

Credit Modelling and Regime-Switching

A thesis presented for the degree of
Doctor of Philosophy of Imperial College
by

Richard Bell

Department of Mathematics
Imperial College
180 Queen's Gate, London SW7 2AZ

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Signed:

A handwritten signature in black ink, consisting of a large, stylized initial 'A' followed by a series of loops and a long horizontal stroke extending to the right.

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In memory of Margaret Newton Bell.

ABSTRACT

This thesis is concerned with some issues arising in relation to the capital structure and pricing of credit risk of a firm partly financed by debt. Three related models are presented. The first extends the existing literature on structural credit models of firms financed by the so called roll-over debt structure to allow for dependence between interest rates, asset volatility and the probability of default by incorporating regimes via the introduction of a Markov chain. The asset returns of a firm are modelled by a regime-switching geometric Brownian motion. An optimised capital structure is generated and the associated credit spreads analysed. The second model adds to recent work on regime-switching in the case of consol, or infinite maturity debt, by incorporating jumps into the asset process. The asset process of the firm is modelled as a phase-type Lévy process which affords a flexible framework capable of accommodating a wide range of stochastic dynamics. An optimal capital structure is identified. The contribution of the first two models is that they are highly flexible and allow for an arbitrary number of market regimes to be combined in an intuitive way. The final model extends the literature on endogenous default to a firm which is partly financed by a single finite maturity bond. The assets of the firm are modelled as a geometric Brownian motion which pays a continuous dividend. By solving the associated optimal stopping problem, a default boundary is characterised in terms of an integral equation.

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NOMENCLATURE

Λ_b	Identity matrix with vector b across the diagonal
$\mathbb{1}_{\{A\}}$	Indicator function of the event A
$\mathbb{E}_A[\cdot A]$	Conditional expectation given the event A
\mathbb{F}	Filtration generated by \mathcal{F} , $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$
\mathbb{L}_X	Infinitesimal generator of the stochastic process X
\mathbb{N}	Set of integers, $\{1, 2, \dots\}$
$\mathbb{P}_A(\cdot)$	Probability given the event A
\mathbb{R}^+	Set of positive real numbers, $(0, \infty)$
\mathcal{F}	σ -algebra
\mathcal{T}_A	Set of \mathbb{F} -measurable stopping times in the set A
Ω	Sample space
$\{X_t\}_{t \geq 0}$	Stochastic process X
$\underline{1}$	Column vector of 1s
$\underline{1}_k$	Column vector with 1 in row k and zero elsewhere
E_a	Matrix $E - \Lambda_a$
I	Identity Matrix

CHAPTER 1 INTRODUCTION

This thesis is concerned with some issues arising in relation to the capital structure and pricing of credit risk of a firm partly financed by debt. This subject remains of topical interest in financial mathematics particularly in view of recent financial market events. In early 2009, observed credit spreads¹ rose to levels not seen since the Great Depression of the 1930s. Poor lending standards, particularly in housing finance, coupled with a high level of leverage in the banking system, itself associated with the introduction of new and innovative financial instruments, resulted in significant financial distress. What had been regarded as *normal* financial market relationships appeared to breakdown, significantly exposing potential weaknesses in both asset pricing models and financial risk systems.

The primary objective of this thesis is to contribute to the understanding of the underlying mechanisms which determine firm capital structure and credit spreads. Ever since Modigliani and Miller [1958], economists and financial mathematicians have devoted considerable effort to understanding firms' financing policies. *Structural models* of credit risk provide a framework within which to explore issues related to corporate capital structure, and, for this reason, occupy a pivotal role in this report. The structural approach to credit risk attempts to model the default event² of a firm under varying conditions upon assuming the assets of the firm obey some given stochastic dynamics. Structural models are characterised by a default event occurring when the value of a firm's assets fall below some boundary. The so called *reduced-form* credit approach, which attempts to model the default probability of a firm directly,

¹The term credit spread refers to interest paid by an entity in excess of the risk free interest rate.

²Default event refers to a situation in which an entity is unable, or unwilling, to repay contractual debt obligations.

will not be considered. In Chapter 2, following a brief description of corporate bonds terminology, a literature review of the major developments in structural models of credit risk is presented.

That historically observed financial market behaviour has been so at odds with recent experience has led to the question as to whether existing models are flexible enough to capture financial market behaviour in a wide range of circumstances. Being necessarily an abstraction, arguably financial models may never be able to capture the complexity of the *real world*. Recent interest in the literature to model economic regime shifts has largely been motivated by the work of Hamilton [1989] in which it was suggested that economies are subject to period regime-shifts between distinct business cycles. In the latter regard, there may be some advantage in viewing financial markets as operating in a variety of different regimes at different times. Allowing model parameters that are otherwise typically fixed, to vary according by regime, may provide a convenient mechanism to capture changing behaviour in an elemental way and provide a framework within which to accommodate a wide range of financial market conditions.

The topic of endogenously determined optimal capital structure also remains an active area of research. Understanding how and when companies refinance and the associated implications for default risk continues to be of significant interest. In an effort to understand long term financing behaviour, and to avoid the technical complications related to time dependencies, many authors have sought to examine optimal capital structure in the context of an assumption of a stationary, or time-independent, debt framework³. The so called *roll-over debt model* introduced by Leland [1994b], which will be outlined in Chapter 2, has become the benchmark upon which many recent contributions in the area of structural credit models have been based. The main advantage of the roll-over debt framework, as will be explained in sequel, is that it affords a mechanism to analyse corporate debt profiles of differing structure or maturity.

³In a stationary debt framework, a firm's debt repayments are constant through time.

Chapter 3 builds on the roll-over debt framework literature by presenting a model of corporate capital structure under a *regime-switching* setting, which, potentially, affords a convenient mechanism to be able to combine a number of different market conditions in a simple and intuitive way. The model will focus on only two regimes, but allows firm asset volatility, risk free interest and dividend rates to be regime-dependent, via the firm's asset value process being modelled as a regime-switching geometric Brownian motion⁴. Tax relief on interest payments is known to be an important element in determining corporate capital structure, since it effectively generates a stream of positive future cash flows that add to a firm's net worth. Indeed, the essence of the contribution of Modigliani and Miller [1958] is that, without any tax advantages, there may be little incentive for corporate entities to leverage their balance sheets at all. In Chapter 3 the case of also allowing tax rates and bankruptcy costs to vary by regime is investigated. Default is modelled as the first time that the asset value process of the firm falls below a barrier, also regime-dependent, whether given exogenously and fixed, or, endogenously determined by the firm in order to optimise equity value. An optimised capital structure and associated financing costs are identified.

In Chapter 4, the model of Chapter 3 is extended to allow the asset price process of the firm to jump in the negative direction. Moreover, the jump process is also allowed to be regime-dependent motivated by the conjecture that during times of economic stress, negative jumps of the asset process might be expected to be larger in magnitude. Unlike Chapter 3, however, the firm in Chapter 4, is financed by a single infinite maturity, or consol, bond. While the latter assumption unfortunately loses the richness of term structure inherent in the roll-over model, it implies that the debt profile is fully stationary, which allows an optimal capital structure to be identified. The model also allows firm asset volatility, risk free interest and dividend rates as well as tax rates and bankruptcy costs to be regime-dependent. An interesting

⁴A geometric Brownian motion where one, or more, of the process parameters are regime-dependent

feature of the model is that the *payout*⁵ a firm might expect to receive on default also becomes regime-dependent, which has the implication that, under certain conditions, it may never be optimal for a firm to endogenously declare default in certain regimes. When an optimal default boundary is known to exist, optimal capital structure and credit spreads are also identified.

Naturally, the assumption of time-homogeneity outlined above affords significant analytical benefits and allows insights to be gained in relation to aspects of corporate capital structure that persist through time. However, the fact that the vast majority of lending agreements mature at a fixed date implies that there may also be significant time dependencies to consider in relation to the determination of capital structure. Not only does virtually all debt issued by corporate entities mature at a fixed date, finite maturity debt contracts are also prevalent in areas such as private equity, infrastructure finance and the insurance industry.

For the above reason, in Chapter 5, attention is turned to focus on a firm financed by finite maturity debt. As in the earlier chapters, the firm is assumed to be endowed with an asset process, but one which is modelled by a single geometric Brownian motion. The firm still receives tax advantages on coupon payments and is assumed to choose a default policy which maximises the equity value of the firm or structure. The main implication of time dependence in the structure of debt for the model in Chapter 5 is that the endogenously determined level of asset at which it is optimal to declare default is also time dependent. The primary focus of the analysis, is therefore, in identifying the time dependent default boundary for a firm's asset process.

It has been typical of structural models in the recent literature to restrict the benefit accrued to a firm in terms of tax relief on coupon payments to situations in which the asset value of a firm is in excess of some boundary level. The latter restriction has typically been motivated by the observation that tax relief on interest payments is only afforded to companies in many cases under the proviso that positive profits

⁵The payout on default will include the value of debt that the firm would otherwise have been obligated to pay.

are being generated. While such an assumption is entirely reasonable, in view of the complications of incorporating regime shifts into the analysis described above, the so called *tax-threshold* restriction on tax relief related to debt servicing costs will not feature in the analysis presented in this thesis, and in the context of regime-switching models, remains an area for future research. Other such potential areas of future study arising from the conclusions of this thesis are offered in Chapter 6 along with some concluding remarks.

CHAPTER 2 STRUCTURAL MODELS OF CREDIT RISK

2.1 Modelling Credit Risk

The term *to default* refers to the event under which a counterpart to a financial contract fails to meet an obligation under the terms of the contract. An obvious example would be that of a debtor failing to meet a payment specified in a loan agreement. More generally, the term *credit risk* refers to an event which directly affects the likelihood of a default occurring. Examples of the latter might be an increase in the amount of debt owed by a company or perhaps a deterioration in the economic performance of a company, both of which may impair the ability to repay debt. Naturally one might expect that higher credit risk should be associated with an increased cost of borrowing.

There have been two distinct approaches taken towards pricing securities which are subject to credit risk. The so called *reduced form* or *hazard rate* approach is based on the idea that default events occur by surprise and therefore focuses on modelling the probability of default directly (see for example Turnbull and Jarrow [1995], Lando et al. [1997], Duffie and Singleton [1999] and the associated literature). Typically, the default event is modelled using a *hazard process* where the *intensity* of the process reflects the infinitesimal likelihood of a default. The intensity associated with the default of a given financial contract may be, for example, estimated by an empirical analysis of historical default events. Such models will not be considered in this report.

By contrast, *structural* models concentrate on modelling financial characteristics of the borrower which may influence whether a default event occurs. The probability

of default is determined within the framework of the model itself, and in this sense, the structural approach attempts to model not only *when* a default will occur but also *why* a default occurs. Section 2.3 onwards of this chapter will focus on the major developments in the literature on structural models as they pertain to pricing corporate bonds. Before doing so, some salient terminology and definitions relating to corporate bonds are outlined.

2.2 Corporate Bonds

A *bond* is a financial contract under which a borrower commits to making future repayments in return for receiving a payment when the contract is issued. The *maturity date* of a bond refers to the date of the final promised payment of the bond and the *maturity* to length of time until that final payment. The *par value* or *face value* of a bond is repaid at maturity. Typically, a bond will pay a sequence of smaller intermediate payments at regular intervals, for example annually, called *coupons*, which may be either a fixed or variable amount. Bonds which pay no coupons are referred to as *discount* bonds. Bonds which never mature and pay coupons in perpetuity are referred to as *perpetual* or *consol* bonds. The price paid to acquire ownership of a bond is referred to as its *market value*.

Corporate bonds are issued by companies to raise funds and form part of the capital structure of a company. Since a corporation may fail to deliver the payments promised under the terms of a bond, corporate bonds are subject to risk of default and therefore to credit risk as well. Clearly, a bond cannot be defaulted upon after its maturity date.

2.2.1 Recovery Rules Under the circumstances of a default a creditor will attempt to recover debt which is owed, the sum recovered being referred to as the *recovery value*. In practice, the procedures surrounding the recovery process can be very complex and will depend upon both the terms of the financial contract which

has been defaulted upon, together with applicable laws in force in the jurisdiction of the default. For example, certain debt may be classified as *senior*, which will receive priority over *junior* debt, and may therefore attract a higher recovery value. Some classes of debt may be *secured* against another asset allowing the debtor to take full or partial control of the asset, while other debt may be *subordinated*, meaning that it is not secured or has a lower priority than another claim.

Factors such as those above will effect the *recovery payment* which is the amount that will be paid to creditors following a default. Typically, the recovery payment is defined in terms of a *recovery rate* which specifies the fraction of some valuation of the debt outstanding which will be paid to the creditor should a default occur. In practice, a wide variety of recovery rules are applied. For example, the recovery amount can be paid for valuation at either maturity of the debt, or at the time of default, and the recovery rate may be applied to either the face or market value of the debt. The *loss given default* refers to loss in value suffered by the creditor following a default and is often specified as one minus the recovery rate.

2.2.2 Exogenous Default and Safety Covenants Borrowers are often forced to incorporate covenants into a bond contract which afford creditors a certain degree of protection over and above the commitment of the borrower to repay its debt. For example, creditors may be afforded the right to take control of a company should the net worth of a company fall below a specified level. In this case, the decision to default is referred to as being *exogenous* to the company. A decision taken by a company to default of its own volition is referred to as an *endogenous* default.

2.3 Structural Models

The rest of this chapter is concerned with the major developments in the literature on structural models of corporate default. The majority of the models in the literature share the same following general framework.

2.3.1 General Framework A single firm is assumed to hold *tradeable*¹ assets, the total value of which can be represented by a single stochastic process, $V = \{V_t\}_{t \geq 0}$. The *value of the firm* refers to the total value of all future contingent claims of the firm. The firm is partly financed by debt with a value process given by $D = \{D_t\}_{t \geq 0}$ which is also a tradeable asset. The *Equity* value of the firm is simply the value of the firm minus the total value of debt and the holders of equity are referred to as the *shareholders*. For the most part, shareholders and the firm can be regarded as synonymous. Firms can raise finance either by issuing more debt or dilute existing shareholders by selling more equity. Some models will assume the existence of taxes and costs associated with bankruptcy as well as restrictions on the ability of firms to raise additional finance.

The structural approach to pricing corporate bonds is concerned with directly modeling the capital structure of a firm. The behaviour of the asset price and the associated implications for the net worth of the firm in relation to the level of debt outstanding, is naturally a key determinant of the probability that a firm will default. However, as will be seen, restrictions on how firms can raise additional finance, bankruptcy costs and taxes all play key roles in the valuation of corporate debt.

2.3.2 Technical Considerations The market is assumed to contain a savings account process with the value at time t of a deposit maturing at time T given by

$$B(t, T) = \mathbb{E} \left[\exp \left(- \int_t^T r(s) ds \right) \middle| \mathcal{F}_t \right], \quad (2.3.1)$$

where $r = \{r_t\}_{t \geq 0}$ is the risk free interest rate process and $\mathbb{E}[\cdot]$ is the expectation taken under the risk neutral measure. In many cases r will be assumed to be either constant or deterministic. It is also assumed that the stochastic processes (V, B) are defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ where $\mathbb{F} = \{F_t\}_{t \geq 0}$ denotes the completed filtration generated by (V, B) . In addition, the financial market is assumed

¹Tradeable assets can be hedged in the financial market and therefore priced using standard no arbitrage pricing techniques.

to be frictionless² and complete³.

2.3.3 Merton (1974) In a seminal paper by Merton [1974], it was shown that the value of a finite maturity discount debt obligation of a firm, which can only default at maturity, can be characterised as the value of a risk free bond minus the value of a European put on the assets of the firm with a strike equal to the nominal value of the debt. The classical solution to the European option problem in the case of asset prices being assumed to follow a log-normal distribution, originally outlined by Black and Scholes [1973], was then applied to provide an explicit formula for the value of a corporate discount bond.

The rich literature on the subject of corporate capital structure is testimony to the insight provided by the model in Merton [1974] which captures the very essence of the problem of pricing corporate debt. In view of the central role played in the literature, the model presented in Merton [1974] is outlined below using the now familiar techniques of risk neutral pricing. Expressions for bond prices, yields and credit spreads are derived and the sensitivity of these metrics to changes in leverage, asset volatility and interest rates, as well as implications for the term structure are examined.

The value of the assets of the firm is assumed to evolve under the risk neutral measure according to a geometric Brownian motion

$$\frac{dV_t}{V_t} = (r - \delta) dt + \sigma dW_t, \quad (2.3.2)$$

where $r > 0$ is a constant risk free rate, $\sigma > 0$, $W = \{W_t\}_{t \geq 0}$ is a standard Brownian motion, $\delta > 0$ represents the dividend rate and $V_t = v$. The firm is partly financed by a single discount bond with face value fixed at K which is non-callable and matures

²A frictionless market supports unlimited borrowing and lending at the same interest rate, allows short sales of all assets with full use of proceeds, has no transactions costs or indivisibilities of assets and trades continuously in time.

³The market is complete if every contingent claim can be hedged by a tradeable asset.

at time $T > 0$. The firm can only default at time T if $V_T < K$, in which case the residual assets become the property of the bond holder.

According to standard financial theory, the value of the firm's debt at time t will be given by

$$\begin{aligned}
 D(v, t, T) &= e^{-r(T-t)} (\mathbb{E}_{t,v} [\mathbb{1}_{\{V_T > K\}} K] + \mathbb{E}_{t,v} [\mathbb{1}_{\{V_T \leq K\}} V_T]) \\
 &= K e^{-r(T-t)} - e^{-r(T-t)} \mathbb{E}_{t,v} [(K - V_T)^+] \\
 &= KB(t, T) - P(v, t, T),
 \end{aligned} \tag{2.3.3}$$

where $(\cdot)^+ = \max(\cdot, 0)$, $\mathbb{1}_{\{A\}}$ is the indicator function of the set A , $\mathbb{E}_{t,v} [\cdot] = \mathbb{E} [\cdot | V_t = v]$ and

$$P(v, t, T) = e^{-r(T-t)} \mathbb{E}_{t,v} [(K - V_T)^+]$$

is the value of a European put option on the assets of the firm with strike equal to the face value of debt, in this case fixed at K , also maturing at time T .

Equation (2.3.3) shows the value of the debt is equal to a risk free discount bond maturing at time T minus the value of a put option on the assets of the firm with a strike equal to the value of the debt with maturity T . If the probability of default is zero, the price of a corporate bond is equal to the price of a risk free discount bond. If the probability of default is one, a corporate bond represents a claim on the assets of the firm. Clearly, as the probability of default increases, a corporate bond becomes more and more like a claim on the firm's assets.

Using (2.3.2), the value of the firm's debt in equation (2.3.3) can be expressed explicitly as

$$D(v, t, T) = V_t (L_t \Phi(\zeta_2) + \Phi(-\zeta_1)), \tag{2.3.4}$$

where leverage⁴ is defined as $L_t = \frac{KB(t,T)}{V_t}$,

$$\zeta_1 = \frac{-\log(L_t) + (r - \delta + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$\zeta_2 = \zeta_1 - \sigma\sqrt{T - t},$$

and $\Phi(\cdot)$ is the standard normal distribution. Merton [1974] worked primarily with an adjusted bond price

$$\begin{aligned}\bar{D}(v, t, T) &= D(v, t, T) / B(t, T) \\ &= \Phi(\zeta_2) + L_t^{-1}\Phi(-\zeta_1),\end{aligned}\tag{2.3.5}$$

which is the discounted value of the risky bond price, a quantity which features in the formula for the credit spread or term premium associated with the firm.

The yield of the firm's debt at time t which matures at time T , $y(v, t, T)$, is defined by the relation $D(v, t, T) = Ke^{-y(t,T)(T-t)}$ from which it follows that

$$y(v, t, T) = -\frac{\log(D(v, t, T) / K)}{T - t}.\tag{2.3.6}$$

The term premium or the credit spread at time t for a bond which matures at time T , $s(v, t, T)$, which is the additional yield a firm must offer on risky debt over and above the risk free interest rate, is given by

$$\begin{aligned}s(v, t, T) &= y(v, t, T) - r \\ &= -\frac{\log(\bar{D}(v, t, T))}{T - t},\end{aligned}\tag{2.3.7}$$

where the latter equality follows directly from (2.3.4).

One of the key characteristics of Merton [1974] is that credit spreads either converge to zero or infinity as time converges to maturity. To see this observe from (2.3.4) and

⁴A firm is typically regarded as leveraged when the value of debt exceeds the value of the firm.

(2.3.7) that

$$\lim_{t \rightarrow T} \Phi(-\zeta_1) = \begin{cases} 1, & \{V_T < K\}, \\ 0, & \{V_T > K\}, \end{cases}$$

$$\lim_{t \rightarrow T} \Phi(\zeta_2) = \begin{cases} 0, & \{V_T < K\}, \\ 1, & \{V_T > K\}, \end{cases}$$

from which it easily follows that

$$\lim_{t \rightarrow T} s(v, t, T) = \begin{cases} +\infty, & \{V_T < K\}, \\ 0, & \{V_T > K\}. \end{cases}$$

The cause of this predicted behaviour of spreads is the assumption of a geometric Brownian motion for the asset price process, (2.3.2), from which the continuity properties of Brownian motion imply that the asset value at maturity will be revealed immediately prior to maturity. Once the asset value at maturity is known, the default event becomes predictable.

Table 2.3.1 details the static sensitivities of bond prices, yields and credit spreads to changes in leverage, asset volatility and interest rates. As would be intuitively expected, bond prices/yields fall/rise in response to increases in leverage, asset volatility and interest rates. Higher interest rates make distant cash flows less valuable and therefore reduce bond prices (increase yields). Greater volatility or leverage both act to make default more likely, and therefore, a higher yield (lower bond price) is required to compensate for increased risk, which is also reflected in higher credit spreads. Contrary to yields however, higher interest rates act to reduce credit spreads.

2.3.4 Critiques of Merton's Approach The approach taken in Merton [1974] is attractive from the perspective of analytic tractability but has been criticised in terms of the unrealistic nature of the required assumptions of the model. In particular, the model assumes that asset price returns are normally distributed and that the asset price process is continuous, implying that for un-leveraged firms credit spreads are

	Bond Price (D)	Yield (y)	Spread (s)
<hr/> <hr/> Leverage <hr/>			
$\partial/\partial L$	≤ 0	≥ 0	≥ 0
<hr/> <hr/> Volatility (σ^2) <hr/>			
$\partial/\partial (\sigma^2)$	≤ 0	≥ 0	≥ 0
<hr/> <hr/> Interest Rate (r) <hr/>			
$\partial/\partial r$	≤ 0	≥ 0	≤ 0

Table 2.3.1: Merton (1974) Comparative Static Analysis

This table shows the directionality of the sensitivity of Merton [1974] bond prices, yields and credit spreads to changes in leverage, asset volatility and interest rates.

approximately zero for very short maturities which is wholly at odds with observed data. Furthermore, default may only occur in the model at maturity, and, only if the asset price is less than outstanding nominal debt. Whereas in practice, firms routinely declare bankruptcy prior to the maturity of their debt. In addition, both volatility and interest rates are assumed to be constant which also implies that the probability of default is independent of the risk free term structure. Since default depends on the economic performance of the firm, which may in turn be related to the economy as a whole, the latter assumption would appear to be unrealistic.

In response to these criticisms, Merton [1974] has been extended in the literature in a number of ways. For example, Geske [1977, 1979] applied methodology associated with valuing compound options to address the issue of multiple bonds, each maturity being viewed as one of the compound options. Ho and Singer [1982] examine the effect of a number of bond indenture provisions, concluding that risk is transferred from stockholders to bondholders, as the tenor and size of debt increase.

One major difficulty is the unobservable nature of the asset value process which may be contrary to the assumption in Merton [1974] that the asset value can be hedged in the market. Moreover, difficulties arise from an empirical standpoint in relation to calibrating models to market data. Since data for the asset process is not normally available, typically some filtering method is used to extract estimates of the volatility

of the asset value from proxy data, normally equity prices. As an alternative to modelling the value of a firm's assets, Buffet [2000] instead proposed to maximise the value of equity based on assumed stochastic dynamics of a firm's profits. As a result, expressions for the value of debt were obtained that depended only on the current value of assets and the stochastic dynamics of profits, both of which are observable in the market.

Perhaps the most important extensions to Merton [1974] have been to allow for both early and endogenously determined default, the addition of stochastic risk free interest rates and for discontinuities in the asset price process. In the subsequent sections some of the major contributions which have been suggested in the literature in these regards are reviewed.

2.4 First Passage Models

First passage models allow default to occur at any time prior to the maturity of debt. Such models focus on the first passage time of a random process related to a relevant financial metric, typically the net worth of the borrower, below some specified barrier level. The barrier may be deterministic or random and either determined exogenously or endogenously to the firm.

The introduction of first passage times greatly enriches the structural approach in allowing not only the probability, but also the timing of the default event to be determined within the context of the model. In addition, more realistic recovery payments can be modelled by specifying that the securities of a firm take on specific values at the time of first passage.

First passage models for credit risk were first introduced by Black and Cox [1976] and have subsequently been extended by many authors. See for example, Longstaff and Schwartz [1995], Cathcart and El-Jahel [1998], Hsu et al. [2004], Zhou [1997] amongst many others. In addition, first passage models laid the foundations for developments

in relation to optimal capital structure⁵. In view of the importance of the contribution of Black and Cox [1976] to the understanding of credit risk, the model is outlined below.

2.4.1 Black and Cox (1976) In Merton [1974] the value of the firm could rise and fall to arbitrarily high and low levels without triggering any reorganisation of the finances of the firm. Black and Cox [1976] suggested that this was unrealistic and argued that some mechanism to force a reorganisation, should the firm perform poorly according to some set standard, should be adopted.

One standard for enforcing a reorganisation of the firm prior to maturity could be the omission of interest payments on the debt. However, such a restriction would not be very effective if the company were allowed to sell assets to meet interest payments. For this reason Black and Cox [1976] introduced boundaries for the value of the firm's assets at which the securities of the firm could take on specific values, and in doing so, extended Merton [1974] to allow for safety covenants.

To incorporate safety covenants, Black and Cox [1976] introduced a time dependent deterministic barrier, $\hat{K}e^{-\gamma(T-t)}$ for some $\hat{K}, \gamma > 0$, and assumed that default occurred if the value of the firm's assets fell below this barrier prior to maturity. Following Merton [1974], it was also assumed that the firm would only default at maturity should the face value of debt exceed the value of the firm.

More specifically, for a firm financed by a single bond with face value K maturing at time T , default occurred the first time, τ , that the value of the firm falls below a barrier $b(t)$

$$\tau_b = \inf \{t \in [0, T] \mid V_t < b(t)\}, \tag{2.4.1}$$

⁵See 2.5 below.

where

$$b(t) = \begin{cases} \hat{K}e^{-\gamma(T-t)} & \text{for } t < T \\ K & \text{for } t = T, \hat{K} \leq K. \end{cases}$$

The usual convention that $\inf \{\emptyset\} = \infty$ is adopted in (2.4.1) meaning that, if the value of the firm does not fall below the barrier, default does not occur. It is natural to assume that the barrier is no greater than the face value of the debt.

Under the assumptions in Black and Cox [1976], the value of a discount bond with face value K maturing at time T is given by

$$\begin{aligned} D(v, t, T) &= \mathbb{E}_{t,v} [Ke^{-r(T-t)} \mathbb{1}_{\{\tau_b \geq T, V_T \geq K\}}] + \mathbb{E}_{t,v} [V_T e^{-r(T-t)} \mathbb{1}_{\{\tau_b \geq T, V_T < K\}}] \\ &+ \mathbb{E}_{t,v} [\hat{K}e^{-\gamma(T-\tau)} e^{-r(\tau-t)} \mathbb{1}_{\{\tau_b < T\}}]. \end{aligned} \quad (2.4.2)$$

The first two terms in (2.4.2) are analogous to Merton's model. The last term in (2.4.2) reflects the payment the debt holder receives should default occur prior to maturity. Under the assumption that the value of the firm obeys (2.3.2), the expectations in (2.4.2) can be readily explicitly calculated using standard results pertaining to the first passage times of Brownian motion.⁶

Black and Cox [1976] observed that the basic properties of company debt such that $D(v, t, T)$ is an increasing function of v and a decreasing function of σ^2 and r remain the same as in Merton [1974]. However, while in Merton [1974], the value of a corporate debt can be zero at maturity, Black and Cox [1976] showed that safety covenants provide a floor for the value of the bond which limits the gains that can be made by stockholders by somehow circumventing other indenture provisions.

Firms frequently issue debt with different levels of seniority whereby the claim of a junior debt holder will be subordinated to that of a senior debt holder. In the event of a default, the junior debt holders will only receive a payment after the senior debt holders have been paid in full. Such an arrangement is often referred to as the

⁶The explicit formula is somewhat complex and is therefore omitted here for purposes of clarity.

Claim	$V_T < K_s$	$K_s \leq V_T \leq K_s + K_j$	$V_T > K_s + K_j$
Senior Bonds	V_T	K_s	K_s
Junior Bonds	0	$V_T - K_s$	K_j
Equity	0	0	$V_T - K_s - K_j$

Table 2.4.1: Debt Subordination: Values of Claims at Maturity

This table shows the shareholder and bondholder claims at maturity where total debt outstanding is comprised of senior, K_s , and junior, K_j , under the strict priority rule.

strict priority rule: The senior debt holders having strict priority over the junior debt holders.

For example, assume that the principal values of senior and junior debt outstanding are K_s and K_j respectively. At maturity, the payoff to the respective classes of bonds will be as shown in Table 2.4.1

Black and Cox [1976] showed the value of junior debt could easily be written in terms of the difference between value of total and senior debt. Using (2.4.2) and the notation $D(v, t, T, K)$ for the value of a $(T - t)$ maturity corporate bond with principle value K , the total value of debt is given by $D(v, t, T, K_s + K_j)$, whereas the value of senior debt is given by $D(v, t, T, K_s)$. Therefore, the value of junior debt is clearly equal to

$$D(v, t, T, K_j) = D(v, t, T, K_s + K_j) - D(v, t, T, K_s).$$

Black and Cox [1976] observed that subordination affords senior bond holders larger value than would be obtained from an undifferentiated bond issue. Junior bonds, however, have characteristics which are different from those normally associated with bonds. While the value of senior bonds is always an increasing function of asset value, the same is not true of junior bonds. This means that bondholders as a group may under certain circumstances have conflicting interests. In highly leveraged circumstances, junior bondholders may be incentivised to act more like shareholders than senior bondholders.

2.5 Optimal Capital Structure

A debt financed firm which is not constrained in any way by safety covenants will only declare bankruptcy when required repayments cannot be made by issuing new equity. The latter case will only occur when the value of equity falls to zero. However, under the conditions of an absolute priority rule, any level of the assets of the firm that triggers bankruptcy will also imply that the value of equity at that asset value is zero.

This latter observation led Black and Cox [1976] to argue that restrictions on a firm's financing arrangements could mean that there is a critical value of the assets of a firm below which no additional equity can be sold. For example, in the case where a firm pays a continuous coupon, should the value of the firm fall below the coupon, under a strict priority rule, equity value may fall to zero even though the value of the firm's assets may still be substantially positive. Since the firm will be unable to sell more equity, if raising funds by selling assets is prohibited, the firm will be forced into bankruptcy.

The level of assets at which a firm is declared bankrupt can either be specified exogenously, for example as a term in the indentures of the bond, or determined endogenously by the firm in the course of some optimisation procedure. Black and Cox [1976] derived the optimal value of assets at which to declare bankruptcy for a firm financed by a single continuous (fixed) coupon perpetual bond for an asset process given by (2.3.2) with $\delta = 0$ based on the assumption of absolute priority. Under these assumptions, Black and Cox [1976] were also able to show that the value of debt is always less when asset sales are permitted than when they are not. Leland [1994a] replicated and extended Black and Cox [1976] in an important paper which will now be outlined in some detail.

2.5.1 Leland (1994) Following Black and Cox [1976], Leland [1994a] also considered the case of a firm financed by a single consol bond, of face value P , paying

a continuous fixed coupon rate, c , so long as the firm remains solvent, where the assets of the firm obey the stochastic dynamics given in (2.3.2) with $\delta = 0$. The latter condition ensures that any net cash outflows from the firm must be financed by selling additional equity, while the chief advantage of assuming a constant perpetual coupon rate is that the value of debt has no time dependency. It will become apparent in sequel that the assumption of a time homogeneous debt structure represents a significant modelling advantage.

Leland [1994a] also extended Black and Cox [1976] in two ways. First, a fraction $\xi \in [0, 1]$ of asset value is lost to costs associated with reorganising the firm if bankruptcy occurs. Second, the firm benefits from tax relief at a rate $\pi \in [0, 1]$ on coupon payments made prior to default. The latter assumption means the the firm receives an additional continuous income stream at a fixed rate equal to $c(1 - \pi)$ where $0 \leq \pi \leq 1$ is a constant tax rate.

Default occurs the first time, τ_b , that the value of assets falls below a constant barrier, b ,

$$\tau_b = \inf \{t \geq 0 \mid V_t < b\}. \quad (2.5.1)$$

The convention that $\inf \{\emptyset\} = \infty$ is adopted in (2.5.1). Default does not occur if the value of the firm does not fall below the barrier.

The value of debt, $D(v, b)$, is the present value of the sum of all coupon payments paid up until the time of bankruptcy plus the residual value of the assets of the firm should default occur,

$$D(v, b) = \mathbb{E}_v \left[\int_0^\infty cP e^{-ru} \mathbb{1}_{\{u < \tau_b\}} du \right] + \mathbb{E}_v \left[(1 - \xi) e^{-r\tau_b} V_{\tau_b} \right] \quad (2.5.2)$$

$$= \frac{cP}{r} + \left((1 - \xi) b - \frac{cP}{r} \right) \left(\frac{v}{b} \right)^{-\frac{2r}{\sigma^2}} \quad (2.5.3)$$

where ξ is cost of reorganising the firm on default.

The value of the firm, $F(v, b)$, is made up of the assets of the firm and the present

value of future tax rebates minus the reorganisation costs incurred in the event of bankruptcy

$$\begin{aligned} F(v, b) &= v + \mathbb{E}_v \left[\int_0^\infty (1 - \pi) cP e^{-ru} \mathbb{1}_{\{u < \tau_b\}} du \right] - \mathbb{E}_v \left[\xi e^{-r\tau_b} V_{\tau_b} \right] \\ &= v + \frac{\pi cP}{r} \left(1 - \left(\frac{v}{b} \right)^{-\frac{2r}{\sigma^2}} \right) - \xi b \left(\frac{v}{b} \right)^{-\frac{2r}{\sigma^2}}, \end{aligned} \quad (2.5.4)$$

where π is the tax rate.

The explicit formulas in (2.5.2) and (2.5.4) follow directly from well known results pertaining to the first passage time of a Brownian motion with drift to a constant boundary ⁷.

The equity value of the firm, $Q(v, b)$, is simply the total value of the firm minus the value of debt

$$\begin{aligned} Q(v, b) &= F(v, b) - D(v, b) \\ &= v - (1 - \pi) \frac{cP}{r} + \left((1 - \pi) \frac{cP}{r} - b \right) \left(\frac{v}{b} \right)^{-\frac{2r}{\sigma^2}}. \end{aligned} \quad (2.5.5)$$

It is interesting to note from (2.5.5) that cost of reorganisation on bankruptcy does not feature in the value of equity and is thus borne entirely by the bond holders.

In the case where there are no protective covenants, the firm will endogenously determine the optimal asset level at which to declare bankruptcy. An inspection of (2.5.5) indicates that equity value will be maximised by setting b as low as possible. However, as discussed above, the strict priority rule prevents b from being arbitrarily small because $Q(v, b)$ must be non-negative for all $v > b$. Leland [1994a] observes that since $Q(v, b)$ is convex in v when $b < (1 - \pi) cP/r$, the lowest possible value for b consistent with positive equity value for all $v > b$ must be consistent with the *smooth pasting* condition, $dQ(v, b)/dv|_{v=b} = 0$, which applied to (2.5.5), yields the

⁷See for example Borodin and Salminen [2002].

optimal default boundary

$$b^* = \frac{(1 - \pi) cP}{r + \frac{1}{2}\sigma^2}. \quad (2.5.6)$$

The optimal default barrier is proportional to the coupon rate, c , decreases in the corporate tax rate, π , the risk free interest rate, r , and the riskiness of the firm, σ^2 , but is independent of the bankruptcy costs, ξ , and the current value of assets, v . The latter fact is significant in that it allows the barrier to be estimated from the coupon payment without needing to know the firm's current asset value. And, as expected, $Q(b^*, b^*) = (1 - \xi)b^*$. Using (2.5.6), (2.5.5) and (2.5.4), and (2.5.2) closed form equations for the values of the firm, equity and debt can also easily be derived⁸.

Table 2.5.1 is an excerpt from Leland [1994a] which summarises the effects on debt value, yields and credit spreads of changes in the coupon rate, volatility of assets, risk free interest rate and current asset value⁹. Since Leland [1994a] is an extension of Black and Cox [1976] and Merton [1974], it is to be expected that the comparative statics of the model are broadly in line with prior results, and, for *low leverage*¹⁰ firms, this is indeed the case. Bond prices/yields fall/rise in response to increases in leverage (i.e. lower asset value), asset volatility and interest rates. Credit spreads also increase as the probability of default rises (higher asset volatility or lower assets) but fall as interest rates increase.

However, as the asset value approaches the default boundary, meaning that the firm becomes more *leveraged*, provided that either bankruptcy costs or the tax rate are positive, the effect of higher asset volatility and risk free interest rates are reversed: higher asset volatility and risk free interest rates increase the value of debt (and reduce yields). In addition, for highly leveraged firms, an increase in volatility can

⁸See the original paper, Leland [1994a] for formulae.

⁹The table in Leland [1994a] also includes the sensitivities of firm and equity value to the same variables as well as to changes in tax rates and bankruptcy costs. Higher/lower bankruptcy costs/taxes reduce the value of the debt.

¹⁰ Low leverage in the sense that v is substantially higher than b^* .

Variable	Sign of Change in Instrument for Increase In			
	c	σ^2	r	v
Bond Price $D(v, b^*)$	> 0	< 0	< 0	> 0
Yield $y(v, b^*)$	> 0	> 0	> 0	< 0
Spread $s(v, b^*)$	> 0	> 0	< 0	< 0

Table 2.5.1: Leland (1994) Comparative Static Analysis

This table shows the directionality of the sensitivity of Leland [1994a] bond prices, yields and credit spreads to changes in coupon rate, asset volatility, interest rates asset value under endogenous bankruptcy.

reduce credit spreads. Moreover, while as would be expected, bond prices are typically increasing in the coupon rate, for highly leveraged firms a higher coupon can cause bond prices to fall. Thus the behaviour of the bond prices of highly leveraged firms can be quite different to that of firms with a low level of leverage when either bankruptcy and/or taxes are positive¹¹.

The reason for these apparently counter intuitive results is that a lowering of the default boundary will make the arrival of bankruptcy costs less imminent and the value of debt will be very sensitive to such changes when v is close to b^* . Higher risk free rates, higher asset volatility and a lower coupon rate all act to lower b^* which has a dominating positive effect on $D(v, b^*)$ at values of v close to b^* . Leveraged debt holders are thus incentivised to act like shareholders.

2.5.2 Leland and Toft (1996) Leland and Toft [1996] extend Leland [1994a] to examine the optimal capital structure of a firm financed by debt with a specific maturity profile, by introducing a model which assumes that a firm refinances itself in a rather ingenuous way. At time t , it is assumed that debt of a given par value p is

¹¹The bonds of highly leveraged firms are typically referred to as 'junk' bonds or high yield bonds and normally trade at much lower prices than less leveraged investment grade bonds

redeemed, and new debt instantaneously issued, with the same par value, according to a specific maturity profile, with the face value of debt issued with maturity s given by $p\Psi(s)$ where $\Psi(s) \leq 1$ and $\int_0^\infty \Psi(s) ds = 1$. The latter approach, which is commonly referred to in the literature as *roll-over* debt, has subsequently been used by many authors. The primary advantage of the roll-over debt model is that, as is the case with a single infinite maturity bond, principal value of outstanding debt remains fixed at a given level P and debt service costs are time invariant¹². However, because debt is being issued with a specific maturity profile, the model is rich enough to permit an analysis of the term structure of debt.

Apart from the assumption of roll-over debt, the firm structure in Leland and Toft [1996] remains the same as Leland [1994a]. The value of assets is assumed to obey (2.3.2) with $\delta > 0$. On default, a proportion of assets $\xi \in [0, 1]$ is lost in the reorganisation of the firm and prior to default, the firm receives a tax relief benefit at a rate of $(1 - \pi)c$, where $0 \leq \pi \leq 1$. Default is again assumed to occur at time τ_b as in (2.5.1).

The price of a risky bond at time 0, paying 1 at maturity time t , is made up of the discounted value of coupon income up until maturity or default, the principle value of the bond if default does not occur before maturity and the recovery value should default occur prior to maturity given by

$$d(v, b, t) = \int_0^t \mathbb{E}_v [e^{-rs} \mathbb{1}_{\{s < \tau_b\}}] ds + \mathbb{E}_v [e^{-rt} \mathbb{1}_{\{t < \tau_b\}}] + \frac{(1 - \xi)}{P} \mathbb{E}_v [e^{-r\tau_b} V_{\tau_b} \mathbb{1}_{\{t \geq \tau_b\}}]. \quad (2.5.7)$$

The total value of all outstanding debt is then given by integrating (2.5.7) over all

¹²A formal description of the roll-over debt approach is outlined in Subsection 3.2.2.

possible maturity times

$$\begin{aligned}
 D(v, b) &= \int_0^\infty p\Psi(s) d(v, b, s) ds \\
 &= pc\mathbb{E}_v \left[\int_0^{\tau_b} e^{-rs} \widehat{\Psi}(s) ds \right] + p\mathbb{E}_v \left[\int_0^{\tau_b} e^{-rs} \Psi(s) ds \right] \\
 &\quad + \frac{(1-\xi)p}{P} \mathbb{E}_v \left[e^{-r\tau_b} V_{\tau_b} \widehat{\Psi}(\tau_b) \right], \tag{2.5.8}
 \end{aligned}$$

where $\widehat{\Psi}(t) := \int_t^\infty \Psi(s) ds$.

Leland and Toft [1996] assumed a form of the maturity profile was given by, δ_T , the Dirac delta function concentrated at time T . Leland [1994a] assumed that the maturity profile given by

$$\psi(t) = me^{-mt} \tag{2.5.9}$$

from which it follows that $\Psi(t) = m\widehat{\Psi}(t) = \exp(-mt)$, and in which case, the total value of debt simplifies to,

$$D(v, b) = \frac{(m+c)}{(m+r)} P\mathbb{E}_v [1 - e^{-(m+r)\tau_b}] + (1-\xi) \mathbb{E}_v [e^{-(m+r)\tau_b} V_{\tau_b}], \tag{2.5.10}$$

with the total value of the firm is given by

$$\begin{aligned}
 F(v, b) &= v + \mathbb{E}_v \left[\int_0^\infty (1-\pi) cPe^{-ru} 1_{\{u < \tau_b\}} du \right] - \mathbb{E}_v [e^{-r\tau_b} V_{\tau_b}] \\
 &= v + \frac{\pi cP}{r} \mathbb{E}_v [1 - e^{-r\tau_b}] - \xi \mathbb{E}_v [e^{-r\tau_b} V_{\tau_b}]. \tag{2.5.11}
 \end{aligned}$$

The firm is assumed to choose the bankruptcy level, b , to maximise the value of equity $Q(v, b) = F(v, b) - D(v, b)$. Leland and Toft [1996] also achieve this by invoking the *smooth fit* condition

$$\left. \frac{\partial Q(v, b)}{\partial v} \right|_{v=b} = 0$$

and solve to derive a constant barrier at which it is optimal for the firm to declare default. Using this barrier Leland and Toft [1996] then proceed to investigate optimal leverage, debt capacity and credit spreads deriving broadly similar conclusions to

Leland [1994a].

2.6 Stochastic Interest Rates

It is well known that empirical observation is broadly consistent with the negative relationship between risk free interest rates and credit spreads which is a feature of Merton [1974]¹³. Attempts to incorporate this feature have led many authors to assume that risk free interest rates obey stochastic dynamics of the general form

$$dr = \alpha(r_t) dt + \beta(r_t) dW_t. \quad (2.6.1)$$

In order to introduce a dependence between the risk free rate and the asset price process it has been suggested that the Brownian motions in (2.3.2) and (2.6.1) can be correlated. It is worth noting that this latter assumption has typically been implemented as a constant instantaneous correlation which has the obvious implication that the relationship between interest rates and asset valuations remains invariant through time.

There have been a number of popular interest rate models which are distinguished by the form of the drift and volatility terms in (2.6.1). Vasicek [1977] assumed that $\alpha(r_t) = \kappa(\theta - r_t)$ where $\theta, \kappa > 0$ and $\beta(r_t) = \sigma$ whereas in Cox et al. [1985], the so called CIR model, the drift term is the same but the volatility term is given by $\beta(r_t) = \sigma\sqrt{r_t}$. In the generalised Vasicek model, $\alpha(r_t) = \kappa(t)(\theta(t) - r_t)$ and $\beta(r_t) = \sigma_r(t)$ where $a, b, \sigma : [0, T] \rightarrow \mathbb{R}^+ \setminus \{0\}$ are deterministic functions.

Many authors have extended structural models to include stochastic interest rates. For example, Jamshidian [1989], Shimko et al. [1993], Briys and de Varenne [1997], Longstaff and Schwartz [1995], all assume the interest rates follow Vasicek dynamics whereas Wang [1999], Kim et al. [1993a], Cathcart and El-Jahel [1998] assume interest

¹³See Table 2.3.1.

rates follow the CIR model. Hsu et al. [2004], on the other hand, assumed that interest rates follow a simple Itô process driven by a Brownian motion.

Some authors have suggested that interest rates have significant effects on credit spreads. Longstaff and Schwartz [1995], for example, numerically verified model predictions that credit spreads exhibited a negative relationship with interest rates and that the durations of risky bonds can significantly depend on the correlation of a firm's assets with interest rates. Other authors have found weaker relationships between interest rate volatility and spreads ¹⁴.

2.7 Discontinuous Asset Price Processes

The assumption that assets follow a diffusive process in Merton [1974] results in credit spreads for solvent firms being close to zero for short maturities, which is at odds with empirically observed behaviour. Brownian motion is a continuous process which in turn implies that the asset value process associated with (2.3.2) is also continuous. As a result, immediately prior to maturity, the value of assets at maturity is already revealed, at which point, if the value of assets exceeds the level of debt, the bond becomes effectively risk free and the credit spread collapses to zero.

In order to overcome this issue many authors have suggested incorporating discontinuities into the asset value process. Intuitively, if the asset process can *jump* at unpredictable times, then even if the asset value exceeds the level of debt immediately prior to maturity, there may still be a chance that the asset process can jump below the level of debt before maturity. Positive credit spreads may be required to compensate for this *gap risk* even for apparently solvent firms. In some respects, structural models with jump risk represent somewhat of a reconciliation with reduced form models in the sense that the default event may come by surprise.

¹⁴See Kim et al. [1993b], Brennan and Schwartz [1980]

While discontinuities in asset price processes were studied as early as Merton [1976] they have only been applied in the context of structural models of credit risk more recently. Typically, in order to introduce jumps into structural models, the asset value process (2.3.2) is replaced by a process of the form

$$\frac{dV_t}{V_t} = (r - \delta) dt + \sigma dW_t + dJ_t, \quad (2.7.1)$$

where $\{J_t\}_{t \geq 0}$ is some *jumping* process. Naturally there is a wide range of modelling possibilities for the jumping process and the choice of such has varied amongst authors and differentiated recent contributions.

In terms of first passage models, Zhou [2001] extends Black and Cox [1976] to incorporate jumps by including a Poisson process into the asset value process and assuming a normal distribution for the value of the jumps. The resulting model is seen to be able to generate credit spreads and credit spread term structures which more closely reflect those observed empirically. In addition, recovery rates, rather than being exogenously specified, are linked to the firm value at default and specifically the *overshoot* of the asset value should default occur as a result of a jump in the asset value process.

The capital structure model of Leland and Toft [1996] was extended by Hilberink and Rogers [2002] and Surya and Kyprianou [2007] to the case of a general Lévy process with only downward jumps, finding that credit spreads do not collapse to zero for short maturities. Chen and Kou [2009] and Dao and Jeanblanc [2012] also extend Leland and Toft [1996] but allow for two-sided jumps where the jump size has a double-exponential density. Optimal leverage was found to be lower in the presence of downward jumps, and indeed, Chen and Kou [2009] observed that highly risky firms would optimally hold very little debt.

2.8 Regime-Switching Models

In the recent literature, there has been considerable interest in so called 'regime-switching' models which allow modelling parameters such as asset volatility to be

dependent of some state of the economy. There has been substantial empirical research reported in the literature going back as far as Hamilton [1989] suggesting that the aggregated economy is subject to period regime-shifts between distinct business cycles. It is natural, therefore, to conjecture that regime-shifts between different economic regimes may have an important influence on the capital decisions and credit profile of a firm. Academic interest in the relationship between capital structure and credit quality has only been increased by the advent of the recent global financial crisis.

Alexander and Kaeck [2008] empirically studied the impact of equity returns and volatility on Credit Default Swap (CDS)¹⁵ spreads and found strong evidence that theoretical determinants of credit spreads has a regime dependent behaviour. It was observed that during volatile regimes, credit spreads were more sensitive to changes in asset volatility, whereas during tranquil regimes, stock market returns were the influence on credit spreads. Similar results were reported by Tang and Yan [2010] who empirically studied the interaction between CDS spreads and GDP¹⁶ growth rates and the volatility of GDP growth rates, identifying investor sentiment at the aggregate level and asset volatility at the firm level as the most important determinants of credit spreads. However, evidence was also reported suggesting an important relationship between economic conditions and firm-specific characteristics. In particular, it was found that firms with cash flows, that had a positive correlation to GDP growth, exhibited lower credit spreads during economic expansions and vice versa during economic contractions.

In terms of capital structure, Lemmon et al. [2008] reported long-term stability and convergence of leverage ratios across sectors of the economy and a weak relationship between a number of firm specific financial metrics and leverage ratios, leaving a large unexplained residual. However, building on the latter study, Akhtar [2012], after incorporating the effect of four different stages of the business cycle, including

¹⁵Credit Default Swap (CDS) is a type of financial instrument where the seller insures the buyer against losses accrued as a result of a default type event effecting a firm.

¹⁶Gross Domestic Product.

expansion, peak, contraction and trough, found business cycles to be an important determinant of the unexplained variation in leverage ratios reported earlier by Lemmon et al. [2008]. In particular, it was found that business cycle phases become far more important in explaining variation in leverage after having controlled for the effect of firm specific financial characteristics. The latter study suggests that the business cycle may indeed have an important impact on firm capital structure decisions and therefore credit quality.

In one of the more extensive empirical studies conducted in relation to corporate default risk, Giesecke et al. [2011] reported on a 150-year of history of corporate default rates. Using a regime-switching framework, the latter study found that both stock market returns and stock market volatility have significant forecasting power in relation to default rates. In addition, it was found that changes in GDP have a strong relationship with subsequent default rates. The latter observations suggest that important relationships may exist between the macro economy and the firm credit quality. In a follow-up study, Giesecke et al. [2014] focused more specifically on the effects of banking crisis and default rates and found that, despite the size of the corporate bond market being in the same order of magnitude as the bank lending market, banking crisis appear to have much graver implications for the macroeconomic environment than crises in corporate bond markets.

Motivated by the above empirical justifications, a number of authors have recently proposed models of capital structure and credit risk which incorporate regime-switching features. Hackbarth et al. [2006] considered the effect of a macroeconomic shock on capital structure and credit risk. The firm's EBIT¹⁷ was modelled as the product of two random variables, one reflecting the aggregate state of the economy and one an idiosyncratic shock reflecting firm-level productivity uncertainty. The aggregate shock was assumed to take one of two values according to two-state Markov chain and firm level uncertainty was reflected as a geometric Brownian motion. Adopting the Leland and Toft [1996] rollover debt structure, Hackbarth et al. [2006] derived

¹⁷The firm's profitability defined as earnings before interest and taxes.

analytic expressions for equity, firm and debt values of a firm and then imposed a smooth-fit condition to derive endogenous default boundaries which maximised equity value. The authors found that the firm's optimal default policy was characterised by different default threshold in each regime. Default was more likely in the recessionary regime and credit spreads were higher.

Siu et al. [2008] investigated the pricing of CDS using a structural model where the asset process of the firm was assumed to be Markov modulated with the interest rate process, firm's asset volatility and drift being driven by a two state Markov chain to distinguish between *good* and *bad* states of the economy. Based on numerical calculations for assumed levels of asset volatility, drift and default boundaries, the authors conclude that CDS spreads increase substantially in the bad state of the economy characterised by higher asset volatility and lower asset returns. In addition, it was argued that the inclusion of regime-switching effects provide a potential mechanism to improve the underestimation of default probabilities which are often associated with the standard Merton structural model.

Optimal capital structure under regime-switching was also investigated by Elliott and Shen [2015] using a model in which the firm's asset process was modelled as a regime-switching geometric Brownian motion. As in the case of Siu et al. [2008], both the firm's asset price volatility and drift, as well as the risk free interest rate, were assumed to be modulated by an finite state Markov chain. The firm was, however, assumed to be financed by a single consol bond and to benefit from tax relief on coupon payments, adding a stream of positive cashflows to the value of the firm. Once again, assuming a two-state economy with regime shifts being controlled by a Markov chain, analytic expressions for optimal endogenously determined default barriers and the optimal capital structure were identified.

2.9 Summary

The initial seminal paper by Merton [1974] has spawned a rich literature on the subject of capital structure and credit risk. There have since been considerable advances made in terms of default at first passage and the inclusion of jumps into the asset value process. Leland [1994a], Leland and Toft [1996] made significant advances in terms of characterising the optimal capital structure of a firm financed by debt and thereby initiated a new line of research enquiry. However, while these initial advances were now made some time ago, in view of continued contributions in literature, it is clear that structural models of credit risk remain an active area of interest in financial mathematics research.

CHAPTER 3 ROLL-OVER DEBT UNDER REGIME-SWITCHING

3.1 Introduction

The topic of endogenously determined optimal capital structure remains an active area of research. Since its introduction by Leland [1994b], the roll-over debt model, outlined in Subsection 3.2.2 below, has been the motivation a large volume of research in the area of structural models of credit. The key facet of the roll-over debt framework is that debt of a given maturity profile is constantly being issued and redeemed in such a way that the outstanding nominal value of debt remains constant. The main advantage of the roll-over debt approach is that it combines the analytical tractability of a time invariant or stationary debt structure with a debt maturity profile. The latter allows the term structure of debt to be analysed.

The roll-over debt model has been adopted by numerous authors and extended in many ways. The approach was first introduced by Leland [1994b] first with an exponentially driven maturity profile and later in Leland and Toft [1996] extended to the case of a finite maturity profile. Originally formulated with the firm's asset price modelled as a geometric Brownian motion Hilberink and Rogers [2002], Surya and Kyprianou [2007] extended the generality of the approach by modelling the asset price process as a spectrally negative Lévy process. More recently, Chen and Kou [2009] and Dao and Jeanblanc [2012] extended further the approach to allow for two-sided jumps where the jump size had a double-exponential density.

The approach taken in this chapter follows one that has been adopted by many authors in the literature. The assertion that, on default, the value of a firm's equity

should be zero, normally referred to as the strict priority rule, provides a boundary condition that the equity price process must satisfy on default. The assumption underlying many implementations of the roll-over debt framework is that default occurs at first passage of the asset price process below a fixed boundary. If using the latter two criteria, the value of equity can be identified in terms of the boundary, then in principle, the boundary can be chosen so as to maximise the value of equity. In many cases, including the original formulation by Leland [1994b], the value of equity in terms of the default boundary can be identified in closed form.

Motivated by the empirical research outlined in Section 2.8, the model presented in this chapter contributes to the literature by presenting the roll-over debt structure in the context of a *regime-switching*. The model will focus on a single-asset firm partly financed by debt in two economic regimes, one *good* and one *recessionary*. It will be assumed that the asset process of the firm follows a regime-switching geometric Brownian motion. Asset volatility, risk free interest and dividend rates will be assumed to be regime-dependent. Default is modelled as the first time that the asset value process of the firm falls below a fixed barrier, also regime-dependent. Capital structure is first identified for fixed boundaries given exogenously before the boundaries are selected in order to maximise the value of equity in line with the comments above. Credit spreads associated with the maximised capital structure are also identified.

The firm will also be assumed to benefit from tax relief on interest payments generating an income stream which is additive to the value of the firm. If the firm defaults, it is assumed to suffer bankruptcy costs modelled as a fixed proportion of asset value at time of default. The existing literature is extended by also allowing tax rates and bankruptcy costs to vary by regime. As already noted, it has been customary on the literature to incorporate a tax threshold for the asset value under which tax relief on interest payments is not accrued. This feature is not modelled in this chapter. It has been noted by Leland and Toft [1996] that without this feature, a firm may be incentivised by the additional cashflows to undertake very high levels of leverage. In the model presented in this chapter, however, lower tax rates in the recessionary

regime will act as a disincentive for the firm to undertake leverage.

The outline of this chapter is as follows. The model formulation is outlined in Section 3.2. In Section 3.3, the capital structure associated with exogenously given regime-dependent boundaries is identified. Endogenously determined regime-dependent boundaries which maximise equity value are characterised along with optimised credit spreads in Section 3.4. Finally, Section 3.5 presents some numerical calculations. Auxiliary results are collected in Section 3.6.

3.2 Model Formulation

3.2.1 Asset Processes Let $W = \{W_t\}_{t \geq 0}$ be a standard Brownian motion and $\{Z_t\}_{t \geq 0}$ a continuous time Markov chain with finite state space $E = \{1, 2\}$ and generator matrix G given by

$$G = \begin{pmatrix} -\gamma_1 & \gamma_1 \\ \gamma_2 & -\gamma_2 \end{pmatrix}$$

with $\gamma_i > 0, i \in E$ which is independent of W . It is assumed the value of a firm's assets, $V = \{V_t\}_{t \geq 0}$, evolve according to a regime-switching geometric Brownian motion with stochastic dynamics given by

$$\frac{dV_t}{V_t} = (r(Z_t) - \delta(Z_t))dt + \sigma(Z_t)dW_t, \quad V_0 = v > 0, \quad Z_0 = i \in E, \quad (3.2.1)$$

where $r(i), \delta(i) : E \mapsto \mathbb{R}^+ \setminus \{0\}$ are the compound risk free interest and dividend rates respectively and $\sigma(i) : E \mapsto \mathbb{R}^+ \setminus \{0\}$ are the volatility coefficients. The value of Z represents the state of the economy at a given time in which the compound interest and dividend rates and the volatility parameter are assumed to be known constants. Where there is no scope for ambiguity, the notation a_i will be used to denote the value of parameter a in regime i .

It is assumed that the stochastic processes V and Z are defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ where $\mathbb{F} = \{F_t\}_{t \geq 0}$ denotes the completed filtration

jointly generated by V and Z . Conditional expectations will be denoted by writing $\mathbb{E}_{v,i}[\cdot] = \mathbb{E}[\cdot | V_0 = v, Z_0 = i]$.

Furthermore, it is assumed that there exists a savings account process $\{B_t\}_{t \geq 0}$ satisfying

$$B_t = \exp\left(-\int_0^t r(Z_s) ds\right). \quad (3.2.2)$$

Defining $X = \log(V)$, Theorem 1 in Mijatovic and Pistorius [Preprint] implies that the discounted moment generating function of the process (X, Z) is given by

$$\mathbb{E}_{0,i} \left[\frac{e^{uX_t}}{B_t} \mathbb{1}_{\{Z_j=j\}} \right] = \underline{1}_i' e^{t(K(u) - \Lambda_r)} \underline{1}_j, \quad u \in \mathbb{R},$$

where $k(u) := G + \Lambda(u)$, Λ_r is a 2×2 diagonal matrix with diagonal elements $r(i)$ for $i \in E$ and $\Lambda(u)$ is a 2×2 diagonal matrix where the i^{th} element is equal to the characteristic exponent of the process X in regime $i \in E$ given by

$$\phi_i(u) = u \left(r_i - \delta_i - \frac{1}{2} \sigma_i^2 \right) + \frac{1}{2} \sigma_i^2 u^2.$$

It follows, therefore, from (3.2.1) that the risk neutral drift of the process V is given by $r_i - \delta_i, i \in E$. As a consequence, it will be assumed that all expectations in sequel are taken under the risk neutral pricing measure.

3.2.2 Roll-over debt structure. The firm is partly financed by debt which is being constantly issued and redeemed in such a way that the *face value* of debt remains constant - the so called *roll-over debt structure*. Originally introduced in Leland [1994b], the description which follows draws heavily on the exposition in Hilberink and Rogers [2002]. In the time interval $(t, t + dt)$, the firm issues debt with face value pdt and maturity profile $\psi > 0$ where $\int_0^\infty \psi(s) ds = 1$. Therefore, debt with face value $p\psi(s) dt ds$ will be issued in the time interval $(t, t + dt)$ which matures in $(t + s, t + s + ds)$. At time 0, the face value of all debt maturing in $(s, s + ds)$ will be given by

$$p \int_{-\infty}^0 \psi(s - v) dv ds := p\Psi(s) ds \quad (3.2.3)$$

where $\Psi(u) := \int_u^\infty \psi(t) dt$. Taking $s = 0$ in (3.2.3), the face value of debt maturing in $(0, du)$ is pdu , which is the same as the face value of newly issued debt, implying the face value of all debt remains constant and given by

$$P = p \int_0^\infty \Psi(s) ds.$$

All debt is of equal seniority and pays a fixed coupon rate of c . The importance of the roll-over debt structure is that it provides a flexible and time-independent debt profile which permits the analysis of debt maturity.

3.2.3 Default Default occurs the first time, $\tau(b)$, that the value of the firm falls to a level which is dependent upon the regime,

$$\tau(b) = \inf \{t \geq 0 \mid V_t \leq b(Z_t)\}, \quad (3.2.4)$$

where the boundaries are defined as $b = (b_1, b_2)$. On default, a regime dependent fraction $\xi_i \in [0, 1], i \in E$, of the firm's assets are lost in the resulting reorganisation. Following the established literature, it is assumed that assets in place cannot be liquidated in order to raise funds to pay debt holders.

In Section 3.3 below, the default levels in (3.2.4) are assumed to be known and given exogenously. Later in Section 3.4, the latter assumption will be relaxed to identify default levels determined endogenously by the firm in order to maximise the value of equity.

3.2.4 Bond Prices and Total Debt The price of a bond issued by the firm at time 0 paying 1 at maturity time t and a fixed coupon rate of c is equal to the discounted value of coupon income and principal payments, which occur prior to maturity, together with the discounted value of the residual value of the assets,

should default occur before maturity of the bond, and is given by

$$\begin{aligned}
 d(v, i, b, t) &= \int_0^t \mathbb{E}_{v,i} \left[ce^{-\int_0^s r(z_u) du} \mathbb{1}_{\{s < \tau(b)\}} \right] ds \\
 &\quad + \mathbb{E}_{v,i} \left[e^{-\int_0^t r(z_u) du} \mathbb{1}_{\{t < \tau(b)\}} \right] \\
 &\quad + \frac{1}{P} \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} r(z_u) du} (1 - \xi(Z_{\tau(b)})) V_{\tau(b)} \mathbb{1}_{\{t \geq \tau(b)\}} \right], \tag{3.2.5}
 \end{aligned}$$

where $\mathbb{E}_{v,i}[\cdot] = \mathbb{E}[\cdot \mid V_0 = v, Z_0 = i]$. The term $\frac{1}{P}$ in the latter display reflects the proportion of assets to which a bond of face value 1, in proportion to a total principal debt outstanding of P , will be entitled on default.

The total value of all outstanding debt at time 0 is then obtained by integrating over all maturity dates using the roll-over maturity profile, Ψ , to obtain

$$\begin{aligned}
 D(v, i, b) &= \int_0^\infty p\Psi(t) d(x, b, t, j) dt \\
 &= pc\mathbb{E}_{v,i} \left[\int_0^{\tau(b)} ce^{-\int_0^t r(Z_u) du} \widehat{\Psi}(t) dt \right] \\
 &\quad + p\mathbb{E}_{v,i} \left[\int_0^{\tau(b)} e^{-\int_0^t r(Z_u) du} \Psi(t) dt \right] \\
 &\quad + \frac{p}{P} \mathbb{E}_{v,j} \left[e^{-\int_0^{\tau(b)} r(Z_u) du} (1 - \xi(Z_{\tau(b)})) V_{\tau(b)} \widehat{\Psi}(\tau(b)) \right], \tag{3.2.6}
 \end{aligned}$$

where $\widehat{\Psi}(t) := \int_t^\infty \Psi(s) ds$.

Following Leland and Toft [1996], Hilberink and Rogers [2002], Dao and Jeanblanc [2012] it is assumed that the form of the roll-over debt maturity profile from Subsection (3.2.2) is given by

$$\psi(t) = m \exp(-mt) \tag{3.2.7}$$

from which it follows that $\Psi(t) = m\widehat{\Psi}(t) = \exp(-mt)$ and that, after some simple

manipulations, the total value of debt at time 0 in (3.2.6) simplifies to

$$\begin{aligned}
 D(v, i, b) &= P(m+c) \mathbb{E}_{v,i} \left[\int_0^{\tau(b)} e^{-\int_0^t (r(Z_u)+m) du} dt \right] \\
 &\quad + \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} (r(Z_u)+m) du} (1 - \xi(Z_{\tau(b)})) V_{\tau(b)} \right] \\
 &= P(m+c) \underline{1}_i' (\Lambda_{r+m} - G)^{-1} \underline{1} \\
 &\quad - \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} (r(Z_u)+m) du} (\tilde{v}(Z_{\tau(b)}) - (1 - \xi(Z_{\tau(b)})) V_{\tau(b)}) \right] \quad (3.2.8)
 \end{aligned}$$

for $i \in E$, where

$$\tilde{v}(k) = P(m+c) \underline{1}_k' (\Lambda_{r+m} - G)^{-1} \underline{1}, \quad k \in E, \quad (3.2.9)$$

Λ_a is a 2×2 diagonal matrix with elements equal to $a(i)$ for $i \in E$, $\underline{1}$ is a 2 element vector of ones and $\underline{1}_k$ is a 2 element vector with 1 in row k and zero in the alternate row. The expectations in the first equality in the above display were calculated using Lemma's 3.6.1 and 3.6.2 which are included in Section 3.6.

3.2.5 Value of the Firm The firm receives tax relief on the value of coupon payments made equal to $\pi(i) \in [0, 1]$ in regime i . It follows that the value of the firm is comprised of the value of assets at time 0 together with the present value of future tax relief minus the proportion of assets lost in reorganisation of the firm on default given by

$$\begin{aligned}
 F(v, i, b) &= v + cP\pi \mathbb{E}_{v,i} \left[\int_0^{\tau(b)} e^{-\int_0^t r(Z_u) du} dt \right] - \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du} \xi(Z_{\tau(b)}) V_{\tau(b)} \right] \\
 &= v + cP \underline{1}_i' (\Lambda_r - G)^{-1} \pi \\
 &\quad - \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du} (\tilde{\chi}(Z_{\tau(b)}) + \xi(Z_{\tau(b)}) V_{\tau(b)}) \right] \quad (3.2.10)
 \end{aligned}$$

for $i \in E$ with

$$\tilde{\chi}(k) = \underline{1}_k' cP (\Lambda_r - G)^{-1} \pi, \quad k \in E, \quad (3.2.11)$$

and $\pi = (\pi(1), \pi(2))'$ where once again Lemma's 3.6.1 and 3.6.2 were used to calcu-

late the expectations in the first equality.

3.2.6 Equity Value The equity value of the firm is simply taken to be the value of the firm in (3.2.10) minus the value of debt in (3.2.8),

$$\begin{aligned}
 Q(v, i, b) = & v + \underline{1}_i \left(cP(\Lambda_r - G)^{-1} \pi - P(m + c)(\Lambda_{r+m} - G)^{-1} \underline{1} \right) \\
 & - \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du} (\tilde{\chi}(Z_{\tau(b)}) + \xi(Z_{\tau(b)}) V_{\tau(b)}) \right] \\
 & + \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} (r(Z_u) + m) du} (\tilde{\nu}(Z_{\tau(b)}) - (1 - \xi(Z_{\tau(b)})) V_{\tau(b)}) \right] \quad (3.2.12)
 \end{aligned}$$

for $i \in E$ where $\tilde{\chi}(\cdot)$ and $\tilde{\nu}(\cdot)$ are given in (3.2.11) and (3.2.9) respectively. It is worth noting that on default the value of equity is identically zero which is in accordance with the strict priority rule.

3.3 Capital Structure with Exogenous Boundaries

3.3.1 Exogenous Boundaries With the boundaries in (3.2.4) given exogenously, in order to determine the value of equity, the firm and debt, the key observation is that at the default boundaries, the value of equity is identically zero, which, together with an application of Dynkin's Formula, implies a constraint on the characteristic operator of the process (V, Z) when applied to $Q(v, i, b)$ in (3.2.12). The latter observations together with Proposition 3.3.2 allows the value of equity, the firm and total debt identified explicitly in the following Theorem when the default boundaries in (3.2.4) are known to satisfy $b_1 > b_2$. An entirely analogous result for the case when $b_2 > b_1$ can be derived by symmetry in exactly the same manner as Theorem 3.3.1 and is therefore omitted. The case when the default boundaries are the same will not be addressed in this chapter.

Theorem 3.3.1. *Let $b_1 > b_2$ in (3.2.4) then the value of equity in (3.2.12), the firm in (3.2.10) and total debt in (3.2.8) are given by*

$$Q(v, i, b) = v + \underline{1}_i (cP(\Lambda_r - G)^{-1} \pi - P(m + c)(\Lambda_{r+m} - G)^{-1} \underline{1}) - w_0(v, i, b, \xi, \tilde{\chi}) + w_m(v, i, b, \xi - \underline{1}, \tilde{\nu}) \quad (3.3.1)$$

$$F(v, i, b) = v + cP\underline{1}_i(\Lambda_r - G)^{-1} \pi - w_0(v, i, b, \xi, \tilde{\chi}) \quad (3.3.2)$$

$$D(v, i, b) = P(m + c)\underline{1}_i'(\Lambda_{r+m} - G)^{-1} \underline{1} - w_m(v, i, b, \xi - \underline{1}, \tilde{\nu}) \quad (3.3.3)$$

where w_λ is specified in (3.3.4), Λ_a is a 2×2 diagonal matrix with elements equal to $a(i)$ for $i \in E$, $\underline{1}$ is a 2 element vector of ones and $\underline{1}_k$ is an 2 element vector with 1 in row k and zero in the alternate row and $\tilde{\chi}$ and $\tilde{\nu}$ are vectors given in (3.2.11) and (3.2.9) respectively.

Proof. The proof follows directly from an application of Proposition 3.3.2 to (3.2.8), (3.2.10) and (3.2.12) upon insisting that $Q(b, i, b) = 0, i \in E$ in (3.2.12). \square

Proposition 3.3.2. *Let $\tilde{h}, \tilde{a} : E \rightarrow \mathbb{R}, \lambda > 0$ and $b_1 > b_2$ in (3.2.4) then*

$$w_\lambda(v, i, b, \tilde{a}, \tilde{h}) = \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} (r(Z_u) + \lambda) du} \left(\tilde{h}(Z_{\tau(b)}) - \tilde{a}(Z_{\tau(b)}) V_{\tau(b)} \right) \right], \quad i \in E, \quad (3.3.4)$$

where $w_\lambda(b_i, i, b, \tilde{a}, \tilde{h}) = \tilde{h}(i) - \tilde{a}(i) b_i, i \in E$, is given by

$$w_\lambda(x, 1, b, \tilde{a}, \tilde{h}) = \begin{cases} C_1 x^{\rho_{\lambda 1}} + C_2 x^{\rho_{\lambda 2}}, & x > b_1, \\ \phi_1(b_1), & b_1 \geq x, \end{cases} \quad (3.3.5)$$

$$w_\lambda(x, 2, b, \tilde{a}, \tilde{h}) = \begin{cases} \frac{\mathcal{G}_{\lambda,1}(\rho_{\lambda 1})}{\gamma_1} C_1 x^{\rho_{\lambda 1}} + \frac{\mathcal{G}_{\lambda,1}(\rho_{\lambda 2})}{\gamma_1} C_2 x^{\rho_{\lambda 2}}, & x > b_1, \\ A_1 x^{\zeta_{\lambda 1}^2} + A_2 x^{\zeta_{\lambda 2}^2} + \varphi_2(x), & b_1 > x > b_2, \\ \phi_2(b_2), & b_2 \geq x, \end{cases} \quad (3.3.6)$$

where $\phi_j(x) = \tilde{h}(j) - \tilde{a}(j)x$, the constants C_1, C_2, A_1 and A_2 satisfy the linear system

given by

$$\begin{bmatrix} b_2^{\zeta_{\lambda 2}^2} & b_2^{\zeta_{\lambda 2}^2} & 0 & 0 \\ 0 & 0 & \frac{\mathcal{G}_{\lambda,1}(\rho_{\lambda 1})}{\gamma_1} b_1^{\rho_{\lambda 1}} & \frac{\mathcal{G}_{\lambda,1}(\rho_{\lambda 2})}{\gamma_1} b_1^{\rho_{\lambda 2}} \\ b_1^{\zeta_{\lambda 1}^2} & b_1^{\zeta_{\lambda 2}^2} & -b_1^{\rho_{\lambda 1}} & -b_1^{\rho_{\lambda 2}} \\ \zeta_{\lambda 1}^2 b_1^{\zeta_{\lambda 1}^2-1} & \zeta_{\lambda 2}^2 b_1^{\zeta_{\lambda 2}^2-1} & -\rho_{\lambda 1} b_1^{\rho_{\lambda 1}-1} & -\rho_{\lambda 2} b_1^{\rho_{\lambda 2}-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \phi_2(b_2) + \varphi_{\lambda,2}(b_2) \\ \phi_1(b_1) \\ \varphi_{\lambda,2}(b_1) \\ \varphi'_{\lambda,2}(b_1) \end{bmatrix} \quad (3.3.7)$$

with

$$\begin{aligned} \varphi_{\lambda,j}(x) &= \gamma_j \left(\frac{\tilde{h}(3-j)}{r(j) + \gamma_j} + \frac{\tilde{a}(3-j)x}{\delta(j) + \gamma_j + \lambda} \right), \quad j \in E, \\ \mathcal{G}_{\lambda,j}(z) &= \frac{1}{2} z^2 \sigma(j)^2 + z \left(r(j) - \delta(j) - \frac{1}{2} \sigma(j)^2 \right) - (\gamma_j + r(j) + \lambda), \quad j \in E, \end{aligned}$$

$\zeta_{\lambda 1}^j$ and $\zeta_{\lambda 2}^j$ are positive and negative roots of the equation $\mathcal{G}_{\lambda,j}(\zeta) = 0$ for $j \in E$ respectively, $\rho_{\lambda 1}$ and $\rho_{\lambda 2}$ are the negative roots of the equation $\mathcal{G}_{\lambda,1}(\rho) \mathcal{G}_{\lambda,2}(\rho) - \gamma_1 \gamma_2 = 0$ and f' denotes the derivative of the function f .

Proof. The infinitesimal generator of the process (V, Z) for $f(\cdot) \in C^2$ is given by

$$\mathbb{L}_{V,Z} f(x, i) = \frac{1}{2} \sigma_i^2 x^2 \frac{\partial^2 f(x, i)}{\partial x^2} + (r_i - \delta_i) x \frac{\partial f(x, i)}{\partial x} + \gamma_i (f(x, 1-i) - f(x, i)) \quad (3.3.8)$$

for $i \in E$. When $x > b_1$, if w_λ solves $\mathbb{L}_{V,Z} w_\lambda(x, i) = 0, i \in E$, for the stated boundary conditions, $w_\lambda(b_i, i, b, \tilde{a}, \tilde{h}) = \tilde{h}(i) - \tilde{a}(i) b_i, i \in E$, then Dynkin's formula implies that (3.3.4) holds. To solve for w_λ , note that (3.3.8) forms two coupled ODE's with characteristic equation given by

$$\mathcal{G}_{\lambda,1}(\rho) \mathcal{G}_{\lambda,2}(\rho) = \gamma_1 \gamma_2, \quad (3.3.9)$$

where

$$\mathcal{G}_{\lambda,i}(\rho) = \frac{1}{2} \sigma_i^2 \rho^2 + (r_i - \delta_i - \frac{1}{2} \sigma_i^2) \rho - (r_i + \gamma_i + \lambda), \quad i \in E, \quad (3.3.10)$$

which has 4 distinct real roots, $\rho_{\lambda 1} < \rho_{\lambda 2} < 0 < \rho_{\lambda 3} < \rho_{\lambda 4}$ with the general form of

the solution when $x > b_1$ given by

$$w_\lambda(x, 1, b, \tilde{a}, \tilde{h}) = \sum_{k=1}^4 C_k x^{\rho_{\lambda k}}, \quad x > b_1, \quad (3.3.11)$$

$$w_\lambda(x, 2, b, \tilde{a}, \tilde{h}) = \sum_{k=1}^4 \frac{\mathcal{G}_{\lambda,1}(\rho_{\lambda k})}{\gamma_1} C_k x^{\rho_{\lambda k}}, \quad x > b_1. \quad (3.3.12)$$

However, since $w_\lambda(x, i, b, \tilde{a}, \tilde{h})$ is bounded by $\max_{i \in E} |\tilde{h}(i) - \tilde{a}(i) b_i|$, it follows that the terms in (3.3.11) and (3.3.12) with positive exponents must be zero.

When $b_1 \geq x > b_2$, $w_\lambda(x, 1, b, \tilde{a}, \tilde{h}) = \tilde{h}(1) - \tilde{a}(1)x$ which implies that $w_\lambda(x, 2, b, \tilde{a}, \tilde{h})$ solves an inhomogeneous ODE with complementary solution reading as

$$w_\lambda^c(x, 2, b, \tilde{a}, \tilde{h}) = A_1 x^{\zeta_{\lambda 1}^2} + A_2 x^{\zeta_{\lambda 2}^2}, \quad (3.3.13)$$

where $\zeta_{\lambda 1}^2$ and $\zeta_{\lambda 2}^2$ are the positive and negative roots of $\mathcal{G}_{\lambda,2}(\zeta) = 0$ given in (3.3.10). The Wronskian for the fundamental set of solutions of (3.3.13) is then given by

$$W = \begin{vmatrix} x^{\zeta_{\lambda 1}^2} & x^{\zeta_{\lambda 2}^2} \\ \zeta_{\lambda 1}^2 x^{\zeta_{\lambda 1}^2 - 1} & \zeta_{\lambda 2}^2 x^{\zeta_{\lambda 2}^2 - 1} \end{vmatrix} = x^{\zeta_{\lambda 1}^2} \zeta_{\lambda 2}^2 x^{\zeta_{\lambda 2}^2 - 1} - x^{\zeta_{\lambda 2}^2} \zeta_{\lambda 1}^2 x^{\zeta_{\lambda 1}^2 - 1} \quad (3.3.14)$$

(which is non-zero since $\zeta_{\lambda 1}^2 \neq \zeta_{\lambda 2}^2$) generating a particular solution

$$\begin{aligned} w_\lambda^p(x, 2, b, \tilde{a}, \tilde{h}) &= x^{\zeta_{\lambda 2}^2} \int \frac{\gamma_2 (\tilde{h}(1) - \tilde{a}(1)x)}{\frac{1}{2} \sigma_2^2 x^2 x^{\zeta_{\lambda 2}^2 - 1} (\zeta_{\lambda 1}^2 - \zeta_{\lambda 1}^2)} dx \\ &\quad - x^{\zeta_{\lambda 1}^2} \int \frac{\gamma_2 (\tilde{h}(1) - \tilde{a}(1)x)}{\frac{1}{2} \sigma_2^2 x^2 x^{\zeta_{\lambda 1}^2 - 1} (\zeta_{\lambda 1}^2 - \zeta_{\lambda 2}^2)} dx \end{aligned} \quad (3.3.15)$$

$$\begin{aligned} &= \frac{2\gamma_2 (\tilde{h}(1) (\zeta_{\lambda 1}^2 - 1) (\zeta_{\lambda 2}^2 - 1) - \zeta_{\lambda 1}^2 \zeta_{\lambda 2}^2 \tilde{a}(1)x)}{\sigma_2^2 (\zeta_{\lambda 1}^2 - 1) \zeta_{\lambda 1}^2 (\zeta_{\lambda 2}^2 - 1) \zeta_{\lambda 2}^2} \\ &= \gamma_2 \left(\frac{\tilde{h}(1)}{\gamma_2 + r_2} - \frac{\tilde{a}(1)x}{\gamma_2 + \delta_2 + \lambda} \right) := \varphi_{2,\lambda}(x). \end{aligned} \quad (3.3.16)$$

Collecting results it follows that the general solution to (3.3.4) when $b_1 > b_2$ may be

written

$$w_\lambda(x, 1, b, \tilde{a}, \tilde{h}) = C_1 x^{\rho_{\lambda 1}} + C_2 x^{\rho_{\lambda 2}}, \quad x > b_1 > b_2, \quad (3.3.17)$$

$$w_\lambda(x, 2, b, \tilde{a}, \tilde{h}) = \frac{\mathcal{G}_{1,\lambda}(\rho_{\lambda 1})}{\gamma_1} C_1 x^{\rho_{\lambda 1}} + \frac{\mathcal{G}_{1,\lambda}(\rho_{\lambda 2})}{\gamma_1} C_2 x^{\rho_{\lambda 2}}, \quad x > b_1 > b_2, \quad (3.3.18)$$

$$w_\lambda(x, 2, b, \tilde{a}, \tilde{h}) = A_1 x^{\zeta_{\lambda 1}^2} + A_2 x^{\zeta_{\lambda 2}^2} + \varphi_{2,\lambda}(x). \quad b_1 \geq x > b_2. \quad (3.3.19)$$

At the boundaries, $w_\lambda(x, i, b, \tilde{a}, \tilde{h}) = \tilde{h}(i) - \tilde{a}(i) b_i := \phi_i(b_i)$ for $i \in E$, which, together with the requirement that (3.3.18) and (3.3.19) are C^0 and C^1 respectively at $x = b_1$, generates the system given in (3.3.7). \square

3.4 Optimised Capital Structure

3.4.1 Optimised Boundaries Following the established literature¹, equity value in (3.3.1) can be maximised (relative to constant boundaries) by enforcing the heuristic of the smooth-pasting condition at the optimised boundaries, $b^* = (b_1^*, b_2^*)$,

$$\left. \frac{\partial}{\partial v} Q(v, i, b^*) \right|_{v=b_i^*} = 0, \quad i \in E, \quad (3.4.1)$$

which in view of (3.3.1), in the context of the model being presented, translates into the requirement that

$$\left. \frac{\partial}{\partial v} (w_m(v, i, b^*, \xi + \underline{1}, \tilde{v}) - w_0(v, i, b^*, \xi, \tilde{\chi})) \right|_{v=b_i^*} = -1, \quad i \in E, \quad (3.4.2)$$

where $\tilde{\chi}(\cdot)$ and $\tilde{v}(\cdot)$ are given in (3.2.11) and (3.2.9) respectively and w_λ is specified in (3.3.4). It follows, therefore, that a $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ which satisfies the two non-linear equations given by (3.4.2), in conjunction with system (3.3.5)-(3.3.7), will satisfy (3.4.1).

¹See for example, Leland [1994b], Hilberink and Rogers [2002], Dao and Jeanblanc [2012].

It is not known *a priori* which boundary will be the larger of the two, and therefore, both possibilities can be numerically checked to see which prior assumption is verified. Furthermore, in view of the complexity of the system of non-linear equations resulting from (3.4.2) and (3.3.5)-(3.3.7), it has not been possible to establish existence and uniqueness properties of the optimised boundaries. To be clear, the term optimised is being used here in the sense that the boundaries satisfy the smooth fit requirement. No claim is being made that the latter boundaries are globally optimal. Once the optimised boundaries, $b^* = (b_1^*, b_2^*)$, have been established, it is then possible to invoke Theorem 3.3.1, or the analogous result for the case when $b_2^* > b_1^*$, to calculate the optimised capital structure in terms of the value of the equity, firm value and debt. In addition, optimised credit spreads can be computed as outlined in the next section.

3.4.2 Credit Spreads Once again following the established literature, *credit spread* is defined as the coupon in excess of the risk-free rate which would encourage an investor to buy debt at face value. Using (3.2.2), the risk free rate in regime i until a finite time t may be defined as

$$\tilde{r}(i, t) = -\frac{\log \underline{\mathbf{1}}_i' \mathbb{E} \left(\left[\exp^{-\int_0^t r(Z_s) ds} \right] \underline{\mathbf{1}} \right)}{t}, \quad i \in E, \quad (3.4.3)$$

which allows the risk free rate associated with a given maturity profile to be defined as

$$\tilde{r}(i, m) = \begin{cases} -m \log \left(\underline{\mathbf{1}}_i' \exp^{\frac{(G-\Lambda_r)}{m}} \underline{\mathbf{1}} \right) & m > 0, i \in E, \\ \left(\underline{\mathbf{1}}_i' (\Lambda_r - G)^{-1} \underline{\mathbf{1}} \right)^{-1} & m = 0, i \in E, \end{cases} \quad (3.4.4)$$

after noting that under an exponential maturity profile, $t = \frac{1}{m}$.

In the context of the roll-over debt framework, credit spreads can be viewed in two distinct ways ². First, the *firm credit spread* is the spread required by the market to buy total debt at face value for a given selection of the firm's setup parameters, including the maturity profile m , and an initial value of assets, v . Second, once the firm

²See for example Hilberink and Rogers [2002].

has been established, and in particular having selected a maturity profile parameter, then the *term credit spread* is the spread required by the market to buy a bond of a given maturity at par. Both types of spreads may be computed as outlined below. The analysis of credit spreads will be restricted to cases where the firm remains solvent,

$$(v, i) \in \{v > b_i^*, i \in E\}. \quad (3.4.5)$$

3.4.3 Firm Credit Spreads When the firm is set up and debt is first issued, the parameters of the firm are typically constrained such that at the coupon rate, c , the market value of debt, $D(v, i, b^*)$ is equal to the face value of debt, P . If the asset value of the firm at startup is $V_0 = v$, the latter constraint requires that the coupon rate, c , be the smallest solution to the equation

$$D(v, i, b^*) = P, \quad i \in E, \quad (3.4.6)$$

where $D(v, i, b^*)$ is defined in (3.3.3).

The regime dependent coupon rate which solves (3.4.6) will be denoted $\hat{c}_m^*(v, i, b^*)$. Unlike Leland [1994b], Hilberink and Rogers [2002], Dao and Jeanblanc [2012], the framework of this chapter does not provide a closed form solution for the barrier b^* . Nevertheless, by numerically calculating b^* for a given set of startup parameters for the firm, including the coupon rate, and then evaluating (3.4.6) it is possible to search for $\hat{c}_m^*(v, i, b^*)$ numerically. Computations based on the latter approach are presented in Section 3.5 below. The firm's credit spread is then given by

$$\hat{s}_m^*(v, i, b^*) = \hat{c}_m^*(v, i, b^*) - \tilde{r}(i, m), \quad (v, i) \in \{v > b_i^*, i \in E\}, \quad (3.4.7)$$

where $\tilde{r}(i, \cdot)$ is defined in (3.4.4).

3.4.4 Term Structure Credit Spreads As noted above, term structure credit spreads are defined as the excess over the coupon rate which would, once the setup parameters of the firm have been selected and the boundaries fixed, allow a newly

issued bond of a specific maturity, t , to price at par for a given value of initial assets, v , in regime i .

Using (3.2.5), the latter calculation amounts to finding the coupon rate which solves

$$\tilde{c}_t^*(v, i, b^*) = \frac{1 - \mathbb{E}_{v,i} \left[e^{-\int_0^t r(Z_u) du} 1_{\{t < \tau(b^*)\}} \right] - \frac{v}{P} \mathbb{E}_{v,i} \left[(1 - \xi(Z_{\tau(b^*)})) e^{X_{\tau(b^*)} - \int_0^{\tau(b^*)} r(Z_u) du} 1_{\{t \geq \tau(b^*)\}} \right]}{\int_0^t \mathbb{E}_{v,i} \left[e^{-\int_0^s r(Z_u) du} 1_{\{s < \tau(b^*)\}} ds \right]}$$

for $(v, i) \in \{v > b_i^*, i \in E\}$. Explicit expressions for the expectations in the latter equality are not available, however, the spread may be computed using inverse Laplace transforms of expressions established in Proposition 3.3.2 as outlined in Proposition 3.4.1 below for the case when $b_1^* > b_2^*$. An entirely analogous result applies for the case when $b_2^* > b_1^*$. Numerical calculations based on the above approach are also presented in Section 3.5.

Term structure credit spreads are then defined as

$$\tilde{s}_t^*(v, i, b^*) = \tilde{c}_t^*(v, i, b^*) - \tilde{r} \left(i, \frac{1}{t} \right), \quad (v, i) \in \{v > b_i^*, i \in E\}, \quad (3.4.8)$$

where $\tilde{r}(i, \cdot)$ is defined in (3.4.4) and $\tilde{c}_t^*(v, i, b^*)$ is defined in (3.4.9).

Proposition 3.4.1. *Given $b_1 > b_2$ in (3.2.4), the coupon rate of a corporate bond in (3.2.5) issued at par with maturity t is given by*

$$\tilde{c}_t^*(v, i, b) = \frac{1 - \mathcal{L}^{-1}(A_1^m(v, i, b), m) - \frac{v}{P} \mathcal{L}^{-1}(A_2^m(v, i, b), m)}{\mathcal{L}^{-1}(A_3^m(v, i, b), m)} \quad (3.4.9)$$

for $(v, i) \in \{v > b_i, i \in E\}$ where $\mathcal{L}^{-1}(\cdot, s)$ is the inverse Laplace transform with Laplace exponent s ,

$$\begin{aligned} A_1^m(v, i, b) &= \underline{1}'_i (\Lambda_r - G)^{-1} \underline{1} - w_m(v, i, b, 0, \tilde{\vartheta}), \\ A_2^m(v, i, b) &= \frac{1}{m} w_m(v, i, b, (\underline{1} - \xi), \underline{1}), \\ A_3^m(v, i, b) &= \frac{1}{m} A_1^m(v, i, b), \end{aligned}$$

w_λ is specified in (3.3.4) and

$$\tilde{\vartheta}(k) = \underline{1}_k' (\Lambda_{r+m} - G)^{-1} \underline{1}, \quad k \in E.$$

Proof. Solving $d(v, i, b, t) = 1$ in (3.2.5) yields

$$\tilde{c}_t^*(v, i, b) = \frac{1 - a_1(v, i, b) - \frac{v}{P} a_2(x, i, b)}{a_3(v, i, b)} \quad (3.4.10)$$

where

$$a_1(v, i, b) = \mathbb{E}_{v,i} \left[e^{-\int_0^t r(Z_u) du} \mathbb{1}_{\{t < \tau(b)\}} \right] \quad (3.4.11)$$

$$a_2(v, i, b) = \mathbb{E}_{v,i} \left[\left(\underline{1} - \xi(Z_{\tau(b)}) \right) e^{-\int_0^{\tau(b)} r(Z_u) du + X_{\tau(b)}} \mathbb{1}_{\{t > \tau(b)\}} \right] \quad (3.4.12)$$

$$a_3(v, i, b) = \int_0^t \mathbb{E}_{v,i} \left[e^{-\int_0^s r(z_u) du} \mathbb{1}_{\{s < \tau(b)\}} \right] ds. \quad (3.4.13)$$

Using Lemmas 3.6.1 and 3.3.2 the Laplace transform of (3.4.11) computes directly as

$$\begin{aligned} A_1^m(v, i, b) &= \int_0^\infty e^{-mt} a_1(x, i, b) dt \\ &= \int_0^\infty e^{-mt} \mathbb{E}_{v,i} \left[e^{-\int_0^t r(Z_u) du} \mathbb{1}_{\{t < \tau(b)\}} \right] dt \\ &= \int_0^\infty e^{-mt} \mathbb{E}_{v,i} \left[e^{-\int_0^t r(Z_u) du} \right] dt - \int_0^\infty e^{-mt} \mathbb{E}_{v,i} \left[e^{-\int_0^t r(Z_u) du} \mathbb{1}_{\{\tau(b) \leq t\}} \right] dt \\ &= \underline{1}_i' (\Lambda_{r+m} - G)^{-1} \underline{1} - \mathbb{E}_{v,i} \left[e^{-\int_0^{\tau(b)} (r(Z_u) + m) du} \tilde{\vartheta}(Z_{\tau(b)}) \right] \\ &= \underline{1}_i' (\Lambda_{r+m} - G)^{-1} \underline{1} - w_m(v, i, b, 0, \tilde{\vartheta}) \end{aligned}$$

where

$$\tilde{\vartheta}(k) = \underline{1}_k' (\Lambda_{r+m} - G)^{-1} \underline{1}, \quad k \in E,$$

and w_λ is specified in (3.3.4). The Laplace transform of (3.4.12) reads

$$\begin{aligned}
 A_2^m(v, i, b) &= \int_0^\infty e^{-mt} a_2(x, i, b) dt \\
 &= \int_0^\infty e^{-mt} \mathbb{E}_{v,i} \left[(1 - \xi(Z_{\tau(b)})) e^{-\int_0^{\tau(b)} r(Z_u) du + X_{\tau(b)}} \mathbb{1}_{\{t \geq \tau(b)\}} \right] dt \\
 &= \frac{1}{m} \mathbb{E}_{v,i} \left[(1 - \xi(Z_{\tau(b)})) e^{-\int_0^{\tau(b)} (r(Z_u) + m) du + X_{\tau(b)}} \right] dt \\
 &= \frac{1}{m} w_m(v, i, b, (\underline{1} - \xi), \underline{1})
 \end{aligned}$$

where $\underline{1}$ is a column vector of ones with the latter equality following directly from Proposition 3.3.2. Finally, the Laplace transform of (3.4.13) is given by

$$\begin{aligned}
 A_3^m(v, i, b) &= \int_0^\infty a_3(x, i, b) dt \\
 &= \int_0^\infty \int_0^t e^{-mt} \mathbb{E}_{v,i} \left[e^{-\int_0^s r(Z_u) du} \mathbb{1}_{\{s < \tau(b)\}} \right] ds dt \\
 &= \frac{1}{m} \int_0^\infty e^{-mt} \mathbb{E}_{v,i} \left[e^{-\int_0^t r(Z_u) du} \mathbb{1}_{\{t < \tau(b)\}} \right] dt \\
 &= \frac{1}{m} A_1^m(v, i, b)
 \end{aligned}$$

which completes the proof. □

3.5 Numerical Analysis

3.5.1 Calculation Methodology All calculations were conducted using Matlab. In order to establish the optimised default boundaries, the non-linear system given (3.4.2) and (3.3.5)-(3.3.7) was first solved using Matlab's `lsqnonlin` routine under the assumption that one of the boundaries was greater than the other. Both the assumption that the regime 1 boundary was greater than the regime 2 boundary and vice versa were calculated. The resulting calculated boundaries were then compared with the initial assumption, and the calculated boundaries which agreed with the initial assumption regarding the relative levels of the boundaries, selected as the

optimised boundary pair. Only one set of calculated boundaries ever agreed with the prior assumption. Once calculated, the latter optimised boundaries were then used to calculate the value of equity, the firm and debt using (3.3.1), (3.3.2) and (3.3.3) respectively. Finally, in order to calculate firm credit spreads, a simple numerical search procedure was used, repeating the calculations above in order to calculate the smallest of coupon which satisfied (3.4.6). The Laplace inversions required to calculate term structure credit spreads associated with (3.4.9) were carried out using the *Euler* method proposed by Abate and Whitt [1995].

The default parameters used for all calculations are illustrated in Table 3.5.1. The parameters which were invariant between regimes and the regime 1 parameter values were taken from Leland [1994b], with the exception of the regime intensities which were set to 0.3 and 1 for regimes 1 and 2 respectively. The regime 2 parameters were selected to reflect a *recessionary* scenario in which interest rates and dividend payments were assumed to fall but asset volatility and bankruptcy costs assumed to rise. In addition, in regime 2, the tax rate was assumed to be lower, reflecting a desire by the fiscal authorities to stimulate growth.

3.5.2 Optimised Capital Structure Table 3.5.2 lists the values of optimised regime dependent default boundaries, for different maturity profiles, but otherwise using the default parameters in Table 3.5.1. For comparison purposes, boundaries from Leland [1994b] are also included, calculated using the regime 1 parameters only. It is notable that regime 1 boundaries are higher than regime 2 boundaries for all maturities. Regime 1 boundaries are also higher than the corresponding Leland [1994b] for the same maturity profile. However, whereas regime 2 boundaries are higher than the corresponding Leland [1994b] for the same maturity profile for maturities less than 5 years, for 5 year maturities and greater, they are lower.

Optimised firm value as a function of leverage for different maturity profiles in each regime is illustrated in Figure 3.5.1. Leverage here, and in sequel, is defined as in Leland [1994b] as total debt value (3.3.3) divided by firm value (3.3.2), $D(v, i, b) / F(v, i, b)$. Higher leverage was generated in the calculations by increasing the nominal value of

debt while holding all other parameters constant. The results reflect the findings observed in the literature which indicate that firm value initially increases with leverage only to reach what appears to be a maximal level and then declines. The levels of leverage and credit spreads which are associated with the maximum values achieved in Figure 3.5.1 are calculated in Subsection 3.5.4 below.

Figure 3.5.1 suggests that maximal firm value is achieved at lower levels of leverage for shorter maturity profiles which is also in line with the findings reported in the literature (Leland [1994b], Leland and Toft [1996], Hilberink and Rogers [2002], Dao and Jeanblanc [2012]). Figure 3.5.1 indicates that in regime 1, firm value is maximised at a leverage equal to approximately 25%, whereas for a 30 year maturity profile, firm value is maximised at approximately 55%. As regards the effect of regimes, firm value is lower in regime 2 than regime 1 for all maturity profiles and declines more quickly as a function of leverage than in regime 1 after having reached its maximal value.

Figure 3.5.2 plots optimised total debt value as a function of leverage in each regime with higher leverage again, simulated by increasing nominal debt value holding other parameters constant. As in the case of firm value above, and also reflective of results reported in the literature, debt value initially increases as a function of leverage only to decline after reaching an apparent maximal value.

Again, mirroring firm value, maximal debt value is reached at lower levels of leverage for shorter maturity profiles. Optimised debt values are lower in regime 2 for all maturity profiles with the value of debt appearing to decline more sharply in regime 2 than regime 1 as leverage increases beyond the level which maximises debt value. In each maturity profile presented in Figure 3.5.2, maximal debt value appears to be achieved at lower levels of leverage in regime 2 than in regime 1.

3.5.3 Firm Spreads Firm credit spreads, as described in Subsection 3.4.3, are presented in Figure 3.5.3 as a function of leverage and by maturity profile and regime. The results are similar to those reported for the geometric Brownian motion case Leland [1994b] but lower than those reported for an asset process driven by a double-

exponential jump diffusion in Dao and Jeanblanc [2012]. Firm spreads tend to increase with leverage for all maturity profiles as would be expected intuitively. Short maturity profile firm spreads are more sensitive to higher leverage and increase faster as leverage rises.

Spreads in regime 2 are higher than in regime 1 and increase more rapidly as a function of leverage. The latter observation is consistent with the familiar *hump-backed* term structure of credit spreads which are often observed in the market for highly leveraged firms, where short dated credit spreads may be observed to be much higher than longer dated spreads. The latter effect is also visible in calculations of the term structure of credit spreads presented in Subsection 3.5.5 below.

Figure 3.5.4 illustrates firm credit spreads as a function of coupon for different maturity profiles in each regime. Firm credit spreads rise as a function of coupon but the latter effect is more pronounced for longer maturity profiles than it is for shorter maturity profiles. Indeed, firm spreads for a 2-year maturity profile exhibit a very low sensitivity to higher coupons. The latter observation is in accordance with intuition since for shorter dated bonds, coupon repayments are a smaller proportion of total repayments than is the case for longer maturity bonds.

Unlike the results reported in Dao and Jeanblanc [2012], positive credit spreads are observed for zero coupons. In the context of the roll-over debt structure, a zero coupon amounts to the case where the firm is constantly refinancing itself with discount bonds of maturity $1/m$. Credit spreads in the latter case are higher for shorter maturity profiles simply because redemptions occur at a higher frequency than is the case for longer maturity profiles.

An interesting feature of Figure 3.5.4 is that the effect of regime change differs by maturity profile. Firm credit spreads increase with coupon in both regimes and increase more steeply with coupon in regime 2. However, the effect of a regime change is not the same for all maturities. For the 2-year and 5-year maturity profiles, firm credit spreads are higher in regime 2 than regime 1. However, for a 30-year maturity profile, firm credit spreads are lower in regime 2 for low coupons but rise

more steeply than firm credit spreads in regime 1 as coupons increase to the point where credit spreads are higher in regime 2 than in regime 1 for higher coupons.

3.5.4 Optimised Leverage Table 3.5.3 illustrates the leverage ratio which maximises firm value for a selection of maturity profiles within a defined range of nominal debt values. The optimised leverage ratios were calculated by a simple search procedure over the range of nominal debt values in the interval $(0, 100]$, which in view of Figure 3.5.1, was regarded as a reasonable search domain. For each level of nominal debt, optimised boundaries and the associated value of the firm were calculated. The maximum firm value was calculated using Matlab's `fminbnd` routine using a function change tolerance of $1e - 15$.

The results illustrated in Table 3.5.3 are consistent with those reported in the literature. For example, the leverage and credit spreads in regime 1 are close to those reported in Leland [1994b]. Optimised leverage increases in both regimes with the maturity profile parameter as does firm value and firm credit spreads. In regime 1, firm value is maximised with a 5 year maturity profile at leverage of 0.356 with a corresponding firm spread of 43bps. In contrast, leverage of 0.545 maximises firm value for a 30 year maturity profile with a firm credit spread of 129bps.

Optimised leverage and firm value is lower in regime 2 than in regime 1 for all maturities. Firm credit spreads are higher in regime 2 for 0.25-year and 1-year maturity, for 5-year maturity and over, firm credit spreads are lower in regime 2 than in regime 1. That optimised firm credit spreads appear to be lower for longer maturities may at first appear surprising in view of the higher asset volatility in regime 2. However, separate calculations, which increased only volatility in regime 2, indicated a lower optimised leverage in regime 2 which was, nevertheless, associated with higher firm credit spreads. The observation that credit spreads are lower in regime 2 for higher maturities under the default parameters is reflective of parameters other than volatility having a damping impact on spreads for longer maturities.

3.5.5 Term Structure of Credit Spreads Figure 3.5.5 illustrates term structure credit spreads, which were described in Subsection 3.4.4, for selected maturities between 0.5 and 30 years for different values of nominal debt (and therefore leverage) but otherwise using the default parameters in Table 3.5.1. Spreads are illustrated for both regimes. Nominal debt values of 55, 60 and 65 corresponded to leverage of 0.515, 0.562 and 0.61 respectively.

For lower levels of leverage, the credit curves illustrated in Figure 3.5.5 have the familiar upward sloping shape as a function of maturity. Credit spreads increase across all maturities in both regimes as leverage increases. Shorter maturity spreads increase more than longer maturity spreads as leverage increases.

As leverage increases, the shape of the spread curve changes. In regime 1 with a nominal value of debt equal to 65, credit spreads increase with maturity up to 10 years but are fairly flat as a function of maturity in excess of 10 years. In regime 2, however, with nominal debt of 65, the term structure of credit spreads rises steeply up to 4 year maturity and then becomes *inverted* with the credit curve sloping down as a function of maturity in excess of 4 years. The latter hump-shaped term structure of credit spreads is a frequently observed market phenomena and is typically associated with firms in financial distress.

Figure 3.5.6 displays for shorter maturities the term structure of credit spreads using the same parameters as above. For shorter maturities the term structure of spreads remains upward sloping in both regimes and as leverage increases. Spreads are higher in regime 2 than in regime 1 and the term structure steepens sharply in regime 2 for higher levels of leverage. For low levels of leverage, calculated credit spreads were close to zero for very short maturities.

Roll-over Debt under Regime-Switching

Fixed Parameters		
Name	Symbol	Default Value
Asset Value	v	100
Coupon Rate	c	0.08
Principal	P	50
Average Maturity	$1/m$	5

(a) Fixed Parameters

Regime Dependent Parameters			
Name	Symbol	Default Value	
		Regime 1	Regime 2
Asset Volatility	σ	0.2	0.35
Tax Rate	π	0.35	0.15
Bankruptcy Costs	ξ	0.5	0.75
Interest Rate	r	0.075	0.03
Dividend Yield	δ	0.07	0.02
Regime Intensity	γ	0.3	1

(b) Regime Dependent Parameters

Table 3.5.1: Default Parameter Values

The table lists default parameter values for all calculations unless otherwise stated. To facilitate comparison, fixed and regime 1 parameter values are taken from Leland [1994b]. Regime intensities were set at 0.3 and 1 for regimes 1 and 2 respectively.

Roll-over Debt under Regime-Switching

Maturity	Leland 94	Regime 1	Regime 2
0.25 Year	77.63	82.63	81.04
1 Year	62.80	67.03	67.25
5 Year	43.29	44.36	42.86
10 Year	36.46	36.90	28.94
20 Year	31.45	32.65	29.71
30 Year	29.38	30.67	28.89

Table 3.5.2: Default Boundaries by Maturity Profile

The table shows optimised boundaries in each regime for selected maturity profiles. For comparison, boundaries using Leland [1994b] are included which are calculated using regime 1 parameters only. Regime 1 boundaries are higher than regime 2 boundaries for all maturity profiles. For lower maturities, both regime boundaries are higher than the corresponding Leland [1994b] boundary but for larger maturities, regime 2 boundaries fall below the corresponding Leland [1994b] boundary.

Roll-over Debt under Regime-Switching

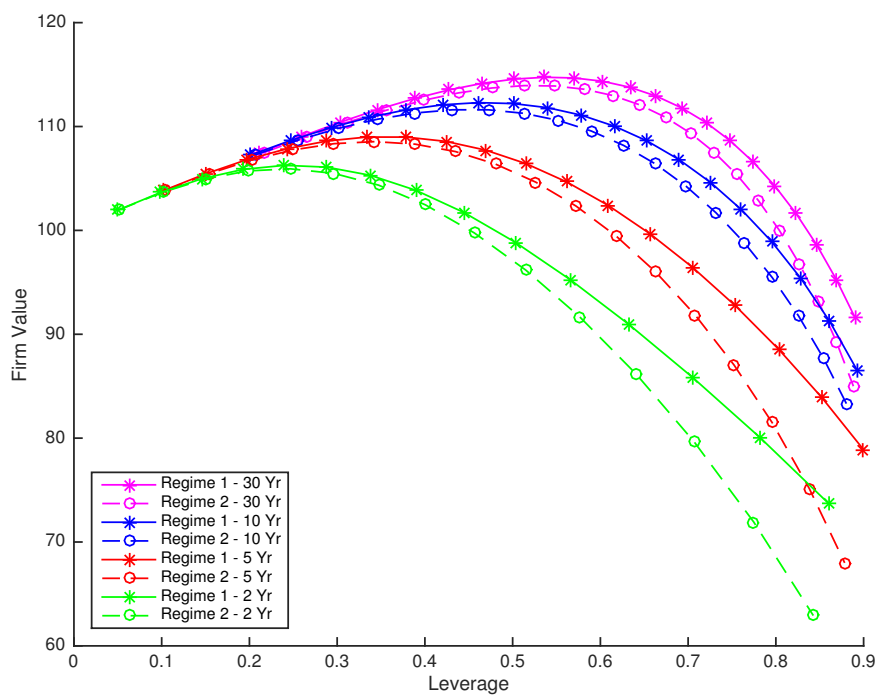


Figure 3.5.1: Firm Value by Leverage

The figure shows optimised firm value by leverage in each regime for different maturity profiles but otherwise using the default parameters in Table 3.5.1. Firm value appears to achieve a maximum value at lower levels of leverage for shorter maturity debt profiles. Firm value is lower in regime 2 than in regime 1 for equivalent levels of leverage.

Roll-over Debt under Regime-Switching

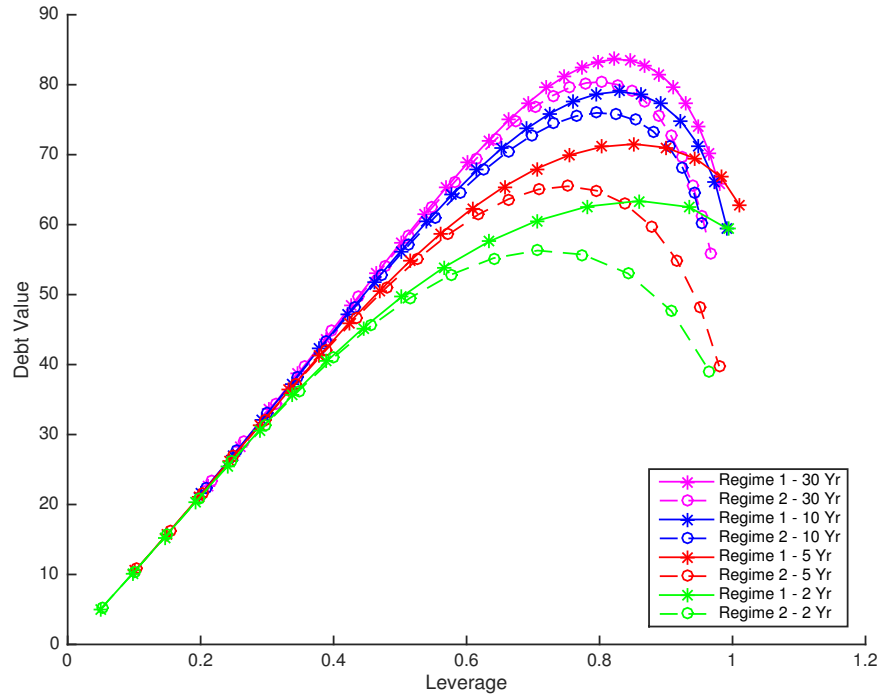


Figure 3.5.2: Debt Value by Leverage

The figure shows optimised total debt value by leverage in each regime for different maturity profiles but otherwise using the default parameters in Table 3.5.1. Debt value appears to achieve a maximum value at lower levels of leverage for shorter maturity debt profiles. Optimised debt value is lower in regime 2 than in regime 1 for the same level of leverage and declines more sharply as a function of leverage in regime 2 than in regime 1 after achieving its maximum value.

Roll-over Debt under Regime-Switching

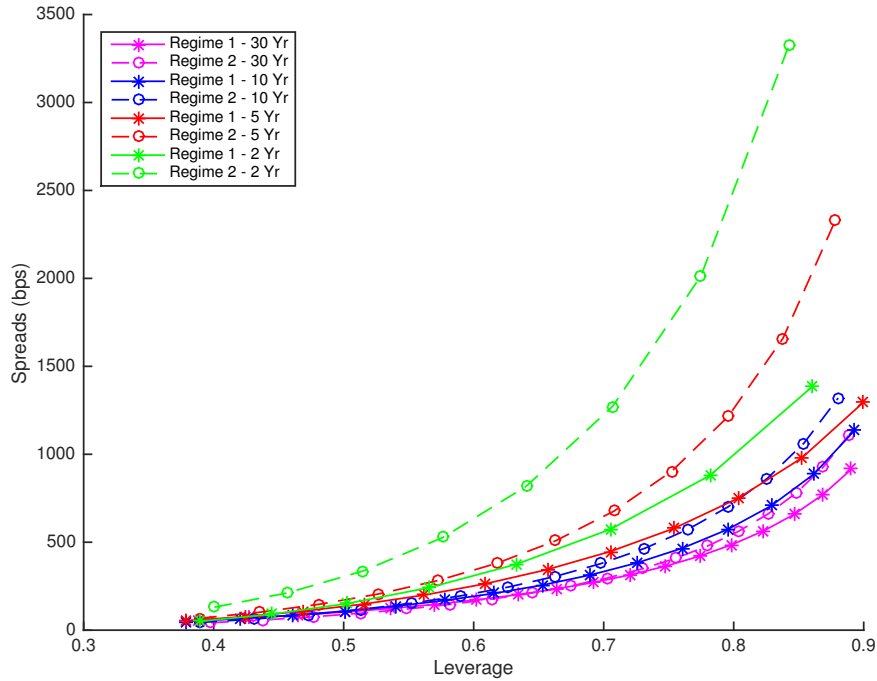


Figure 3.5.3: Firm Spreads by Leverage

The figure shows optimised firm spreads as a function of leverage in each regime for different maturity profiles but otherwise using the default parameters in Table 3.5.1. Firm spreads are higher for a shorter dated maturity profile and higher in regime 2 than in regime 1. Firm spreads increase rapidly in regime 2 as a function of leverage indicating that firms with shorter maturity profiles are subject to increased default risk during periods of high volatility.

Roll-over Debt under Regime-Switching

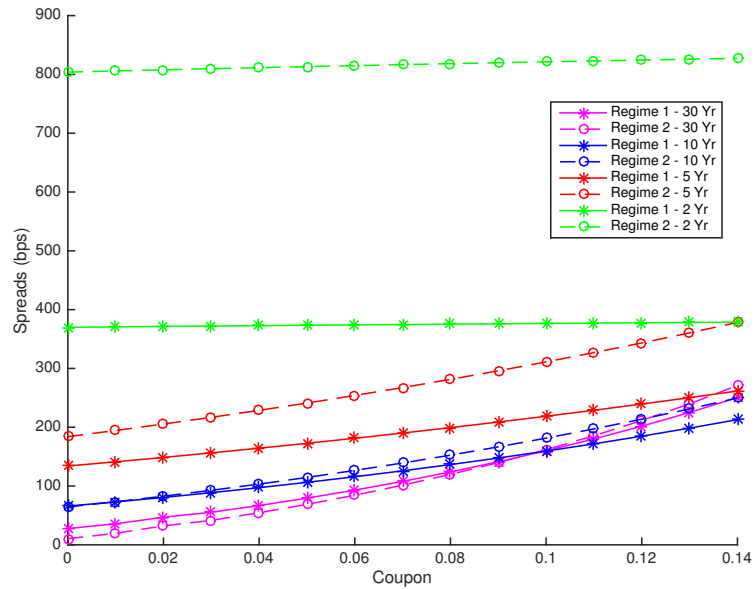


Figure 3.5.4: Firm Spreads by Coupon

The figure shows optimised firm spreads as a function of coupon in each regime for different maturity profiles but otherwise using the default parameters in Table 3.5.1. Firm spreads are higher for a shorter dated maturity profile and higher in regime 2 than in regime 1. For longer maturity profiles, spreads increase as the coupon rate increases while for shorter maturity profiles, spreads are relatively insensitive to the level of the coupon rate. Spreads are positive for discount bonds (zero coupon).

Roll-over Debt under Regime-Switching

Maturity	Regime 1			Regime 2		
	Leverage	Firm Value	Spread	Leverage	Firm Value	Spread
0.25 Year	0.142	103.54	0	0.133	103.35	33
1 Year	0.189	104.70	1	0.179	104.45	40
5 Year	0.356	109.03	43	0.341	108.55	41
10 Year	0.470	112.30	89	0.452	111.65	75
20 Year	0.518	113.85	117	0.500	113.10	95
30 Year	0.545	114.75	129	0.526	113.96	105

Table 3.5.3: Numerically Optimised Firm Value by Maturity

The table shows optimised leverage, firm value and associated firm credit spread for different maturity profiles. Optimised leverage increases with maturity as does firm value and spread. Optimised leverage is lower in regime 2 than regime 1. Firm credit spreads are higher for shorter maturities but are lower as the maturity profile increases beyond 5 years. Optimised leverage was calculated for nominal debt in the range $(0, 100]$.

Roll-over Debt under Regime-Switching

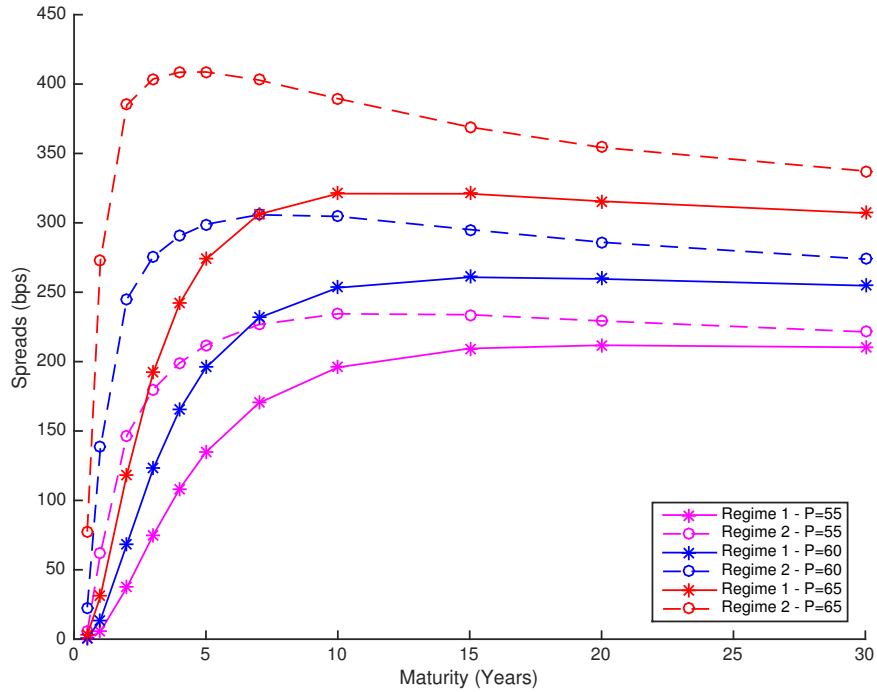


Figure 3.5.5: Term Structure Credit Spreads by Leverage

The figure shows optimised term structure credit spreads in each regime for different levels of nominal debt but otherwise using the default parameters in Table 3.5.1. Higher levels of nominal debt reflect higher levels of leverage. Spreads increase across all maturities with higher leverage, increasing more for shorter maturity debt. Spreads are higher in regime 2 compared to regime 1. For nominal debt equal to 65, the curve *inverts* and becomes downward sloping starting from 4-years to maturity which is a familiar market phenomena for firms in financial distress.

Roll-over Debt under Regime-Switching

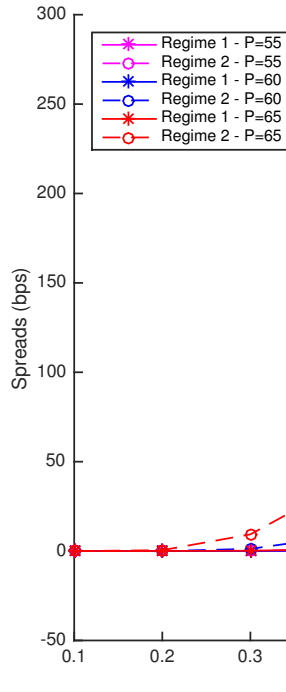


Figure 3.5.6: Short Maturity Term Credit Spreads by Leverage

The figure shows optimised short dated term structure credit spreads in each regime for different levels of nominal debt but otherwise using the default parameters in Table 3.5.1. Higher levels of nominal debt reflect higher levels of leverage. Spreads increase across all maturities with higher leverage and are higher in regime 2 than regime 1. The term structure of spreads retains a positive slope as a function maturity even at higher levels of leverage.

3.6 Auxiliary Results and Proofs

Lemma 3.6.1 below is used in the calculation of total debt and firm value in equations (3.2.8) and (3.2.10) respectively. Supporting Lemma's 3.6.2 and 3.6.3 were taken from Mijatovic and Pistorius [2009].

Lemma 3.6.1. *Let $Z = \{Z_t\}_{t \geq 0}$ be a Markov chain on a finite state space $E = \{1, 2, \dots, N\}$, τ a stopping time measurable with respect to the filtration generated by Z , $h : E \rightarrow \mathbb{R}$ and $a : E \rightarrow \mathbb{R}^+$. If G denotes the generator of Z and Λ_a is a diagonal matrix of size N with diagonal elements $a(i), i = 1, 2, \dots, N$, then*

$$\int_0^\infty \mathbb{E}_j \left[e^{-\int_0^t a(Z_u) du} h(Z_t) \mathbb{1}_{\{\tau \leq t\}} \right] dt = \mathbb{E}_j \left[e^{-\int_0^\tau a(Z_u) du} \tilde{h}(Z_\tau) \right] \quad (3.6.1)$$

where

$$\tilde{h}(k) = \int_0^\infty \mathbb{E}_k \left[e^{-\int_0^u a(Z_s) ds} \right] du = \mathbf{1}_k' (\Lambda_a - G)^{-1} h, \quad k \in E,$$

where $h = (h(1), h(2), \dots, h(N))$, $\mathbf{1}$ is an N element vector of ones and $\mathbf{1}_k$ is a column vector of size N with one in the k^{th} row equal to one and zero otherwise.

Proof. First note that by the tower property of conditional expectation and the strong Markov property

$$\mathbb{E}_j \left[e^{-\int_0^t a(Z_u) du} h(Z_t) \mathbb{1}_{\{\tau \leq t\}} \right] = \mathbb{E}_j \left[e^{-\int_0^\tau a(Z_u) du} \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{Z_\tau, \tau} \left[e^{-\int_0^{t-\tau} a(Z_u) du} h(Z_t) \right] \right]$$

so that by Lemma 3.6.2 and Lemma 3.6.3

$$\begin{aligned}
 \hat{h}(k) &= \int_{\tau}^{\infty} \mathbb{E} \left[e^{-\int_0^{t-\tau} a(Z_u) du} h(Z_t) \mid Z_{\tau} = k \right] dt \\
 &= \int_{\tau}^{\infty} \underline{1}_k' e^{(G-\Lambda_a)(t-\tau)} h dt \\
 &= \int_0^{\infty} \underline{1}_k' e^{(G-\Lambda_a)t} h dt \\
 &= \underline{1}_k' (\Lambda_a - G)^{-1} h
 \end{aligned}$$

where $h' = (h(1), h(2), \dots, h(N))$ with the application of Lemma 3.6.3 in the latter equality justified by Theorem 2 in Mijatovic and Pistorius [2009] which implies that all the eigenvalues $(\Lambda_a - G)^{-1}$ have non-positive real part, completing the proof. \square

Lemma 3.6.2. *Let Z be a Markov chain on a finite state space $E = \{1, 2, \dots, N\}$ and let $a, h : E \rightarrow \mathbb{R}$. If G denotes the generator of Z and Λ_a is a diagonal matrix of size N with diagonal elements $a(i), i = 1, 2, \dots, N$, then*

$$\mathbb{E}_j \left[e^{-\int_0^t a(Z_s) ds} h(Z_t) \right] = \underline{1}_j' e^{(G-\Lambda_a)t} h \tag{3.6.2}$$

where $\underline{1}_j$ is a column vector of size N with one in the j^{th} row and zero otherwise and $h' = (h(1), \dots, h(N))$.

Proof. See Mijatovic and Pistorius [2009] \square

Lemma 3.6.3. *Let $a \in \mathbb{C}$ and M a matrix whose eigenvalues all have non-positive real part. Then the matrix $\Lambda_a - M$ is invertible and the following formula holds*

$$\int_0^{\infty} e^{-at} \exp(tM) dt = (\Lambda_a - M)^{-1}.$$

Proof. See Mijatovic and Pistorius [2009] \square

CHAPTER 4 OPTIMAL CAPITAL STRUCTURE UNDER REGIME-SWITCHING WITH JUMPS

4.1 Introduction

In this chapter, the model of Chapter 3 is extended to allow the asset price process of the firm to jump in the negative direction. Moreover, the jump process is also allowed to be regime-dependent motivated by the conjecture that during times of economic stress, negative jumps of a firm's asset process might be expected to be larger in magnitude. In this chapter, however, the firm is assumed to be financed by a single infinite maturity, or consol, bond. The latter assumption unfortunately loses the richness of term structure analysis inherent in the roll-over model, but it nevertheless implies that the debt profile is fully stationary, which permits an optimal capital structure to be identified.

The model of this chapter also allows firm asset volatility, risk free interest and dividend rates as well as tax rates and bankruptcy costs to be regime-dependent. As in Chapter 3, two regimes are considered. Maximisation of equity on the part of the firm manifests itself an optimal stopping problem which takes the form of a perpetual American put option. An interesting feature of the model is, however, that the payout of the option depends upon the regime. Indeed, the strikes of the put option become regime-dependent which has the implication that, under certain conditions, it may never be optimal for a firm to endogenously declare default in certain regimes. The intuition behind the latter finding is that if the strike in one regime is very far below

the strike in the other regime, then it may be optimal not to declare bankruptcy in the former regime, but simply wait for the regimes to switch to collect an expected larger payout. When an optimal default boundary is known to exist, optimal capital structure and credit spreads are identified.

The problem of a firm financed by a single infinite maturity bond was first considered by Black and Cox [1976]. Later, Leland [1994a] examined optimal capital structure for a firm, financed by an infinite maturity bond, which received tax benefits on interest payments and where the asset process followed a geometric Brownian motion. More recently, dynamic capital structure was considered by Elliott and Shen [2015] incorporating tax benefits and bankruptcy costs with regime-switching dynamics in the case where the firm's asset process was assumed to be a geometric regime-switching Brownian motion. The model presented in this paper builds on the literature by adding negative jumps to a regime-switching asset process of a firm financed by a single infinite maturity bond where both bankruptcy costs and tax relief on interest payments are also assumed to be regime-dependent.

The outline of this chapter is as follows. The formulation of the model is outlined in Section 4.2. In Section 4.3, some Wiener-Hopf factorisation results which are required in sequel to calculate first-passage probabilities are collected. Endogenously determined optimal default boundaries which maximise equity value are determined in Section 4.4 and an optimal capital structure identified. Finally, in Section 4.5 some numerical calculations based on the results are presented. Auxiliary results are collected in Section 4.6.

4.2 Model Formulation

4.2.1 Market Structure The market is assumed to be composed of a savings account and a single-asset firm whose price processes, $\{B_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$, are given

by

$$B_t = \exp\left(-\int_0^t r(Z_s) ds\right), \quad (4.2.1)$$

$$V_t = \exp(X_t), \quad (4.2.2)$$

where $\mathbb{E}[V_1] < \infty$, $r(i) : E^0 \mapsto \mathbb{R}^+ \setminus \{0\}$ is the instantaneous risk free interest rate, $\{Z_t\}_{t \geq 0}$ is an irreducible Markov process with finite state space $E^0 = \{1, 2\}$ and intensity matrix G given by

$$G = \begin{pmatrix} -\gamma_1 & \gamma_1 \\ \gamma_2 & -\gamma_2 \end{pmatrix}$$

with $\gamma_i > 0, i \in E$ and $\{X_t\}_{t \geq 0}$ is a regime-switching Lévy process given by

$$X_t = x + \int_0^t \mu(Z_s) ds + \int_0^t \sigma(Z_s) dW_s + \sum_{i \in E^0} \int_0^t 1_{\{Z_s=i\}} dJ_i(s), \quad x \in \mathbb{R}, \quad (4.2.3)$$

where $\{W_t\}_{t \geq 0}$ is a Wiener process independent of Z , $\{J_i(t), t \geq 0\}$ are independent compound Poisson processes with *negative* jumps arriving at rate λ_i , $\mu(i), \sigma(i) : E^0 \mapsto \mathbb{R}$ with $\sigma(\cdot) > 0$ and $1_{\{A\}}$ is the indicator of the set A . Where there is no scope for ambiguity, the notation a_i will be used to denote the value of parameter a in regime i . The infinitesimal generator of the process (X, Z) is given by

$$\begin{aligned} \mathbb{L}_{X,Z} w(x, i) &= \frac{1}{2} \sigma_i^2 \frac{\partial^2 w(x, i)}{\partial x^2} + \mu_i \frac{\partial w(x, i)}{\partial x} \\ &\quad + \gamma_i (w(x, 3-i) - w(x, i)) \\ &\quad + \lambda_i \int_{-\infty}^{\infty} (w(x+y) - w(x)) f_{Y_i}(y) dy, \quad i \in E^0, \end{aligned} \quad (4.2.4)$$

for C^2 functions w where f_{Y_i} is the probability density function of the jump process in regime i .

Jump sizes are assumed to be *phase-type* distributed (see Asmussen [2000]). A *phase-type* distribution F on $(0, \infty)$ is the distribution of the absorption time of a finite state

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Markov chain with one state absorbing and the remaining states transient. Restricted to the transient states with generator matrix T and initial distribution given by the column vector α , it follows from Markov chain theory that the density of F is given by

$$f(x) = \alpha' e^{Tx} t \quad (4.2.5)$$

for $x > 0$, where $t = (-T)^{-1} \underline{1}$, $'$ denotes transpose and $\underline{1}$ is a column vector of ones.

It is assumed that the stochastic processes V and Z are defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denotes the completed filtration jointly generated by V and Z . Conditional expectations will be denoted by writing $\mathbb{E}_{x,i}[\cdot] = \mathbb{E}[\cdot | X_0 = x, Z_0 = i]$. Furthermore, since it is known (see Jiang and Pistorius [2008]) that there exists an equivalent martingale measure which is structure preserving, in sequel all expectations can be understood to be taken in the risk neutral measure.

Theorem 1 in Mijatovic and Pistorius [Preprint] implies that the discounted moment generating function of the process (X, Z) is given by

$$\mathbb{E}_{0,i} \left[\frac{e^{uX_t}}{B_t} \mathbb{1}_{\{Z_j=j\}} \right] = \underline{1}_i' e^{t(K(u) - \Lambda_r)} \underline{1}_j, \quad u \in \mathbb{R},$$

where $k(u) := G + \Lambda(u)$, Λ_r is a 2×2 diagonal matrix with diagonal elements $r(i)$ for $i \in E^0$ and $\Lambda(u)$ is a 2×2 diagonal matrix where the i^{th} element is equal to the characteristic exponent of the process X in regime $i \in E^0$ given by

$$\phi_i(u) = u\mu_i + \frac{1}{2}\sigma_i^2 u^2 + \lambda_i (\alpha_i' (\Lambda_u - T_i)^{-1} t_i - 1).$$

It follows, therefore, that if the drift vector μ is chosen such that $\Lambda(1) + \Lambda_\delta - \Lambda_r = 0$ then

$$\mu_i = r_i - \delta_i - \frac{1}{2}\sigma_i^2 - \lambda_i (\alpha_i' (I - T_i)^{-1} t_i - 1), \quad i \in E^0, \quad (4.2.6)$$

which implies that the discounted risk neutral drift will be $-\delta_i, i \in E^0$. In sequel,

therefore, it will be assumed that condition (4.2.6) holds.

4.3 Matrix Wiener-Hopf Factorisation

In this section some Wiener-Hopf factorisation results related to finite state Markov chains are briefly collected, which are required in sequel to calculate first-passage probabilities. This section draws heavily on the presentation in Jiang and Pistorius [2008] which related to the more general case of both positive and negative jumps. The results have therefore been tailored to the case of negative jumps only since this will be the focus of this chapter.

4.3.1 Fluid Embedding In sequel applications it will be of great interest to characterise the boundary crossing times of X given by

$$\tau(b) = \inf \{t \geq 0 | X_t \leq b(Z_t)\}.$$

The first step in the latter regard is to reformulate the setting into one of the first hitting time for a related continuous Markov process. The latter is achieved by a process of *fluid embedding*, outlined below, which transforms the path of X by replacing jumps with linear stretches. As a result, the transformed process always touches any boundary at first passage. An additional advantage of the latter approach is that a mechanism is afforded by which to calculate any shortfall or overshoot should X jump across a boundary.

Define $\{Y_t\}_{t \geq 0}$ to be an irreducible continuous Markov chain with finite state space $E \cup \partial$, where ∂ is an absorbing cemetery state, and denote by $\{A_t\}_{t \geq 0}$ the Markov modulated Brownian motion given by

$$A_t = A_0 + \int_0^t m(Y_s) ds + \int_0^t s(Y_s) dW_s \tag{4.3.1}$$

where $s, m : E \cup \partial \mapsto \mathbb{R}$ with $s(\partial) = m(\partial) = 0$.

The process A is the *fluid-embedding* of X if the generator of Y restricted to E is equal to Q_0 where, in block notation,

$$Q_a = \begin{pmatrix} G - D_a & B \\ t & T \end{pmatrix}, \quad (4.3.2)$$

with D_a an $N \times N$ diagonal matrix where $(D_a)_{ii} = \lambda_i + a_i$ and,

$$B = \begin{pmatrix} \lambda_1 \alpha'_1 & \\ & \lambda_2 \alpha'_2 \end{pmatrix}, T = \begin{pmatrix} T_1 & \\ & T_2 \end{pmatrix}, t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}.$$

The functions $m(\cdot)$ and $s(\cdot)$ are specified as:

$$m(j) = \begin{cases} \mu_j & \text{if } j \in E^0 \\ -1 & \text{if } j \in E^- \end{cases}, \quad s(j) = \begin{cases} \sigma_j & \text{if } j \in E^0 \\ 0 & \text{otherwise} \end{cases}. \quad (4.3.3)$$

It is clear from (4.3.2) that Q_a admits the partition $E = E^0 \cup E^-$ where E^0 is the state space inherited from Z and E^- are the states in which the path of A is linear and with slope -1 . In sequel, the subset of E which corresponds to the i^{th} regime of X will be denoted as $E_i = \{i\} \cup E_i^-$. The elements of the matrix Q_a will be denoted q_{ij} for $i, j \in E^0$.

4.3.2 Path Transformation While the chain Y is in $i \in E^0$, A evolves according to a Brownian motion with drift $m(i)$ and volatility $\sigma(i)$. However, when Y takes values in E^- , the path of A is linear with slope -1 , with the length of the linear stretches corresponding to the size of the jumps in X . Informally, therefore, a path of X can be thought of as being obtained from a path of A by simply replacing the latter linear stretches with negative jumps in X of the same size as the length of the linear stretch in E^- .

More formally, a stochastic process which has the same law as (X, Z) can be obtained by first time-changing the process (A, Y) so that A is observed only when Y takes

values in E^0 . To proceed, first define

$$T_0(t) = \int_0^t 1_{\{Y_s \in E^0\}} ds \quad \text{and} \quad T_0^{-1}(u) = \inf \{t \geq 0 \mid T_0(t) > u\},$$

which correspond to the time spent by chain Y in E^0 before time t and its right continuous inverse.

From the definition of the generator in (4.3.2) it is clear that when Y jumps from any state in E^- to state $i \in E^0$, it must have been in state $i \in E^0$ immediately prior to jumping from E^0 . It is obvious, therefore, from the generator of Y in (4.3.2) that the process $Y \circ T_0^{-1}$ is a Markov chain with generator Z because it simply ignores all excursions from E^0 . Furthermore, the process $A \circ T_0^{-1}$ is a regime-switching jump diffusion and the process $(A \circ T_0^{-1}, Y \circ T_0^{-1})$ is equal in law to (X, Z) under $\mathbb{P}_{x,i}$ for all $x \in \mathbb{R}$ and $i \in E_0$. Furthermore, it holds that for

$$\hat{\tau}(b) = \inf \left\{ t \geq 0 \mid Y_t \in E^0, A_t \leq \hat{b}(Y_t) \right\} \quad (4.3.4)$$

that $(T(\hat{\tau}(b)), A_{\hat{\tau}(b)}, Y_{\hat{\tau}(b)}) \stackrel{d}{=} (\tau(b), X_{\tau(b)}, Z_{\tau(b)})$ and $\hat{b} : E \mapsto \mathbb{R}$ given by $\hat{b}(j) = b(i)$ for $j \in E_i$.

State dependent discounting can be incorporated by replacing the generator Q_0 with Q_r from which it follows that

$$\mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du + X_{\tau(b)}} h(Z_{\tau(b)}) \right] = \mathbb{E}_{x,i} \left[e^{A_{\hat{\tau}(b)}} h(Y_{\hat{\tau}(b)}) 1_{\{\hat{\tau}(b) < \zeta\}} \right], \quad (4.3.5)$$

where

$$\zeta = \inf \{t \geq 0 \mid Y_t \notin E\} \quad (4.3.6)$$

with $\inf \{\emptyset\} = \infty$ and Y evolves according to the generator Q_r .

4.3.3 Matrix Factorisation The solution of the first-passage problem of the Markov process (A, Y) across a constant level is closely related to the up and down-crossing ladder processes, \hat{Y}^+ and \hat{Y}^- respectively, of (A, Y) . The up and down-

crossing ladder processes are time changes of Y constructed such that Y is observed only when A is at its maximum and minimum respectively. The up and down-crossing ladder processes \hat{Y}^+ and \hat{Y}^- of (A, Y) are defined by

$$\hat{Y}_a^\pm = \begin{cases} Y(\tau_a^\pm), & \text{if } \tau_a^\pm < \infty, \\ \partial, & \text{otherwise,} \end{cases} \quad (4.3.7)$$

where ∂ is a graveyard state and $\tau_a^\pm = \inf \{s \geq 0 \mid A_s \geq a\}$ with $\inf \{\emptyset\} = \infty$.

The ladder processes \hat{Y}^+ and \hat{Y}^- are again Markov processes with state spaces E^0 and $E^0 \cup E^-$ respectively. Matrix Wiener-Hopf factorisation, in the presence of negative jumps only, characterises the generator matrices of \hat{Y}^+ and \hat{Y}^- denoted Q_a^+ and Q_a^- along with the initial distribution defined by

$$\eta^+(i, j) = \mathbb{P}_{0,i}[\hat{Y}_0^+ = j, \tau_0^+ < \zeta] \quad \text{for } i \in E^-, j \in E^0. \quad (4.3.8)$$

In the definition of a matrix Wiener-Hopf factorisation given below $\mathcal{Q}(n)$ denotes the set of irreducible $n \times n$ generator matrices, $\mathcal{P}(n, m)$ the set of $n \times m$ matrices whose rows are sub-probability vectors and Σ and V denote the 2×2 diagonal matrices with $\Sigma_{ii} = s(i)$ and $V_{ii} = m(i)$ for $i \in E^0$ respectively. Since the rows of the matrix Q_a do not sum to zero, the matrix is referred to as transient. Let N^- be the number of elements in E^- and set $N_0^- = 2 + N^-$.

Definition 4.3.1. Let G^+, C^+ and G^- be elements of of the sets $\mathcal{Q}(2)$, $\mathcal{P}(N^-, 2)$, and $\mathcal{Q}(N_0^-)$ respectively. A triple (C^+, G^+, G^-) is called a *matrix Wiener-Hopf factorisation* of (A, Y) if

$$\Xi(-G^+, W^+) = O \quad \text{and} \quad \Xi(G^-, W^-) = O, \quad (4.3.9)$$

where

$$\Xi(S, W) = \frac{1}{2}\Sigma^2 W S^2 + V W S + Q_a W \quad (4.3.10)$$

and W^+ and W^- are given in block notation by

$$W^+ = \begin{pmatrix} I_0 \\ C^+ \end{pmatrix}, \quad W^- = \begin{pmatrix} I_0 & O \\ O & I^- \end{pmatrix}, \quad (4.3.11)$$

where I_0 and I^- are identity matrices of sizes 2×2 and $N^- \times N^-$ respectively and O denotes a zero matrix of appropriate size.

The following Theorem from Jiang and Pistorius [2008] links the previous definition to a characterisation of the Wiener-Hopf factorisation of (A, Y) .

Theorem 4.3.2. *If Q is transient then the quadruple (η^+, Q^+, Q^-) is a unique Wiener-Hopf factorisation of (A, Y) .*

4.3.4 First Exit from an Interval The two-sided exit problem of A from an interval $[k, l]$ with $-\infty < k < l < +\infty$ is to find the distribution of the process (A_τ, Y_τ) at the first exit time given by

$$\tau = \inf \{t \geq 0 \mid A_t \notin [k, l]\}.$$

The two-sided exit problem can be solved in terms of (η^+, Q^+, Q^-) , and the matrices,

$$Z^+ = \begin{pmatrix} I_0 \\ \eta^+ \end{pmatrix} e^{Q_r^+(l-k)}, \quad Z^- = (1, O^+) e^{Q_r^-(l-k)}, \quad (4.3.12)$$

and

$$\Psi^+(x) = \left(W^+ e^{Q_r^+(l-x)} - e^{Q^-(x-k)} Z^+ \right) (I - Z^- Z^+), \quad (4.3.13)$$

$$\Psi^-(x) = \left(e^{Q^-(x-k)} - W^+ e^{Q_r^+(l-x)} Z^- \right) (I - Z^+ Z^-), \quad (4.3.14)$$

$$\Psi^0(a, x) = \left(e^{ax} I - e^{al} \Psi^+(x) J^+ - e^{ak} \Psi^-(x) \right) (-K(a)^{-1}), \quad (4.3.15)$$

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where I is an identity matrix, W^+ is given in (4.3.11) with $C^+ = \eta^+$, J^+ is the transpose of W^+ with C^+ replaced by zeros and

$$K(a) = \frac{1}{2}\Sigma^2 a^2 + Va + Q_r. \quad (4.3.16)$$

The following Proposition from Jiang and Pistorius [2008] outlines the solution to the two-sided exit problem.

Proposition 4.3.3. *Let h^+ , h^- and h^0 be functions that map E^0 , $E^0 \cup E^-$ and E to \mathbb{R} . If*

$$\frac{1}{2}\sigma(i)^2 + \mu(i) < -q_{ii}, \quad i \in E,$$

then it holds for $x \in [k, l]$ and $i \in E$ that

$$\mathbb{E}_{x,i} [h^+(Y_\tau) 1_{\{A_\tau=l, \tau<\zeta\}}] = \underline{1}_i \Psi^+(x) h^+, \quad (4.3.17)$$

$$\mathbb{E}_{x,i} [h^-(Y_\tau) 1_{\{A_\tau=k, \tau<\zeta\}}] = \underline{1}_i \Psi^-(x) h^-, \quad (4.3.18)$$

$$\mathbb{E}_{x,i} [e^{aA_\zeta - h^0(Y_{\zeta^-})} 1_{\{\zeta < \tau\}}] = \underline{1}_i \Psi^0(a, x) \Lambda_{h^0} q_r, \quad (4.3.19)$$

where $q_r = (-Q_r) \underline{1}$, ζ is defined in (4.3.6) and Λ_{h^0} is a diagonal matrix with elements h^0 .

4.4 Optimal Capital Structure

4.4.1 Debt Structure The firm is partly financed by a single consol bond with face value P upon which the firm may default. The time of default, τ , is assumed to be endogenously chosen by the firm and will be described further below. On default a proportion $\xi_i \in [0, 1]$ of the asset value is lost in regime i during the resulting reorganisation of the firm and the residual value of the firm's assets become the property of the debt holders. The firm's debt attracts a continuous coupon rate of c and therefore has total value given by

$$\begin{aligned}
 D(x, i) &= cP \mathbb{E}_{x,i} \left[\int_0^\tau e^{-\int_0^t r(Z_u) du} dt \right] + \mathbb{E}_{x,i} \left[e^{-\int_0^\tau r(Z_u) du} (1 - \xi(Z_\tau)) V_\tau \right] \\
 &= \underline{1}_i' cP (\Lambda_r - G)^{-1} \underline{1} - \mathbb{E}_{x,i} \left[e^{-\int_0^\tau r(Z_u) du} (\hat{v}(Z_\tau) - (1 - \xi(Z_\tau)) V_\tau) \right], \quad (4.4.1)
 \end{aligned}$$

where

$$\hat{v}(k) = \underline{1}_k' cP (\Lambda_r - G)^{-1} \underline{1}, \quad k \in E^0, \quad (4.4.2)$$

Λ_a is a 2×2 diagonal matrix with elements equal to $a(i), i \in E^0$, $\underline{1}$ is a 2 element vector of ones and $\underline{1}_k$ is a 2 element vector with 1 in row k and zero in the alternate row. The expectations in (4.4.1) were calculated using Lemma's 3.6.1 and 3.6.2.

4.4.2 Value of the Firm The economy's tax policy is assumed to be regime dependent and therefore, the firm receives tax relief on the value of coupon payments made equal to $\pi(i) \in [0, 1]$, for each regime $i \in E^0$. It follows that the value of the firm is comprised of the value of assets at time 0 together with the present value of future tax relief minus the proportion of assets lost in reorganisation of the firm on default given by

$$\begin{aligned}
 F(x, i) &= e^x + cP \mathbb{E}_{x,i} \left[\int_0^\tau e^{-\int_0^t r(Z_u) du} \pi(Z_t) dt \right] - \mathbb{E}_{x,i} \left[e^{-\int_0^\tau r(Z_u) du} \xi(Z_\tau) V_\tau \right] \\
 &= e^x + \underline{1}_i' cP (\Lambda_r - G)^{-1} \pi - \mathbb{E}_{x,i} \left[e^{-\int_0^\tau r(Z_u) du} (\hat{\chi}(Z_\tau) + \xi(Z_\tau) V_\tau) \right], \quad (4.4.3)
 \end{aligned}$$

where

$$\hat{\chi}(k) = \underline{1}_k' cP (\Lambda_r - G)^{-1} \pi, \quad k \in E^0, \quad (4.4.4)$$

$\pi' = \{\pi(1), \pi(2)\}$ and where again, Lemma's 3.6.1 and 3.6.2 were used in the calculations.

4.4.3 Equity Value The equity value of the firm is taken to be the value of the firm in (4.4.3) minus the value of debt in (4.4.1),

$$Q(x, i) = e^x - \underline{1}_i' cP (\Lambda_r - G)^{-1} \hat{\pi} + \mathbb{E}_{x,i} \left[e^{-\int_0^\tau r(Z_u) du} (\hat{\kappa}(Z_\tau) - V_\tau) \right], \quad (4.4.5)$$

where

$$\hat{\kappa}(k) = \mathbb{1}_k cP (\Lambda_r - G)^{-1} \hat{\pi}, \quad k \in E^0, \quad (4.4.6)$$

and $\hat{\pi} = \mathbb{1} - \pi$. Note that the recovery rates, $\xi = (\xi(1), \xi(2))'$ do not feature in the value of equity which is in contrast to the roll-over debt structure. Any residual value left on default becomes the property of the bond holders and affords no benefit to the equity holders. The value of equity on default is also identically zero which is in accordance with the strict priority rule.

4.4.4 Default Decision The time of default, τ , is assumed to be endogenously chosen by the firm so as to maximise the value of equity in (4.4.5). Since the decision to default is entirely at the discretion of the firm, there is always the option to wait (and receive zero). It therefore follows that maximising equity value in (4.4.5) amounts to solving an optimal stopping problem of the form

$$\begin{aligned} w^*(x, i) &= \sup_{\tau \in \mathcal{T}_{0, \infty}} \mathbb{E}_{x, i} \left[e^{-\int_0^\tau r(Z_u) du} (\hat{\kappa}(Z_\tau) - e^{X_\tau}) \mathbb{1}_{\{\tau < \infty\}} \right] \\ &= \sup_{\tau \in \mathcal{T}_{0, \infty}} \mathbb{E}_{x, i} \left[e^{-\int_0^\tau r(Z_u) du} (\hat{\kappa}(Z_\tau) - e^{X_\tau})^+ \right] \\ &= \sup_{\tau \in \mathcal{T}_{0, \infty}} \mathbb{E}_{x, i} \left[e^{-\int_0^\tau r(Z_u) du} g(X_\tau, Z_\tau) \right], \end{aligned} \quad (4.4.7)$$

where $z^+ = \max(0, z)$, $\hat{\kappa}$ is given in (4.4.6), $V_0 = v = e^x$, $\mathcal{T}_{0, \infty}$ represents the set of \mathbb{F} -measurable finite stopping times and $g(i, x) = (\hat{\kappa}(i) - e^x)^+$. Following the established literature, it is assumed that assets in place cannot be liquidated in order to raise funds to pay debt holders.

Note that since $x \mapsto w^*(x, i), i \in E^0$ is convex, implying that $w^*(x, \cdot)$ is C^0 in \mathbb{R} , and (X, Z) is a Markov process, the general theory of optimal stopping in Peskir and Shiryaev [2006] implies a *continuation region* associated with the optimal stopping problem in (4.4.7) reads as

$$\mathcal{C} = \{(x, i) \in \mathbb{R} \times E^0 \mid w^*(x, i) > g(x, i)\} \quad (4.4.8)$$

with the *stopping region* defined as

$$\mathcal{S} = \{(x, i) \in \mathbb{R} \times E^0 \mid w^*(x, i) = g(x, i)\} \quad (4.4.9)$$

and the stopping time τ_B defined by

$$\tau_B = \inf \{t \geq 0 \mid X_t \in \mathcal{S}\} \quad (4.4.10)$$

being optimal in (4.4.7) providing τ_B is finite. The following Theorem characterises a candidate value function for (4.4.7) and stopping time for (4.4.10).

Theorem 4.4.1. *Let $(b_i^*, i \in E^0) \in \mathbb{R}$ and $w : \mathbb{R} \times E^0 \rightarrow \mathbb{R}$ be bounded such that $w(\cdot, i), i \in E^0$ is C^1 on \mathbb{R} and C^2 on $\mathbb{R} \setminus \{b_i^*\}$ and the following system is satisfied:*

$$\mathbb{L}_{X,Z}w - rw \leq 0 \quad x \leq b_i^*, i \in E^0, \quad (4.4.11)$$

$$w(x, i) \geq g(x, i) \quad x > b_i^*, i \in E^0, \quad (4.4.12)$$

$$\mathbb{L}_{X,Z}w - qw = 0 \quad x > b_i^*, i \in E^0, \quad (4.4.13)$$

$$w(x, i) = g(x, i) \quad x \leq b_i^*, i \in E^0, \quad (4.4.14)$$

then $w = w^*$ and a stopping time that achieves the supremum in (4.4.7) is given by

$$\tau(b^*) = \inf \{t \geq 0 \mid X_t \leq b^*(Z_t)\}. \quad (4.4.15)$$

Proof. The optimality of w and b^* can be proved by standard means as follows¹. An application of a generalised version of Itô's formula to $e^{-\int_0^t r(Z_u)du}w(x, i)$ obtains

$$e^{-\int_0^t r(Z_u)du}w(X_t, Z_t) = w(x, i) + \int_0^t e^{-\int_0^s r(Z_u)du}\Pi_{X,Z}w(X_s, Z_s) \mathbb{1}_{\{X_s \neq b(Z_s)\}}ds + \mathcal{M}_t \quad (4.4.16)$$

where $\Pi_{X,Z}f := \mathbb{L}_{X,Z}f - rf$ and \mathcal{M}_t can be shown to be a martingale. Using (4.4.13),

¹See for example Peskir [2005], Theorem 25.1.

(4.4.11), (4.4.12) and (4.4.14) it follows that

$$e^{-\int_0^t r(Z_u)du} g(X_t, Z_t) \leq e^{-\int_0^t r(Z_u)du} w(X_t, Z_t) \leq w(x, i) + \mathcal{M}_t.$$

Defining $\{\tau_n\}_{n \geq 1}$ as a localisation sequence of bounded stopping times for \mathcal{M} , then for every stopping time τ of (X, Z) it follows that

$$e^{-\int_0^{\tau \wedge \tau_n} r(Z_u)du} g(X_{\tau \wedge \tau_n}, Z_{\tau \wedge \tau_n}) \leq w(x, i) + \mathcal{M}_{\tau \wedge \tau_n}, \quad n \geq 1,$$

taking $\mathbb{P}_{x,i}$ expectations, noting that $\mathbb{E}_{x,i}[\mathcal{M}_{\tau \wedge \tau_n}] = 0$ for all n (as a consequence of the optional sampling theorem) and letting $n \rightarrow \infty$ yields by Fatou's lemma that

$$\mathbb{E}_{x,i} \left[e^{-\int_0^\tau r(Z_u)du} g(X_\tau, Z_\tau) \right] \leq w(x, i).$$

Finally, taking the supremum over all stopping times obtains that $w^*(x, i) \leq w(x, i)$. Next, using (4.4.16) and again using (4.4.13) along with the optional sampling theorem it follows that

$$\mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b^*) \wedge \tau_n} r(Z_u)du} w(X_{\tau(b^*) \wedge \tau_n}, Z_{\tau(b^*) \wedge \tau_n}) \right] = w(x, i), \quad n \geq 1.$$

Letting $n \rightarrow \infty$ and noting that

$$e^{-\int_0^{\tau(b^*)} r(Z_u)du} w(X_{\tau(b^*)}, Z_{\tau(b^*)}) = e^{-\int_0^{\tau(b^*)} r(Z_u)du} g(X_{\tau(b^*)}, Z_{\tau(b^*)})$$

it follows by dominated convergence that

$$\mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b^*)} r(Z_u)du} g(X_{\tau(b^*)}, Z_{\tau(b^*)}) \right] = w(x, i)$$

which shows that $\tau(b^*)$ in (4.4.15) is optimal in (4.4.7) and that $w(x, i) = w^*(x, i)$ for all $(x, i) \in \mathbb{R} \times E^0$ completing the proof. \square

Remark 4.4.2. It is intuitive that if the regime dependent strikes of the put option payoff in (4.4.7) are far away from each other, that in the regime with the lower strike,

for low asset values, it may be optimal to defer declaring default until the regime with the higher strike to achieve a much higher payoff. The parameters effecting the regime strike values are the risk free interest rate and the tax rate. The latter would suggest that monetary and fiscal policies targeted at reducing interest and tax rates may well be effective in deferring default decisions. It is worth noting that if the risk free interest rate and tax rate are the same in both regimes, the strike is constant across regimes and (4.4.7) reduces to the case of a standard perpetual put with a regime-switching asset process. In sequel, the focus is to the case that the lowest boundary is finite.

4.4.5 First Passage Below Regime Dependent Levels In view of Theorem 4.4.1, solving for the optimal value of equity in (4.4.5) amounts to identifying the first passage time of X below a regime dependent level and finding a function of the form

$$\Theta_{a,b}(x, i, h) = \mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du + aX_{\tau(b)}} h(Z_{\tau(b)}) \right] \quad (4.4.17)$$

with $h : E^0 \mapsto \mathbb{R}$ where

$$\tau(b) = \inf \{t \geq 0 | X_t \leq b(Z_t)\}. \quad (4.4.18)$$

The solution (4.4.17) is identified in the following Lemma by considering the fluid embedding of the process X and the associated matrix Wiener-Hopf factorisation results outlined in Section 4.3. Note that since the regime dependent levels in (4.4.18) are given, without loss of generality, it will be assumed that $b_1 > b_2$.

Lemma 4.4.3. *Let $X = \{X_t\}_{t \geq 0}$ be defined in (4.2.3) and $\tau(b)$ be defined in (4.4.18) with $b_1 > b_2$ and assume that $\frac{1}{2}\sigma(i)^2 + \mu(i) < -q_{ii}$ for $i \in E^0$ then*

$$\mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du + aX_{\tau(b)}} h(Z_{\tau(b)}) \right] := \Theta_{a,b}(x, i, h) \quad (4.4.19)$$

with

$$\Theta_{a,b}(x, i, h) = \begin{cases} \underline{\mathbb{1}}'_i e^{Q_r^-(x-b_1)} \hat{h}, & x > b_1, \\ \underline{\mathbb{1}}'_i \Psi_2^+(x) C + \Psi_2^-(x) \tilde{h} + \Psi_2^0(a, x) \Lambda_{\tilde{h}} q^{(2)} & b_1 \geq x > b_2 \end{cases} \quad (4.4.20)$$

where Q_r is defined in (4.3.2), $q^{(i)} = \left(-Q_r^{(i)} \underline{\mathbb{1}}\right)$ with $Q_r^{(i)}$ the restriction of Q_r to state i , Q_r^- satisfies $\Xi(Q_r^-) = O$ with Ξ given in (4.3.10), Ψ_2^-, Ψ_2^+ and Ψ_2^0 are defined in (4.4.27)-(4.4.29), the vectors \tilde{h} and \hat{h} are given in block notation by

$$\tilde{h} = \left(h_2 e^{b_2} \left(1, (I - T_2)^{-1} t_2\right)\right)', \quad (4.4.21)$$

$$\hat{h} = \left\{ \left(h_1 e^{b_1}, C, h_1 e^{b_1} (I - T_1)^{-1} t_1, D\right)\right\}', \quad (4.4.22)$$

C is a constant and D a vector of constants of appropriate size jointly determined by

$$\Theta'_{a,b}(b_1+, 2, h) = \Theta'_{a,b}(b_1-, 2, h), \quad (4.4.23)$$

$$\Theta_{a,b}(b_1+, k, h) = \Theta_{a,b}(b_1-, k, h), \quad k \in E_2^- \quad (4.4.24)$$

where $f(x+)$ and $f(x-)$ represent the right- and left- limits of the function f at x and f' denotes differentiation of f with respect to x .

Proof. By embedding state-dependent discounting at rate $r(i)$ when $Y_t = i \in E^0$ into the generator of A . it follows from (4.3.5) that

$$\begin{aligned} \Theta_{a,b}(x, i) &:= \mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b)} r(Z_u) du + a X_{\tau(b)}} h \left(Z_{\tau(b)} \right) \right] \\ &= \mathbb{E}_{x,i} \left[e^{a A_{\hat{\tau}_b}} h \left(Y_{\hat{\tau}_b} \right) \mathbb{1}_{\{\hat{\tau}_b < \zeta\}} \right] \end{aligned} \quad (4.4.25)$$

where

$$\hat{\tau}_b = \hat{\tau}(b) = \inf \left\{ t \geq 0 \mid Y_t \in E^0, A_t \leq \hat{b}(Y_t) \right\},$$

$Y = \{Y_t\}_{t \geq 0}$ is a Markov process which evolves according to the generator $Q_r =$

$Q - \Lambda_r, \hat{b} : E \mapsto \mathbb{R}$ given by $\hat{b}(j) = b(i)$ for $j \in E_i$ and

$$\zeta = \inf\{t \geq 0 \mid Y_t \notin E\}$$

with $\inf\{\emptyset\} = \infty$. Note that the expectations in (4.4.25) are of the form $\mathbb{E}_{x,i}[\cdot] = \mathbb{E}[\cdot \mid A_0 = x, Y_0 = i]$.

Using the down-crossing ladder process \tilde{Y}^- of (A, Y) defined in (4.3.7), it follows from Theorem 4.3.2 that, since Q_r is evidently transient, the generator of \tilde{Y}^- is given by the unique matrix, Q_r^- , which solves $\Xi(Q_r^-) = O$ defined in (4.3.10). The strong Markov property then implies that when $x > b_1$, the payoff of hitting the level b_1 , having started in state i , will be given by

$$\begin{aligned} \Theta_{a,b}(x, i, h) &= \mathbb{E}_{x,i} \left[\Theta_{a,b}(b_1, Y_{\hat{\tau}(b_1)}, h) \mathbb{1}_{\{\hat{\tau}(b_1) < \zeta\}} \right] \\ &= \mathbb{E}_{x,i} \left[h^- (Y_{\hat{\tau}(b_1)}) \mathbb{1}_{\{\hat{\tau}(b_1) < \zeta\}} \right] \\ &= \underline{1}'_i e^{Q_r^-(x-b_1)} h^- \end{aligned} \tag{4.4.26}$$

which yields the first part of (4.4.19) for some vector h_j^- .

For the second part of (4.4.19), note that the first passage over b can only happen when $Y \in E^0$ and it therefore follows that when $b_1 > x > b_2$, Y can only exit the interval $[b_2, b_1]$ in one of two ways: Either A hits the level b_2 while in state 2, or, Y jumps back into state 1. Thus when $b_1 > x > b_2$, one is led to consider the generator of the process $Y^{(2)}$ being Y restricted to state 2. Clearly, $Y^{(2)}$ is also a Markov process with an associated generator $Q_r^{(2)}$ being Q_r restricted to state 2. The associated two-sided exit probabilities of x from the interval $[b_2, b_1]$ are identified in Lemma 4.4.4 below. Then once again applying the strong Markov property yields for $b_1 > x > b_2$

$$\begin{aligned} \Theta_{a,b}(x, i, h) &= \mathbb{E}_{x,i} \left[\Theta_{a,b}(b_2, Y_{\hat{\tau}}, h) \mathbb{1}_{\{\hat{\tau} < \zeta, A_{\hat{\tau}} = b_2\}} \right] \\ &\quad + \mathbb{E}_{x,i} \left[\Theta_{a,b}(b_1, Y_{\hat{\tau}}, h) \mathbb{1}_{\{\hat{\tau} < \zeta, A_{\hat{\tau}} = b_1\}} \right] \\ &\quad + \mathbb{E}_{x,i} \left[\Theta_{a,b}(A_{\zeta-}, Y_{\zeta-}, h) \mathbb{1}_{\{\zeta < \hat{\tau}\}} \right] \end{aligned}$$

with $\hat{\tau} = \inf \{t \geq 0 \mid Y_t \in E^0, A_t \notin [b_2, b_1]\}$ from which invoking Proposition 4.3.3 yields for $b_1 > x > b_2$

$$\Theta_{a,b}(x, i, h) = \underline{1}'_i \left(\Psi_2^+(x) h_2^+ + \Psi_2^-(x) h_2^- + \Psi_2^0(a, x) \Lambda_{h_2^0} q^{(2)} \right)$$

where Ψ_2^-, Ψ_2^+ and Ψ_2^0 are defined in (4.4.27)-(4.4.29) and for some vectors h_2^-, h_2^+ and h_2^0 .

To verify that the vectors in (4.4.21) and (4.4.22) are in the correct form, first note that in regime k , if A_t exits the interval at b_k in E_k^0 , the payoff will be $h(k) e^{b_k}$. If, however, A_t exits the interval at b_k in $l \in E_k^-$, the payoff will be $\underline{1}'_l h(k) e^{b_k} (I - T_k)^{-1} t_k$ which accounts for $\tilde{h}(l)$ and $\hat{h}(l)$ for $l \in E_k = E_k^0 \cup E_k^-$. The remaining constants are determined by insisting on appropriate smoothness across the boundaries in (4.4.23) and (4.4.24). The fact that $\Theta_{a,b}(\cdot, i, h), i \in E$ is C^1 on \mathbb{R} at b was proved in Jiang and Pistorius [2008] and is therefore omitted. \square

Lemma 4.4.4. *Let $X = \{X_t\}_{t \geq 0}$ be defined in (4.2.3), $\tau(b)$ be defined in (4.4.18) with $b_1 > b_2$ then, assuming that $\frac{1}{2}\sigma(2)^2 + \mu(2) < -q_{22}$, restricted to state 2, it follows that the two-sided exit probabilities of X from the interval $[b_2, b_1]$ satisfy*

$$\Psi_2^+(x) = \left(W^+ e^{Q_r^{(2)+}(b_1-x)} - e^{Q^{(2)-(x-b_2)} Z^+} \right) (I - Z^- Z^+), \quad (4.4.27)$$

$$\Psi_2^-(x) = \left(e^{Q^{(2)-(x-b_2)} Z^-} - W^+ e^{Q^{(2)+}(b_1-x)} Z^- \right) (I - Z^+ Z^-), \quad (4.4.28)$$

$$\Psi_2^0(a, x) = \left(e^{ax} I - e^{ab_1} \Psi^+(x) J^+ - e^{ab_2} \Psi^-(x) \right) (-K(a)^{-1}), \quad (4.4.29)$$

where $Q_r^{(2)-}$ satisfies $\Xi \left(Q_r^{(2)-} \right) = 0$ with Ξ given in (4.3.10), $Q_r^{(2)+} = -y$ where y is the scalar given by the positive root of

$$\frac{1}{2}\sigma_2^2 y^2 + \mu_2 x - q^{(2)} - B_2 (T_2 - xI)^{-1} t_2 = 0 \quad (4.4.30)$$

with $\eta^+ = - \left(T_2 + Q_r^{(2)+} I \right)^{-1} t_2$.

Proof. Restricted to state 2, the generator of the process $Y^{(2)}$ is given by

$$Q_a = \begin{pmatrix} \gamma_2 - r_2 - \lambda_2 & B_2 \\ t_2 & T_2 \end{pmatrix}$$

from which, using Definition 4.3.1, it is clear that $Q_r^{(2)+} = -y$ where y is a scalar given by the positive root of

$$\frac{1}{2}\sigma_2^2 y^2 + \mu_2 y - q^{(2)} - B_2 \eta^+ = 0,$$

where η^+ satisfies

$$-\eta^+ (-Q_r^{(2)+}) = t_2 + T_2 \eta^+$$

which yields (4.4.30). The fact that $Q_r^{(2)-}$ satisfies $\Xi(Q_r^{(2)-}) = 0$ with Ξ given in (4.3.10) follows directly from Theorem 4.3.2. Equations (4.4.27)-(4.4.29) follow from (4.3.12) and (4.3.13)-(4.3.15) with $Q^+ = Q_r^{(2)+}$ and $Q^- = Q_r^{(2)-}$. \square

4.4.6 Optimal Capital Structure Drawing on the results above, conditions under which the optimal default boundaries in (4.4.15) exist are identified below in Theorem 4.4.5 and the optimal value of equity value in (4.4.5) identified in Theorem 4.4.6 when optimal boundaries $b^* = (b_1^*, b_2^*)$ exist and $b_1^* > b_2^*$. The results where $b_2^* > b_1^*$, which follow by symmetry, are entirely analogous and are therefore omitted.

Theorem 4.4.5. *Let $w_{b^*} : \mathbb{R} \times E^0 \rightarrow \mathbb{R}$ be given in*

$$w_{b^*}(x, k, \hat{\kappa}) = \mathbb{E}_{x,i} \left[e^{-\int_0^{\tau(b^*)} r(Z_u) du} (\hat{\kappa}(Z_{\tau(b^*)}) - V_{\tau(b^*)}) \right], \quad k \in E^0, \quad (4.4.31)$$

and assume that there exists a solution $b^ = (b_1^*, b_2^*)$ to the system*

$$w_{b^*}(x, k, \hat{\kappa}) - g(x, k) \geq 0, \quad x > b_k^*, k \in E^0, \quad (4.4.32)$$

$$w'_{b^*}(b_k^*, k, \hat{\kappa}) = -e^{b_k^*}, \quad k \in E^0, \quad (4.4.33)$$

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which satisfies

$$\delta_k e^{b_k^*} - r_k \hat{\kappa}_k + \gamma_k (\hat{\kappa}_{3-k} - e^{b_{3-k}^*} - \hat{\kappa}_k + e^{b_k^*}) < 0, \quad k \in E^0, \quad (4.4.34)$$

where $\hat{\kappa}$ is defined in (4.4.6), then if $w_{b^*}(\cdot, k, \hat{\kappa})$, $k \in E^0$ is bounded, C^1 on \mathbb{R} and C^2 on $\mathbb{R} \setminus \{b_i^*\}$ then $w_{b^*}(x, k, \hat{\kappa}) = w^*(x, k)$ for $x \in \mathbb{R}$ and $k \in E^0$.

Proof. First note from the form of the function in (4.4.31) and by a standard Dynkin's formula type argument, (4.4.13) and (4.4.14) are satisfied. Next fix $k \in E^0$ and assume that $b_k^* > b_{3-k}^*$ from which using (4.2.4) and (4.2.6), it follows that for $b_{3-k}^* < x \leq b_k^*$,

$$\Pi_{X,Z}g(x, k) = \delta_k e^x - r_k \hat{\kappa}_k + \gamma_k (w_{b^*}(x, 3-k, \hat{\kappa}) - \hat{\kappa}_k + e^x),$$

where $g(k, x) = (\hat{\kappa}(k) - e^x)$ $k \in E^0$ and $\Pi_{X,Z}g := \mathbb{L}_{X,Z}g - rg$, which implies,

$$\begin{aligned} \Pi_{X,Z}g(b_k^*, k) &= \delta_k e^{b_k^*} - r_k \hat{\kappa}_k + \gamma_k (w_{b^*}(b_k^*, 3-k, \hat{\kappa}) - \hat{\kappa}_k + e^{b_k^*}) \\ &\leq \delta_k e^{b_k^*} - r_k \hat{\kappa}_k + \gamma_k (w_{b^*}(b_{3-k}^*, 3-k, \hat{\kappa}) - \hat{\kappa}_k + e^{b_k^*}) \\ &= \delta_k e^{b_k^*} - r_k \hat{\kappa}_k + \gamma_k (\hat{\kappa}_{3-k} - e^{b_{3-k}^*} - \hat{\kappa}_k + e^{b_k^*}) \end{aligned}$$

since $w_{b^*}(b_k^*, 3-k, \hat{\kappa}) < w_{b^*}(b_{3-k}^*, 3-k, \hat{\kappa})$. Therefore, if

$$\delta_k e^{b_k^*} - r_k \hat{\kappa}_k + \gamma_k (\hat{\kappa}_{3-k} - e^{b_{3-k}^*} - \hat{\kappa}_k + e^{b_k^*}) \leq 0$$

then, since $x \mapsto \delta_k e^x + \gamma_k (w_{b^*}(x, 3-k, \hat{\kappa}) - \hat{\kappa}_k + e^x)$ is increasing in x , it follows that $\Pi_{X,Z}w_b^*(x, k, \hat{\kappa}) \leq 0$ for all $x \leq b_k^*$, $k \in E^0$, which verifies that (4.4.34) holds true when $b_{3-k}^* < x \leq b_k^*$. When $x \leq b_{3-k}^*$, by an entirely analogous argument to the above,

$$\Pi_{X,Z}g(b_{3-k}^*, 3-k) \leq \delta_k e^{b_{3-k}^*} - r_{3-k} \hat{\kappa}_{3-k} + \gamma_{3-k} (\hat{\kappa}_k - e^{b_k^*} - \hat{\kappa}_{3-k} + e^{b_{3-k}^*}),$$

which implies that (4.4.34) also holds when $x \leq b_{3-k}^*$. It follows that $\mathbb{L}_{X,Z}w_{b^*} - rw_{b^*} \leq 0$ for $x \leq b_k^*$, $k \in E^0$ which proves (4.4.11). Finally, (4.4.12) holds true by assumption

which completes the proof. \square

Theorem 4.4.6. *Assuming that $\frac{1}{2}\sigma(j)^2 + \mu(j) < -q_{jj}, j \in E^0$ and that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ to the system (4.4.32)-(4.4.33) which satisfies (4.4.34), the value of equity in (4.4.5) under the default time $\tau(b^*)$ in (4.4.15) is given by*

$$Q_{b^*}(x, i) = e^x - \underline{1}'_i cP(\Lambda_r - G)^{-1} \hat{\pi} + w_{b^*}(x, i, \hat{\kappa}), \quad i \in E^0, \quad (4.4.35)$$

where $\hat{\pi} = \underline{1} - \pi$,

$$w_b(x, 1, \hat{\kappa}) = \begin{cases} \underline{1}'_1 e^{Q_r^-(x-b_1)} \hat{h}, & x > b_1, \\ \underline{1}'_1 cP(\Lambda_r - G)^{-1} \hat{\pi} - e^x, & x \leq b_1, \end{cases}$$

$$w_b(x, 2, \hat{\kappa}) = \begin{cases} \underline{1}'_2 e^{Q_r^-(x-b_1)} \hat{h}, & x > b_1, \\ \underline{1}'_2 \left(\Psi_2^+(x) C + \Psi_2^-(x) \tilde{h} \right. \\ \quad \left. + (\Psi_2^0(0, x) \Lambda_{\hat{\kappa}_2} - \Psi_2^0(1, x)) q^{(2)} \right), & b_1 \geq x > b_2, \\ \underline{1}'_2 cP(\Lambda_r - G)^{-1} \hat{\pi} - e^x, & x \leq b_2, \end{cases}$$

Q_r is defined in (4.3.2), $q^{(i)} = (-Q_r^{(i)} \underline{1})$ with $Q_r^{(i)}$ the restriction of Q_r to state i , Q_r^- satisfies $\Xi(Q_r^-) = O$ with Ξ defined in (4.3.10), Ψ_2^-, Ψ_2^+ and Ψ_2^0 are defined in (4.4.27)-(4.4.29), the vectors \tilde{h} and \hat{h} are given in block notation as

$$\tilde{h} = (\hat{\kappa}_2 - e^{b_2} (1, (I - T_2)^{-1} t_2))',$$

$$\hat{h} = \left\{ (\hat{\kappa}_1 - e^{b_1}, C, \hat{\kappa}_1 - e^{b_1} (I - T_1)^{-1} t_1, D) \right\}',$$

$\hat{\kappa}$ is defined in (4.4.6), C is a constant and D a vector of constants of appropriate size jointly satisfying (4.4.23) and (4.4.24) with $w_b(x, \cdot, \hat{\kappa}) := \Theta_{0,b}(x, \cdot, \hat{\kappa}) - \Theta_{1,b}(x, \cdot, \underline{1})$.

Proof. Upon assuming that a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ to Theorem 4.4.5 exists, the result then follows from an application of Lemma 4.4.3 to (4.4.31) with

$\tau(b^*)$ given in (4.4.15) together with (4.4.5) . □

Providing the optimal default boundaries (b_1^*, b_2^*) with $b_1^* > b_2^*$ exist, an application of Lemma 4.4.3 to (4.4.1) yields the optimal value of debt as outlined in the following Corollary. The optimal firm value in (4.4.3) when $b_1^* > b_2^*$ is outlined in Corollary 4.4.8 below.

Corollary 4.4.7. *Assuming that $\frac{1}{2}\sigma(j)^2 + \mu(j) < -q_{jj}, j \in E^0$ and that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ to the system (4.4.32)-(4.4.33) which satisfies (4.4.34), the optimal value of debt in (4.4.1) under the default time $\tau(b^*)$ in (4.4.15) is given by*

$$D_{b^*}(x, i) = \underline{1}'_i cP(\Lambda_r - G)^{-1} \underline{1} - d_{b^*}(x, i, \hat{\nu}), \quad i \in E^0, \quad (4.4.36)$$

where

$$d_b(x, 1, \hat{\nu}) = \begin{cases} \underline{1}'_1 e^{Q_r^-(x-b_1)} \hat{h}, & x > b_1, \\ \underline{1}'_1 cP(\Lambda_r - G)^{-1} \underline{1} - (1 - \xi_1) e^x, & x \leq b_1, \end{cases}$$

$$d_b(x, 2, \hat{\nu}) = \begin{cases} \underline{1}'_2 e^{Q_r^-(x-b_1)} \hat{h}_1, & x > b_1, \\ \underline{1}'_2 \left(\Psi_2^+(x) C + \Psi_2^-(x) \tilde{h} \right. \\ \quad \left. + (\Psi_2^0(0, x) \Lambda_{\hat{\nu}_2} - \Psi_2^0(1, x) \Lambda_{\underline{1}}) q^{(2)} \right), & b_1 \geq x > b_2, \\ \underline{1}'_2 cP(\Lambda_r - G)^{-1} \underline{1} - (1 - \xi_2) e^x, & x \leq b_2, \end{cases}$$

where Q_r is defined in (4.3.2), $q^{(i)} = (-Q_r^{(i)} \underline{1})$ with $Q_r^{(i)}$ the restriction of Q_r to state i , Q_r^- satisfies $\Xi(Q_r^-) = O$ with Ξ defined in (4.3.10), Ψ_2^-, Ψ_2^+ and Ψ_2^0 are defined in (4.4.27)-(4.4.29), the vectors \tilde{h} and \hat{h} are given in block notation as

$$\tilde{h} = (\hat{\nu}_2 - (1 - \xi_2) e^{b_2} (1, (I - T_2)^{-1} t_2))',$$

$$\hat{h} = \left\{ (\hat{\nu}_1 - (1 - \xi_1) e^{b_1}, C, \hat{\nu}_1 - (1 - \xi_1) e^{b_1} (I - T_1)^{-1} t_1, D) \right\}',$$

$\hat{\nu}$ is defined in (4.4.2), C is a constant and D a vector of constants of appropriate size jointly satisfying (4.4.23) and (4.4.24) with $d_b(x, \cdot, \hat{\nu}) := \Theta_{0,b}(x, \cdot, \hat{\nu}) - \Theta_{1,b}(x, \cdot, \underline{1})$.

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Proof. Upon assuming that a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ to Theorem 4.4.5 exists, the result follows from an application of Lemma 4.4.3 to (4.4.1) with $\tau = \tau(b^*)$ in (4.4.15). \square

Corollary 4.4.8. *Assuming that $\frac{1}{2}\sigma(j)^2 + \mu(j) < -q_{jj}, j \in E^0$ and that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ to the system (4.4.32)-(4.4.33) which satisfies (4.4.34), the optimal value of the firm in (4.4.3) under the default time $\tau(b^*)$ in (4.4.15) is given by*

$$F_{b^*}(x, i) = e^x + \mathbf{1}'_i cP(\Lambda_r - G)^{-1} \pi - f_{b^*}(x, i, \hat{\chi}), \quad i \in E^0, \quad (4.4.37)$$

where

$$f_b(x, 1, \hat{\chi}) = \begin{cases} \mathbf{1}'_1 e^{Q_r^-(x-b_1)} \hat{h}, & x > b_1, \\ \mathbf{1}'_1 cP(\Lambda_r - G)^{-1} \pi + \xi_1 e^x, & x \leq b_1, \end{cases}$$

$$f_b(x, 2, \hat{\chi}) = \begin{cases} \mathbf{1}'_2 e^{Q_r^-(x-b_1)} \hat{h}, & x > b_1, \\ \mathbf{1}'_2 \left(\Psi_1^+(x) C + \Psi_2^-(x) \tilde{h}_1 \right. \\ \quad \left. + (\Psi_2^0(0, x) \Lambda_{\hat{\chi}_2} - \Psi_2^0(1, x)) q^{(2)} \right), & b_1 \geq x > b_2 \\ \mathbf{1}'_2 cP(\Lambda_r - G)^{-1} \pi + \xi_2 e^x, & x \leq b_2, \end{cases}$$

where Q_r is defined in (4.3.2), $q^{(i)} = \left(-Q_r^{(i)} \mathbf{1}\right)$ with $Q_r^{(i)}$ the restriction of Q_r to state i , Q_r^- satisfies $\Xi(Q_r^-) = O$ with Ξ defined in (4.3.10), Ψ_2^-, Ψ_2^+ and Ψ_2^0 are defined in (4.4.27)-(4.4.29), the vectors \tilde{h} and \hat{h} are given in block notation as

$$\tilde{h} = \left(\hat{\chi}_2 + \xi_2 e^{b_2} \left(\mathbf{1}, (I - T_2)^{-1} t_2 \right) \right)',$$

$$\hat{h} = \left\{ \left(\hat{\chi}_1 + \xi_1 e^{b_1}, C, \hat{\chi}_1 + \xi_1 e^{b_1} (I - T_1)^{-1} t_1, D \right) \right\}',$$

$\hat{\chi}$ is defined in (4.4.4), C is a constant and D a vector of constants of appropriate size jointly satisfying (4.4.23) and (4.4.24) with $f_b(x, \cdot, \hat{\chi}) := \Theta_{0,b}(x, \cdot, \hat{\chi}) + \Theta_{1,b}(x, \cdot, \mathbf{1})$.

Proof. Upon assuming that a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* > b_2^*$ to Theorem 4.4.5 exists, the result follows from an application of Lemma 4.4.3 to (4.4.3) with $\tau = \tau(b^*)$

in (4.4.15). □

The optimal values of equity, debt and firm value are given in Propositions 4.6.1, 4.6.2 and 4.6.3 of Section 4.6 respectively when the optimal default boundaries identified in Theorem 4.4.6 are identical.

4.4.7 Firm Credit Spreads Unlike the firm endowed with the roll-over debt structure in Chapter 3, a firm financed by consol debt provides no information in regards to the term structure of debt. However, in an analogous manner to Subsection 3.4.2, it is possible to calculate the coupon rate, c , which would encourage investors to purchase the firm's debt at face value, P , when the firm is initially created. The credit spread of the firm's debt can be regarded as the excess of the latter coupon over the risk free rate.

When the firm is set up and debt is first issued, it is assumed that the parameters of the firm are constrained such that at the coupon rate, c , the market value of debt, $D_{b^*}(v, i)$ given in (4.4.36) is equal to the face value of debt, P . If the asset value when the firm is $V_0 = e^x$, the latter constraint requires that the coupon rate, c , be the smallest solution to the equation

$$D_{b^*}(x, i) = P, \quad i \in E^0, \tag{4.4.38}$$

where D_{b^*} is defined in Corollary 4.4.8

The regime dependent coupon which solves (4.4.38) will be denoted $\tilde{c}^*(x, i, b^*)$, $i \in E^0$. Similarly to Chapter 3, the framework of this chapter does not provide a closed form solution for the optimal barrier b^* . However b^* may be calculated numerically for a given set of startup parameters for the firm, including the coupon rate, and then used to evaluate (4.4.38) from which it is possible to search for \tilde{c}^* numerically. Calculations based on the latter procedure are presented in the next section. The

firm's credit spread when the firm is solvent is then given by

$$\tilde{s}^*(v, i, b^*) = \tilde{c}^*(v, i, b^*) - \hat{r}(i), \quad (x, i) \in \{x > b_i^*, i \in E^0\}, \quad (4.4.39)$$

where $\hat{r}(i)$ is defined as

$$\hat{r}(i) = (\underline{1}_i' (\Lambda_r - E)^{-1} \underline{1})^{-1}, \quad i \in E^0. \quad (4.4.40)$$

4.5 Numerical Analysis

4.5.1 Calculation Methodology The default parameters used in the calculations in this section are described in Table 4.5.1. As in Chapter 3, the parameter values invariant between regimes and the non-jump related parameter values in regime 1 were taken from Leland [1994b]. The regime intensities, set to 0.3 and 1 for regimes 1 and 2 respectively, were also taken from Chapter 3. The only exception was the principal value, P , which was set to a value of 100. The jump intensities and the jump distribution in regime 1 were taken from Kou and Wang [2003].

Regime 2 parameters were selected to reflect a *recessionary* scenario with interest rates, tax rates and dividend rates assumed to fall and bankruptcy costs assumed to rise. Asset volatility and the size of jumps was also assumed to rise in regime 2. The average jump sizes in regime 2 were assumed to be -5% and -10% with probability 0.75 and 0.25 respectively, compared with a single jump of average size equal to -3% in regime 1. Default parameter values were used in all calculations unless otherwise stated.

To identify the optimal boundaries, the system of equations given by (4.4.23), (4.4.24) and the smooth fit equation, (4.4.32), was solved using Matlab's `lsqnonlin` routine. Each of the three assumptions, $b_1^* < b_2^*$, $b_1^* > b_2^*$ and $b_1^* = b_2^*$ were tested and the resulting calculated boundaries which agreed with the prior assumption selected as the optimal pair. In no cases, did more than one pair of calculated optimal boundaries

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agree with the prior assumption. It was found that the speed of convergence in the latter calculation was helped by first excluding jumps and calculating optimal boundaries for the geometric Brownian motion case, and then using the calculations as an initial guess for the calculation including jumps. The drift of the asset process for all calculations was calibrated to the martingale condition required in (4.2.6).

Calculated optimal boundaries were tested against conditions (4.4.32) and (4.4.34) and to verify optimality. Figure 4.5.1 plots numerical verifications of condition (4.4.32) for selected combinations of parameter values. In each subplot of Figure 4.5.1 calculations were based on the default parameter values in Table 4.5.1 unless specified in the sub-title of each respective plot. All of the cases presented in Figure 4.5.1 indicate that condition (4.4.32) was verified. The parameter values chosen for display in Figure 4.5.1 represent the extreme of the parameter values used for the rest of the calculations in this section. Condition (4.4.34) was also tested for every calculation reported in this section and found to hold. However, the existence of optimal boundaries was found to be sensitive to asset volatility. For example, condition (4.4.34) failed with an asset volatility of 0.35 in regime 2 where other parameters were set to their default values. The latter indicates that the mixture of parameters, some which may encourage a firm to default sooner and some later, may be important in terms of the existence of optimal boundaries.

Once established, the optimal boundaries were then used to calculate optimal equity, firm and debt values using (4.4.5), (4.4.3) and (4.4.1) using Theorem 4.4.6, Corollary 4.4.8 and Corollary 4.4.7 respectively. The smallest of coupons which satisfied (4.4.38) were calculated by a simple numerical search procedure which repeated the calculations above from which firm credit spreads were calculated using (4.4.39).

The generator matrices of the up-crossing and down-crossing ladder processes required in Lemmas 4.4.3 and 4.4.4 for the above calculations were obtained as follows. Noting that if h is an eigenvector of the down-crossing generator matrix, Q_r^- , which

is associated with an eigenroot θ , then since Q_r^- satisfies (4.3.10),

$$\frac{1}{2}\Sigma^2 (Q_r^-)^2 + VQ_r^- + Q_r = 0, \quad (4.5.1)$$

it follows that post-multiplying (4.5.1) by h implies

$$\left(\frac{1}{2}\Sigma^2\theta^2 + V\theta + Q_r\right) h = 0. \quad (4.5.2)$$

The eigenroots and eigenvectors of the polynomial eigenvalue problem (4.5.2) were then calculated using Matlab's `polyeig` function. The positive root of (4.4.30) was identified using Matlab's `fsolve` routine.

4.5.2 Optimal Capital Structure The optimal boundaries for the default parameters in Table 4.5.1 were calculated to be 48.13 and 42.29 in regime 1 and regime 2 respectively. Note that unlike Chapter 3 for a finite maturity debt profile, for consol debt, the level of bankruptcy cost has no effect on the optimal boundary, since bankruptcy costs do not enter into the optimal value of equity in (4.4.35). The optimal value of debt and of the firm are, however, impacted by changes in bankruptcy costs.

Figure 4.5.3 illustrates firm value as a function of leverage and 3 different assumed levels of bankruptcy costs, 0.25, 0.5 and 0.75, in regime 2. Following the literature, leverage was defined as total debt value (4.4.36) divided by firm value in (4.4.37). Leverage was generated by modulating the principal value over the range [30, 180]. The results illustrated in Figure 4.5.3 follow a similar pattern to those reported in Chapter 3 for finite maturity debt and to those reported in the literature. Firm value initially increases with leverage only to reach what appears to be a maximal value in a leverage range of approximately 0.6 to 0.8 before declining sharply as leverage approaches 1. The results appear broadly consistent with those reported by Leland [1994a] which indicated firm value being maximised at a leverage of approximately 0.7 for default costs equal to 0.5, albeit with slightly differing parameters. As regards regimes, in Figure 4.5.3, firm value reaches a higher maximum value in regime 1 as compared to regime 2. After reaching it's maximum value in Figure 4.5.3, firm

value declines more quickly in regime 1 as compared to regime 2 as leverage increases further for a given level of bankruptcy costs. Higher bankruptcy costs in regime 2 lower firm value relative to regime 1 for the same level of leverage.

Figure 4.5.3 illustrates firm value for different levels of coupon and bankruptcy costs. Coupons were modulated in a range of $[0.02, 0.14]$. Firm value initially increases with coupon and then declines as coupons increase. As reported in Leland [1994a], there appears to be a coupon level which maximises firm value. In Subsection 4.5.4 below, the coupon rate which maximises firm value over a range of coupons is calculated. Higher bankruptcy costs in regime 2 reduce firm value for a given level of coupon. Firm value reaches a maximum value in Figure 4.5.3 at lower levels of coupon in regime 2 than in regime 1. However, when bankruptcy costs are 0.25, after reaching its maximum value, firm value declines more slowly in regime 2 than in regime 1. The latter does not appear to be the case for a bankruptcy level of 0.75.

Debt value as a function of coupon for different levels of bankruptcy costs is illustrated in Figure 4.5.5. Once again, the relationship between coupon and debt value is similar to that which has been reported in the literature. In Figure 4.5.5, debt value initially increases in value as coupons rise only to reach a maximum value and then decline. Debt value is lower for higher bankruptcy costs in regime 2 for equivalent levels of coupon. Mirroring the results for firm value, after achieving its maximum in Figure 4.5.5 in regime 2, for lower bankruptcy costs, debt value diminishes more slowly in regime 2 than is the case in regime 1. Similar results are reported for debt value as a function of leverage and bankruptcy costs in Figure 4.5.5. Debt value achieves a higher maximum level in regime 1 and is lower in regime 2 for the same level of bankruptcy costs. In addition, Figure 4.5.5 suggests that for bankruptcy costs of 0.25, debt value achieves its maximum at higher levels of leverage in regime 2 than in regime 1. For bankruptcy costs of 0.75, however, debt value achieves its maximum in Figure 4.5.5 at lower levels that is the case in regime 1.

4.5.3 Firm Credit Spreads Figure 4.5.7 illustrates firm credit spreads as a function of leverage for different levels of bankruptcy costs. Leverage was again

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generated by modulating the principal value over the range [30, 180]. As might be expected, firm credit spreads exhibit an increasing relationship with leverage in both regimes. Firm credit spreads were higher in regime 2 than regime 1 for equivalent levels of leverage.

Firm credit spreads by coupon rates between 0.05 and 0.14 for different levels of bankruptcy costs are displayed in Figure 4.5.6. For bankruptcy costs of either 0.5 or 0.75 in regime 2, firm credit spreads are higher in regime 2 than in regime 1 and increase with coupon rate. However, interestingly, with bankruptcy costs of 0.25 in regime 2, while firm credit spreads are higher in regime 1 than regime 2 for low coupon rates, for a coupon rate of 0.14, firm credit spreads are higher in regime 1 than in regime 2. This is consistent with Figures 4.5.4 and 4.5.3 which indicate that, with bankruptcy costs of 0.25, at a coupon rate of 0.14, both debt value and firm value is higher in regime 2 than regime 1. The latter observation may also be consistent with findings reported in the literature² that suggest for highly leveraged firms, or so called *junk* or high-yield firms, bond holders may start to *behave more like equity investors* as the recovery rate on default becomes an important determinant of the value of debt.

Figure 4.5.7 illustrates firm credit spreads as a function of leverage for different levels of bankruptcy costs. Firm credit spreads demonstrate an increasing relationship with leverage as might be expected. Firm credit spreads are higher in regime 2 than in regime 1 and increase more quickly in regime 2 as a function of leverage than in regime 1.

4.5.4 Optimised Leverage As mentioned above, Figure 4.5.3 suggests that there may be a coupon rate which maximises firm value. In order to examine this, a simple search procedure was conducted over the range the of coupon rates in Figure 4.5.3, and the coupon which maximised firm value identified. The results are reported

²Black and Cox [1976] identified that the level of bankruptcy costs could have an important impact on the value of bonds and the behaviour of a firm's securities. Leland [1994a] reported that the behaviour of junk bonds can differ significantly from the behaviour of investment grade bonds.

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in Table 4.5.2 for 2 different levels of bankruptcy costs. The results indicate that in general, optimised firm and debt value is higher and leverage lower in regime 1 than in regime 2. With bankruptcy costs of 0.25, optimised leverage falls from 0.699 in regime 1 to 0.679 in regime 2 and the coupon rate falls from 0.072 to 0.067. Credit spreads are also lower, falling from 218bps in regime 1 to 215bps in regime 2. As indicated in Table 4.5.2, conducting the optimisation while in regime 1 versus conducting the optimisation in regime 2 generates different optimal boundaries, which are associated with the optimised coupon rates for each regime. The optimised boundaries for both regimes are lower in regime 2 than in regime 1.

When bankruptcy costs are 0.75 in regime 2, the results are similar but more exaggerated. Debt and firm value fall further between regime 1 and regime 2 than is the case with lower bankruptcy costs. Leverage also falls more, 0.646 to 0.621, between regime 1 and regime 2 respectively. With lower leverage, firm credit spreads are lower in regime 2 than in regime 1, falling from 203bps to 194 bps. Optimised coupon rates fall from 0.064 to 0.059. The findings are consistent with what might be expected with firms tending to reduce balance sheet exposure in difficult economic circumstances.

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Name	Symbol	Default Value
Asset Value	v	100
Coupon Rate	c	0.08
Principal	P	100

(a) Fixed Parameters

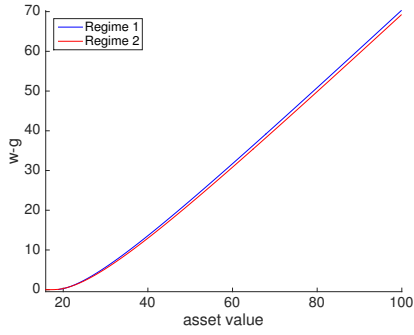
Name	Symbol	Default Value	
		Regime 1	Regime 2
Asset Volatility	σ	0.15	0.30
Tax Rate	π	0.35	0.25
Bankruptcy Costs	ξ	0.5	0.75
Interest Rate	r	0.075	0.03
Dividend Yield	δ	0.07	0.02
Regime Intensity	γ	0.3	1
Jump Intensity	λ	3	3
Phase Jump Generator	T	$-1/.03$	$\begin{pmatrix} -1/.05 & 0 \\ 0 & -1/.1 \end{pmatrix}$
Initial Distribution	α	1	$(.75, .25)'$

(b) Regime Dependent Parameters

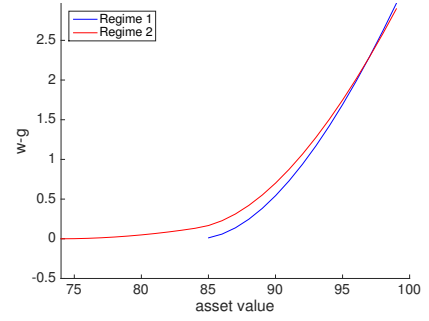
Table 4.5.1: Default Parameter Values

Default parameter values for all calculations unless otherwise stated. All fixed and non-jump related regime 1 parameter values are taken from Leland [1994b]. The jump intensities and jump distribution in regime 1 is taken from Kou and Wang [2003]. Regime 2 parameter values were selected to reflect a *recessionary* regime.

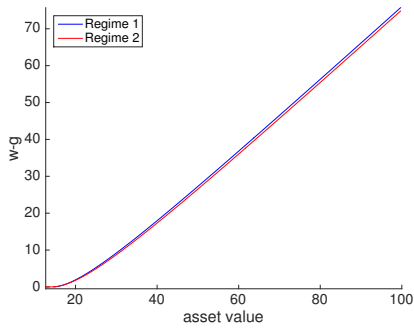
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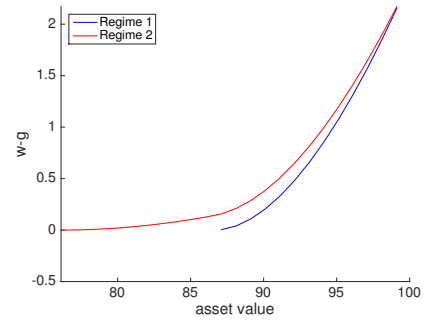
(a) $c = .03$



(b) $c = 0.14$



(c) $P = 30$



(d) $P = 180$

Figure 4.5.1: Numerical Optimality Verification

The above figures plot selected numerical verifications of condition (4.4.32) in the calculation of optimal boundaries. Each plot evaluates the left-hand side of condition (4.4.32) for default parameters with the exception the parameter value indicated in the subtitle of each respective figure. The display was chosen to reflect the range of parameter values utilised for numerical calculations in this chapter. The charts indicate that optimality (4.4.32) was satisfied in the above cases.

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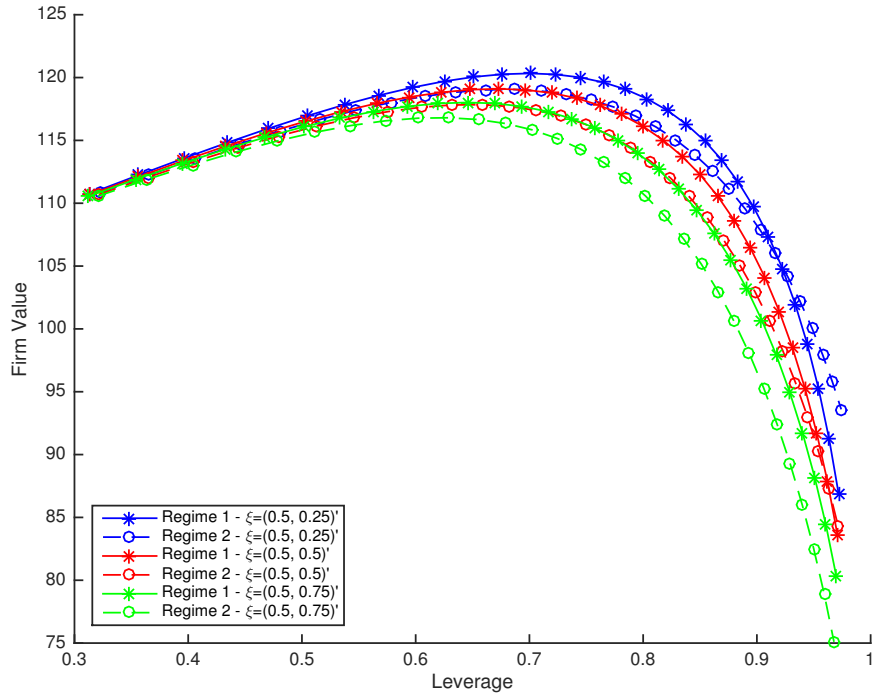


Figure 4.5.2: Firm Value by Leverage and Bankruptcy Costs

The figure shows optimal firm value for different levels of bankruptcy costs and leverage, simulated by modulating P , but otherwise using the default parameters in Table 4.5.1. Higher bankruptcy costs tend to reduce the leverage at which firm value achieves its maximum as a function of leverage. Firm value reaches a higher maximum value in regime 1 as compared to regime 2 for the same level of leverage and bankruptcy costs.

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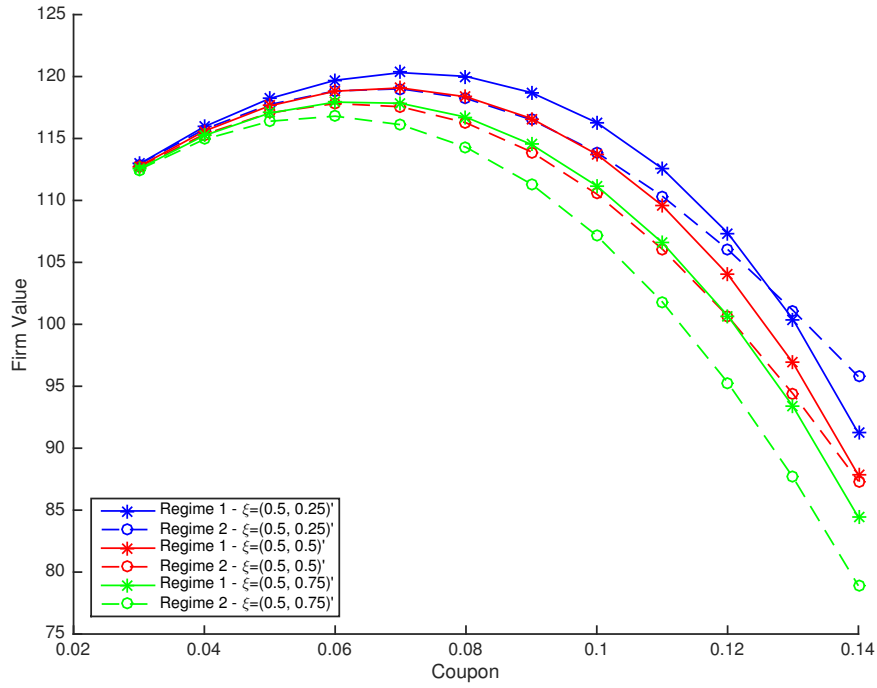


Figure 4.5.3: Firm Value by Coupon and Bankruptcy Costs

The figure shows optimal firm value for different levels of coupon and bankruptcy costs but otherwise using the default parameters in Table 4.5.1. Higher bankruptcy costs tend to reduce the leverage at which firm value achieves its maximum as a function of coupon. Firm value reaches a higher maximum value in regime 1 as compared to regime 2 for the same level of coupon and bankruptcy costs.

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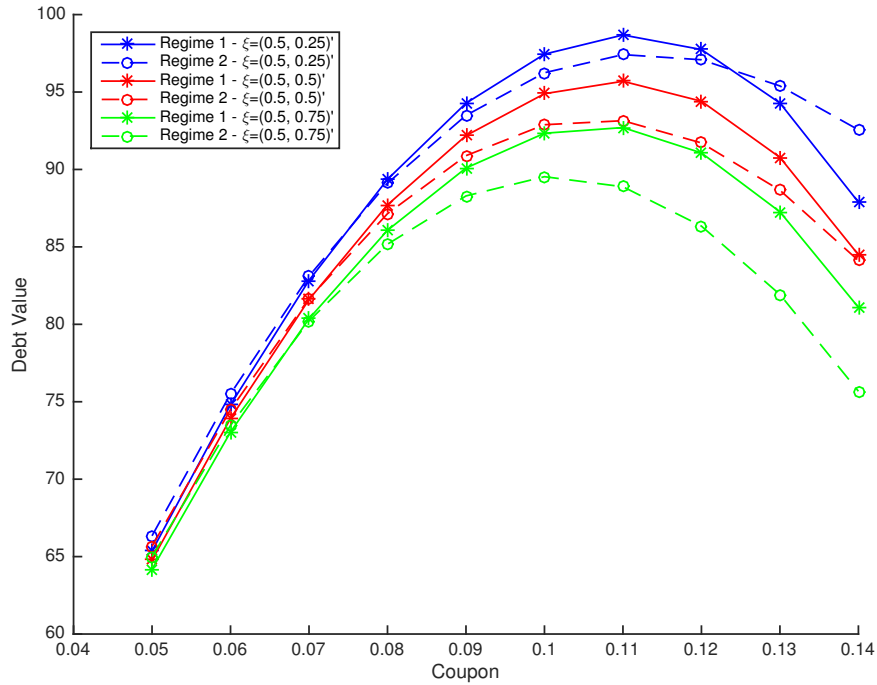


Figure 4.5.4: Debt Value by Coupon and Bankruptcy Costs

The figure shows the optimal value of debt for different levels of coupon under different regimes for three different regime 2 bankruptcy cost values. Other parameters are set at their default values in Table 4.5.1. Debt value reaches a higher maximum value in regime 1 as compared to regime 2 for the same level of coupon and bankruptcy costs. As in Leland [1994a], there appears to be a coupon level which maximises debt value.

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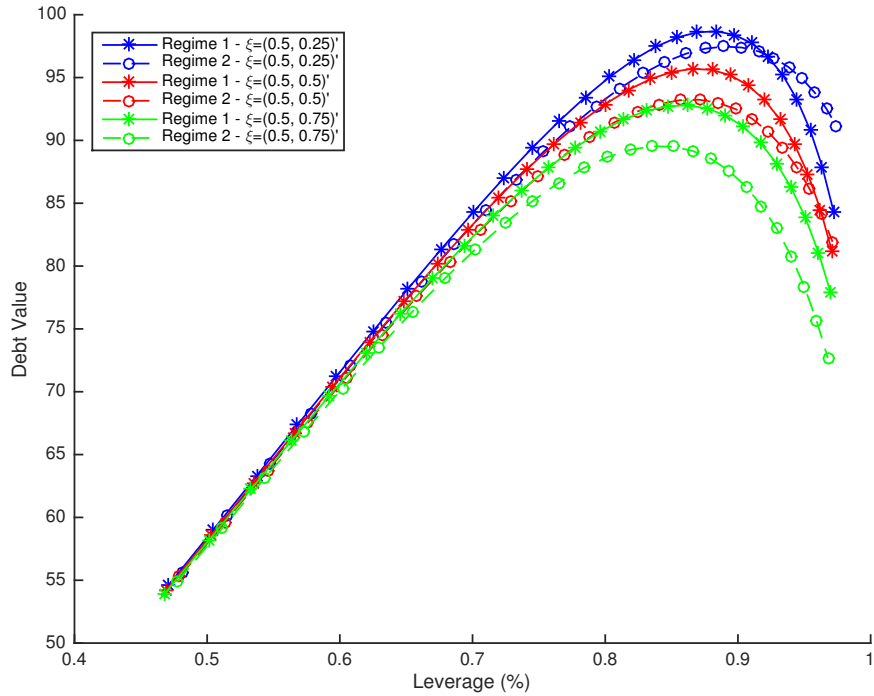


Figure 4.5.5: Debt Value by Leverage and Bankruptcy Costs

The figure shows optimal debt value for different levels of bankruptcy costs and leverage, simulated by modulating P , but otherwise using the default parameters in Table 4.5.1. Higher bankruptcy costs tend to reduce the leverage at which debt value achieves its maximum value. Debt value reaches a higher maximum value in regime 1 as compared to regime 2 for the same level of leverage and bankruptcy costs.

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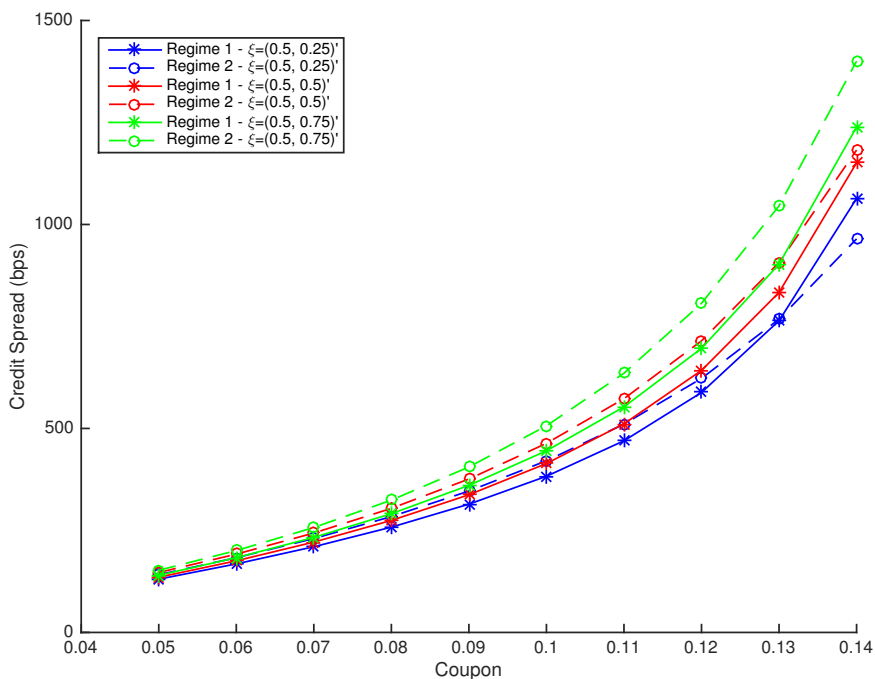


Figure 4.5.6: Firm Credit Spreads by Coupon and Bankruptcy Costs

The figure shows optimal credit spreads for different levels of coupon and bankruptcy costs but otherwise using the default parameters in Table 4.5.1. Higher coupons and leverage lead to increased credit spreads in both regimes. Spreads are typically larger in regime 2 than in regime 1.

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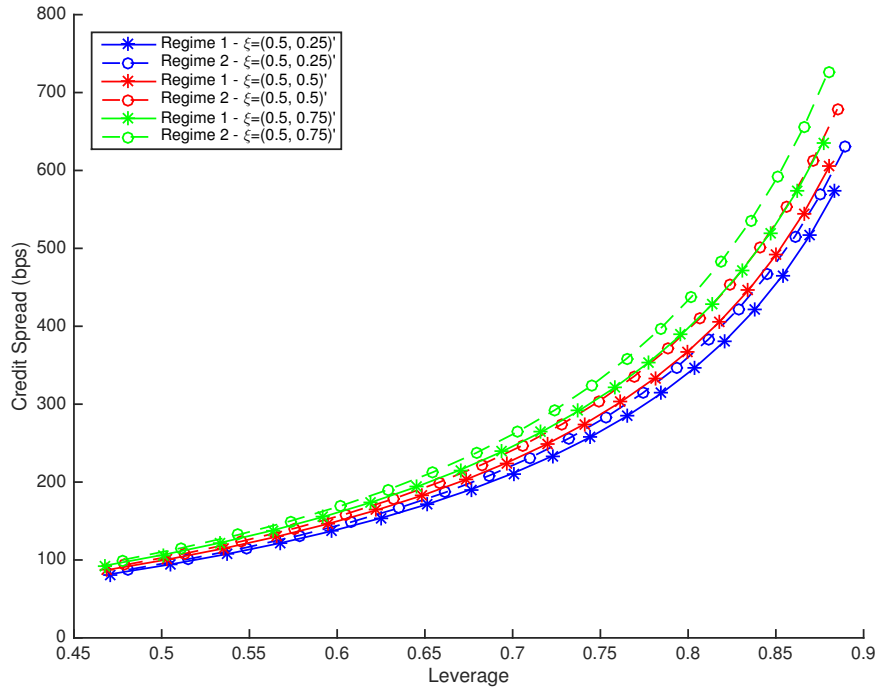


Figure 4.5.7: Firm Credit Spreads by Leverage and Bankruptcy Costs

The figure shows optimal firm credit spreads for different levels of bankruptcy costs and leverage, simulated by modulating P , but otherwise using the default parameters in Table 4.5.1. Higher bankruptcy costs and leverage naturally tend to increase credit spreads in both regimes. Spreads in regime 2 are typically higher than in regime 1 for higher levels of leverage.

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	Regime 2 $\xi = 0.25$		Regime 2 $\xi = 0.75$	
	Regime 1	Regime 2	Regime 1	Regime 2
Firm Value	120.33	119.06	118.02	116.81
Debt Value	84.13	80.88	76.21	72.53
Leverage	0.699	0.679	0.646	0.621
Credit Spread (bps)	218	215	203	194
Coupon	0.072	0.067	0.064	0.059
Regime 1 Boundary	43.20	40.21	38.55	35.36
Regime 2 Boundary	40.21	35.33	33.87	31.08

Table 4.5.2: Optimised Leverage

The table shows the coupon rates which maximise firm value in each respective regime along with associated debt value, leverage, credit spreads and default boundaries for different levels of bankruptcy costs but otherwise using the default parameter values. The maximal firm value was calculated for coupon rates in the range $[\cdot03;14]$.

4.6 Auxiliary Results and Proofs

The following results relate to the special case where the default boundaries are the same.

Proposition 4.6.1. *Assuming that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* = b_2^*$ to the system (4.4.32)-(4.4.33) and which satisfies (4.4.34), the optimal value of equity in (4.4.5) under the default time $\tau(b^*)$ in (4.4.15) is given by*

$$\tilde{Q}_{b^*}(x, i) = e^x - \underline{1}'_i cP(\Lambda_r - G)^{-1} \hat{\pi} + \tilde{w}_{b^*}(x, i, \hat{\kappa}), \quad i \in E^0,$$

where $\hat{\pi} = \underline{1} - \pi$,

$$\tilde{w}_b(x, i) = \begin{cases} \underline{1}'_i e^{Q_r^-(x-b_i)} \hat{h}, & x > b_i, \\ \underline{1}'_i cP(\Lambda_r - G)^{-1} \hat{\pi} - e^x, & x \leq b_i, \end{cases}$$

Q_r is defined in (4.3.2) and satisfies $\Xi(Q_r^-) = O$ with Ξ given in (4.3.10), the vector \hat{h} is given in block notation by $\hat{h} = (\hat{h}_1, \hat{h}_2)'$ where

$$\hat{h}_k = (\hat{\kappa}_k - e^{b_k} (1, (I - T_k)^{-1} t_k))', \quad k = 1, 2,$$

$\hat{\kappa}$ is defined in (4.4.6).

Proof. Upon assuming that a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* = b_2^*$ to Theorem 4.4.5 exists, the result then follows from an application of Lemma 4.4.3 to (4.4.5) with $\tau = \tau(b^*)$ in (4.4.15). □

Proposition 4.6.2. *Assuming that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* = b_2^*$ to the system (4.4.32)-(4.4.33) and which satisfies (4.4.34), the optimal value of debt in (4.4.1) is given by*

$$\tilde{D}_{b^*}(x, i) = \underline{1}'_i cP(\Lambda_r - G)^{-1} \underline{1} - \tilde{d}_{b^*}(x, i, \hat{\nu}), \quad i \in E^0,$$

where

$$\tilde{d}_b(x, i, \hat{\nu}) = \begin{cases} \underline{1}'_i e^{Q_r^-(x-b_i)} \hat{h}, & x > b_i, \\ \underline{1}'_i cP(\Lambda_r - G)^{-1} \underline{1} - (1 - \xi_i) e^x, & x \leq b_i, \end{cases}$$

where Q_r is defined in (4.3.2) and satisfies $\Xi(Q_r^-) = O$ with Ξ defined in (4.3.10), the vector \hat{h} is given in block notation by $\hat{h} = (\hat{h}_1, \hat{h}_2)'$ where

$$\hat{h}_k = (\hat{\nu}_k - (1 - \xi_k) e^{b_k} (1, (I - T_k)^{-1} t_k))', \quad k = 1, 2,$$

and $\hat{\nu}$ is defined in (4.4.2).

Proof. After assuming that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* = b_2^*$ to Theorem 4.4.5 the result follows from an application of Lemma 4.4.3 to (4.4.1) with $\tau = \tau(b^*)$ in (4.4.15). \square

Proposition 4.6.3. *Assuming that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* = b_2^*$ to the system (4.4.32)-(4.4.33) and which satisfies (4.4.34), the optimal value of the firm in (4.4.3) is given by*

$$\tilde{F}_{b^*}(x, i) = e^x + \underline{1}'_i cP(\Lambda_r - G)^{-1} \pi - \tilde{f}_{b^*}(x, i, \hat{\chi}), \quad i \in E^0,$$

where

$$\tilde{f}_b(x, i, \hat{\chi}) = \begin{cases} \underline{1}'_i e^{Q_r^-(x-b_i)} \hat{h}, & x > b_i, \\ \underline{1}'_i cP(\Lambda_r - G)^{-1} \pi + \xi_i e^x, & x \leq b_i, \end{cases}$$

Q_r is defined in (4.3.2) and satisfies $\Xi(Q_r^-) = O$ with Ξ defined in (4.3.10), the vector \hat{h} is given in block notation by $\hat{h} = (\hat{h}_1, \hat{h}_2)'$ where

$$\hat{h}_k = (\hat{\chi}_k + \xi_k e^{b_k} (1, (I - T_k)^{-1} t_k))', \quad k = 1, 2,$$

and $\hat{\chi}$ is defined in (4.4.4).

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Proof. After assuming that there exists a solution $b^* = (b_1^*, b_2^*)$ with $b_1^* = b_2^*$ to Theorem 4.4.5 the result follows from an application of Lemma 4.4.3 to (4.4.3) with $\tau = \tau(b^*)$ in (4.4.15). □

CHAPTER 5 OPTIMAL DEFAULT FOR FINITE MATURITY DEBT

5.1 Introduction

In Chapters 3 and 4 the assumption of a time-homogeneous debt structure was used to analyse aspects of corporate capital structure that persist through time. Such approaches have the objective of gaining insights into what determines an optimal long-term stationary capital structure for a firm financed by debt. However, while models which assume a stationary debt structure afford analytical tractability, the fact of the matter is, that the vast majority of lending agreements observed in the financial market mature at a fixed date. Virtually all debt issued by corporate entities mature at a fixed finite maturity date. Indeed, finite maturity debt contracts are prevalent in many other areas of financial interest such as private equity, infrastructure finance and the insurance industry. The implication is that there may also be significant time dependencies to consider in relation to the determination of capital structure. Motivated by the latter observation, the focus of this chapter turns to the default decision of a firm, or possibly other financial entities, financed by finite maturity debt.

As has been the case in the earlier chapters, the firm is assumed to be endowed with a single asset value process, which is, however, unlike the earlier chapters, modelled by a geometric Brownian motion. The firm pays a positive dividend, receives tax relief on interest payments made during the lifetime of its debt and is assumed to choose a default policy which maximises the equity value up until maturity of the debt. On default, the firm incurs bankruptcy costs. The main implication of time dependence

in the structure of debt is that the endogenously determined level of asset at which to declare default is also time dependent. The primary focus of the analysis in this chapter, is therefore, to identify the time dependent default boundary.

The setting of a firm financed by a finite maturity bond paying a fixed coupon was first studied by Black and Cox [1976] and later considered by Leland and Toft [1996] as a building block for the roll-over debt structure. Chen et al. [2009] extended the model of Black and Cox [1976] to include jumps in the asset process of a firm to derive an integral equation satisfied by a finite maturity bond. More recently, Elliott and Shen [2015] studied the price of a finite maturity bond for a firm not facing tax benefits and bankruptcy costs in a regime-switching setting. The optimal default boundary of a firm financed by finite maturity debt was considered Surya [2012] by in the general setting with the firm's asset price process driven by a Lévy process. The case of a firm financed by a roll-over debt structure was treated as a numerical example.

The contribution of this chapter is that the case where a firm is financed by finite maturity debt, is analysed and conditions identified under which the existence of an optimal boundary can be verified. In addition, it is assumed that the firm can default and incur bankruptcy costs at maturity. The motivation for the latter assumption is that the model might later be extended to one where debt is refinanced at maturity.

The outline of this chapter is as follows. The model structure is outlined in Section 5.2. In Section 5.3, the setting articulated by Leland [1994a] of firm financed by a consol bond where the firm's asset process is modelled as a geometric Brownian is presented as a precursor to the analysis of finite maturity debt. In Section 5.4, the case of finite maturity debt is analysed. Finally, Section 5.5 presents some numerical calculations.

5.2 Model Structure

Throughout this Chapter it will be assumed that there exists a firm endowed with a single asset having a value process, $V = \{V_t\}_{t \geq 0}$, which evolves under the risk neutral measure according to a geometric Brownian motion

$$V_{t+s} = v e^{(r-\delta-\frac{1}{2}\sigma^2)s + \sigma W_s} \quad (5.2.1)$$

where $r > 0$ is the compound risk free interest rate, $\delta > 0$ is the dividend rate, $W = \{W_t\}_{t \geq 0}$ is a standard Brownian motion, $V_t = v$ and $\sigma > 0$ is the volatility coefficient.

The process V is well known to be a strong Markov diffusion with infinitesimal generator given by

$$\mathbb{L}_V f(x) = (r - \delta) x \frac{\partial f(x)}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f(x)}{\partial x^2}. \quad (5.2.2)$$

The firm is partly financed by debt, with notional value, P , which attracts a continuous coupon rate of c and the firm receives tax relief the value of coupon payments equal to $\pi \in [0, 1]$. The firm may choose to default on its debt by ceasing to make scheduled repayments, in which case, a proportion $\xi \in [0, 1]$ of the asset value is lost in the resulting reorganisation and the residual value of the firms assets become the property of the debt holders. The firm is assumed to choose a default strategy, in the form of a default time τ , with the objective of maximising the value of equity which is defined as equal to the value of the firm's assets less the value of the firm's debts. Following the established literature, it is assumed that assets in place cannot be liquidated in order to raise funds to pay debt holders.

It is assumed that the stochastic process V is defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denotes the completed filtration generated by V .

5.3 Consol Debt

The results in this section, in the case of consol debt, have already been derived by Black and Cox [1976] and Leland [1994a]. A brief review of these results is presented in the context of an optimal stopping problem as a precursor to the the main focus of this chapter which is to study the case of a firm financed by finite maturity debt.

The value of the firm is given by the value of the assets together with the discounted value of tax rebates earned on coupon payments, prior to default minus the expected loss in the value of assets on default and is given by

$$F(v) = v + \mathbb{E}_v \left[\int_0^\infty cP\pi e^{-ru} \mathbb{1}_{\{u < \tau\}} du \right] - \xi \mathbb{E}_v [e^{-r\tau} V_\tau] \quad (5.3.1)$$

where $\mathbb{E}_v[\cdot] = \mathbb{E}[\cdot \mid V_0 = v]$ and $\mathbb{1}_{\{A\}}$ is the indicator function of the event A .

In the case of consol or perpetual debt, the firm is partly financed by debt paying a continuous fixed coupon which never matures (and therefore pays no principal). In these circumstances the value of the firm's debt is equal to the discounted value of coupon income paid before default plus the residual value of the firm's assets, should default occur

$$D(v) = \mathbb{E}_v \left[\int_0^\infty cP e^{-ru} \mathbb{1}_{\{u < \tau\}} du \right] + (1 - \xi) \mathbb{E}_v [e^{-r\tau} V_\tau]. \quad (5.3.2)$$

The firm is assumed to choose the default strategy which maximises equity value given by

$$\begin{aligned} Q(v) &= F(v) - D(v) \\ &= v - \hat{c} + \mathbb{E}_v [e^{-r\tau} (\hat{c} - V_\tau)], \end{aligned} \quad (5.3.3)$$

where $\hat{c} := \frac{cP(1-\pi)}{r}$.

Note from (5.3.3) that, on default, the value of equity is zero, in accordance with the strict priority rule. Since the firm always has the option not to declare default and

wait for a payoff of zero, it follows that the consol debt case therefore amounts to solving an infinite horizon optimal stopping problem of the form

$$\begin{aligned} w^*(v) &= \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_v [e^{-r\tau} (\hat{c} - V_\tau) \mathbb{1}_{\{\tau < \infty\}}] \\ &= \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_v [e^{-r\tau} g(V_\tau)] \end{aligned} \quad (5.3.4)$$

where the gain function $g(v) = (\hat{c} - v)^+$ and $\mathcal{T}_{0,\infty}$ denotes the set of \mathbb{F} measurable finite stopping times with the usual assumption that $\inf \{\emptyset\} = \infty$.

The optimal stopping problem in (5.3.4) can be solved by first making a guess at a solution which will then be verified. A solution will be sought under the assumption that there is a level b such that the stopping time

$$\tau_b = \inf \{t \geq 0 \mid V_t \leq b\} \quad (5.3.5)$$

is optimal in (5.3.4). Note that by well known first passage probabilities for Brownian motion to a fixed boundary¹, $\mathbb{E}_v [e^{-\gamma\tau_b}] > 0$ for $\gamma > 0$, and hence τ_b is finite P-a.s. for $v > b$.

Standard arguments based on the strong Markov property, lead to the formulation of the following *free-boundary problem* for the unknown value function $w := w^*$ in (5.3.4) and the unknown level b in (5.3.5),

$$\mathbb{L}_V w = rw \quad \text{for } v > b \quad (5.3.6)$$

$$w(v) = \hat{c} - v \quad \text{for } v = b \quad (5.3.7)$$

$$w'(v) = -1 \quad \text{for } x = b \quad (\text{smooth fit}) \quad (5.3.8)$$

$$w(v) > \hat{c} - v \quad \text{for } v > b \quad (5.3.9)$$

$$w(v) = \hat{c} - v \quad \text{for } 0 < v < b \quad (5.3.10)$$

where w' and w'' denote the first and second derivative respectively.

¹See for example Borodin and Salminen [2002]

Using (5.2.2) and (5.3.6) it can be seen that the value function solves the Cauchy-Euler equation given by

$$\frac{1}{2}\sigma^2v^2w''(v) + (r - \delta)vw'(v) - rw = 0 \quad (5.3.11)$$

and seeking a solution of the form $w(v) = v^p$ obtains a quadratic equation

$$p^2 - \left(1 - \frac{2(r - \delta)}{\sigma^2}\right)p - \frac{2r}{\sigma^2} = 0 \quad (5.3.12)$$

which has two roots

$$p^\pm = \frac{\left(1 - \frac{2(r - \delta)}{\sigma^2}\right) \pm \sqrt{\left(1 - \frac{2(r - \delta)}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}}{2}. \quad (5.3.13)$$

It follows that the general solution to (5.3.11) may be written

$$w(v) = A_1v^{p^+} + A_2v^{p^-} \quad (5.3.14)$$

where A_1 and A_2 are undetermined constants.

The fact that $w(v) \leq \hat{c}$ for all $x > 0$ combined with the observation that $p^+ > 0$ implies that $A_1 = 0$. Equations (5.3.7) and (5.3.8) then form a system of two algebraic equations in two unknowns, the boundary level b and the unknown constant A_2 , which have the solutions

$$b = \frac{\hat{c}}{1 - \frac{1}{p}} \quad (5.3.15)$$

$$A_2 = -\frac{1}{p} \left(\frac{k}{1 - \frac{1}{p}} \right) \quad (5.3.16)$$

where $p := p^-$ has been used for notational convenience. Note that $b < \hat{c}$.

Inserting (5.3.16) into (5.3.14) and using that $A_1 = 0$ obtains the solution to (5.3.4),

$$w^*(v) = \begin{cases} \frac{v^p}{p} \left(\frac{\hat{c}}{\frac{1}{p}-1} \right) & \text{if } v \in (b, \infty) \\ \hat{c} - v & \text{if } v \in (0, b]. \end{cases} \quad (5.3.17)$$

It is easily verified by a simple calculation that (5.3.15) is the same boundary derived by Black and Cox [1976], Leland [1994a]. The optimality of (5.3.17) and (5.3.5) may be verified by standard means².

Plugging (5.3.17) into (5.3.3) obtains an explicit expression for the value of equity

$$Q(v) = \begin{cases} v - \hat{c} + \frac{v^p}{p} \left(\frac{\hat{c}}{\frac{1}{p}-1} \right) & \text{if } v \in (b, \infty) \\ 0 & \text{if } v \in (0, b]. \end{cases} \quad (5.3.18)$$

It is again easily verified that (5.3.18) accords with Black and Cox [1976], Leland [1994a]. Explicit expressions for the value of the firm, (5.3.1), and the value of the firm's debt, (5.3.2), can also easily be obtained using well known first passage probabilities for Brownian motion.

5.4 Finite Maturity Debt

The firm is partly financed by debt with face value fixed at P which pays a continuous fixed coupon rate, c , until maturity, is not callable and matures at time $T > 0$ whereupon the firm is obligated to repay the face value. The value of the firm at time t is given by the value of the assets together with tax rebates earned on coupon payments, at rate $\pi \in [0, 1]$, prior to default minus the expected loss in the value of the assets on default

$$F(t, v) = v + cP\pi \mathbb{E}_{t,v} \left[\int_t^T e^{-r(u-t)} \mathbb{1}_{\{u < \tau\}} du \right] - \xi E_{t,v} \left[e^{-r(\tau-t)} V_\tau \mathbb{1}_{\{\tau \leq T\}} \right]. \quad (5.4.1)$$

²See for example Theorem 25.1 in Peskir [2005]

The value of the firm's debt at time t is equal to the discounted value of the face value of the debt and coupon payments made before default, plus the residual value of the firm's assets should default occur before time T

$$D(t, v) = Pe^{-r(T-t)}\mathbb{E}_{t,v}[\mathbb{1}_{\{T < \tau\}}] + cP\mathbb{E}_{t,v}\left[\int_t^T e^{-r(u-t)}\mathbb{1}_{\{u < \tau\}}du\right] + (1 - \xi)\mathbb{E}_{t,v}\left[e^{-r(\tau-t)}V_\tau\mathbb{1}_{\{\tau \leq T\}}\right]. \quad (5.4.2)$$

The firm chooses the default strategy (the default time τ) which maximises equity value, $Q(t, x) = F(t, x) - D(t, x)$, which after some simple calculations reduces to

$$Q(t, v) = v - \hat{c}(1 - e^{-r(T-t)}) - Pe^{-r(T-t)} + \mathbb{E}_{t,v}\left[e^{-r(\tau-t)}(\hat{c}(1 - e^{-r(T-\tau)}) + Pe^{-r(T-\tau)} - V_\tau)\mathbb{1}_{\{\tau \leq T\}}\right] \quad (5.4.3)$$

where $\hat{c} = \frac{cP(1-\pi)}{r}$. Note that on default the value of equity is always zero which is consistent with the strict priority rule. If the firm does not choose to default, the value of the firm at time T would be $Q(T, V_T) = V_T - P$.

It follows from (5.4.3) that in order to maximise equity value, the firm's default strategy should be chosen to solve the optimal stopping problem given by

$$\begin{aligned} w^*(t, v) &= \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_{t,v}\left[e^{-r(\tau-t)}(\hat{c}(1 - e^{-r(T-\tau)}) + Pe^{-r(T-\tau)} - V_\tau)\mathbb{1}_{\{\tau \leq T\}}\right] \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,v}\left[e^{-r(\tau-t)}g(\tau, V_\tau)\right] \end{aligned} \quad (5.4.4)$$

where $g(t, v) = (\hat{c}(1 - e^{-r(T-t)}) + Pe^{-r(T-t)} - v)^+$, $a^+ = \max(a, 0)$ and \mathcal{T}_B the set of \mathbb{F} measurable stopping times in the set B . The latter equality follows from the fact that the firm can always choose not to default in the interval $[0, T]$ and receive a payoff of at least zero.

The optimal stopping problem in (5.4.4) is similar to that of an American put option with the exception that the *strike* price of the option is time dependent. In order to analyse and solve the problem, it will be convenient to convert the payoff into a standard American call option on a stochastic process with a time inhomogeneity

embedded into its drift, as outlined in the following proposition.

Proposition 5.4.1. *The stockholders endogenous bankruptcy problem in the case of finite maturity debt is given by $w^*(t, v) = vv^*(t, x)$ with $x = \hat{p}(t)/v$ where $v^*(t, x)$ is given by*

$$v^*(t, x) = \sup_{\tau \in \mathcal{T}_{[0, T-t]}} \tilde{\mathbb{E}}_{t,x} [e^{-\delta\tau} g(X_{t+\tau})] \quad (t, x) \in E \quad (5.4.5)$$

with

$$g(x) = (x - 1)^+ \quad (5.4.6)$$

where $E = \{(t, x) \in [0, T] \times (0, \infty)\}$, τ is a stopping time of the process $X = \{X_t\}_{t \geq 0}$ which has dynamics under $\tilde{\mathbb{P}}$ for $t < s$ and $0 \leq s \leq T - t$ given by

$$X_{t+s} = x \frac{\hat{p}(t+s)}{\hat{p}(t)} e^{(\delta-r)s - \frac{1}{2}\sigma^2 s + \sigma \tilde{W}_s} \quad (5.4.7)$$

with $\tilde{W} = \{\tilde{W}_t\}_{t \geq 0}$ a standard Brownian motion under $\tilde{\mathbb{P}}$, $\delta, r, \sigma > 0$, $X_t = x$ and extended infinitesimal generator given by

$$\mathbb{L}_X f(x) = \left(\delta - \left(r - \frac{\hat{p}'(t)}{\hat{p}(t)} \right) \right) x \frac{\partial f(t, x)}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f(t, x)}{\partial x^2} \quad (5.4.8)$$

for $C^{1,2}$ functions f where $\hat{p}(s) = \hat{c}(1 - e^{-r(T-s)}) + P e^{-r(T-s)}$ and $\hat{p}' = \partial \hat{p} / \partial t$.

Proof. Defining the probability measure $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}} = \exp(\sigma W_T - \frac{1}{2}\sigma^2 T) d\mathbb{P}$ so that by Girsanov's theorem $\tilde{W}_t = \sigma t - W_t$ is a standard Brownian motion under $\tilde{\mathbb{P}}$ for $0 \leq t \leq T$ it follows from (5.4.4) that

$$\begin{aligned} w^*(t, v) &= \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,v} [e^{-r(\tau-t)} (\hat{p}(\tau) - V_\tau)^+] \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,v} \left[V_\tau e^{-r(\tau-t)} \left(\frac{\hat{p}(\tau)}{V_\tau} - 1 \right)^+ \right] \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} v \tilde{\mathbb{E}}_{t,x} [e^{-\delta(\tau-t)} (X_\tau - 1)^+] \end{aligned} \quad (5.4.9)$$

$$= \sup_{\tau \in \mathcal{T}_{0, T-t}} v \tilde{\mathbb{E}}_{t,x} [e^{-\delta\tau} (X_{t+\tau} - 1)^+] \quad (5.4.10)$$

where $X = \{X_t\}_{t \geq 0}$ is defined by $X_t = \hat{p}(t)/V_t$ for $0 \leq t \leq T$. An application of Itô's quotient rule yields the stochastic dynamics of X to be

$$\frac{dX_t}{X_t} = \left(\delta - \left(r - \frac{\hat{p}'(t)}{\hat{p}(t)} \right) \right) dt + \sigma d\tilde{W}_t \quad (5.4.11)$$

from which Itô's formula implies that

$$X_{t+s} = x \frac{\hat{p}(t+s)}{\hat{p}(t)} e^{(\delta-r)s - \frac{1}{2}\sigma^2 s + \sigma \tilde{W}_s},$$

with $X_t = x$, which completes the proof. □

Remark 5.4.2. Note that since $\hat{c} \geq 0$ it follows that,

$$r - \frac{\hat{p}'(s)}{\hat{p}(s)} = r \left(1 - \frac{e^{-r(T-s)}(P - \hat{c})}{e^{-r(T-s)}(P - \hat{c}) + \hat{c}} \right) > 0,$$

implying that the quantity $r - \frac{\hat{p}'(t)}{\hat{p}(t)}$ in (5.4.11), which may be given the interpretation of an instantaneous rate of dividend yield, cannot be negative. That means to say, that the drift of X under $\tilde{\mathbb{P}}$ when discounted by δ is never positive.

Remark 5.4.3. The dynamics of X in (5.4.11) may be rewritten as

$$\frac{dX_t}{X_t} = \frac{\delta \hat{p}(t) - c(1 - \pi)/r}{\hat{p}(t)} dt + \sigma \tilde{W}_t \quad (5.4.12)$$

from which it follows that if $c = 0$, the stock holders' problem reduces to the form of a standard American call option on a non-dividend paying asset. Since it is well known that an American call option on a zero dividend asset under positive discounting is never optimally exercised early, if $c = 0$, the option to declare bankruptcy prior to maturity will never be taken.

In the following propositions, some properties of the value function, v^* , are established.

Proposition 5.4.4. *The map $x \mapsto v^*(t, x)$ is (a) increasing and (b) convex.*

Proof. For part (a), let τ_0^* to be optimal for (t, x_0) in (5.4.5) and define

$$X_s(t, x) := x \frac{\hat{p}(t+s)}{\hat{p}(t)} e^{(\delta-r)s - \frac{1}{2}\sigma^2 s + \sigma \tilde{W}_s}$$

noting that $x \mapsto X_s(t, x)$ is obviously increasing and convex. Then, for $x_0 < x_1$, it follows that

$$\begin{aligned} v^*(t, x_0) &= \tilde{\mathbb{E}}_{t, x_0} \left[e^{-\delta(\tau_0^* - t)} g(X_{\tau_0^*}(x_0, t)) \right] \\ &< \tilde{\mathbb{E}}_{t, x_1} \left[e^{-\delta(\tau_0^* - t)} g(X_{\tau_0^*}(x_1, t)) \right] \\ &\leq v^*(t, x_1), \end{aligned}$$

where the first inequality follows from the observation that $X_{\tau^*}(t, x_0) < X_{\tau^*}(t, x_1)$, and the second, from the fact that τ_0^* is may be sub-optimal starting from x_1 implying that $x \mapsto v^*(t, x)$ is increasing.

For (b), define $x_\theta = \theta x_0 + (1 - \theta) x_1$ for $\theta \in [0, 1]$ and let τ_θ^* be optimal for (t, x_θ) in (5.4.5). Noting that convexity of the map $x \mapsto X_s(t, x)$ implies that $x \mapsto g(X_{\tau^*}(t, x_0))$ is also convex yields,

$$g(X_{\tau_\theta^*}(t, x_\theta)) \leq \theta g(X_{\tau_\theta^*}(t, x_0)) + (1 - \theta) g(X_{\tau_\theta^*}(t, x_1)),$$

from which, multiplying by $e^{-\delta(\tau^* - t)}$ and taking expectations, obtains

$$v^*(t, x_\theta) \leq \theta v^*(t, x_0) + (1 - \theta) v^*(t, x_1),$$

where the latter inequality results from the observation that, starting from either (t, x_0) or (t, x_1) , τ_θ^* is possibly sub-optimal. The latter inequality shows that the map $x \mapsto v^*(t, x)$ is convex, which completes the proof. \square

Proposition 5.4.5. *The value function in (5.4.5) (a) dominates the gain function: $v^*(t, x) \geq g(x)$, and (b), continuous on E .*

Proof. Part (a) follows from the fact that one can always take $\tau = t$. For part (b), continuity follows directly from Proposition 2.2 in Jaillet et al. [1990] since the process X is continuous and therefore bounded over a finite interval. Note also that continuity of the map $x \mapsto v^*(t, x)$ is also implied by convexity in Proposition 5.4.4. \square

Proposition 5.4.6. (*Smooth Fit*) *The function $x \mapsto v^*(t, x)$ is C^1 at $b(t)$ and $v_x^* = g_x$.*

Proof. The proof follows from well known arguments. First fix (t, x^*) on the boundary so that $X_t^* = b(t) > 1$ then (5.4.13) and (5.4.14) imply that for all $\varepsilon > 0$

$$\begin{aligned} \frac{\partial^+ v^*(t, x^*)}{\partial x} &= \lim_{\varepsilon \downarrow 0} \frac{v^*(t, x^* + \varepsilon) - v^*(t, x^*)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{g(x^* + \varepsilon) - g(x^*)}{\varepsilon} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^- v^*(t, x^*)}{\partial x} &= \lim_{\varepsilon \downarrow 0} \frac{v^*(x^* - \varepsilon) - v^*(x^*)}{-\varepsilon} \\ &\geq \lim_{\varepsilon \downarrow 0} \frac{g(x^* - \varepsilon) - g(x^*)}{-\varepsilon} \\ &= 1, \end{aligned}$$

where the derivatives, $\frac{\partial^\pm v^*(t, x)}{\partial x}$, exist by virtue of the convexity of the mapping $x \mapsto v^*(t, x)$ established in Proposition 5.4.4. Since v^* is convex, it is left and right differentiable and therefore $\partial_x^- v^* \leq \partial_x^+ v^*$. Since $\partial_x^+ v^* = 1$ and $\partial_x^- v^* \geq 1$, it follows that, $\partial_x^- v^* = \partial_x^+ v^* = 1$. \square

Proposition 5.4.7. *The value function satisfies $\partial_x^+ v^*, \partial_x^- v^* \in [0, 1]$*

Proof. The derivative in x of v^* is increasing, since v^* is convex, and non-negative, since v^* is increasing. Moreover for $(t, x) \in E$ such that $x > b(t)$, $v_x^* = 1$. \square

Since v^* and g are continuous Corollary 2.9 Peskir and Shiryaev [2006] implies a *stopping region*, defined as the set of pairs (t, x) at which immediate declaration of bankruptcy is an optimal policy, is given by

$$\bar{\mathcal{S}} = \{(t, x) \in [0, T] \times (0, \infty) \mid v^*(t, x) = g(x)\} \quad (5.4.13)$$

with the *continuation region* defined as

$$\mathcal{C} = \{(t, x) \in [0, T] \times (0, \infty) \mid v^*(t, x) > g(x)\}. \quad (5.4.14)$$

Furthermore, the stopping time τ_B defined by

$$\tau_B = \inf \{u \in [t, T] \mid X_t \in \bar{\mathcal{S}}\} \quad (5.4.15)$$

is optimal in (5.4.5). The following proposition establishes the existence of a boundary which separates the stopping and continuation regions.

Proposition 5.4.8. *There exists a boundary $t \mapsto b(t)$ such that the continuation region equals*

$$\mathcal{C} = \{(t, x) \in [0, T] \times (0, \infty) \mid x < b(t)\} \quad (5.4.16)$$

and the stopping region $\bar{\mathcal{S}}$ is the closure of the set

$$\mathcal{S} = \{(t, x) \in [0, T] \times (0, \infty) \mid x > b(t)\} \quad (5.4.17)$$

together with the remaining points (T, x) for $x \geq b(T)$.

Proof. Fix $t \in [0, T)$ and suppose $0 < x < 1$, then since $\mathbb{P}_{t,x}(X_{t+s} > 1) > 0$ for some $s \in [0, T - t)$, immediate exercise is suboptimal implying $(t, x) \in [0, T] \times (0, 1)$ are in the continuation region. The solution to the finite horizon problem (5.3.4) implies that there exists a boundary $b > 1$ such that the stopping time $\tau = \inf \{s > 0 \mid X_s \geq b\}$ is optimal in (5.3.4), which implies that all points (t, x) with $x \geq b$ for $t \in [0, T]$

belong to the stopping region. Furthermore, the considerations above, and the fact that $x \mapsto v^*(t, x)$ is convex on $(0, \infty)$, implies that there exists a function $t \mapsto b(t)$ satisfying $1 \leq b(t) \leq b$ for all $t \in [0, T)$ which separates the continuation and stopping regions. Since v^* is continuous, \mathcal{C} is open. \square

Proposition 5.4.9. *At time T , the exercise boundary in Proposition 5.4.8 equals 1.*

Proof. Assume that $b(T) > 1$ and consider (T, x^*) such that $x^* \in (1, b(T))$. Since $x^* \in \mathcal{C}$, it follows from (5.4.14) that $v^*(T, x^*) > g(T, x^*)$. However, the latter is contradicted by fact that $v^*(T, x^*) = g(x^*)$ because $x^* > 1$ by assumption. The observation that, by the form of the gain function, $B(T)$ is clearly not less than 1, completes the proof. \square

Remark 5.4.10. Note that since the instantaneous benefit of stopping is given by $(r - \hat{p}'(t) / \hat{p}(t))x - \delta$, it might be conjectured that for,

$$\begin{aligned} X(T-) &< \lim_{t \uparrow T} \frac{\delta}{\left(r - \frac{\hat{p}'(t)}{\hat{p}(t)}\right)} \\ &= \frac{\delta}{c(1 - \pi)} \end{aligned}$$

stopping may be suboptimal, since the instantaneous gain would be negative. The latter suggest that there might be a discontinuity in the boundary at maturity. Numerical calculations presented in Section 5.5 are suggestive of such a discontinuity.

Collecting the results above, together with the fact that the value function in (5.4.5) is the solution to a PDE with constant coefficients³ in \mathcal{C} , and is therefore $C^{1,2} \in \mathcal{C}$, the following verification theorem presents a free boundary formulation satisfied by the value function. Proposition 5.4.12 below then presents an integral equation which the boundary satisfies. Finally, Theorem 5.4.13 verifies that the boundary is the unique solution to the latter identified integral equation under certain assumptions.

³See for example Friedman [1964].

Theorem 5.4.11. *Let w be continuous on E , $C^{1,2}$ on $E \setminus \Gamma(b(t))$, for every $t \in [0, T]$ $w(t, \cdot)$ be C^1 on \mathbb{R}^+ , $w_x \in [0, 1]$ and solve the following system of partial differential equations,*

$$w_t + \mathbb{L}_X w - \delta w \leq 0 \quad (t, x) \in \mathcal{S} \quad (5.4.18)$$

$$w(t, x) \geq g(x) \quad (t, x) \in \mathcal{C} \quad (5.4.19)$$

$$w_t + \mathbb{L}_X w - \delta w = 0 \quad (t, x) \in \mathcal{C} \quad (5.4.20)$$

$$w(t, x) = g(x) \quad (t, x) \in \bar{\mathcal{S}} \quad (5.4.21)$$

where $\Gamma(b(t)) = \{(t, x) \in E \mid x = b(t)\}$ and

$$b(t) = \inf \{x \in \mathbb{R}^+ \mid w(t, x) < g(x)\}$$

then $w = v^*$ in (5.4.5) and the stopping time given by

$$\tau_b = \inf \{s \in [0, T - t] \mid X_{t+s} > b(t + s)\} \quad (5.4.22)$$

is optimal in (5.4.5).

Proof. It follows from an application of a generalised version of Itô's formula that, for $\sigma \in \mathcal{T}_{0, T-t}$,

$$\begin{aligned} e^{-\delta\sigma} g(X_{t+\sigma}) &\leq e^{-\delta\sigma} w(t + \sigma, X_{t+\sigma}) \\ &= w(t, x) + \int_0^\sigma e^{-\delta s} \Pi_w(X_{t+s}) ds + \mathcal{M}_\sigma \end{aligned}$$

where $\Pi_f = f_t + \mathbb{L}_X f - qf$ and \mathcal{M}_σ is a $\mathbb{P}_{t,x}$ martingale by Proposition 5.4.6 with the first inequality following from (5.4.19) and (5.4.21). Taking $\mathbb{P}_{t,x}$ expectations and using (5.4.18) and (5.4.20) obtains

$$\begin{aligned} \mathbb{E}_{t,x} [e^{-\delta\sigma} g(X_{t+\sigma})] &= w(t, x) + \int_0^\sigma \mathbb{E}_{t,x} [e^{-\delta s} \Pi_w(X_{t+s}) ds] \\ &\leq w(t, x). \end{aligned}$$

As a consequence, since σ was chosen arbitrarily, taking the supremum over all stopping times in $\mathcal{T}_{0,T-t}$ yields

$$\sup_{\sigma \in \mathcal{T}_{0,T-t}} \mathbb{E}_{t,x} [e^{-\delta\sigma} g(X_{t+\sigma})] = v^*(t, x) \leq w(t, x)$$

implying $v^*(t, x) \leq w(t, x)$.

For the alternate inequality, first note that by applying a version of Itô's formula to the function $e^{-\delta s} w(t+s, X_{t+s})$, setting $s = u \wedge \tau_b$, where $u \in [0, T-t]$ and τ_b is defined in (5.4.22) yields

$$w(t, x) = e^{-\delta(u \wedge \tau_b)} w(t + u \wedge \tau_b, X_{t+u \wedge \tau_b}) - \int_0^{u \wedge \tau_b} e^{-\delta s} \Pi_w(X_{t+s}) ds - \mathcal{M}_{u \wedge \tau_b}.$$

Taking $\mathbb{P}_{t,x}$ expectations obtains

$$\begin{aligned} w(t, x) &= \mathbb{E}_{t,x} [e^{-\delta(u \wedge \tau_b)} w(t + u \wedge \tau_b, X_{t+u \wedge \tau_b})] \\ &= \mathbb{E}_{t,x} [\mathbb{E} [e^{-\delta\tau_b} g(X_{t+\tau_b}) \mid \mathcal{F}_u]] \\ &= \mathbb{E}_{t,x} [e^{-\delta\tau_b} g(X_{t+\tau_b})] \\ &\leq \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}_{t,x} [e^{-\delta\tau} g(X_{t+\tau})] = v^*(t, x) \end{aligned}$$

where the first equality follows from (5.4.20) and the fact that $\mathcal{M}_{u \wedge \tau_b}$ is again a $\mathbb{P}_{t,x}$ martingale, the second equality from an application of the Markov property and (5.4.21) implying $w = g$ at τ_b , and the final equality from the tower property. Taking supremum over all stopping times in $\mathcal{T}_{0,T-t}$ therefore obtains that $v^*(t, x) \geq w(t, x)$ completing the proof.

□

Proposition 5.4.12. *The boundary in Theorem 5.4.11 admits a representation as a solution to the free boundary equation given by*

$$\begin{aligned}
 b(t) - 1 &= \rho(t, T-t) e^{-r(T-t)} \Phi(\xi_1(\rho(t, T-t), 1, T-t)) \\
 &- e^{-\delta(T-t)} \Phi(\xi_2(\rho(t, T-t), 1, T-t)) \\
 &+ \int_0^{T-t} e^{-rs} \rho(t, s) \gamma(t+s) \Phi(\xi_1(\rho(t, s), b(t+s), s)) ds \\
 &- \delta \int_0^{T-t} e^{-\delta s} \Phi(\xi_2(\rho(t, s), b(t+s), s)) ds
 \end{aligned} \tag{5.4.23}$$

with $b(T) = 1$ where

$$\begin{aligned}
 \rho(u, s) &= x \frac{\hat{p}(u+s)}{\hat{p}(u)}, \\
 \gamma(u) &= r - \frac{\hat{p}'(u)}{\hat{p}(u)}, \\
 \xi_1(x, y, u) &= \frac{\log(x/y) + (\delta - r + \frac{1}{2}\sigma^2)u}{\sigma\sqrt{u}}, \\
 \xi_2(x, y, u) &= \xi_1(x, y, u) - \sigma\sqrt{u}
 \end{aligned}$$

and $\Phi(\cdot)$ is the standard normal distribution.

Proof. The optimal default boundary in (5.4.5) may be characterised by applying a version of Itô's formula to the function $e^{-\delta u} w(t+u, X_{t+u})$ where w solves (5.4.18)-(5.4.21) whereupon taking $\tilde{\mathbb{P}}_{t,x}$ expectations obtains that

$$\begin{aligned}
 w(t, x) &= e^{-\delta u} w(t+u, X_{t+u}) - \int_0^u e^{-\delta s} \tilde{\mathbb{E}}_{t,x} [\Pi_w(X_{t+s})] ds \\
 &+ \tilde{\mathbb{E}}_{t,x} \left[\int_0^u e^{-\delta s} \sigma X_{t+s} w_x d\tilde{W}_{t+s} \right]
 \end{aligned} \tag{5.4.24}$$

where the operator Π is defined as $\Pi_f = f_t + \mathbb{L}_X f - \delta f$.

Noting that (5.4.20) implies $\Pi_w = 0$ in \mathcal{C} and (5.4.21) implies $w(t, x) = g(x)$ in $\bar{\mathcal{S}}$, obtains from (5.4.8) that $\Pi_g(x) = \delta - (r - \hat{p}'/\hat{p})x$ for $X_t \geq b(t)$. It follows that by

setting $u = T - t$ and $\gamma(u) := r - \frac{\hat{p}'(u)}{\hat{p}(u)}$, and noting that, since (5.4.21) implies that $w(t, x) = g(x) = x - 1$ for $x \in \bar{\mathcal{S}}$, results in the so called *early exercise premium representation* of the value function given by

$$\begin{aligned} x - 1 &= e^{-\delta(T-t)} \tilde{\mathbb{E}}_{t,x} [(X_T - 1)^+] \\ &+ \int_0^{T-t} e^{-\delta s} \gamma(t+s) \tilde{\mathbb{E}}_{t,x} [X_{t+s} \mathbb{1}_{\{X_{t+s} \geq b(t+s)\}}] ds \\ &- \delta \int_0^{T-t} e^{-\delta s} \tilde{\mathbb{E}}_{t,x} [\mathbb{1}_{\{X_{t+s} \geq b(t+s)\}}] ds \end{aligned} \quad (5.4.25)$$

holds for $x \in (0, b(t)]$ and $t \in [0, T]$.

Equation (5.4.25) represents a family of boundaries $t \mapsto b(t)$, one for each value of x . A natural candidate *free boundary equation* can be obtained by inserting $x = b(t)$ into (5.4.25) whereupon using (5.4.7) obtains (5.4.23). The fact that $b(T) = 1$ is contained in Proposition 5.4.9 which completes the proof. \square

Theorem 5.4.13. *Assuming that the boundary (5.4.22) is continuous and*

$$\Pi_g(x) = (g_t + \mathbb{L}_X g - \delta g)(x) < 0 \in \bar{\mathcal{S}}, \quad (5.4.26)$$

the boundary in Theorem 5.4.12 is the unique solution to the free boundary equation (5.4.23).

Proof. The following argument is an adaption of one found in Peskir [2005]. Suppose that (v^c, c) is another solution pair to the free boundary problem (5.4.18)-(5.4.21) and

$$\tau_c = \inf \{s \in [0, T - t] \mid X_{t+s} \geq c(t+s)\}$$

then an application of a generalised version of Itô's formula to $e^{-\delta u} v^c(t+u, X_{t+u})$ implies that

$$\begin{aligned} e^{-\delta u} v^c(t+u, X_{t+u}) &= v^c(t, x) + \int_0^u e^{-\delta s} \Pi_g(X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq c(t+s)\}} ds \\ &+ \int_0^u e^{-\delta s} \sigma X_{t+s} v_x^c dW_{t+s}, \end{aligned} \quad (5.4.27)$$

where $\Pi_g = g_t + \mathbb{L}_X g - \delta g$ in $\bar{\mathcal{S}}$. After taking $\mathbb{P}_{t,x}$ expectations, the final martingale term in the latter equality vanishes, whereupon inserting $u = \tau_c$ obtains

$$\begin{aligned} v^c(t, x) &= \mathbb{E}_{t,x} \left[e^{-\delta \tau_c} v^c(t + \tau_c, X_{t+\tau_c}) \right] \\ &\quad - \mathbb{E}_{t,x} \left[\int_0^{\tau_c} e^{-\delta s} \Pi_g(X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq c(t+s)\}} ds \right] \\ &= \mathbb{E}_{t,x} \left[e^{-\delta \tau_c} v^c(t + \tau_c, X_{t+\tau_c}) \right] \\ &= \mathbb{E}_{t,x} \left[e^{-\delta \tau_c} g(X_{t+\tau_c}) \right] \end{aligned}$$

for all $(t, x) \in E$, where the latter equality follows from the fact that, by (5.4.21), $v^c(t, x) = g(x)$ for $X_t \geq c(t)$. Note that it follows immediately from (5.4.5) that

$$v^c(t, x) \leq v^*(t, x) \tag{5.4.28}$$

for $(t, x) \in E$.

By an analogous argument to that above, applying a generalised version of Itô's formula to $e^{-\delta u} v(t + u, X_{t+u})$ obtains

$$\begin{aligned} e^{-\delta u} v^*(t + u, X_{t+u}) &= v^*(t, x) + \int_0^u e^{-\delta s} \Pi_g(X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq b(t+s)\}} ds \\ &\quad + \int_0^u e^{-\delta s} \sigma X_{t+s} v_x^* dW_{t+s}. \end{aligned} \tag{5.4.29}$$

Letting $(t, x) \in E$ such that $x > \max(b(t), c(t))$ and inserting the stopping time defined by

$$\sigma_b = \inf \{s \in [0, T - t] \mid X_{t+s} \leq b(t + s)\}$$

into (5.4.27) and (5.4.29) and after taking $\mathbb{P}_{t,x}$ expectations obtains

$$\begin{aligned} \mathbb{E}_{t,x} \left[e^{-\delta \sigma_b} v^c(t + \sigma_b, X_{t+\sigma_b}) \right] &= g(t, x) \\ &\quad + \mathbb{E}_{t,x} \left[\int_0^{\sigma_b} e^{-\delta s} \Pi_g(X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq c(t+s)\}} ds \right] \end{aligned} \tag{5.4.30}$$

$$\mathbb{E}_{t,x} \left[e^{-\delta \sigma_b} v^*(t + \sigma_b, X_{t+\sigma_b}) \right] = g(t, x) + \mathbb{E}_{t,x} \left[\int_0^{\sigma_b} e^{-\delta s} \Pi_g(X_{t+s}) ds \right]. \tag{5.4.31}$$

Combining (5.4.30) and (5.4.31) with (5.4.28) yields

$$\mathbb{E}_{t,x} \left[\int_0^{\sigma_b} e^{-\delta s} \Pi_g (X_{t+s}) ds \right] \geq \mathbb{E}_{t,x} \left[\int_0^{\sigma_b} e^{-\delta s} \Pi_g (X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq c(t+s)\}} ds \right],$$

which, since c and b are continuous and $\Pi_g < 0$, by assumption, implies that $c(t) \leq b(t)$.

To show that $c(t) = b(t)$, first assume that there exists $t \in (0, T)$ such that $c(t) < b(t)$ and let $x \in (c(t), b(t))$. Insert τ_b from (5.4.22) into (5.4.27) and (5.4.29) and take $\mathbb{P}_{t,x}$ expectations to obtain

$$\begin{aligned} v^c(t, x) &= \mathbb{E}_{t,x} [e^{-\delta \tau_b} g(X_{t+\tau_b})] \\ &\quad - \mathbb{E}_{t,x} \left[\int_0^{\sigma_b} e^{-\delta s} \Pi_g (X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq c(t+s)\}} ds \right] \\ v^*(t, x) &= \mathbb{E}_{t,x} [e^{-\delta \tau_b} g(X_{t+\tau_b})] \end{aligned}$$

from which, using again (5.4.28), implies that

$$\mathbb{E}_{t,x} \left[\int_0^{\sigma_b} e^{-\delta s} \Pi_g (X_{t+s}) \mathbb{1}_{\{X_{t+s} \geq c(t+s)\}} ds \right] \geq 0,$$

which cannot be true since, c and b are continuous and $\Pi_g < 0$ by assumption, implying that (t, x) cannot exist, which, by the contradiction together with the facts above, implies $c(t) = b(t)$, completing the proof. \square

5.5 Numerical Analysis

5.5.1 Method of calculation In this section, numerical calculations of default boundaries are presented for varying parameter values. The calculations are included for illustrative purposes, recognising that the uniqueness of the boundary characterised in (5.4.23) has not been verified. The equation in (5.4.23), which the boundary associated with (5.4.5) satisfies, is a *non-linear Volterra integral equation of the second*

kind. Numerical calculations for the boundaries presented in this section are derived using a simple bootstrap method where the starting point for calculation at time T is taken from Proposition 5.4.9 to be $b(T) = 1^4$. Starting from the latter known point, one can work backwards in time by successive approximations. The boundary derived from (5.4.23) is in terms of the process X . The firm's underlying default boundary associated with (5.4.4) in terms of the process V is then calculated from the call boundary simply using the definition of $X_t = \hat{p}(t)/V_t$ in Proposition 5.4.1. The default parameter values used for the calculations were $P=100$, $r=7.5\%$, $\sigma=20\%$, $\pi=20\%$ and $\delta=7\%$ and $c = 10\%$. These values are in accordance with Leland [1994a] with the exception of the tax rate.

5.5.2 Effect of coupons Figure 5.5.1 illustrates the effect of higher coupons on the calculated underlying call option boundary. As can be observed in Figure 5.5.1, the call boundary tends to be higher, and decrease even more rapidly, as a function of time as the coupon rate increases.

Figure 5.5.2 displays the calculated default boundary associated with the same parameter values as Figure 5.5.1. As might be expected, the default boundary shifts upwards with higher coupons, which would naturally tend to increase the probability that a firm would declare bankruptcy prior to maturity and would therefore be expected to be associated with higher credit spreads. For the default parameter values, the net coupon rate exceeds the dividend rate and therefore the calculated boundary at maturity is equal to the principal. However, for coupon rates less than 14%, the default boundary appears to be upward sloping as a function of time, naturally implying an increasing likelihood of default as time passes towards maturity.

However, as Figure 5.5.3 indicates, that for high coupon rates relative to other parameters, the calculated default boundary initially declines as a function of time before increasing as a function of time prior to maturity. The latter observation has the obvious consequence that firms financed by high coupon bonds may be encouraged

⁴See Linz [1985].

to declare default well before maturity of their debt.

5.5.3 Effect of dividends and interest rates Numerical calculations suggest that lower dividends raise the default boundary as illustrated in Figure 5.5.4. Naturally, higher cash flows to equity holders are likely to encourage equity holders to wait longer before declaring bankruptcy. As noted in Remark 5.4.10, numerical calculations also suggest that, if the dividend rate exceeds the net of tax coupon rate, there may be a discontinuity in the default boundary at maturity, with the default boundary being below the principal value of the debt immediately prior to maturity.

Figure 5.5.5 displays the impact of different levels of risk free interest rate on the calculated default boundary. Higher interest rates tend to lower the default boundary, although the sensitivity of the boundary to interest rate changes appears to be relatively small. The latter observation is consistent with credit spreads being inversely related to risk free interest rates which is often observed in market data.⁵

Finally, Figure 5.5.6 displays a calculated boundary with a net coupon rate which is less than the risk free interest rate. It is interesting to observe that numerical calculations suggest that if the net coupon rate is lower than the risk free interest rate in this model, the default boundary is very low when there is a long time left to maturity, and then steepens dramatically prior to maturity. The obvious implication of the latter is, that default would be far more likely as maturity approaches, which is consistent with the finding in Remark 5.4.3, in which it was noted that, it would never be optimal to declare bankruptcy prior to maturity for a zero coupon finite maturity bond.

⁵ The structural model of Merton [1974] indicates such a negative relationship between interest rates and credit spreads.

Optimal Default for Finite Maturity Debt

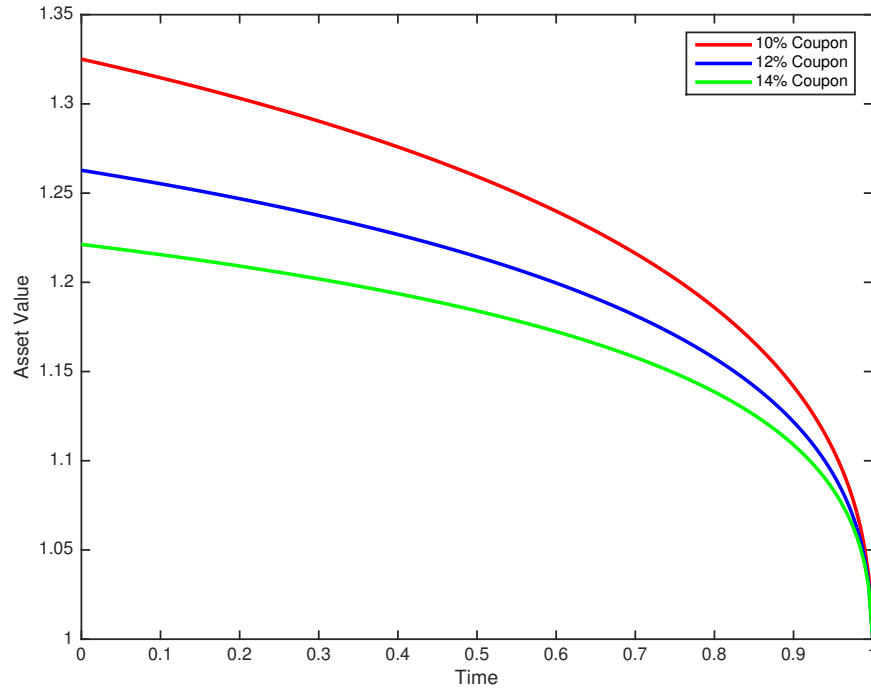


Figure 5.5.1: Call Boundary Calculations for Different Levels of Coupon

The figure shows calculated call boundaries for $P=100$, $r=7.5\%$, $\sigma=20\%$, $\pi=2\%$ and $\delta=7\%$ for different levels of coupon, c . The boundary appears to flatten as a function of time as the coupon rate increases.

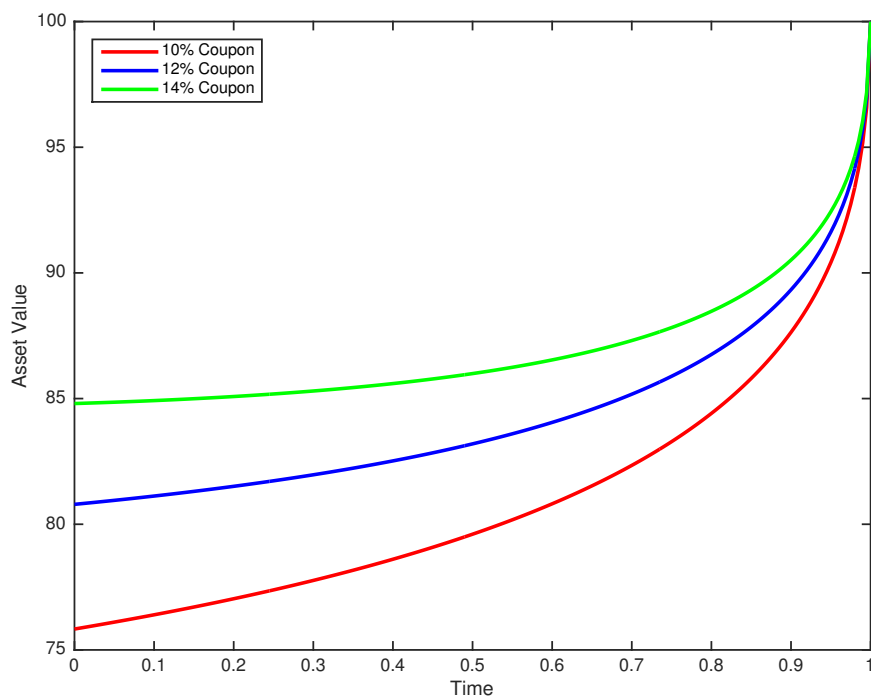


Figure 5.5.2: Default Boundary Calculations for Different Levels of Coupon
The figure shows calculated default boundaries for $P=100$, $r=7.5\%$, $\sigma=20\%$, $\pi=2\%$ and $\delta=7\%$ for different levels of coupon rate, c .

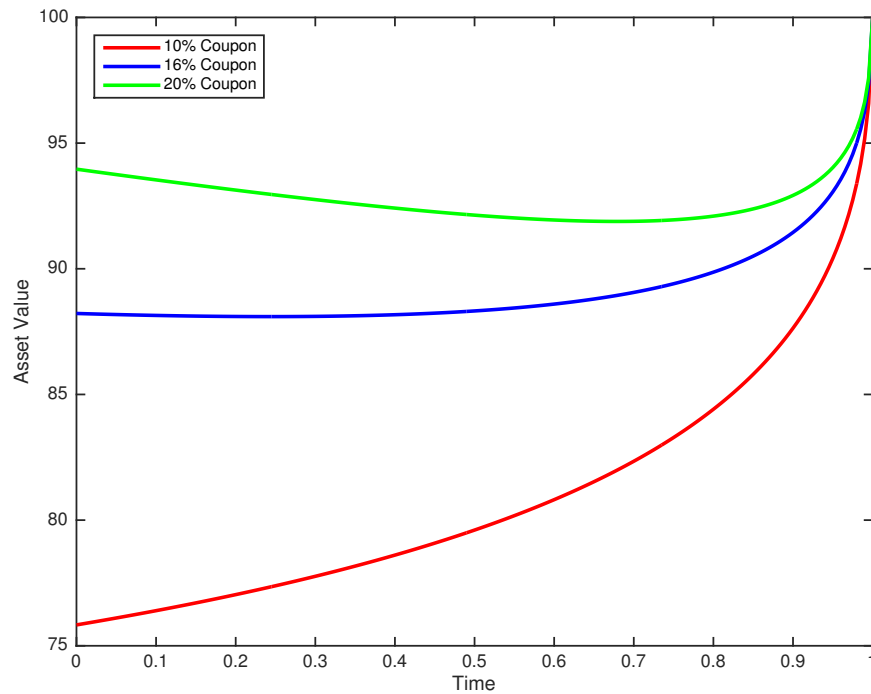


Figure 5.5.3: Default Boundary Calculations for Different Levels of Coupon
The figure shows calculated default boundaries for $P=100$, $r=7.5\%$, $\sigma=20\%$, $\pi=20\%$ and $\delta=7\%$ for different levels of coupon, c . When the coupon is relatively very high, the boundary as a function of time can initially slope downwards, only to slope upwards immediately prior to maturity.

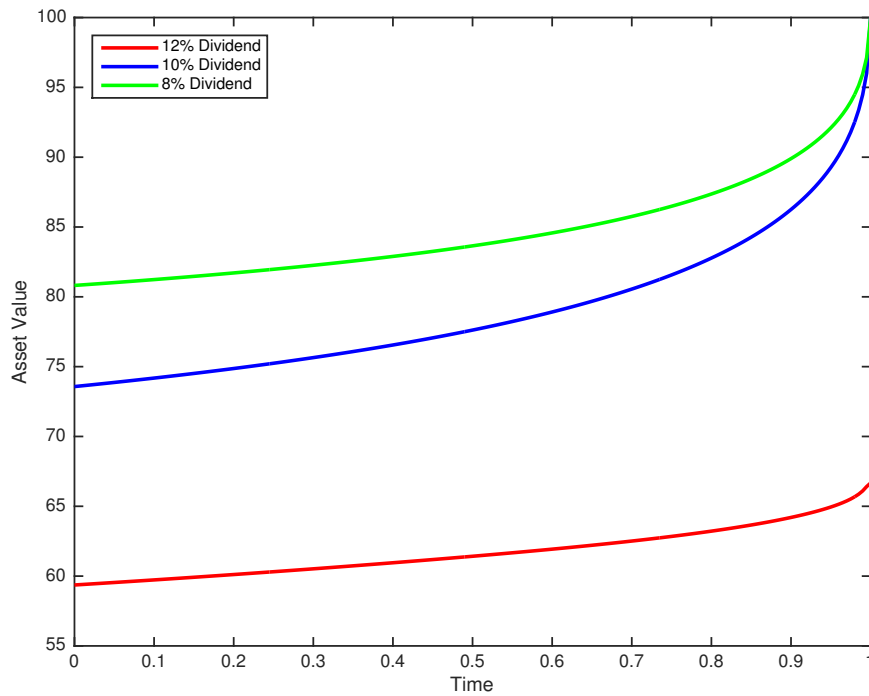


Figure 5.5.4: Default Boundary Calculations for Different Levels of Dividend Yield
 The figure shows calculated default boundaries for $P=100$, $r=7.5\%$, $\sigma=20\%$, $\pi=20\%$ and $c=10\%$ for different levels of dividend yield, δ . An increasing level of dividend yield on the underlying asset may make it less likely for the stockholders to declare default. When the dividend yield on the asset is greater than the net coupon rate, calculations suggest that there is a discontinuity at maturity with $b(T) = P$.

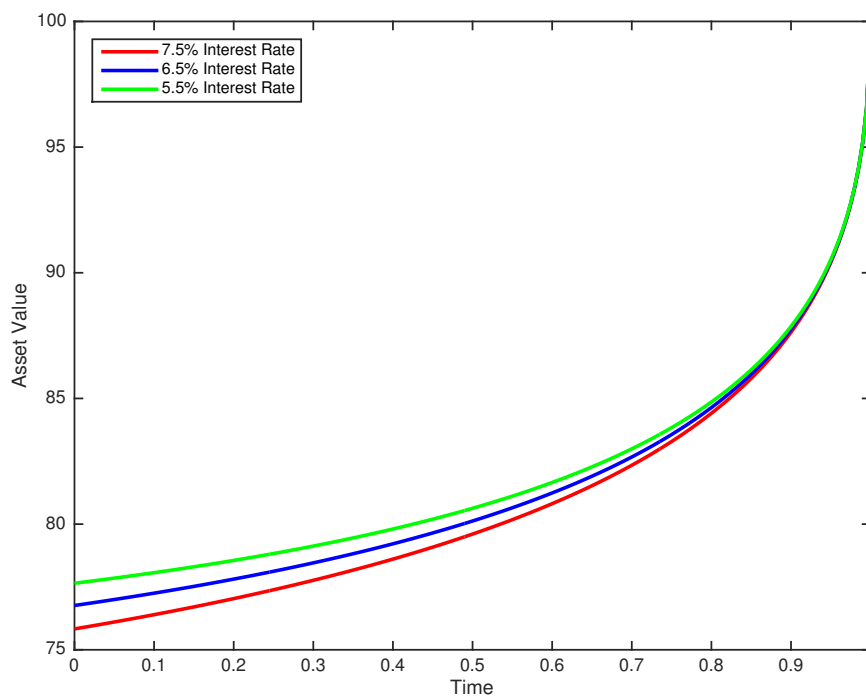


Figure 5.5.5: Default Boundary Calculations for Different Levels of Interest Rate
The figure shows calculated default boundaries for $P=100$, $\delta=7\%$, $\sigma=20\%$, $\pi=20\%$ and $c=10\%$ for different levels of interest rate, r . A falling interest rate tends to increase the level of the boundary with the effect greatest the longer the time left until maturity. The latter would be consistent with lower interest rates increasing credit spreads.

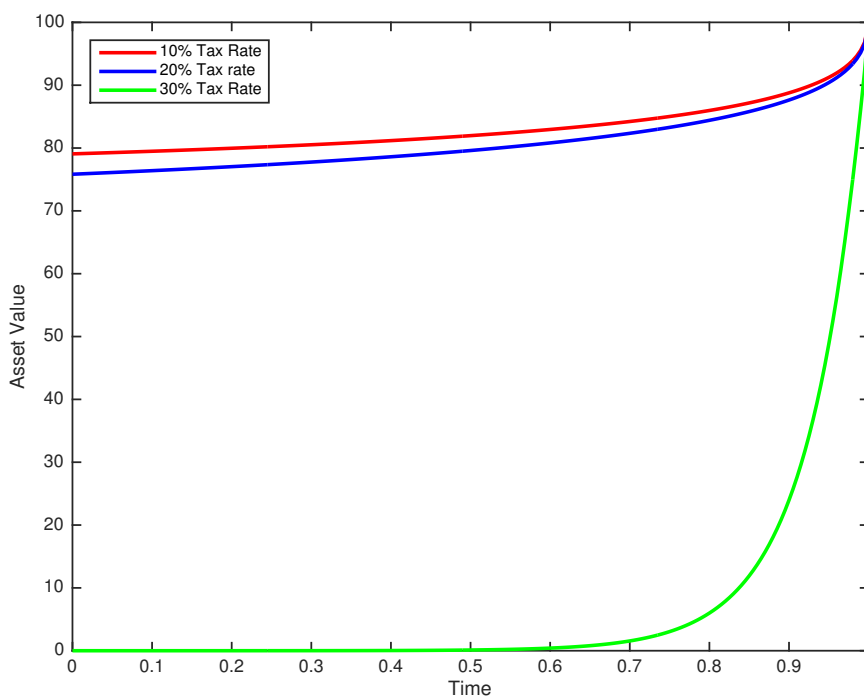


Figure 5.5.6: Default Boundary Calculations for Different Levels of Tax Rate
 The figure shows the numerical boundary calculations for $P=100$, $r=7.5\%$, $\sigma=20\%$, $c=10\%$ and $\delta=7\%$ for different levels of tax rate, π . An increasing tax rate lowers the boundary since the net coupon rate is lower. If the net coupon rate is less than the interest rate, numerical calculations suggest the boundary may steepen dramatically prior to maturity if the net coupon rate falls below the interest rate, the boundary.

CHAPTER 6 CONCLUDING REMARKS AND FUTURE RESEARCH

The models presented in Chapters 3 and Chapter 4 have demonstrated that regime shifts can have significant effects on firm capital structure decisions and associated financing costs. Numerical calculations suggest that in a recessionary economic environment, where interest, dividend, and tax rates are assumed to fall, and bankruptcy costs and asset volatility assumed to rise, corporate entities would generally choose to adopt lower level of leverage, and, equity, firm and debt value would tend to be lower.

For firms starting up in a recessionary environment, at a lower level of leverage, debt financing costs are also likely to be lower. However, for existing firms, with a set capital structure, debt financing costs in a recessionary environment might well be higher as a result of increased asset volatility. The latter effects would naturally be exacerbated in the presence of negative asset value jumps. Certainly the results of Chapter 3, in relation to the roll-over debt structure, suggest that shifting regimes may have a substantial impact on the term structure of credit spreads for existing firms.

However, it is clear from the results of Chapter 4, that further work is required, to fully understand the optimal behaviour of firms in a regime-switching setting. While conditions were identified under which optimal boundaries exist in the case of infinite maturity debt, questions still remain in generality. Since the strikes from the American put-style payout of the optimal stopping problem, associated with equity maximisation under regime-switching are regime dependent, it is not guaranteed that an optimal stopping time will exist in all regimes.

Concluding Remarks and Future Research

In the case of two regimes, where the strike in one regime is far below the strike in the other regime, it is intuitively plausible that it might be optimal to wait for a regime-change before exercising the option to default. Understanding the latter behaviour might be important for policy makers who may be motivated in guiding macroeconomic policies in such a way, as to create a regime in which to avoid defaults occurring. However, once the regime switches again, there could potentially be unintended consequences.

Regime-switching models arguably, therefore, provide a rich framework for modelling complex financial behaviour in a straightforward manner. It is also clear that, the general mixing of parameters in a regime-switching setting, can potentially accommodate a wide variety of scenarios. Ultimately, however, it will be required to calibrate the models to market data to determine the capital structure decisions that would be made under empirically identifying regimes. Future research activity should include calibrating the models to market data.

In the above regard, while the focus of this thesis has been to try and identify default boundaries, which, in some sense maximise equity value, in many practical applications, exogenously determined boundaries are used by practitioners to calibrate credit models to financial market data. Well known examples include the KMV model. Typically, such models are based on the assumption that a firm's asset price follows a geometric Brownian motion. However, clearly the results in relation to exogenously determined boundaries contained in this thesis may well find applications in the latter regard.

All the models presented in this thesis could also be enhanced further by incorporating tax thresholds for asset value, below which firms do not benefit from tax relief on interest payments, which has been typical in the literature. While in a regime-switching setting, a lower tax rate in a regime, will, to a certain extent, discourage firms from undertaking leverage, modelling the availability of tax relief on interest payments directly is an important feature of structural models of credit risk. Therefore, enhancing the models presented in this thesis to incorporate a tax threshold is

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also an intended future area of research.

Adding positive jumps to the models presented is also an objective for future research. While, for default purposes, negative jumps are arguably more important than positive jumps, there may be circumstances in which positive jumps have a meaningful role to play. For investment grade bonds, the primary risk is to the downside in terms of the value of debt should the credit quality of a firm deteriorate. In the latter circumstance, negative jumps clearly play a positive role. However, for more highly leveraged firms, the positive surprise of a credit quality improvement may also have an important implications for capital structure valuation.

There is further work to be done in Chapter 5 to verify analytically the properties of the default boundary and the framework could also be extended by incorporating jumps into the asset value process. In addition, the effects of the introduction of regime switching into the asset value process could be examined. Furthermore, the question as to whether the firm would refinance at maturity and how this would be achieved needs to be explored. Efficient methods for the valuing the capital structure associated with the finite maturity debt boundary, along with the associated credit spreads, is an additional area of potential future research.

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