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## Determinantal Measures Related to Big $q$ -Jacobi polynomials\*

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**ABSTRACT.** We define a novel combinatorial object—the extended Gelfand–Tsetlin graph with cotransition probabilities depending on a parameter  $q$ . The boundary of this graph admits an explicit description. We introduce a family of probability measures on the boundary and describe their correlation functions. These measures are a  $q$ -analogue of the spectral measures studied earlier in the context of the problem of harmonic analysis on the infinite-dimensional unitary group.

**KEY WORDS:** Gelfand–Tsetlin graph, determinantal measures, big  $q$ -Jacobi polynomials, basic hypergeometric series.

Determinantal measures form a special class of probability measures on spaces of locally finite point configurations (see Borodin’s survey [1]). The key property of a determinantal measure is that its correlation functions of any order are expressed in a simple way through a function in two variables, called the correlation kernel.

A family of determinantal measures, called the *zw-measures*, was studied by Borodin and Olshanski [2] in connection with the problem of harmonic analysis on the infinite-dimensional unitary group  $U(\infty)$ , posed by Olshanski [12]. Analogues of the *zw-measures* also exist for other infinite-dimensional classical groups and for infinite-dimensional symmetric spaces (see Olshanski and Osinenko’s paper [14]). The *zw-measures* play a fundamental role in infinite-dimensional harmonic analysis, because their scaling limits govern the spectral decomposition of certain distinguished unitary representations.

On the other hand, the *zw-measures* are nice combinatorial objects and have much in common with the so-called *z-measures*, which are a particular case of Okounkov’s Schur measures on partitions. Like Schur measures, the *zw-measures* admit a generalization involving an additional deformation parameter similar to Dyson’s  $\beta$ -parameter in random matrix theory or the continuous parameter of the Jack symmetric functions (see Olshanski’s paper [13]).

Our aim is to show that the notion of *zw-measures* can be extended in another direction; namely, there exists a (nonevident)  $q$ -analogue of the *zw-measures*. Our first result says that the “ $N$ -particle  $q$ -*zw-measures*” have a large- $N$  limit, and the limiting probability measure is a determinantal measure on a space of infinite point configurations. We find the corresponding correlation kernel—it is expressed through the basic hypergeometric function  ${}_2\phi_1$ . The large- $N$  limit transition is related to another result, which is of independent interest—a description of the  $q$ -boundary for an extended version of the Gelfand–Tsetlin graph.

**1. The  $q$ -analogue of *zw-measures*.** We fix a triple  $(\zeta_+, \zeta_-, q)$  of real parameters, where  $\zeta_+ > 0$ ,  $\zeta_- < 0$ , and  $0 < q < 1$ . The corresponding *double  $q$ -lattice* is a subset  $\mathfrak{L} \subset \mathbb{R} \setminus \{0\}$  of the form

$$\mathfrak{L} = \mathfrak{L}_- \cup \mathfrak{L}_+, \quad \mathfrak{L}_\pm := \{\zeta_\pm q^n : n \in \mathbb{Z}\}.$$

By a *configuration* we mean an arbitrary subset  $X \subset \mathfrak{L}$ . For  $N = 1, 2, \dots$ , let  $\mathbb{G}_N$  denote the countable set consisting of all  $N$ -point configurations. We enumerate the points of every  $X \in \mathbb{G}_N$  in increasing order and write  $X = (x_1 < \dots < x_N)$ .

We are going to introduce, for every  $N = 1, 2, \dots$ , a probability measure  $M_N^{\alpha, \beta, \gamma, \delta}$  on  $\mathbb{G}_N$ . It depends on a quadruple  $(\alpha, \beta, \gamma, \delta)$  of parameters subject to constraints specified below and is given

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by

$$M_N^{\alpha,\beta,\gamma,\delta}(X) = \frac{1}{\mathcal{Z}_N(\alpha,\beta,\gamma,\delta)} \prod_{i=1}^N |x_i| \frac{(\alpha x_i; q)_\infty (\beta x_i; q)_\infty}{(\gamma q^{1-N} x_i; q)_\infty (\delta q^{1-N} x_i; q)_\infty} \prod_{1 \leq i < j \leq N} (x_j - x_i)^2, \quad (1)$$

where  $\mathcal{Z}_N(\alpha,\beta,\gamma,\delta)$  is a normalization constant and the standard notation  $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$  of  $q$ -analysis is used.

The constraints on the parameters are as follows. We assume that the pair  $(\alpha, \beta)$  is  $\mathfrak{L}$ -admissible in the sense that the product  $(\alpha x; q)_\infty (\beta x; q)_\infty$  is real and strictly positive for every  $x \in \mathfrak{L}$ ; this means that either  $\alpha = \bar{\beta} \in \mathbb{C} \setminus \mathbb{R}$  or both  $\alpha$  and  $\beta$  are nonzero reals such that  $\alpha^{-1}$  and  $\beta^{-1}$  lie in an open interval between two neighboring nodes of  $\mathfrak{L}$ . Next, we assume that  $(\gamma, \delta)$  is  $\mathfrak{L}$ -admissible, too. Finally, we assume that  $\gamma\delta q^2 > \alpha\beta$ , but we believe that all results presented below hold under the weaker condition  $\gamma\delta q > \alpha\beta$ .

Under all these assumptions, the measure on  $\mathbb{G}_N$  defined by (1) but without the constant factor has strictly positive weights and the sum of all weights is finite. Then the normalization constant is equal to the sum of weights. For this constant, there exists an explicit but rather complicated expression.

**Remark 1.** Our definition of  $M_N^{\alpha,\beta,\gamma,\delta}$  fits into the general formalism of *orthogonal polynomial ensembles*; see Borodin's paper [1] and König's paper [8]. In our case, the orthogonal polynomials that we need (let us denote them by  $P_n(x; a, b, c, d)$ ,  $n = 0, 1, 2, \dots$ ) are close relatives of the classical big  $q$ -Jacobi polynomials. Koornwinder [10] calls them *pseudo big  $q$ -Jacobi polynomials*. The weight function of the polynomials  $P_n(x; a, b, c, d)$  has the form

$$w(x; a, b, c, d) = |x| \frac{(ax; q)_\infty (bx; q)_\infty}{(cx; q)_\infty (dx; q)_\infty}, \quad x \in \mathfrak{L},$$

and for the  $N$ th measure, we choose the parameters  $(a, b, c, d) = (\alpha, \beta, \gamma q^{1-N}, \delta q^{1-N})$ .

**Remark 2.** The papers [12] by Olshanski and [2] by Borodin and Olshanski deal with the “zw-measures” which live on the levels  $\mathbb{GT}_N$  of the Gelfand–Tsetlin graph. The zw-measures depend on a quadruple of parameters  $(z, z', w, w')$  and are related to certain discrete orthogonal polynomials on the ordinary lattice  $\mathbb{Z}$ , termed the *Askey–Lesky polynomials* in [2]. In a suitable limit regime as  $q \nearrow 1$ , the polynomials  $P_n(x; a, b, c, d)$  degenerate into the Askey–Lesky polynomials and the measures  $M_N^{\alpha,\beta,\gamma,\delta}$  degenerate into zw-measures. This is one of the reasons to regard the measures  $M_N^{\alpha,\beta,\gamma,\delta}$  as a  $q$ -analogue of the zw-measures.

More precisely, in the limit transition we take

$$\alpha = q^{w+1}, \quad \beta = q^{w'+1}, \quad \gamma = q^{-z}, \quad \delta = q^{-z'}$$

and assume that  $\zeta_+ = 1$ , while  $\zeta_- = \zeta_-(q)$  may vary arbitrarily with the only condition that it remains bounded away from  $-\infty$ . Note that in this limit the weight function  $w(x; a, b, c, d)$  (after a suitable renormalization) disappears on  $\mathfrak{L}_-$ , so that we are left with the positive part of  $\mathfrak{L}$ , which can be identified with  $\mathbb{Z}$ .

**2. Large- $N$  limit transition.** Let  $\bar{\mathbb{G}}_\infty$  denote the set of all configurations on  $\mathfrak{L}$  that are bounded as subsets of  $\mathbb{R}$ , and let  $\mathbb{G}_\infty \subset \bar{\mathbb{G}}_\infty$  consist of countable bounded configurations. Evidently,

$$\bar{\mathbb{G}}_\infty = \{\emptyset\} \cup \mathbb{G}_1 \cup \mathbb{G}_2 \cup \dots \cup \mathbb{G}_\infty.$$

Given  $\varepsilon > 0$ , we say that two configurations in  $\bar{\mathbb{G}}_\infty$  are  $\varepsilon$ -close if they coincide outside the interval  $(-\varepsilon, \varepsilon)$ . This makes  $\bar{\mathbb{G}}_\infty$  a uniform space and hence a topological space, and in fact a locally compact topological space.

**Theorem 1.** Fix a quadruple  $(\alpha, \beta, \gamma, \delta)$  of parameters subject to the constraints stated above. The measures  $M_N^{\alpha,\beta,\gamma,\delta}$ ,  $N = 1, 2, \dots$ , treated as probability measures on  $\bar{\mathbb{G}}_\infty$  weakly converge as  $N \rightarrow \infty$  to a probability measure  $M^{\alpha,\beta,\gamma,\delta}$  on  $\mathbb{G}_\infty$ .

**Theorem 2.** *The limit measure  $M^{\alpha,\beta,\gamma,\delta}$  is a determinantal measure with correlation kernel  $K^{\alpha,\beta,\gamma,\delta}$  expressed through the basic hypergeometric series  ${}_2\phi_1$  as*

$$K^{\alpha,\beta,\gamma,\delta}(x, y) = \text{const}(\alpha, \beta, \gamma, \delta) \cdot \frac{F_0(x)F_1(y) - F_1(x)F_0(y)}{x - y}, \quad x, y \in \mathfrak{L},$$

where, for  $m = 0, 1$ ,

$$F_m(x) = \sqrt{|x| \frac{(\alpha x; q)_\infty (\beta x; q)_\infty}{(\gamma x; q)_\infty (q/(\gamma x; q)_\infty (\delta x; q)_\infty (q/(\delta x; q)_\infty)} \cdot x^{1-m}} \\ \times \frac{(\beta\gamma^{-1}q^{m-1}; q)_\infty (\delta^{-1}x^{-1}q^m; q)_\infty}{(\alpha\beta\gamma^{-1}\delta^{-1}q^{2m-2}; q)_\infty} \cdot {}_2\phi_1 \left( \begin{matrix} \alpha\delta^{-1}q^{m-1}, \beta^{-1}x^{-1}q \\ \delta^{-1}x^{-1}q^m \end{matrix} \middle| \beta\gamma^{-1}q^{m-1} \right),$$

and the constant factor also admits a closed expression.

Theorem 1 is derived from Theorems 3 and 4. Theorem 2 is obtained by a direct (but tedious) computation using the known explicit formula for the squared norm of the polynomials  $P_n(x; a, b, c, d)$ .

**3. The boundary and the coherency property.** We define the interval  $I(a, b)$  between two points  $a < b$  of  $\mathfrak{L}$  as  $(a, b]$  if  $0 < a < b$ ;  $[a, b]$  if  $a < 0 < b$ ; and  $[a, b)$  if  $a < b < 0$ . Then we say that two configurations  $X = (x_1 < \dots < x_{N+1}) \in \mathbb{G}_{N+1}$  and  $Y = (y_1 < \dots < y_N) \in \mathbb{G}_N$  *interlace* if  $y_i \in I(x_i, x_{i+1})$  for  $i = 1, \dots, N$ .

For every  $N = 1, 2, \dots$ , we introduce a matrix  $\Lambda_N^{N+1}$  of format  $\mathbb{G}_{N+1} \times \mathbb{G}_N$ : its nonzero entries correspond to interlacing pairs  $(X, Y)$  and are given by

$$\Lambda_N^{N+1}(X, Y) = \prod_{i=1}^N |y_i| \cdot \prod_{i=1}^N (1 - q^i) \cdot \frac{\prod_{1 \leq i < j \leq N} (y_j - y_i)}{\prod_{1 \leq i < j \leq N+1} (x_j - x_i)}.$$

When both  $X$  and  $Y$  lie in  $\mathfrak{L}_+$ , this definition essentially reduces to the definition of cotransition probabilities in Gorin's paper [4, Sec. 4]. We prove that the matrices  $\Lambda_N^{N+1}$  are stochastic, which makes it possible to apply Borodin and Olshanski's version ([2], [3]) of the Vershik–Kerov formalism and define the (minimal) *boundary* of the chain  $\{\mathbb{G}_N, \Lambda_N^{N+1}\}$ . The *approximation* of a boundary point by a sequence  $X(N) \in \mathbb{G}_N$  is understood in the sense of Okounkov and Olshanski [11, Theorem 6.1].

**Theorem 3.** *The boundary of the chain  $\{\mathbb{G}_N, \Lambda_N^{N+1}\}$  can be identified with  $\mathbb{G}_\infty$ . Under this identification, a sequence  $\{X(N) \in \mathbb{G}_N\}$  approximates a boundary point  $X \in \mathbb{G}_\infty$  if and only if  $X(N) \rightarrow X$  in the topology of the ambient space  $\tilde{\mathbb{G}}_\infty$ .*

This result generalizes (a part of) Gorin's Theorem 5.1 in [4], but the proof is different.

**Theorem 4.** *The measures  $M_N^{\alpha,\beta,\gamma,\delta}$ ,  $N = 1, 2, \dots$ , form a coherent system in the sense that they satisfy the coherency relation*

$$M_{N+1}^{\alpha,\beta,\gamma,\delta} \Lambda_N^{N+1} = M_N^{\alpha,\beta,\gamma,\delta},$$

or, in more detail,

$$\sum_{X \in \mathbb{G}_{N+1}} M_{N+1}^{\alpha,\beta,\gamma,\delta}(X) \Lambda_N^{N+1}(X, Y) = M_N^{\alpha,\beta,\gamma,\delta}(Y) \quad \forall Y \in \mathbb{G}_N.$$

According to a general theory, Theorems 3 and 4 imply the existence of a probability measure  $M^{\alpha,\beta,\gamma,\delta}$  on the boundary  $\mathbb{G}_\infty$ , and then we use additional arguments to show that  $M^{\alpha,\beta,\gamma,\delta}$  is the weak limit of the measures  $M_N^{\alpha,\beta,\gamma,\delta}$ ; this leads to Theorem 1.

Theorem 4 is a  $q$ -analogue of Proposition 7.7 in Olshanski's paper [12], which establishes the coherency property for the  $zw$ -measures. The present paper stemmed from the desire to find such an analogue. At first we thought that this can be achieved in the framework of Gorin's paper [4], where a  $q$ -version of the notion of coherent systems on the levels of the Gelfand–Tsetlin graph was

suggested. But it turned out that, for our purposes, the formalism of [4] is insufficient and the very notion of the Gelfand–Tsetlin graph has to be extended.

In a different representation-theoretic context, the polynomials  $P_n(x; a, b, c, d)$  earlier appeared in a series of works by Groenevelt and Koelink (see, e.g., [5]–[7] and the references therein). We thank Yuri Neretin for bringing our attention to these works; their ideas helped us very much. We are also grateful to Erik Koelink for valuable comments.

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