall x_j \in Y_1 \quad \text{and} \quad a \cdot T(y_j) > 0, \; \forall y_j \in Y_2,$$

$$b \cdot T(y_j) \leq 0, \; \forall y_j \in Y_2 \quad \text{and} \quad b \cdot T(x_j) > 0, \; \forall x_j \in Y_1.$$

Combining these we get

$$(a - b) \cdot T(x_j) < 0, \; \forall x_j \in Y_1 \tag{3.20}$$

and

$$(b - a) \cdot T(y_j) < 0, \; \forall y_j \in Y_2. \tag{3.21}$$

Let $w \in \operatorname{conv} A_1 \cap \operatorname{conv} A_2$. Then

$$w = \sum_{T(x_j) \in A_1} \lambda(x_j) T(x_j) = \sum_{x_j \in Y_1} \lambda(x_j) T(x_j) = \sum_{T(y_j) \in A_2} \mu(y_j) T(y_j) = \sum_{y_j \in Y_2} \mu(y_j) T(y_j),$$

for some $0 \leq \lambda(x_j), \mu(y_j) \leq 1$, $j = 1, \ldots, k$, $\sum_{x_j \in Y_1} \lambda(x_j) = \sum_{y_j \in Y_2} \mu(y_j) = 1$.

Multiply (3.20) by $\lambda(x_j) \geq 0$, $x_j \in Y_1$ and sum over all $x_j \in Y_1$ and (3.21) by $\mu(y_j) \geq 0$, $y_j \in Y_2$ and sum over all $y_j \in Y_2$. Then we get

$$\begin{aligned} 0 &> (a-b) \cdot \sum_{x_j \in Y_1} \lambda(x_j) T(x_j) = (a-b) \cdot w = (a-b) \cdot \sum_{y_j \in Y_2} \mu(y_j) T(y_j) \\ &= -(b-a) \cdot \sum_{y_j \in Y_2} \mu(y_j) T(y_j) > 0. \end{aligned}$$

This contradiction completes the proof. □

Remark 3.5.3.

The number of colour classes in Theorem 3.5.2 can be improved under certain assumptions on the functions f_i. We may assume that $f_m = 1$, since the set $\{f_m(x),\ x \in X\}$ takes positive, negative or 0 values for at least a third of the points of X. If it is 0, then T maps X to a subspace of dimension $m-1$. If it is positive (or negative) and we replace the functions f_i by f_i/f_m we can map X to an affine subspace of dimension $m-1$. The same is true if we consider sets of the form

$$F = \left\{ x \in \mathbb{R}^d \ :\ \sum_{i=1}^{m-1} \alpha_i f_i(x) \leq \alpha \right\},$$

i.e the sets induced by all halfspaces in $\mathbb{R}^{m-1}$. In this case the number of colour classes required is $\lfloor \frac{m+2}{2} \rfloor$.

Remark 3.5.4.

Let $X \subset \mathbb{R}^d$ be a finite set and $\mathcal{F} \subset \mathcal{P}(\mathbb{R}^d)$ the family defined by the following $(d+2)$ functions $f_i : \mathbb{R}^d \to \mathbb{R}$.

$$\begin{aligned} f_i &: (x_1, \ldots, x_d) \mapsto x_i, \quad \text{for } i = 1, \ldots, d, \\ f_{d+1} &: (x_1, \ldots, x_d) \mapsto x_1^2 + \cdots + x_d^2 \\ f_{d+2} &: (x_1, \ldots, x_d) \mapsto 1. \end{aligned}$$

Clearly $\mathcal{F}$ contains the family of Euclidean balls and their complements, which we denote by $\mathcal{B}_c$. According to Theorem 3.1.1, if X is partitioned into $\lfloor \frac{d+3}{2} \rfloor$ colour classes, we can find $Y \subset X$ that intersects each colour class in a single point and so that if $B \in \mathcal{B}$ contains Y, then B contains a positive fraction of the points of X. Since $\mathcal{B} \subset \mathcal{F}$, according to Theorem 3.5.2 (and Remark 3.5.3) the number of colour classes required to partition X is $\lfloor \frac{d+4}{2} \rfloor$ (which, in the case where d is even, is one more than the best possible according to Theorems 3.1.1 and 3.1.3). It is easy to see that the proofs of Theorem 3.1.1 and Lemma 3.1.2 apply to the case of $\mathcal{B}_c$ as well. But if a family contains both the balls and their complements, $\lfloor \frac{d+3}{2} \rfloor$ colour classes do not suffice. What fails to be true in this case is Lemma 3.1.2. For example, in the case $d = 2$, any 6 point set X with 3 points coloured red and 3 blue, must have two rainbow subsets $A_1 = \{r_1, b_1\}$ and $A_2 = \{r_2, b_2\}$ such that the segments $S_1 = \text{conv}\{r_1, b_1\}$, $S_2 = \text{conv}\{r_2, b_2\}$ intersect. Clearly the points r_1, r_2, b_1, b_2 form the vertices of a convex quadrilateral Π with diagonals S_1, S_2. Then one of the diagonals S_1, S_2 has the property that any disc that contains it, contains at least another vertex of Π. Assume that it is the longest one and let it be S_2. Then S_1 has the same property, for the case of the complements of discs. Now choose two discs $B_1 \subset B_2$, with $S_1 \subset B_1$ and $S_2 \subset B_2^c$. Then, for the family $\mathcal{F}$ there are $F_1 = B_1 \in \mathcal{F}$ and $F_2 = B_2^c \in \mathcal{F}$ such that $A_1 \subset F_1$, $A_2 \subset F_2$ and $A_1 \cap F_2 = A_2 \cap F_1 = \varnothing$.

Bibliography

[An63] G.E. Andrews. A lower bound for the volumes of strictly convex bodies with many boundary points, *Trans. AMS*, **106** (1963), 270–279.

[Ap76] T. M. Apostol, *Introduction to Analytic number theory*, Springer, Berlin, (1976).

[Ar80] V.I. Arnold. Statistics of integral lattice polytopes, *Func. Anal. Appl.*, **14** (1980), 1–3.

[Bá95] I. Bárány, The limit shape of convex lattice polygons, *Discrete Comput. Geom.*, **13** (1995) no. 3-4, 279–295.

[Bá97] I. Bárány. Affine perimeter and limit shape, *J. reine angew. Math.*, **484** (1997), 71–84.

[Bá99] I. Bárány. Sylvester's question: The probability that n points are in convex position, *The Annals of Probability*, **27** (1999), 2020–2034.

[Bl23] W. Blaschke. *Vorlesungen über Differenzialgeometrie II. Affine Differenzialgeometrie.*, Springer, Berlin 1923.

[BL90] I. Bárány, D.G. Larman, A Combinatorial Property of Points and Ellipsoids, *Discrete Comput. Geom.*, **5** (1990), 375–382.

[BL92] I. Bárány, D.G. Larman, A Colored Version of Tverberg's Theorem, *J. London Math. Soc.*, **45** (2) (1992), 314–320.

[BP] I. Bárány, M. Prodromou, On maximal convex lattice polygons inscribed in a plane convex set, *Israel J. Math.*, (to appear).

[BSSU89] I. Bárány, J.H. Schmerl, S.J. Sidney, J. Urrutia, A Combinatorial Result about Points and Balls in Euclidean Space, *Discrete Comput. Geom.*, **4** (1989), 259–262.

[BT04] I. Bárány, N. Tokushige. The minimum area convex lattice n-gon, *Combinatorica*, **24** (2004), 171–185.

[Ha89] R. Hayward, A Note on the Circle Containment Problem. *Discrete Comput. Geom.*, **4** (1989), 263–264.

[HRW89] R. Hayward, D. Rappaport, R. Wenger, Some Extremal Results on Circles Containing Points, *Discrete Comput. Geom.*, **4** (1989), 253–258.

[HW79] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, Claredon Press, Oxford (1979).

[Ja25] V. Jarník. Über Gitterpunkte und konvex Kurven, *Math. Z.* **2**, (1925), 500–518.

[KLS95] R. Kannan, L. Lovász, M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, *Discrete Comput. Geom.*, **13** (1995), 541–559.

[Lo78] L. Lovász. Kneser's conjecture, chromatic number and homotopy, *J. Combin. Theory*, Series A, **25** (1978), 319–324.

[Lu91] E. Lutwak. Extended affine surface area, *Adv. Math.*, **85** (1991), 39–68.

[Ma] Jiří Matoušek, *Lectures on Discrete Geometry*, Springer-Verlag, New York, (2002).

[NU88] V. Neumann-Lara, J. Urrutia, A Combinatorial Result on Points and Circles on the Plane, *Discr. Math.*, **69** (1988), 173–178.

[P1] M. Prodromou, Limit shape of convex lattice polygons with minimal perimeter, *Discr. Math.*, (to appear).

[P2] M. Prodromou, On a Combinatorial Property of Points and Balls - A Coloured Version, *Discr. Comp. Geom.*, (submitted).

[Ra93] S. Rabinowitz: $O(n^3)$ bounds for the area of a convex lattice n-gon, *Geombinatorics*, **2** (1993), 85–89.

[RS63] A. Rényi, R. Sulanke. Über die konvexe Hülle von n zufällig gewählten Punkten, *Z. Wahrshc. Verw. Geb.*, **2** (1963), 75–84.

[Sch93] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia Math. Appl. vol 44. Cambridge University Press, Cambridge (1993).

[Si] Ya. G. Sinai, Probabilistic approach to analyse the statistics of convex polygonal curves, *Funct. Anal. Appl.*, **28** (1994), 41–48 (in Russian).

[St01] M. Stojaković, The perimeter of optimal convex lattice polygons in the sense of different metrics, *Bull. Austral. Math. Soc.* **63** (2001) no. 2, 229–242.

[St03] M. Stojaković, Limit shape of optimal convex lattice polygons in the sense of different metrics, *Discrete Math.*, **271** (2003) no. 2, 235–249.

[Ve94] A. M. Vershik. The limit shape for convex lattice polygons and related topics, *Funk. Anal. Appl.*, **28** (1994), 16–25. (in Russian)

[VŽ94] S.T. Vrećica, R.T. Živaljević, New cases of the Colored Tverberg Theorem, H. Barcelo and G. Kalai, editors, *Jerusalem Combinatorics '93, Contemp. Math.*, Amer. Math. Soc., Providence, **178** (1994), 325–334.

[Ži97] R.T. Živaljević, Topological Methods, J.E. Goodman, J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, Chapter 11, CRC Press LLC, Boca Raton, FL, (1997), 209–224.

[ŽV92] R.T. Živaljević, S.T. Vrećica, The Colored Tverberg's Problem and Complexes of Injective Functions, *J. Combin. Theory*, Series A, **61** (1992), 309–318.

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