CALCULATION OF A SELF-CONSISTENT, LOW
FREQUENCY ELECTROSTATIC FIELD IN THE
DRIFT-KINETIC APPROXIMATION

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ABSTRACT

We derive an asymptotic series in ω_p^{-2}, the inverse-square plasma

frequency, for the self-consistent, low frequency electrostatic field

in tori. The derivation is consistent with the drift-kinetic ordering

and may be used in either instability or equilibrium calculations. We

find that in a time-dependent formalism, the electric field is completely

determined to first order in a drift-kinetic expansion.

I. INTRODUCTION

The creation of magnetically confined plasmas naturally results in the occurrence of spatial inhomogeneities in densities, currents, and temperatures, as well as in the confining magnetic field. These inhomogeneities drive species-dependent particle currents, causing charge separations which set up electric fields. These electric fields in turn influence the particle motions and currents, completing the cycle.

The physically interesting phemomena relating to equilibrium or many types of instability behavior in such plasmas often occur on relatively long drift, collision, or bounce time scales. However, the formal mathematical coupling of the kinetic equations for the individual species and the electric field, by use of Poisson's equation, leads to the prediction of extremely rapid oscillations at or near the plasma frequency ω_p. Aside from the mathematical and numerical difficulties such oscillations introduce, they are of little physical interest in a wide variety of plasma problems. The classical physical statement of this fact[1] for ionized gases, virtually tantamount to the definition of a plasma, is that one is only rarely interested in a time resolution sufficiently fine to require accounting for any significant build up of charge density.

An alternative statement of this physical concept is that many plasma phenomena (including many instabilities) occur over distances long compared to the Debye lengths. There are important exceptions, for which one must account in detail for the instantaneous charge density (and/or current) build-up in a relatively localized way. This process is essential, for example, in the study of a whole host of microinstabilities[2], especially since the latter very frequently involve plasma oscillations.

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If we consider plasmas confined in tori, both for quasi-equilibria (including neoclassical transport theory) as well as for typical plasma instability phenomena in toroidal discharges, we usually find a situation in which at least one species responds on a much faster time scale than the one of physical interest. It is then pointless to keep track in detail of the time-space build-up of electrical charge, and Poisson's equation is relegated to a position of only secondary interest. It often suffices to state simply that the electron and ion densities are "nearly" equal. In fact, it is often sufficient to determine the charge density of one species directly from its dynamics, and then use that information (see Spitzer's discussion of the ion-acoustic mode in Ref. 1, for example) to determine the (coupled) kinetic behavior of the second species. The "quasi-neutral" limit of the ion acoustic dispersion relation emerges in this way.[1]

The physical ideas sketched briefly above are naturally, extremely well-known. They are used, in one form or another, in virtually all of the plasma physics literature. This is especially true in the treatment of both neoclassical transport and instability problems in toroidally confined plasmas. The concept is generally referred to as quasineutrality.[1] Thus, at this point, we are not attempting to discuss anything new, only to review it.

Not only is this basic idea used in many situations; it is also deceptively simple. A clue to this latter fact can be obtained by a survey of the wide variety of the means by which the principle is actually employed in practice (compare, for example, Refs. 3-7). When appropriate, quasineutrality allows one to determine the required components of the

electric field relatively simply.

In the next section we discuss briefly the variety of useful formulations of the quasineutrality concept. Following that, we give a somewhat more complete discussion of the mathematical and physical implications of the more complete governing equations. We indicate as we go the relationship of quasineutrality, as normally employed, to the more exact statement. It will be seen that earlier treatments represent useful special cases of the general analysis. Our particular motivation in developing this more precise treatment lies in our need to be able to compute the full electric field relatively precisely for use with the CPM.[8,9] However, we feel also that the extended treatment offers some increased insight into the more subtle implications of quasineutrality.

II. UNDERLYING PRINCIPLES - QUASINEUTRALITY REVISITED

It is widely accepted that an adequate mathematical statement of the physical notion of quasineutrality (applied, for example, to neoclassical transport; see Ref. 4) is simply

$$\vec{\nabla} \cdot \vec{j} = 0 \ , \tag{2.1}$$

$\vec{j}$ being the total electrical current density in the plasma. Eq. (2.1) implies, of course, that the rate of build-up of charge density is negligible or zero. Note that in general one cannot use the rather tempting equation, $\operatorname{div} \vec{E} = 0$. The recognition of this distinction is as old as the theory of MHD itself.[10]

First let us note, from Maxwell's equations alone, that if $\operatorname{div} \vec{E}$ is and remains zero (which implies a certain initial condition for a time-dependent problem), then $\operatorname{div} \vec{j} = 0$ automatically. However, the reverse

is not true. For a simple conducting MHD fluid, for example, if one puts div $\vec{j}$ = 0, one cannot, simultaneously, put div $\vec{E}$ = 0, without implying generally unacceptable constraints on the flow field. [In ideal MHD, for example, $\vec{E}$ = $-\vec{V} \times \vec{B}$ and div ($\vec{V} \times \vec{B}$) $\neq$ 0 in general]. Instead, a simple MHD fluid is effectively "polarized" by its motion through the $\vec{B}$-field.

For a physically more complex plasma the concept of polarization is not normally adequate. Still, we will be able to show in later sections that the general treatment leads to a statement of the "polarizability" of toroidally confined plasmas appropriate to phenomena occurring on a sufficiently long time scale. We will also be able to establish the regions of applicability of the statement div $\vec{j}$ = 0 as an appropriate description of quasineutrality.

In Ref. 4 the relation div $\vec{j}$ = 0 is used to determine those components of $\vec{E}$ actually required in the portion of neoclassical transport theory treated there. Similarly, in Ref. 5, a flux-averaged version of the corresponding statement is used. Throughout the development of neoclassical theory, various methods have been employed to determine those portions of the electric field needed for the particular problem being studied (see, for example, Refs. 11 & 12). Nevertheless, we have yet to discover in the literature a unified treatment adequate to treat all cases that can arise. For example, no formulation previously stated is adequate to study non-linear instabilities. For these reasons we offer in the following sections a systematic method of determining all components of $\vec{E}$ needed in the general toroidal problem. As it turns out, this development is also crucial for the further use of the CPM[8,9].

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III. DETERMINATION OF $\vec{E}$, AND THE QUASINEUTRAL LIMIT

By recognizing that in toroidally confined plasmas most of the electrical current is aligned with the magnetic field, and by employing momentum and particle number conservation, the authors in Refs. 4 and 5 were able to convert the div $\vec{j} = 0$ requirement into the more appealing statement (s here is the coordinate along $\vec{B}$)

$$\frac{\partial \rho}{\partial s} = 0 \ , \tag{3.1}$$

i.e., quasineutrality leads to the expectation that no significant charge variation can develop along the lines of force. To obtain this result from the stricter requirement, one must not only assume that the electrons behave adiabatically,[*] one must also ignore order Ω^{-1} contributions, where Ω is the ion gyrofrequency.

The importance of (3.1), however, lies primarily in its relatively simple physical nature and the corresponding insight thereby offered into the true meaning of quasineutrality. Nevertheless, its use in Refs. 4 and 5 was sufficient, because of its approximate nature, to allow determination of the electric field only to zeroth-order in the drift-kinetic expansion (expansion in powers of Ω^{-1}, made appropriately dimensionless). In addition, it gave no hint of the possible appearance of a high-frequency component of $\vec{E}$. For application to problems solvable via the CPM,[8,9] in which quantities of (at least) order Ω^{-1} are frequently required, Eq. (3.1) provides only a beginning. It has therefore turned out to be necessary for our purposes to re-examine the problem of determining $\vec{E}$ from first

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[*] We use here the classical definition of "adiabatic" behavior of the electrons; viz., that their inherent response time is much shorter than any time of interest in the plasma problem at hand. The low-order electron density perturbations remain almost exactly in phase with any electric potential which may arise.

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principles.

In many toroidal problems, the electric field consists of the externally applied ring field, $\vec{E}_A$, and the plasma self-consistent field $\vec{E}$, which is still to be regarded as the solution of Poisson's equation. Equivalently, the time-derivative equation (Gaussian units)

$$\vec{\nabla} \cdot \frac{\partial^2 \vec{E}}{\partial t^2} = 4\pi \frac{\partial^2 \rho}{\partial t^2} \, , \tag{3.2}$$

coupled with the initial conditions

$$[\vec{\nabla} \cdot \vec{E} = 4\pi\rho]_{t=0} \, ,$$

$$\left[\vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}\right]_{t=0} \, ,$$

and charge continuity,

$$\frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{j} \, , \tag{3.3}$$

can be combined to eliminate ρ from (3.2):

$$\vec{\nabla} \cdot \left(\frac{\partial^2 \vec{E}}{\partial t^2} + 4\pi \frac{\partial \vec{j}}{\partial t}\right) = 0 \, . \tag{3.4}$$

At this point, one uses the momentum equations to eliminate $\partial\vec{j}/\partial t$ from (3.4). (Again, the particle dynamics are essential in determining $\vec{E}$ [1].) The momentum balance equations for each species, multiplied by charge and added together, provide a relation often referred to as the generalized Ohm's Law (see, for example, Refs. 1 and/or 13).

The process is a familiar one; for our purposes we choose to write the general result in the form

$$4\pi \frac{\partial \vec{j}}{\partial t} = \omega_p^2 \ \vec{E} - \vec{G} \tag{3.5}$$

where $\omega_p^2 \equiv 4\pi \sum_s (e_s^2 N_s/m_s)$, and the vector $\vec{G}$ is written out in detail both in Refs. 13 and 14 (using different but obvious notations). Note in particular that $\vec{G}$ generally contains $\sum_s e_s (\vec{J}_s \times \vec{\Omega}_s)$, i.e. the Hall terms, and $\sum_s \int d^3v \ \vec{v} \ C_s$, the collisional momentum transfer terms. Thus, insertion of (3.5) into (3.4) can lead to the emphasis of either effect (as well as many others), depending on the parameter regime in question.

Combining (3.5) immediately with (3.4) seems to provide us straightforwardly with exactly what we have been seeking, namely, an equation sufficient to determine $\vec{\nabla} \cdot \vec{E}$. We see that that equation is indeed time dependent, indicating the occurrence of a rapidly oscillating portion of $\vec{E}$. It also demonstrates explicitly that div $\vec{E} = 0$ is not in general an acceptable solution, even in the quasineutral limit.

However, in a toroidally confined plasma, as emphasized in the discussion above Eq. (3.1), the parallel component of the drift-kinetic version of $\vec{j}$ dominates its perpendicular part [again, the ordering is such that $|\vec{j}_\perp|/|\vec{j}_\parallel| = O(\Omega^{-1})$]. It is important to take advantage of this fact in (3.5) before insertion in (3.4), adopting a procedure consistent with the drift-kinetic ordering,[8,15] standardly used in describing the particle kinematics in toroidal discharges. (See, however, Ref. 16). Introducing this natural ordering in (3.5), before combination with (3.4), is in fact crucial at this point, because the latter does not naturally break up into ‖ and ⊥ parts [see, for example, its origin, Eq. (3.2)].

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Moreover, because of the drift-kinetic expansion, the appropriate form of $\partial \vec{j}/\partial t$, at each order in Ω^{-1}, is somewhat different[14] from that obtained directly from the unexpanded version of the momentum equations. For example, the Hall terms are treated separately in the drift-kinetic limit. The result of using the drift-kinetic expansion from the beginning in (3.5) then leads to

$$\vec{\nabla} \cdot \left(\frac{\partial^2 \vec{E}}{\partial t^2} + \hat{n} \omega_p^2 \, E_{\parallel} \right) = H_0 + H_1 \qquad (3.6)$$

on insertion in (3.4). Here, (see Ref. 14 for details) $E_{\parallel} = \hat{n} \cdot \vec{E}$, with $\hat{n} = \vec{B}/B$, and

$$H_0 = \vec{\nabla} \cdot (\hat{n} G_0) \quad , \qquad H_1 = \vec{\nabla} \cdot \vec{G}_1 \quad . \qquad (3.7)$$

The notation with the 0 and 1 subscripts is used to emphasize the order-zero and order-one terms in a drift-kinetic (Ω^{-1}) expansion. The vectors $\hat{n} G_0$ and $\vec{G}_1$ are obtained from the indicated moments of the drift-kinetic equation, leading to a corresponding modified version of (3.5) in the form

$$4\pi \, \frac{\partial \vec{j}}{\partial t} = \omega_p^2 \, E_{\parallel} \hat{n} - \hat{n} G_0 - \vec{G}_1 \quad . \qquad (3.5a)$$

It is easy to show that

$$\left. \begin{array}{l} G_0 = -\omega_p^2 \, \hat{n} \cdot \vec{E}_A - 4\pi \sum_s e_s g_{0s} \\[20pt] \vec{G}_1 = -4\pi \sum_s e_s \vec{g}_{1s} \end{array} \right\} \qquad (3.8)$$

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where for each species, s,

$$g_0 = \int d^3v \; v \cos \alpha \; \bar{C} - \hat{n} \cdot \vec{\nabla} K_\parallel + (\hat{n} \cdot \vec{b})(K_\parallel - \tfrac{1}{2} K_\perp)$$

with $\vec{b} = (\vec{\nabla} B)/B$. By definition, $\vec{g}_1 \equiv \hat{n}(\partial J_\parallel / \partial t)_1 + \partial \vec{J}_\perp / \partial t$ (and is written out in Ref. 14) and the moments, $\vec{J} = \int d^3v \; \vec{v} \; \bar{f}$, $K_\perp = \int d^3v \; v^2 \sin^2 \alpha \; \bar{f}$, $K_\parallel = \int d^3v \; v^2 \cos^2 \alpha \; \bar{f}$, are obtained from the indicated integrals of the solution, $\bar{f}$, of the drift-kinetic equation[15,8]. We write the latter schematically as

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} + \vec{A} \cdot \vec{\nabla}_v \right) \bar{f} = \bar{C} \quad . \tag{3.9}$$

The details of the drift and effective acceleration vectors, $\vec{V}$, $\vec{A}$, are written out completely (in $v - \alpha$ space) in Ref. 8. $\bar{f}(v,\alpha)$ is the guiding-center distribution (e.g., see Hazeltine, Ref. 15), satisfying the first-order drift kinetic equation. Finally $v = \sqrt{v_\parallel^2 + v_\perp^2}$, $\alpha = \tan^{-1}(v_\perp/v_\parallel)$, for use in $(v - \alpha)$ space.

Once again, it is a standard property of (3.9) that $|J_\perp|/|J_\parallel| = 0(\Omega^{-1})$, a priori. As already stated, this is why one uses (3.5a), rather than (3.5), for problems of this type. In comparing (3.5a) with (3.5), it is helpful to recall that $\vec{j}$, in (3.5a) is actually obtained from the guiding-center-, rather than the particle-, current, and is subject to finite Lamor radius corrections. It is also apparent from (3.6) that any plasma oscillation effects in the perpendicular direction occur as a result of the $\vec{E}$ dependence in H_1. These are therefore reduced in strength by a factor of order Ω^{-1} compared to any parallel plasma oscillations.

Eq. (3.6) displays all the features we attempted to anticipate in our earlier discussion, including, as already mentioned, the fact that $\vec{\nabla} \cdot \vec{E} = 0$ is not (normally) a solution. In addition, the expected natural rapid oscillations in $\vec{E}$ emerge. In the next section, we will sketch the procedure for obtaining (formally) both the "fast" and the less-rapidly-varying-parts of $\vec{E}$. In the process, whenever we focus on the "slow" portion of $\vec{E}$, we shall in effect be discarding (in the lowest approximation) $\partial \rho / \partial t$, and this is tantamount to putting $\vec{\nabla} \cdot \vec{j} = 0$, which is, again, the classical statement of quasineutrality. For general problems,[8,9] however, it is necessary to have the ability to compute $\vec{E}$ with the increased precision allowed by a systematic treatment of (3.6).

IV. DETERMINATION OF $\vec{E}$ IN TOROIDAL DISCHARGES

Recently, Hazeltine and Ware[11], in a calculation of neoclassical equilibrium to first order in a drift-kinetic approximation, obtained the complete time-independent electric field except for the θ-averaged radial component. Hazeltine and Ware emphasized that in that case, to obtain the remaining portion of the equilibrium radial electric field it would be necessary to go to a second order drift-kinetic approximation. This had already been carried out by Rosenbluth et al.[17]

Plasma oscillations were, of course, precluded in the examples quoted above. Further, the authors were not concerned with the question of the extent to which quasineutrality serves as a substitute for Poisson's equation. The results of Ref. 11 imply that the slow part of the electric quantities obtained from (3.6) must in some sense be expandable in powers of ω_p^{-2}. We shall see later that this is true.

We find that we can usefully make an expansion of this type, in

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addition to the more standard drift-kinetic expansion. By carefully noting which components of the electric field play an important role in these two (complementary) expansion schemes, we find that all components of the time-varying electric field can be determined to first order in the drift-kinetic expansion. This could turn out to have wide practical use.

An additional important consequence of following the drift-kinetic expansion is that by imposing toroidal periodicities and by defining a flux surface in which $\vec{E}_\parallel$ lies, we can determine $\vec{E}$ completely on a flux surface (for a low-β plasma), up to an arbitrary constant. The latter constant reflects simply the total charge enclosed within this flux surface. By contrast, using the fully kinetic description [e.g., (3.5) combined with (3.4)], one would be required to impose radial boundary conditions in the usual, more global sense.

We assume a static magnetic field in the present discussion; in this paper the electrostatic case is then used for illustration:

$$\vec{E} = -\vec{\nabla}\phi \; .$$

(4.1)

(Recall that the total electric field is then $-\vec{\nabla}\phi + \vec{E}_A$.) Let ϕ be separated into its drift-kinetic ordered parts:

$$\phi = \phi_0 + \phi_1 + \cdots$$

If we Fourier-analyze ϕ_m and H_m with respect to both position, $\vec{x}$, and time, t, (m refers to the Ω^{-1} ordering) we can define

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$$\phi_m(x,\vec{t}) = \int d^3k \int d\omega \, e^{i(\vec{k}\cdot\vec{x} + \omega t)} \phi_m(\vec{k},\omega) \quad , \qquad (4.2)$$

$$H_m(\vec{x},t) = \int d^3k \int d\omega \, e^{i(\vec{k}\cdot\vec{x} + \omega t)} H_m(\vec{k},\omega) \quad . \qquad (4.3)$$

Application of this procedure, together with (4.1), readily yields from (3.6) the relationship in $(\vec{k},\omega)$ space:

$$(k_{\parallel}^2\omega_p^2 - k^2\omega^2) \, \phi_m(\vec{k},\omega) = H_m(\vec{k},\omega,\phi_{m-1}) \quad . \qquad (4.4)$$

For reference, the appropriately ordered explicit ϕ dependence of each H_m is indicated, with $\phi_{-1} \equiv 0$. In writing (4.4) at this point, fluctuations in time and space of ω_p^2 are neglected, for convenience. In addition, the variation of $B^{-1}\vec{\nabla}B$ along the magnetic field lines is omitted. These approximations at this point do not affect the basic content of the ω_p^{-2} ordering. However, for those later steps where these dependencies are important, we provide an exact treatment (see section V).

The first step beyond Eq. (4.4) is to note that its general solution in $(\vec{k},\omega)$ space is of the form

$$\phi_m(\vec{k},\omega) = P \frac{H_m(k,\omega,\phi_{m-1})}{k^2(\tilde{\omega}_p-\omega)(\tilde{\omega}_p+\omega)} + A(\vec{k},\omega) \, \delta(\tilde{\omega}_p + \omega) + B(\vec{k},\omega) \, \delta(\tilde{\omega}_p - \omega), \qquad (4.5)$$

where $\tilde{\omega}_p \equiv (k_{\parallel}/k)\omega_p$, and P represents the Cauchy principal value in the generalized function sense, and $\delta(x)$ is the usual unit Dirac delta function. $A(\vec{k},\omega)$ and $B(\vec{k},\omega)$ are arbitrary in (4.5), but are immediately determined in this case by application of the two initial conditions stated below Eq. (3.2). Using the latter step, and applying the standard $\omega - t$ inversion

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of (4.2) and (4.3), one obtains for $\phi_m(\vec{k},t)$:

$$k^2 \phi_m(\vec{k},t) = P \int d\omega \, \frac{H_m(\vec{k},\omega,\phi_{m-1}) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2}$$

$$+ \left[4\pi \rho_m(\vec{k},0) - P \int d\omega \, \frac{H_m(\vec{k},\omega,\phi_{m-1})}{\tilde{\omega}_p^2 - \omega^2} \right] \cos \tilde{\omega}_p t \qquad (4.6)$$

$$+ \frac{1}{\tilde{\omega}_p} \left[4\pi \dot{\rho}_m(\vec{k},0) - P \int d\omega \, \frac{i\omega H_m(\vec{k},\omega,\phi_{m-1})}{\tilde{\omega}_p^2 - \omega^2} \right] \sin \tilde{\omega}_p t$$

where $(\dot{\ }) \equiv \frac{\partial (\ )}{\partial t}$. Note that in practically all cases of interest in

toroidal problems $\tilde{\omega}_p$, as well as ω_p, is very large compared to Ω_i, because

there is a minimum natural value of $k_{\parallel}$. (The exceptional case, namely,

the flux average of (3.6) with (4.1), which corresponds to the $k_{\parallel} = 0$

component of the above, is discussed separately in section V.)

The properties of the transforms $H_m(\vec{k},t)$ are, of course, best

understood through their more physical counterparts, $H_m(\vec{x},t)$. However,

it is through the transforms that we are able most easily to develop

the formal expansion of (4.6) necesary to separate ϕ explicitly into

its rapidly-oscillating and its slowly-varying (or even quasi-steady)

parts. For example, the function $H_0(\vec{x},t)$ is determined principally by

a balance of the ring field, collision, and pressure terms, and it therefore

will vary on a time scale that is much longer than $\tilde{\omega}_p^{-1}$. $H_0(\vec{k},\omega)$ will

consequently die off rapidly at frequencies near $\tilde{\omega}_p$ and above. Therefore,

we can obtain from (4.6) the low frequency (zeroth order) portion of the

potential straightforwardly in the form

$$k^2 \, \bar{\phi}_0(\vec{k},t) = 4\pi \, \bar{\rho}_0(\vec{k},t) = P \int d\omega \, \frac{H_0(\vec{k},\omega) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2} \quad , \qquad (4.7)$$

where $\bar{\phi}_0$ and $\bar{\rho}_0$ are in effect time averages over several modified plasma periods. Except for its dependence on the high frequency part of ϕ_0, H_1 is also a low frequency function, and therefore the low frequency, first-order-correction potential is given by

$$k^2 \, \bar{\phi}_1(\vec{k},t) = 4\pi \, \bar{\rho}_1(\vec{k},t) = P \int d\omega \, \frac{H_1(\vec{k},\omega,\bar{\phi}_0) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2} \quad . \qquad (4.8)$$

A formal asymptotic form exists for this expansion of $\phi(\vec{k},t)$. Because each of the $H_m(\vec{x},t)$ vary slowly in time, the numerators of the integrands will tend to become vanishingly small unless $|\omega/\tilde{\omega}_p| \ll 1$. Thus,

$$P \int d\omega \, \frac{H_m(\vec{k},\omega) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2} = P \int d\omega \left[\sum_{j=0}^{N} \frac{\omega^{2j}}{\tilde{\omega}_p^{2j+2}} + \frac{\omega^{2N}}{\tilde{\omega}_p^{2N+2}(\tilde{\omega}_p^2 - \omega^2)} \right] H_m(\vec{k},\omega) e^{i\omega t}$$

$$(4.9)$$

$$= \sum_{j=0}^{N} \frac{(-1)^j}{\tilde{\omega}_p^{2j+2}} \frac{\partial^{2j}}{\partial t^{2j}} H_m(\vec{k},t) + R_N(\vec{k},t),$$

where the remainder R_N will be negligibly small, provided N is not too large. Finally,

$$k^2 \bar{\phi}_m(\vec{k},t) = 4\pi \, \bar{\rho}_m(\vec{k},t) = \sum_{j=0}^{N} \frac{(-1)^j}{\tilde{\omega}_p^{2j+2}} \frac{\partial^{2j}}{\partial t^{2j}} H_m(\vec{k},t) \quad . \qquad (4.10)$$

Differentiating with respect to time, we readily obtain the (transformed) conservation equation

$$i\vec{k} \cdot \overline{\vec{j}}_m(\vec{k},t) = -\dot{\rho}_m(\vec{k},t) = -\frac{1}{4\pi} \sum_{j=0}^{N} \frac{(-1)^j}{\tilde{\omega}_p^{2j+2}} \frac{\partial^{2j+1}}{\partial t^{2j+1}} H_m(\vec{k},t) \quad . \tag{4.11}$$

Note that, even in this rather formal expansion, the classical statement of quasineutrality ($i\vec{k} \cdot \vec{j} \cong 0$) tends to emerge in lowest order (in the two expansions taken together):

$$i\vec{k} \cdot \overline{\vec{j}}_0 = -\frac{1}{4\pi} \frac{1}{\tilde{\omega}_p^2} \frac{\partial H_0}{\partial t} (\vec{k},t) + \ldots \quad . \tag{4.12}$$

With $\tilde{\omega}_p^{-2}$ typically very small, and with $H_0(\vec{k},t)$ also slowly varying, $i\vec{k} \cdot \vec{j}_0$ (div $\vec{j}_0$ in real space) tends to zero as anticipated. However, (4.11) provides a systematic procedure for improving this approximation, whenever needed. Perhaps even more usefully, the formulas determining $H_0(\underline{x},t)$ [Eq. (3.7) and below] reveal its fundamentally simple physical nature.

If we turn our attention to Poisson's equation, in the form given in (4.10), we see that $\vec{E}$ is determined to order (N+1) in an ω_p^{-2} expansion. Writing out, for example, the leading term in (4.11) for m = 0, we find as illustration

$$k^2 \overline{\phi}_0(\vec{k},t) = \frac{H_0(\vec{k},t)}{\tilde{\omega}_p^2} \quad . \tag{4.13}$$

Inspection of (3.5a) then shows the corresponding current density (given appropriate initial conditions) to be determined only to zeroth-order in $\tilde{\omega}_p^{-2}$. These relations provide a useful illustration of how an electric field arises in a "neutral" plasma. (In this sense, the plasma is indeed polarized.)

Note further that $H_0(\vec{k},t)$ (as well as each H_m) itself depends on $\vec{E}$ or ϕ. In this way (4.13) (or higher-order versions) is capable of generating all of the appropriate dispersion relations expected for the familiar low-frequency modes common, for example, to toroidal discharges. In addition, in the quasi-steady (equilibrium) limit, (4.13) provides the lowest-order information regarding $\vec{E}$, including the familiar ambipolarity statement occuring in that case.

For the application considered in the next section, we take $N = 0$: it follows from (4.10), with (4.1) and (3.7), that the low frequency portion of the electrostatic field must satisfy the equations

$$\vec{\nabla} \cdot \left[\hat{n}(\omega_p^2 \overline{E}_{\parallel,0} - G_0) \right] = 0, \tag{4.14}$$

$$\vec{\nabla} \cdot \left[\hat{n} \, \omega_p^2 \overline{E}_{\parallel,1} - \vec{G}_1 \right] = 0. \tag{4.15}$$

We can calculate $\vec{G}_1$ in (4.15) with $\overline{E}_0 = 0$ because $\overline{E}_0$ would contribute a term of second order in ω_p^{-2} to $\overline{E}_{\parallel,1}$. With $N = 0$ such higher order terms are to be neglected. Equations (4.14) and (4.15) can be combined to give

$$\vec{\nabla} \cdot \left[\hat{n}(\omega_p^2 \overline{E}_{\parallel} - G_0) - \vec{G}_1 \right] = 0. \tag{4.16}$$

Comparing this with (3.6) and (3.7), we see that we can obtain the ($N = 0$) low frequency field simply by neglecting $\partial^2 \vec{E}/\partial t^2$ in (3.6). However, as we shall show, a breakdown of (4.16) occurs for its flux-surface average. This problem is considered in section V, in the framework of toroidal geometry with a specified, static model $\vec{B}$ field.

V. FLUX-AVERAGED FIELD-COMPONENTS; TOROIDAL GEOMETRY

The $k_{\parallel} = 0$ (flux-averaged) part of (4.16), referred to at the end of section IV, reduces simply to

$$<\vec{\nabla} \cdot \vec{G}_1> = 0. \tag{5.1}$$

However, from the definition of $\vec{G}_1$ [see (3.8) and Ref. 14], we find (5.1) cannot be satisfied in general. We therefore return to the exact (time-dependent) equation (3.6) for the flux-surface average, obtaining the more complete low-frequency relation for the perpendicular part:

$$<\vec{\nabla} \cdot \frac{\partial^2 \vec{E}_{\perp}}{\partial t^2} - \vec{G}_1 > = 0 . \tag{5.2}$$

where $\vec{E}_{\perp} = \vec{E} - \hat{n}E_{\parallel}$. Equations (4.16) and (5.2) together can now be used for the complete solution of the electrostatic field, including low-frequency effects arising from $\perp$ components which contribute to the flux average. An explicit solution (for a toroidal geometry) is developed in the following. The extension of (4.16) and (5.2) to higher orders in Ω^{-1} is straightforward.

We model the static magnetic field by the vacuum toroidal field and the plasma current poloidal field, taking care to keep $\vec{\nabla} \cdot \vec{B} = 0$. In the usual toroidal geometry (R_0 = major radius, r = minor radius, θ = poloidal angle, ζ = toroidal angle), the model field is given by

$$\vec{B} = \frac{\hat{n}B_0}{\eta h} = \hat{n} \, B, \tag{5.3}$$

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where

$$h = 1 + \varepsilon \cos \theta \ ,$$

$$\hat{n} = \eta\sigma\hat{\theta} + \eta\hat{\zeta} \ ,$$

$$\eta = (1 + \sigma^2)^{-1/2} \ ,$$

$$\sigma = \frac{\varepsilon}{q(r)} \ , \quad \varepsilon = \frac{r}{R_0} \ .$$

Here, q is the safety factor.

In this geometry the improved low-frequency equation (5.2) reduces to

$$\frac{\partial}{\partial r} \left[r < \hat{r} \cdot \left(\frac{\partial^2 \vec{E}}{\partial t^2} - \vec{G}_1 \right) > \right] = 0 \ , \tag{5.4}$$

which relates $<E_r>$ to $<j_r>$ and properly generalizes Eq. (5.1).

This is also consistent with using an improved verson of (4.16):

$$\vec{\nabla} \cdot \left[\hat{n}(\omega_p^2 \overline{E}_\parallel - G_\parallel) - \vec{G}_\perp \right] = 0 \ , \tag{5.5}$$

where

$$G_\parallel = G_0 + \hat{n} \cdot \vec{G}_1 \ ,$$

$$\vec{G}_\perp = \vec{G}_1 - \hat{n}(\hat{n} \cdot \vec{G}_1) - \hat{r} <\hat{r} \cdot \vec{G}_1> \ .$$

We may write (5.4) and (5.5) in terms of $\overline{\phi}$ by using (4.1) with (5.3), yielding

$$\frac{\partial}{\partial r} \left[r < \frac{\partial^2}{\partial t^2} \frac{\partial \overline{\phi}}{\partial r} + \hat{r} \cdot \vec{G}_1 > \right] = 0 \ , \tag{5.6}$$

and, in the model field, (5.5) becomes

$$\frac{\partial^2 \bar{\phi}}{\partial \theta^2} + \left(\frac{2q}{1 + \varepsilon \cos \theta}\right)\frac{\partial^2 \bar{\phi}}{\partial \theta \partial \zeta} - \frac{\varepsilon \sin \theta}{1 + \varepsilon \cos \theta} \frac{\partial \bar{\phi}}{\partial \theta} + \left[\frac{q^2}{(1 + \varepsilon \cos \theta)^2}\right]\frac{\partial^2 \bar{\phi}}{\partial \zeta^2}$$

$$= -\frac{1}{<\omega_p^2>} \left(\frac{qR_0}{\eta}\right)^2 \vec{\nabla} \cdot (\hat{n}G_\parallel + \vec{G}_\perp) , \tag{5.7}$$

provided we continue to neglect fluctuations about the flux-average of ω_p^2. Because $\bar{\phi}$ must satisfy toroidal periodicity, we can expand it in a Fourier series,

$$\bar{\phi} = \sum_{\mu\lambda} \bar{\phi}_{\mu\lambda} \exp [i(\mu\theta - \lambda\zeta)] , \tag{5.8}$$

and solve (5.7) for all $\bar{\phi}_{\mu\lambda}$. Geometrical effects partially neglected in Sec. IV are now included in Eqs. (5.7) and (5.8). Furthermore, an exact form of (5.5), including any fluctuations in ω_p^2, can be used to write (5.7) in a more complete form if necessary or desired. One proceeds by solving (5.5) first for $h\omega_p^2 E_\parallel \equiv \chi$. One can then solve $\hat{n} \cdot \vec{\nabla}\phi = -(h\omega_p^2)^{-1}\chi$ to obtain completely general form for (5.7).

The equation for $\bar{\phi}$ can be used to determine all the $\bar{\phi}_{\mu\lambda}$ except $\bar{\phi}_{00}$. For most practical purposes, however, we need only $\partial\phi_{00}/\partial r$; the latter can be obtained by integrating the equation following from (5.6) and (5.8)

$$\frac{\partial^2}{\partial t^2} \left[\frac{\partial\bar{\phi}_{00}}{\partial r} + \frac{1}{2} \varepsilon \frac{\partial(\bar{\phi}_{10} + \bar{\phi}_{-10})}{\partial r}\right] = - <\hat{r} \cdot \vec{G}_\perp>. \tag{5.9}$$

Equations (5.7) and (5.9) allow complete determination of the first order (in both Ω^{-1} and ω_p^{-2}) time-varying electrostatic electric fields. The results are valid for any plasma distribution function and do not imply

some of the restrictions cited from the earlier literature. We also have

shown how one obtains the higher order terms in the ω_p^{-2} expansion [Eq. (4.10)].

In the zeroth-order limit in the Ω^{-1} expansion, the electric field reduces

to the form used by Hazeltine and Hinton.[4,5]

By contrast, if one assumes a priori an equilibrium $(\partial\bar{\phi}/\partial t) = 0$,

(5.9) is not useful in determining $\partial\bar{\phi}_{00}/\partial r$, since $\hat{r} \cdot \vec{G}_1$ does not depend

on E_r. Under such an equilibrium constraint it is then indeed necessary[11]

to extend the drift-kinetic expansion to second order[17]. In this limit,

our results agree with those of Ref. 11.

Finally, we note that Eq. (5.9) for $\partial\phi_{00}/\partial r$ can be integrated once

in time because of the simple definition of $\vec{G}_1$ [see Eq. (3.5a) and below].

We obtain readily

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\bar{\phi}_{00} + \frac{1}{2}\varepsilon\left(\frac{\partial(\bar{\phi}_{10} + \bar{\phi}_{-10})}{\partial r}\right)\right] = -4\pi <j_r> + const. \qquad (5.10)$$

where $<j_r> \equiv <\hat{r} \cdot \vec{j}> = <\hat{r} \cdot (\sum_s e_s \vec{J}_s)>$. This result is potentially of

special interest in toroidal discharges because it is related to any

spin-up or rotation[18-22] which may occur in such devices. [In a static

$\vec{B}$-field, determination of the flux-averaged radial component of $\vec{E}$ is

essentially tantamount to the determination of the mean poloidal rotation.]

Equation (5.10) is consistent with known results in the Pfirsch-

Schlüter regime for the spin-up rate.[18,21,22] Moreover, it appears to be

useful as a unifying expression, valid in all regimes. It illustrates

further that one can expect, as $<j_r>$ varies, important differences in

rotational tendencies of the plasma when one compares motions near the

center and toward the edges of the plasma.

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