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The q-PushASEP: A New Integrable Model for Traffic in $1+1$ Dimension

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Abstract We introduce a new interacting (stochastic) particle system q-PushASEP which interpolates between the q-TASEP of Borodin and Corwin (Probab Theory Relat Fields 158(1–2):225–400, 2014; see also Borodin et al., Ann Probab 42(6):2314–2382, 2014; Borodin and Corwin, Int Math Res Not 2:499–537, 2015; O'Connell and Pei, Electron J Probab 18(95):1–25, 2013; Borodin et al., Comput Math, 2013) and the q-PushTASEP introduced recently (Borodin and Petrov, Adv Math, 2013). In the q-PushASEP, particles can jump to the left or to the right, and there is a certain partially asymmetric pushing mechanism present. This particle system has a nice interpretation as a model of traffic on a one-lane highway. Using the quantum many body system approach, we explicitly compute the expectations of a large family of observables for this system in terms of nested contour integrals. We also discuss relevant Fredholm determinantal formulas for the distribution of the location of each particle, and connections of the model with a certain two-sided version of Macdonald processes and with the semi-discrete stochastic heat equation.

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1 Introduction and Main Results

1.1 Definition of the Process

The N-particle *q-PushASEP* (q-deformed pushing asymmetric simple exclusion process) is a continuous-time interacting particle system with the state space consisting of ordered configurations $x_1 > x_2 > \cdots > x_N$, $x_i \in \mathbb{Z}$ (we assume that $N \geq 1$ is fixed). For convenience, we add two "virtual" particles $x_0 = +\infty$ and $x_{N+1} = -\infty$, and denote the state of the system as

$$X^N := \Big\{ \mathbf{x} = (-\infty = x_{N+1} < x_N < \cdots < x_2 < x_1 < x_0 = +\infty) \colon x_1, \ldots, x_N \in \mathbb{Z} \Big\}. \tag{1.1}$$

Let us also denote by $\mathrm{gap}_i := x_{i-1} - x_i - 1$ the ith gap between the particles. Throughout the paper, q is a parameter belonging to $(0, 1)$.

The dynamics of q-PushASEP $\{\mathbf{x}(t)\}_{t\geq 0}$ depend on positive parameters $a_1, \ldots, a_N$ and also on $\mathsf{R}, \mathsf{L} \geq 0$ such that R and L are not simultaneously zero. It is described as follows (see Fig. 1):

- (*right jumps*) Each particle $x_i(t)$, $1 \leq i \leq N$, jumps to the right by one (i.e., instantaneously moves to position $x_i(t) + 1$) at rate $a_i \mathsf{R}\big(1 - q^{\mathrm{gap}_i(t)}\big)$, independently of other particles. The jump rate of $x_i(t)$ vanishes if $\mathrm{gap}_i(t) = 0$, which means that a particle cannot jump onto a site which is already occupied (this is the *exclusion mechanism*).
- (*left jumps*) Each particle $x_i(t)$, $1 \leq i \leq N$, jumps to the left by one (i.e., moves to position $x_i(t) - 1$) at rate $a_i^{-1}\mathsf{L}$, independently of other particles. There is also a *mechanism of instantaneous pushes* present in left jumps. Namely, if any particle $x_j(t)$ has moved to the left, i.e., if $x_j(t + dt) = x_j(t) - 1$, then $x_j(t)$ has a chance to instantaneously (long-range) push its left neighbor $x_{j+1}(t)$ to the left by one with probability $q^{\mathrm{gap}_{j+1}(t)}$. If particle $x_{j+1}(t)$ is pushed then it also has the possibility to push its own left neighbor $x_{j+2}(t)$, and so on. When $\mathrm{gap}_{j+1}(t) = 0$, the probability of a push becomes one, which means that a particle moving to the left always pushes a (possibly empty) cluster of its *immediate* left neighbors.

Clearly, the q-PushASEP preserves the order of particles, so we will always speak about the dynamics of labeled particles $x_1(t) > \cdots > x_N(t)$. We assume that the q-PushASEP $\mathbf{x}(t)$ starts from the *step initial condition* defined as $x_i(0) = -i$, $i = 1, \ldots, N$.

It is worth noting that the first particle $x_1(t)$ performs a very simple dynamics: it jumps to the right or to the left by one (independently of other particles) at rates $\mathsf{R}a_1$ and $\mathsf{L}a_1^{-1}$,

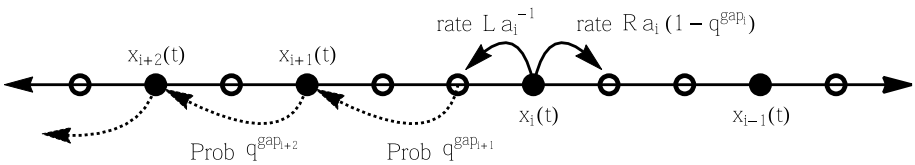

Fig. 1 Jump rates and probabilities of pushes in q-PushASEP

respectively. Likewise, the first $n < N$ particles $x_1(t), \dots, x_n(t)$ evolve without any dependence on those particles $x_{n+1}(t), \dots, x_N(t)$ to their left. Therefore, even though we have restricted our attention to an N-particle system, we may also think of this as the evolution of the N rightmost particles in a system with more than N particles.

1.2 Traffic Model

The q-PushASEP may be viewed as a model of traffic on a one-lane highway in the following sense. Let $v \gg 0$, and set $c_j(t) := vt + x_j(t)$, $j = 1, \dots, N$, where $x_1(t) > \cdots > x_N(t)$ evolve according to the q-PushASEP. Viewing $c_1(t) > \cdots > c_N(t)$ as positions of cars on the highway (i.e., we understand their positions relative to a moving reference frame), one can interpret the dynamics as follows.

The jump of a particle x_j to the right by one (under the q-PushASEP) may be viewed as a brief acceleration of the car c_j, after which c_j becomes closer to c_{j-1}, and after that continues to maintain the constant global speed v. Chances that c_j will briefly accelerate are lower if the car c_{j-1} is already close ahead because of the rate $a_j \mathsf{R}(1-q^{\mathrm{gap}_j})$ of right jumps.

The left jump of x_j may be interpreted as a brief slowdown of the jth car, after which it continues to maintain the constant global speed v. When such a slowdown happens, the car c_{j+1} behind c_j sees the brake lights of c_j, and may also quickly slow down. The probability of the latter event is higher when c_{j+1} is closer to c_j because of the pushing probability $q^{\mathrm{gap}_{j+1}}$ in the q-PushASEP. If c_{j+1} decides to slow down, then c_{j+2} in turn sees the brake lights of c_{j+1}, and may also decide to brake, and so on.

1.3 Relation to Other Models

When $\mathsf{L} = 0$ (i.e., only right jumps are allowed), the q-PushASEP turns into *q-TASEP* (q-deformation of the totally asymmetric simple exclusion process), which is an interacting particle system introduced in [5], see also [6,9,10,22,26].

On the other hand, when $\mathsf{R} = 0$ (i.e., we permit only left jumps), our process essentially becomes the *q-PushTASEP* introduced in [12] as a one-dimensional marginal of a certain stochastic dynamics on two-dimensional arrays of interlacing particles.

Thus, the q-PushASEP interpolates between the q-TASEP and the q-PushTASEP. See also Appendix 1 for an explanation of how the q-PushASEP is also related to a dynamics on two-dimensional interlacing arrays.

Under the $q \to 0$ degeneration, our process becomes *PushASEP* — a two-sided particle system (in the sense that particles can jump to the left and to the right) which interpolates between TASEP and PushTASEP, see [1,11]. The two latter processes appeared in [27] (in that paper the PushTASEP was called the *long-range TASEP*), see also [17,18].

See also [24,25] for related developments.

Remark 1.1 Similarly to [11], one can make the parameters R and L in the definition of the q-PushASEP depend on time (in a sufficiently nice way). This will lead to replacement of the quantities $\mathsf{R}t$ and $\mathsf{L}t$ in our final formulas (e.g., (1.3) or (1.8) below) by $\int_0^t \mathsf{R}(s)ds$ and $\int_0^t \mathsf{L}(s)ds$, respectively. To make exposition clearer, we will consider only constant R and L.

1.4 Moments

To formulate one of our main results, define the Weyl chamber (of type A) as

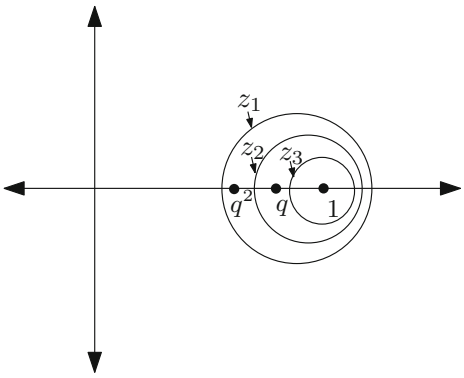

Fig. 2 Nested contours for $k = 3$ and $a_i \equiv 1$

$$\mathbb{W}^{k,N}_{\geq 0} := \{\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k \colon N \geq n_1 \geq \cdots \geq n_k \geq 0\}. \tag{1.2}$$

We compute joint q-moments (or q-exponential moments) of positions of several particles under the q-PushASEP:

Theorem 1.2 *For any* $\mathbf{n} \in \mathbb{W}^{k,N}_{\geq 0}$,

$$\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right) = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{i=1}^{n_j} \frac{a_i}{a_i - z_j}\right) \frac{\Pi_t(q z_j)}{\Pi_t(z_j)} \frac{dz_j}{z_j}, \tag{1.3}$$

where

$$\Pi_t(z) := e^{t(\mathsf{R} z + \mathsf{L} z^{-1})}. \tag{1.4}$$

Here $\mathbf{x}(t)$ *denotes the* q*-PushASEP started from the step initial condition* $\{x_i(0) = -i\}_{i=1}^{N}$. *The contour for* z_A *contains* $a_1, \dots, a_N$ *and all of the contours* $\{q z_B\}_{B>A}$, *but not zero (see Fig. 2 for an example of contours).*

A simple argument bounding the q-PushASEP by Poisson processes shows that the moments in the left-hand side of (1.3) are indeed finite (see Sect. 2.1).

Remark 1.3 It is worth noting that while the left-hand side of (1.3) is symmetric in $n_1, \dots, n_k$, the right-hand side is *not*. Theorem 1.2 states the equality of the two expressions only for $\mathbf{n} = (n_1, \dots, n_k)$ belonging to the Weyl chamber.

1.5 True and Free Evolution Equations

Our strategy of the proof of Theorem 1.2 is the following. We observe (see Sect. 2) that the expectations of $\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}$ for $\mathbf{n} \in \mathbb{W}^{k,N}_{\geq 0}$ evolve according to closed systems of coupled ODEs, which we call the *true evolution equations*. The equations' right-hand sides include as a summand the right-hand sides of [10, (3)] which corresponds to the q-TASEP, and also new terms corresponding to the q-PushTASEP governing the left jumps.

Let us first recall [10] (see also [6]) how one could solve the true evolution equations in the case $\mathsf{L} = 0$ (i.e., when our particle system reduces to the q-TASEP). For the q-TASEP, the true evolution equations are constant coefficient and separable away from the boundary

of the Weyl chamber $\mathbb{W}^{k,N}_{\geq 0}$ (but not on the boundary). In this case, extending the constant coefficient, separable equations to all of $\mathbb{Z}^k_{\geq 0}$ results in the free evolution equations on a function $u(t, \mathbf{n})$, where $t \geq 0$ and $\mathbf{n} \in \mathbb{Z}^k_{\geq 0}$. One of the results about the q-TASEP in [10] is that the restriction to $\mathbb{W}^{k,N}_{\geq 0}$ of a solution of the free evolution equations satisfying certain boundary conditions (resulting from the difference between the free and the true evolution equations) and with the right initial data in $\mathbb{W}^{k,N}_{\geq 0}$ coincides with the solution of the true evolution equations.

In principle, there could be a boundary condition for any possible combination of *clusters* (=stings of equal coordinates) in the vector $\mathbf{n}$. A remarkable property of the q-TASEP (*integrability* in the language of (quantum) many body systems, cf. [3]) is that it suffices to consider only the following $k-1$ two-body boundary conditions: for all $\mathbf{n} \in \mathbb{Z}^k_{\geq 0}$ such that for some $i \in \{1, 2, \ldots, k-1\}$ one has $n_i = n_{i+1}$,

$$(\nabla_i - q\nabla_{i+1})u(t, \mathbf{n}) = 0. \tag{1.5}$$

Here for a function $f : \mathbb{Z} \to \mathbb{R}$, we denote $(\nabla f)(n) := f(n-1) - f(n)$, and ∇_j above means that the difference operator acts in the j-th coordinate.

Let us now explain how the q-PushASEP situation differs from that of the q-TASEP. For $\mathsf{L} > 0$, the corresponding true evolution equations for any $\mathbf{n} \in \mathbb{W}^{k,N}_{\geq 0}$ involve linear combinations of expectations of $\prod_{i=1}^k q^{x_{n_i}(t)+n_i}$ with $\mathbf{n}$ running up to the boundary of $\mathbb{W}^{k,N}_{\geq 0}$. Thus, it is not a priori clear how to write down the free evolution equations (in $\mathbb{Z}^k_{\geq 0}$) for the q-PushASEP such that their solutions satisfying the same $k-1$ boundary conditions (1.5) coincide with solutions of the true evolution equations (in $\mathbb{W}^{k,N}_{\geq 0}$).

A way to write down the free evolution equations which we employ instead is to introduce *another* set of $k-1$ conditions which we call *cumulative*. For simpler notation, assume now that $a_i = 1$ for all $i = 1, \ldots, N$ (see Sect. 3 for a general case). The cumulative conditions are the following: for all $\mathbf{n} \in \mathbb{Z}^k_{\geq 0}$ such that for some $i \in \{1, 2, \ldots, k-1\}$ one has $n_i = n_{i+1}$,

$$\left(\nabla_i^{-1} - q^{-1}\nabla_{i+1}^{-1}\right) u(t, \mathbf{n}) = 0. \tag{1.6}$$

Here by ∇^{-1} we mean the operator acting on $f : \mathbb{Z} \to \mathbb{R}$ as $(\nabla^{-1} f)(n) := -f(n) - f(n-1) - \cdots - f(1)$. Note that $(\nabla\nabla^{-1} f)(n) = f(n)$, but $(\nabla^{-1}\nabla f)(n) = f(n) - f(0)$. As before, ∇_j^{-1} means the application of the operator in the jth coordinate.

We then obtain the *free evolution equations* for the q-PushASEP which are constant coefficient and separable in $\mathbb{Z}^k_{\geq 0}$, and prove that solutions of the free evolution equations satisfying (1.5)–(1.6) and having the right initial data inside $\mathbb{W}^{k,N}_{\geq 0}$ coincide with solutions of the true evolution equations for the q-PushASEP.

The emergence of the cumulative conditions (1.6) which might seem somewhat mysterious from the Bethe ansatz point of view (cf. the treatment of the q-TASEP in [9]) appeared due to a certain "symmetry" of formulas responsible for the right (q-TASEP) and left (q-PushTASEP) jumps. We plan to investigate deeper reasons behind these cumulative conditions in a future work.

1.6 Solving Evolution Equations for the q-PushASEP

One readily sees that there exists a general class of solutions to the free evolution equations for the q-PushASEP, but it is not immediately clear how one should combine them in the right way so that they satisfy (1.5)–(1.6). However, when the q-PushASEP starts from the step initial configuration, it is possible to *check* that the nested contour integral expression in

the right-hand side of (1.3) satisfies the free evolution equations, $k-1$ boundary and $k-1$ cumulative conditions (1.5)–(1.6), and has the right initial data, thus producing the desired moment formula.

The moment formula (1.3) for q-TASEP was initially proved for all $n_i = n, i = 1, \dots, k$, from the Macdonald process framework of [5]. The general $\mathbf{n}$ formula was guessed and checked in [10] via the many body system approach, and reproved in the Macdonald process framework in [8]. Our formula (1.3) for the q-PushASEP differs only in the function $\Pi_t(z)$ which was equal to e^{tz} for the q-TASEP. Discrete-time q-TASEPs of [6] also admit nested contour integral formulas for moments with other choices of $\Pi_t(z)$ (in [6, Theorem 2.1] these functions are denoted by $f_t^{\mathrm{geo}}(z)$ and $f_t^{\mathrm{Ber}}(z)$). The concrete form (1.4) of $\Pi_t(z)$ for the q-PushASEP can be guessed from any of the three sources:

(1) Applying the nested contour integral ansatz for solving the free evolution equations.
(2) By analogy with the PushASEP (i.e., the $q = 0$ case), e.g., see [11, Prop. 2.1]. Presence of factors of the form $e^{t(\mathsf{R}z+\mathsf{L}z^{-1})}$ in the PushASEP is a manifestation of its connection to the algebraic framework of the *two-sided* Schur processes [4].
(3) The q-PushASEP should fit into a more general framework of the *two-sided Macdonald processes* extending the theory of [5,8]. The present paper provides a motivation for a further investigation of the two-sided Macdonald processes. See also Appendix 1.

1.7 Fredholm Determinant

If $\mathsf{L} > 0$, observables of the form $\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right)$ grow rapidly in k, namely, as $c_1 \exp\left\{c_2 e^{c_3 k}\right\}$ (for suitable positive constants). This means that the moments are not sufficient to identify the distribution of the process (at any given positive time).

However, using (1.3) and the rigorously proved result for $\mathsf{L} = 0$, one can *formally* write down a conjectural Fredholm determinantal formula for the q-Laplace transform of $q^{x_n(t)+n}$ (for any $1 \le n \le N$). For simplicity, assume that all $a_i \equiv 1$. We will use the notation

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i), \qquad (a;q)_k := \prod_{i=0}^{k-1}(1-aq^i).$$

Conjecture 1.4 *For all $\zeta \in \mathbb{C} \setminus \mathbb{R}_{>0}$,*

$$\mathbb{E}\left(\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right) = \det(I + K_\zeta). \tag{1.7}$$

Here $\det(I + K_\zeta)$ is the Fredholm determinant of $K_\zeta : L^2(C_1) \to L^2(C_1)$, where C_1 is a small positively oriented circle containing 1, and K_ζ is an integral operator with kernel

$$K_\zeta(w, w') = \frac{1}{2\pi \mathbf{i}} \int_{-\mathbf{i}\infty+1/2}^{\mathbf{i}\infty+1/2} \frac{\pi}{\sin(-\pi s)} (-\zeta)^s \frac{G(q^s w)}{G(w)} \frac{1}{q^s w - w'} ds, \tag{1.8}$$

with (see (1.4))

$$G(w) := (w;q)_\infty^n \Pi_t(w).$$

A formal approach to establish (1.7) is to expand $\mathbb{E}\left(1/(\zeta q^{x_n(t)+n};q)_\infty\right)$ by means of the q-Binomial theorem, and interchange the expectation and the summation in the resulting series. A general scheme of doing this is explained in [10, Sect. 3]. However, for $\mathsf{L} > 0$, our moments of the q-PushASEP lead to a *divergent* series after one interchanges the expectation and the summation.

This is quite similar to the issue which arises in the polymer replica method, in which one attempts to recover the Laplace transform of the solution to the stochastic heat equation from a divergent moment generating series [13,15]. We believe that for the q-PushASEP this issue of divergence can be resolved (and thus (1.8) can be rigorously justified) by passing to a suitable discrete-time approximation (one may call it *regularization*) possessing nested contour integral formulas similar to those of Theorem 1.2. In this approximation, the derivation of a Fredholm determinantal formula would be rigorous, and then a rather straightforward continuous-time limit would yield the proof of Conjecture 1.4. Constructing suitable discrete-time approximations is the subject of a future work [20].

For $\mathsf{L} = 0$, the Fredholm determinantal formula (1.8) corresponds to the q-TASEP and a proof of the conjecture appeared in [5], see also [10]. It was established by interchanging the expectation and the summation, which is perfectly valid in this case. Indeed, for $\mathsf{L} = 0$ (and the step initial configuration), all coordinates $x_n(t)+n$ are nonnegative. Thus, the expectations $\mathbb{E}(q^{k(x_n(t)+n)})$ are all bounded by one, and thus the series $\sum_{k=0}^{\infty} \zeta^k \, \mathbb{E}(q^{k(x_n(t)+n)})/(q;q)_k$ is convergent for small enough values of ζ.

1.8 Outline

In Sect. 2 we discuss the q-PushASEP in detail, and write down the true evolution equations for the observables in the left-hand side of (1.3). We also suggest a Markov process dual to the q-PushASEP. In Sect. 3 we reduce the true evolution equations to the free evolution equations with $k-1$ boundary and $k-1$ cumulative conditions. We show that a solution of the free evolution equations also satisfies the true evolution equations. In Sect. 4 we check that the nested contour integral formula in the right-hand side of (1.3) satisfies the free evolution equations, and thus prove Theorem 1.2. We also discuss the Fredholm determinantal formula (Conjecture 1.4). In Appendix 1 we describe how the q-PushASEP is related to (two-sided) Macdonald processes. In Appendix 2 we briefly discuss connections of our model with the semi-discrete stochastic heat equation.

2 True Evolution Equations

In this section we write down closed systems of coupled ODEs (true evolution equations) which are satisfied by the expectations of the observables of the form $\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}$, where $\mathbf{n} = (n_1, \ldots, n_k)$ belongs to the Weyl chamber $\mathbb{W}^{k,N}_{\geq 0}$ (1.2).

2.1 Finiteness of Moments

Lemma 2.1 *Let $\mathbf{x}(t)$ be the position at time $t \geq 0$ of the q-PushASEP started from any fixed initial condition, i.e., from any point of X^N defined in (1.1). Then for any $\mathbf{n} \in \mathbb{W}^{k,N}_{\geq 0}$, the expectation $\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right)$ is finite.*

Proof Left jumps of the q-PushASEP introduce factors of q^{-1} into $\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right)$, and right jumps lead to factors of q. Since $0 < q < 1$, we need to estimate only the left jumps.

Observe that the leftmost particle $x_N(t)$ has the possibility to be pushed to the left by any of the particles, so it can go to the left as far as a Poisson process with rate $\mathsf{L}(a_1^{-1} + \ldots + a_N^{-1})$. Since the Poisson distribution has finite exponential moments (i.e., $\mathbb{E}(e^{z\xi}) < \infty$ for all $z \in \mathbb{C}$, where ξ has Poisson distribution), we see that the claim holds. □

2.2 Markov Generator of q-PushASEP

It is readily seen from the definition (Sect. 1.1) that the Markov generator of the q-PushASEP (acting on functions $f\colon X^N \to \mathbb{R}$) has the form

$$(\mathscr{L}^{\text{q-PushASEP}} f)(\mathbf{x}) = \sum_{i=1}^{N} \mathsf{R} a_i \left(1 - q^{x_{i-1}-x_i-1}\right)\left(f(\mathbf{x}_i^+) - f(\mathbf{x})\right) + \sum_{i=1}^{N} \mathsf{L} a_i^{-1} \sum_{j=i}^{N} q^{x_i - x_j - (j-i)} \left(1 - q^{x_j - x_{j+1} - 1}\right)\left(f(\mathbf{x}_{j,i}^-) - f(\mathbf{x})\right). \tag{2.1}$$

Here we have denoted for all $1 \le i \le j \le N$:

$$\mathbf{x}_i^+ := (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_N);$$
$$\mathbf{x}_{j,i}^- := (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1} - 1, \ldots, x_{j-1} - 1, x_j - 1, x_{j+1}, \ldots, x_N).$$

That is, $\mathbf{x}_i^+$ corresponds to the configuration in which the ith particle has jumped to the right by one, and $\mathbf{x}_{j,i}^-$ means the configuration in which the particles with indices $m = i, i+1, \ldots, j$ have jumped to the left by one. Note that if any of these jumps breaks the strict order of the particles, then the coefficient in (2.1) by the corresponding term vanishes. This reflects the fact that the q-PushASEP preserves the order of the particles.

2.3 Remark: Stationary Distributions

Here let us present a calculation which suggests how stationary measures of the q-PushASEP with infinitely many particles $-\infty < \cdots < x_1 < x_0 < x_{-1} < \cdots < +\infty$ look like (without discussing the existence of this process or proving that these measures are indeed stationary). Assume translation invariance, i.e., that $a_i = 1$ for all $i \in \mathbb{Z}$.

The case of the q-TASEP (i.e., when $\mathsf{L} = 0$) is discussed in [5, Sect. 3.3.3]. There the stationary measures are those for which the *gaps* $x_{i-1} - x_i - 1 = \mathrm{gap}_i$ between particles are independent and have the *q-geometric distribution* $\mathrm{qGeo}\left(\alpha \mathsf{R}^{-1}\right)$, where $\alpha \in [0, \mathsf{R})$ is arbitrary:

$$\mathrm{Prob}\left(x_{i-1} - x_i - 1 = k\right) = (\alpha \mathsf{R}^{-1}; q)_\infty \frac{(\alpha \mathsf{R}^{-1})^k}{(q;q)_k}, \qquad k = 0, 1, \ldots.$$

One can perform a formal calculation suggesting that this distribution is also stationary for the q-PushTASEP part of the dynamics. Indeed, during a small time interval dt, each gap_i can increase by one with probability $\mathsf{L}\,dt$ (which corresponds to x_i jumping to the left). Next, observe that the particle x_{i-1} moves to the left at total rate (accounting for all possible pushes that x_{i-1} can receive from the left)

$$\mathsf{L}\left(1 + (1 - \alpha \mathsf{R}^{-1}) + (1 - \alpha \mathsf{R}^{-1})^2 + \ldots\right) = \frac{\mathsf{L}\mathsf{R}}{\alpha}, \tag{2.2}$$

because $1 - \alpha \mathsf{R}^{-1} = \mathbb{E}(q^{\mathrm{gap}_i})$ for all $i \in \mathbb{Z}$ (which readily follows from the q-Binomial theorem). This means that during a small time interval dt, the value of gap_i can decrease by one with probability $\frac{\mathsf{L}\mathsf{R}}{\alpha}(1 - q^{\mathrm{gap}_i})dt$. Here the factor $1 - q^{\mathrm{gap}_i}$ in the latter expression is the probability that the moved particle x_{i-1} did *not* push x_i. One can readily check that the law $\mathrm{gap}_i \sim \mathrm{qGeo}\left(\alpha \mathsf{R}^{-1}\right)$ is invariant for the one-dimensional Markov chain on $\mathbb{Z}_{\ge 0}$ which

we have just described. This suggests that this law should be preserved by the q-PushASEP evolution.[1]

In the non-translation invariant case i.e., when the a_i's are different,[2] the consideration of the right jumps (i.e., the q-TASEP dynamics) leads to the following distributions of the gaps: $\mathrm{gap}_i \sim \mathrm{qGeo}\left(\alpha \mathsf{R}^{-1} a_i^{-1}\right)$. Then the series in (2.2) is no longer a geometric progression, but it still sums to $\frac{\mathsf{LR}}{\alpha}$, which suggests that the independent q-geometric gaps $\mathrm{qGeo}\left(\alpha \mathsf{R}^{-1} a_i^{-1}\right)$ are preserved by the left (q-PushTASEP) jumps in the non-translation invariant setting as well.

It would be interesting to generalize the coupling approach of [2] to the two-sided setting.

2.4 $\mathbf{y}$-Variable Notation

Our aim now is to understand how the generator (2.1) acts on moments $\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right)$. It is convenient to pass from the coordinates $\mathbf{n} \in \mathbb{W}_{\geq 0}^{k,N}$ to a new set of coordinates. Denote

$$Y^N := \left\{\mathbf{y} = (y_0, y_1, \ldots, y_N) \in \mathbb{Z}_{\geq 0}^{N+1}\right\}, \qquad Y_k^N := \left\{\mathbf{y} \in Y^N : \sum_{i=0}^{N} y_i = k\right\}. \tag{2.3}$$

To each $\mathbf{n} \in \mathbb{W}_{\geq 0}^{k,N}$ associate $\mathbf{y}(\mathbf{n}) \in Y_k^N$ defined by $y_i(\mathbf{n}) := |\{j : n_j = i\}|$. In the reverse direction, for any $\mathbf{y} \in Y_k^N$, denote by $\mathbf{n}(\mathbf{y})$ the unique $\mathbf{n} \in \mathbb{W}_{\geq 0}^{k,N}$ for which $\mathbf{y}(\mathbf{n}) = \mathbf{y}$. To illustrate, if $\mathbf{n} = (5, 5, 4, 2, 1, 1, 1)$, then $\mathbf{y}(\mathbf{n}) = (0, 3, 1, 0, 1, 2)$. We will call the number of nonzero coordinates of $\mathbf{y}$ the number of *clusters* of $\mathbf{n}$ (the present example has four clusters).

Let us define, for each $x \in X^N$ and $\mathbf{y} \in Y^N$,

$$H(\mathbf{x}, \mathbf{y}) := \prod_{i=0}^{N} q^{(x_i+i)y_i}. \tag{2.4}$$

The product above starts from zero which means that, by agreement, $H(\mathbf{x}, \mathbf{y}) = 0$ if $y_0 > 0$.

2.5 Action of $\mathscr{L}^{q\text{-PushASEP}}$ on $H(\mathbf{x}, \mathbf{y})$

For all $0 \leq i \leq j \leq N$ denote

$$\mathbf{y}^{j,i} := (y_0, y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_{j-1}, y_j - 1, y_{j+1}, \ldots, y_N).$$

That is, in $\mathbf{y}^{j,i}$ the jth coordinate is decreased by one, and the ith coordinate is increased by one. Clearly, $\mathbf{y}^{i,i} = \mathbf{y}$.

Denote by $\mathscr{L}^{\mathrm{dual}}$ the following operator acting on functions $g : Y^N \to \mathbb{R}$:

$$\begin{aligned}(\mathscr{L}^{\mathrm{dual}} g)(\mathbf{y}) := &\sum_{i=1}^{N} \mathsf{R} a_i (1 - q^{y_i}) \left(g(\mathbf{y}^{i,i-1}) - g(\mathbf{y})\right) \\ &+ \sum_{i=1}^{N} \mathsf{L} a_i^{-1} \sum_{j=i}^{N} (q^{-y_j} - 1) q^{-y_i - \ldots - y_{j-1}} g(\mathbf{y}^{j,i}).\end{aligned} \tag{2.5}$$

Remark 2.2 Note that the first sum (containing the parameter R) is the Markov generator of the *q-Boson particle system* (a certain totally asymmetric zero range process) which is

[1] One needs to additionally justify that gap_i indeed evolves according to this one-dimensional Markov chain.

[2] One should also impose reasonable growth and decay assumptions on the a_i's.

dual to the q-TASEP, see [10] (where this process was called q-TAZRP) and also [9]. The second summand is new and it is responsible for the left jumps (which are governed by the q-PushTASEP, cf. Sect. 1.3). See also Sect. 2.7 below.

Proposition 2.3 *For any* $\mathbf{x} \in X^N$ *and* $\mathbf{y} \in Y^N$ *we have*

$$\mathscr{L}_{\mathbf{x}}^{\mathrm{q-PushASEP}} H(\mathbf{x}, \mathbf{y}) = \mathscr{L}_{\mathbf{y}}^{\mathrm{dual}} H(\mathbf{x}, \mathbf{y}),$$

where the subscripts $\mathbf{x}$ *and* $\mathbf{y}$ *in the operators mean the variables in which the operators act.*

Proof This follows from the observations

$$\begin{aligned}
H(\mathbf{x}_i^+, \mathbf{y}) - H(\mathbf{x}, \mathbf{y}) &= (q^{y_i} - 1) H(\mathbf{x}, \mathbf{y});\\
H(\mathbf{x}_{j,i}^-, \mathbf{y}) - H(\mathbf{x}, \mathbf{y}) &= (q^{-y_i - y_{i+1} - \ldots - y_j} - 1) H(\mathbf{x}, \mathbf{y});\\
(1 - q^{x_{i-1} - x_i - 1}) H(\mathbf{x}, \mathbf{y}) &= H(\mathbf{x}, \mathbf{y}) - H(\mathbf{x}, \mathbf{y}^{i,i-1});\\
q^{x_i - x_j - (j-i)} \bigl(1 - q^{x_j - x_{j+1} - 1}\bigr) H(\mathbf{x}, \mathbf{y}) &= H(\mathbf{x}, \mathbf{y}^{j,i}) - H(\mathbf{x}, \mathbf{y}^{j+1,i}).
\end{aligned}$$

To get (2.5) after applying the above identities to (2.1), one should also regroup summands in the second sum (which contains the parameter L) by collecting the coefficients by each $g(\mathbf{y}^{j,i})$. □

2.6 True Evolution Equations

Proposition 2.3 motivates the following definition:

Definition 2.4 A function $h(t, \mathbf{y})$, $t \geq 0$, $\mathbf{y} \in Y^N$, is said to satisfy the *true evolution equations* with initial conditions $h_0(\mathbf{y})$ if

(1) For all $\mathbf{y} \in Y^N$ and $t \geq 0$:

$$\frac{d}{dt} h(t, \mathbf{y}) = \mathscr{L}^{\mathrm{dual}} h(t, \mathbf{y}), \tag{2.6}$$

where the operator $\mathscr{L}^{\mathrm{dual}}$ given by (2.5) acts in the variables $\mathbf{y}$.

(2) (boundary conditions) For all $\mathbf{y} \in Y^N$ such that $y_0 > 0$, $h(t, \mathbf{y}) \equiv 0$ for all $t \geq 0$.

(3) (initial conditions) For all $\mathbf{y} \in Y^N$, $h(0, \mathbf{y}) = h_0(\mathbf{y})$.

Lemma 2.5 *The above true evolution equations have unique solutions.*

Proof The proof is the same as for the q-TASEP, see [10, Lemma 3.5].

The operator $\mathscr{L}^{\mathrm{dual}}$ maps the space of functions $g\colon Y_k^N \to \mathbb{R}$ onto itself. Therefore, the true evolution equations reduce to a collection of finite closed systems of ODEs indexed by $k \geq 1$.

Moreover, for each fixed k, the system of the true evolution equations is *triangular*. Namely, the derivative $\frac{d}{dt} h(t, \mathbf{y})$ depends only on those $h(t, \mathbf{y}')$ for which $y_i' + \cdots + y_N' \leq y_i + \cdots + y_N$ for all $0 \leq i \leq N$. The existence and uniqueness of solutions to each finite, closed, triangular system of linear ODEs is justified by standard methods, e.g., see [14]. □

Theorem 2.6 *For any* $\mathbf{x} \in X^N$*, and for the* q*-PushASEP* $\{\mathbf{x}(t)\}_{t \geq 0}$ *started from an arbitrary (non-random) initial condition* $\mathbf{x}(0) = \mathbf{x}$*, the function* $h(t, \mathbf{y}) := \mathbb{E}(H(\mathbf{x}(t), \mathbf{y}))$ *solves the true evolution equations with initial data* $h_0(\mathbf{y}) = H(\mathbf{x}, \mathbf{y})$.

By linearity, one can also consider good enough random initial configurations for the q-PushASEP. In this case, one should take the initial data to be $h_0(\mathbf{y}) = \mathbb{E}(H(\mathbf{x}, \mathbf{y}))$, where the expectation is with respect to the initial configuration $\mathbf{x}$.

Proof Due to Lemma 2.5, it suffices to check that the function $h(t, \mathbf{y}) := \mathbb{E}^{\mathbf{x}}\left(H(\mathbf{x}(t), \mathbf{y})\right)$ (the superscript $\mathbf{x}$ means that the expectation is taken with respect to the q-PushASEP starting from $\mathbf{x}$) satisfies (2.6) (boundary and initial conditions are straightforward).

One has

$$\frac{d}{dt}\,\mathbb{E}^{\mathbf{x}}\left(H(\mathbf{x}(t), \mathbf{y})\right) = \mathscr{L}^{q\text{-PushASEP}}\,\mathbb{E}^{\mathbf{x}}\left(H(\mathbf{x}(t), \mathbf{y})\right) = \mathbb{E}^{\mathbf{x}}\left(\mathscr{L}^{q\text{-PushASEP}} H(\mathbf{x}(t), \mathbf{y})\right).$$

The first equality is the backwards Kolmogorov equations (essentially, the definition of a Markov generator), and the second one follows from the fact that the generator $\mathscr{L}^{\text{q-PushASEP}}$ of the Markov semigroup of the q-PushASEP commutes with the operators from this semigroup.

Next, using Proposition 2.3, we can continue the above equalities ($\mathscr{L}^{q-PushASEP}$ and $\mathscr{L}^{\text{dual}}$ act on $\mathbf{x}$ and $\mathbf{y}$ variables, respectively)

$$\mathbb{E}^{\mathbf{x}}\left(\mathscr{L}^{\text{q-PushASEP}} H(\mathbf{x}(t), \mathbf{y})\right) = \mathbb{E}^{\mathbf{x}}\left(\mathscr{L}^{\text{dual}} H(\mathbf{x}(t), \mathbf{y})\right) = \mathscr{L}^{\text{dual}}\,\mathbb{E}^{\mathbf{x}}\left(H(\mathbf{x}(t), \mathbf{y})\right).$$

The last equality is due to the fact that the expectation is taken with respect to the $\mathbf{x}$ variables while the operator $\mathscr{L}^{\text{dual}}$ acts in the $\mathbf{y}$ variables. □

2.7 Remark: Markov Process Dual to the q-PushASEP

For $\mathsf{L} > 0$, the operator $\mathscr{L}^{\text{dual}}$ (2.5) is *not* a generator of any continuous-time Markov process on the space Y^N (cf. Remark 2.2 about the $\mathsf{L} = 0$ case). Indeed, applying this operator to the identity function, one has

$$\mathscr{L}^{\text{dual}}\mathbf{1} = \sum_{i=1}^{n} \mathsf{L} a_i^{-1}(q^{-y_i-\ldots-y_N}) := C(\mathbf{y}).$$

However, the fact that $C(\mathbf{y})$ is not zero is the only obstacle preventing $\mathscr{L}^{\text{dual}}$ from being a Markov generator. Thus, let us define the following operator acting on functions $g\colon Y^N \to \mathbb{R}$ by

$$\begin{aligned}(\mathscr{L}^{\text{dual Markov}} g)(\mathbf{y}) &:= (\mathscr{L}^{\text{dual}} g)(\mathbf{y}) - C(\mathbf{y})g(\mathbf{y})\\ &= \sum_{i=1}^{N} \mathsf{R} a_i (1 - q^{y_i})\left(g(\mathbf{y}^{i,i-1}) - g(\mathbf{y})\right)\\ &\quad + \sum_{i=1}^{N} \mathsf{L} a_i^{-1} \sum_{j=i+1}^{N} (q^{-y_j} - 1) q^{-y_i-\ldots-y_{j-1}}\left(g(\mathbf{y}^{j,i}) - g(\mathbf{y})\right). \end{aligned} \tag{2.7}$$

One readily sees that this operator can serve as a generator of a continuous-time Markov process on Y^N; denote this process by $\mathbf{y}(t)$. Representing the state space Y^N as in Fig. 3, we see that the transitions in $\mathbf{y}(t)$ look as follows:

(1) (q-TASEP part) For each $i \in \{2, \ldots, N\}$, the coordinate $y_i(t)$ decreases by one and simultaneously $y_{i-1}(t)$ increases by one (= a particle jumps from site i to site $i-1$) at rate $\mathsf{R} a_i (1 - q^{y_i(t)})$.

(2) (q-PushTASEP part) For each $1 \le i < j \le N$, the coordinate $y_j(t)$ decreases by one and simultaneously $y_i(t)$ increases by one (= a particle jumps from site j to site i) at rate $\mathsf{L} a_i^{-1}(q^{-y_j(t)} - 1) q^{-y_i(t)-\ldots-y_{j-1}(t)}$.

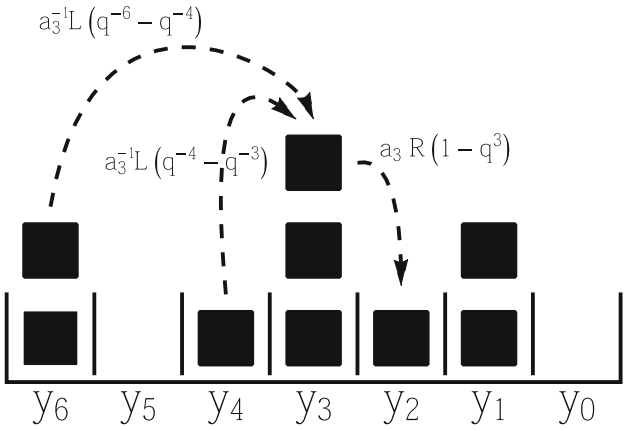

Fig. 3 Markov process $\mathbf{y}(t)$ dual to q-PushASEP. Indicated are all possible jumps involving the parameter a_3

All these transitions occur independently. Note that for $\mathsf{L} > 0$, the process $\mathbf{y}(t)$ is not zero range.

Clearly, Proposition 2.3 implies that the Markov generators of the q-PushASEP $\mathbf{x}(t)$ (2.1) and of the above process $\mathbf{y}(t)$ (2.7) satisfy the following generalized duality relation with respect to the same function $H(\mathbf{x}, \mathbf{y})$ (2.4):

$$\mathscr{L}_{\mathbf{x}}^{\text{q-PushASEP}} H(\mathbf{x}, \mathbf{y}) = \mathscr{L}_{\mathbf{y}}^{\text{dual Markov}} H(\mathbf{x}, \mathbf{y}) + C(\mathbf{y}) H(\mathbf{x}, \mathbf{y}).$$

Consequently, the expectations of $H(\mathbf{x}, \mathbf{y})$ with respect to evolution of the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are related as

$$\mathbb{E}^{\mathbf{x}}\left(H(\mathbf{x}(t), \mathbf{y})\right) = \mathbb{E}^{\mathbf{y}}\left(H(\mathbf{x}, \mathbf{y}(t)) e^{\int_0^t C(\mathbf{y}(s))ds}\right). \tag{2.8}$$

Here in the left-hand side we have the expectation under $\mathbf{x}(t)$ started from $\mathbf{x}$, and on the right there is the expectation under the law of the process $\mathbf{y}(t)$ started from $\mathbf{y}$. About (generalized) duality of Markov processes, e.g., see [16, chap. 4.4] and references therein.

One could use the generalized duality (2.8) to provide a probabilistic insight into Theorem 2.6. However, from the many body systems point of view the process $\mathbf{x}(t)$ is not required to be dual to any Markov process. One only needs the fact that the observables $\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right)$ evolve according to a closed system of ODEs.

3 Free Evolution Equations with $k-1$ Boundary and $k-1$ Cumulative Conditions for the q-PushASEP

The goal of this section is to reduce the true evolution equations for the two-sided q-PushASEP (Theorem 2.6) to free evolution equations which are constant coefficient and separable (see the discussion in Sects. 1.5 and 1.6 for more detail).

Let $\mathbf{a} := (a_1, \ldots, a_N)$, and recall that all a_i are positive. Define the following operators acting on functions $f : \mathbb{Z} \to \mathbb{R}$:

$$(\nabla_{\mathbf{a}} f)(n) := a_n\big(f(n-1) - f(n)\big),$$
$$(\nabla_{\mathbf{a}}^{-1} f)(n) := -a_n^{-1} f(n) - a_{n-1}^{-1} f(n-1) - \ldots - a_1^{-1} f(1).$$

By agreement, let us add "dummy parameters" a_n, $n > N$. They do not enter main formulas of this section, but it is convenient to include them to avoid the requirement that $n \le N$. Equivalently, one may think of dealing with the process with infinitely many particles to the left of the origin (and finitely many particles to the right of the origin), cf. the end of Sect. 1.1.

Clearly,

$$(\nabla_{\mathbf{a}} \nabla_{\mathbf{a}}^{-1} f)(n) = f(n), \qquad (\nabla_{\mathbf{a}}^{-1} \nabla_{\mathbf{a}} f)(n) = f(n) - f(0).$$

For a function on $\mathbb{Z}^k$, let $[\nabla_{\mathbf{a}}]_j$ and $[\nabla_{\mathbf{a}}^{-1}]_j$ denote the application of the corresponding operators in the j-th variable.

Definition 3.1 We say that a function $u\colon \mathbb{R}_{\ge 0} \times \mathbb{Z}^k_{\ge 0} \to \mathbb{R}$ satisfies the free evolution equations with $k-1$ boundary conditions, $k-1$ cumulative conditions, and (partial) initial conditions h_0 inside the Weyl chamber $\mathbb{W}^{k,N}_{\ge 0} \subseteq \mathbb{Z}^k_{\ge 0}$, if

(1) For all $\mathbf{n} \in \mathbb{Z}^k_{\ge 0}$ and $t \ge 0$,

$$\frac{d}{dt} u(t, \mathbf{n}) = \mathsf{R} \cdot (1-q) \sum_{i=1}^{k} [\nabla_{\mathbf{a}}]_i \, u(t; \mathbf{n}) + \mathsf{L} \cdot (1 - q^{-1}) \sum_{i=1}^{k} [\nabla_{\mathbf{a}}^{-1}]_i \, u(t; \mathbf{n}). \tag{3.1}$$

(2) For all $\mathbf{n} \in \mathbb{Z}^k_{\ge 0}$ such that for some $i \in \{1, 2, \dots, k-1\}$ one has $n_i = n_{i+1}$,

$$\left([\nabla_{\mathbf{a}}]_i - q \cdot [\nabla_{\mathbf{a}}]_{i+1}\right) u(t, \mathbf{n}) = 0; \qquad \left([\nabla_{\mathbf{a}}^{-1}]_i - q^{-1} \cdot [\nabla_{\mathbf{a}}^{-1}]_{i+1}\right) u(t, \mathbf{n}) = 0; \tag{3.2}$$

(3) For all $\mathbf{n} \in \mathbb{Z}^k_{\ge 0}$ such that $n_k = 0$, $u(t, \mathbf{n}) \equiv 0$ for all $t \ge 0$;

(4) For all $\mathbf{n} \in \mathbb{W}^{k,N}_{\ge 0}$, $u(0, \mathbf{n}) = h_0(\mathbf{y}(\mathbf{n}))$.

Note that the boundary conditions in (3.2) coincide with the ones for the q-TASEP [10] (and it discrete variants, see [6]), which involve the usual difference operators $[\nabla]_i$ and $[\nabla]_{i+1}$. This is because $n_i = n_{i+1}$ implies $a_{n_i} = a_{n_{i+1}}$. We write the boundary conditions as in (3.2) to emphasize their certain similarity with the cumulative conditions.

Theorem 3.2 *If a function $u\colon \mathbb{R}_{\ge 0} \times \mathbb{Z}^k_{\ge 0} \to \mathbb{R}$ satisfies the free evolution equations with $k-1$ boundary and $k-1$ cumulative conditions (Definition 3.1), then for all $\mathbf{y} \in Y_k^N$, we have $h(t, \mathbf{y}) = u(t, \mathbf{n}(\mathbf{y}))$, where h is the solution to the true evolution equations (Definition 2.4) with initial condition $h_0(\mathbf{y})$.*

Proof Conditions (3) and (4) of Definition 3.1 directly lead to conditions (2) and (3) of the solution to the true evolution equations (Definition 2.4).

It remains to check that condition (1) of Definition 2.4 is satisfied by $u(t; \mathbf{n}(\mathbf{y}))$, where $u(t; \mathbf{n})$ solves the free evolution equations with $k-1$ boundary and $k-1$ cumulative conditions. We will use (3.2) to rewrite (3.1) in the form (2.6) (with $\mathscr{L}^{\mathrm{dual}}$ given by (2.5)). Fix $\mathbf{n} \in \mathbb{W}^{k,N}_{\ge 0}$ and let throughout the proof $\mathbf{y} = \mathbf{y}(\mathbf{n})$ and $\mathbf{n} = \mathbf{n}(\mathbf{y})$, see Sect. 2.4.

First, let us briefly recall (see [10]) how one deals with the summands in (3.1) corresponding to the right jumps. Fix any cluster of $\mathbf{n}$ of size, say, $c \ge 1$, i.e.,

$$\mathbf{n} = (n_1 \ge \cdots \ge n_b > \underbrace{n_{b+1} = \cdots = n_{b+c}}_{\text{cluster}} > n_{b+c+1} \ge \cdots \ge n_k \ge 0).$$

Clearly, $c = y_i$, $b = y_N + \cdots + y_{i+1}$, and $n_{b+1} = \cdots = n_{b+c} = i$ for some $i = 1, \ldots, N$. Combining summands corresponding to $r = b+1, \ldots, b+c$ in the first sum in (3.1) and using the boundary conditions in (3.2), we obtain

$$\mathsf{R}(1-q)\sum_{r=b+1}^{b+c}[\nabla_{\mathbf{a}}]_r u(t;\mathbf{n}) = \mathsf{R}(1-q)\sum_{r=b+1}^{b+c} q^{b+c-r}[\nabla_{\mathbf{a}}]_{b+c} u(t;\mathbf{n})$$
$$= \mathsf{R}(1-q^c)[\nabla_{\mathbf{a}}]_{b+c} u(t;\mathbf{n}).$$

We readily see that in terms of the $\mathbf{y}$ variables, the above expression is equal to

$$\mathsf{R}(1-q^{y_i})a_i\big(u\big(t;\mathbf{n}(\mathbf{y}^{i,i-1})\big) - u\big(t;\mathbf{n}(\mathbf{y})\big)\big),$$

which is one of the summands in the first sum in (2.5) corresponding to the cluster of components of $\mathbf{n}$ which are equal to i.

Now let us explain how one can rewrite the second sum in (3.1) (which corresponds to the left jumps). Fix any $j \geq i$ for which $y_j \geq 1$. Let us calculate the coefficient by $u\big(t;\mathbf{n}(\mathbf{y}^{j,i})\big)$ in the right-hand side of (3.1). This coefficient can come only from the part of the second sum corresponding to the cluster of components of $\mathbf{n}$ which are equal to j. Using the cumulative conditions (3.2), we can rewrite it as (below $b = y_N + \cdots + y_{j+1}$)

$$\mathsf{L}(1-q^{-1})\sum_{r=b+1}^{b+y_j}[\nabla_{\mathbf{a}}^{-1}]_r u(t;\mathbf{n}) = \mathsf{L}(1-q^{-1})\sum_{r=b+1}^{b+y_j} q^{r-b-y_j}[\nabla_{\mathbf{a}}^{-1}]_{b+y_j} u(t;\mathbf{n})$$
$$= \mathsf{L}(1-q^{-y_j})[\nabla_{\mathbf{a}}^{-1}]_{b+y_j} u(t;\mathbf{n}).$$

If $j = i$, then we readily see from the above expression that the coefficient by $u\big(t;\mathbf{n}(\mathbf{y}^{i,i})\big) = u\big(t;\mathbf{n}(\mathbf{y})\big)$ is $\mathsf{L}a_i^{-1}(q^{-y_j}-1)$, as it should be according to (2.5).

Assume now that $i < j$, and also that $y_{j-1} \geq 1$. This means that we can rewrite the above expression as

$$\mathsf{L}(1-q^{-y_j})[\nabla_{\mathbf{a}}^{-1}]_{b+y_j} u(t;\mathbf{n}) = \mathsf{L}(q^{-y_j}-1)\Big(a_j^{-1}u(t;\mathbf{n}) + [\nabla_{\mathbf{a}}^{-1}]_{b+y_j+1} u\big(t;\mathbf{n}(\mathbf{y}^{j,j-1})\big)\Big). \tag{3.3}$$

Indeed, we have simply removed one of the summands from the expression $[\nabla_{\mathbf{a}}^{-1}]_{b+y_j} u(t;\mathbf{n})$ using the definition of $\nabla_{\mathbf{a}}^{-1}$. Now, by (3.2), we clearly can write the application of $[\nabla_{\mathbf{a}}^{-1}]_{b+y_j+1}$ to $u\big(t;\mathbf{n}(\mathbf{y}^{j,j-1})\big)$ as the application of $[\nabla_{\mathbf{a}}^{-1}]_{b+y_j+y_{j-1}}$ times the factor of $q^{-y_{j-1}}$. This observation together with (3.3) implies that the coefficient by $u\big(t;\mathbf{n}(\mathbf{y}^{j,j-1})\big)$ in the right-hand side of (3.1) is equal to $\mathsf{L}a_{j-1}^{-1}(q^{-y_j}-1)q^{-y_{j-1}}$, as it should be by (2.5).

One can check in a similar manner that for any $i < j$, the coefficient by $u\big(t;\mathbf{n}(\mathbf{y}^{j,i})\big)$ in the right-hand side of (3.1) is the same as dictated by (2.5). This concludes the proof. □

The next statement is a straightforward consequence of Theorems 2.6 and 3.2:

Corollary 3.3 *For q-PushASEP started from any fixed or random initial configuration* $\mathbf{x}(0) = \mathbf{x}$, $\mathbb{E}\left(\prod_{i=1}^k q^{x_{n_i}(t)+n_i}\right) = u(t;\mathbf{n})$, *where* $u(t;\mathbf{n})$ *solves the free evolution equations with* $k-1$ *boundary and* $k-1$ *cumulative conditions (Definition 3.1) with initial data inside the Weyl chamber* $\mathbf{n} \in \mathbb{W}_{\geq 0}^{k,N}$ *given by* $u(0;\mathbf{n}) = \mathbb{E}\left(\prod_{i=1}^k q^{x_{n_i}(0)+n_i}\right)$.

4 Nested Contour Integral Formulas for the q-PushASEP

4.1 Moments: Proof of Theorem 1.2

Here we will use Corollary 3.3 to prove Theorem 1.2. It suffices to check that the expression for the moments of the q-PushASEP given by the right-hand side of (1.3) (denote it by $m(t;\mathbf{n})$) satisfies conditions (1)–(4) of Definition 3.1. Let us verify these conditions:

(1) The time derivative in the left-hand side of (3.1) affects only the factor $\prod_{j=1}^{k}\frac{\Pi_t(qz_j)}{\Pi_t(z_j)}$ inside the nested integral in $m(t;\mathbf{n})$, which leads to the multiplication of the integrand by

$$\mathsf{R}(q-1)\sum_{j=1}^{k} z_j + \mathsf{L}(q^{-1}-1)\sum_{j=1}^{k} z_j^{-1}. \tag{4.1}$$

Let us check that the application of the operators in the right-hand side of (3.1) also gives the factor (4.1).

First, note that for each $j=1,\dots,k$, the application of $[\nabla_{\mathbf{a}}]_j$ leads to the replacement of $\prod_{i=1}^{n_j}\frac{a_i}{a_i-z_j}$ by

$$a_{n_j}\left(\prod_{i=1}^{n_j-1}\frac{a_i}{a_i-z_j}-\prod_{i=1}^{n_j}\frac{a_i}{a_i-z_j}\right)=-z_j\prod_{i=1}^{n_j}\frac{a_i}{a_i-z_j}.$$

We see that we have matched summands involving the parameter R in (4.1).

Now let us consider the summands in the right-hand side of (3.1) involving the parameter L. For simpler notation denote $n=n_j$ and $z=z_j$, and consider the application of the operator $\nabla_{\mathbf{a}}^{-1}$ to $\prod_{i=1}^{n}\frac{a_i}{a_i-z}$. It is given by

$$\begin{aligned}&-a_n^{-1}\prod_{i=1}^{n}\frac{a_i}{a_i-z}-a_{n-1}^{-1}\prod_{i=1}^{n-1}\frac{a_i}{a_i-z}-\cdots-a_1^{-1}\frac{a_1}{a_1-z}\\&=-\prod_{i=1}^{n}\frac{a_i}{a_i-z}\cdot\sum_{j=1}^{n}a_j^{-1}\prod_{r=j+1}^{n}\frac{a_r-z}{a_r}.\end{aligned}$$

Let, by agreement, $a_0=a_{-1}=a_{-2}=\cdots=1$. Let us add to the above sum over j more summands corresponding to j running from $-\infty$ to 0, that is, the expression

$$\sum_{j=-\infty}^{0}\prod_{r=j+1}^{n}\frac{a_r-z}{a_r}=\frac{1}{z}\prod_{r=1}^{n}\frac{a_r-z}{a_r}.$$

In view of the nested contour integration in (1.3), we see that these additional summands (when multiplied by $\prod_{i=1}^{n}\frac{a_i}{a_i-z}$) do not introduce any residues in z. Thus, modulo the contour integration, we can rewrite the application of $\nabla_{\mathbf{a}}^{-1}$ to $\prod_{i=1}^{n}\frac{a_i}{a_i-z}$ as

$$-\prod_{i=1}^{n}\frac{a_i}{a_i-z}\left(\frac{1}{z}\prod_{r=1}^{n}\frac{a_r-z}{a_r}+\sum_{j=1}^{n}a_j^{-1}\prod_{r=j+1}^{n}\frac{a_r-z}{a_r}\right).$$

To finish the check of (1) by matching summands involving the parameter L in (4.1), it suffices to establish the following lemma:

Lemma 4.1 *We have*

$$\frac{1}{z}\prod_{r=1}^{n}\frac{a_r-z}{a_r}+\sum_{j=1}^{n}a_j^{-1}\prod_{r=j+1}^{n}\frac{a_r-z}{a_r}=\frac{1}{z}. \tag{4.2}$$

Proof Denote by S_n the left-hand side of (4.2). Then one can readily see that

$$S_0=1/z;\qquad S_{n-1}\frac{a_n-z}{a_n}=S_n-a_n^{-1},\quad n\geq 1,$$

which implies the claim. □

(2) The argument is almost the same for the boundary and the cumulative conditions. As we have seen in the above check of (1), the (boundary condition) operator $[\nabla_{\mathbf{a}}]_i - q\cdot[\nabla_{\mathbf{a}}]_{i+1}$ applied to $m(t;\mathbf{n})$, multiplies the integrand by $-(z_i-qz_{i+1})$. The (cumulative condition) operator $[\nabla_{\mathbf{a}}^{-1}]_i - q^{-1}\cdot[\nabla_{\mathbf{a}}^{-1}]_{i+1}$ leads to the multiplication of the integrand in $m(t;\mathbf{n})$ by

$$-\left(\frac{1}{z_i}-q^{-1}\frac{1}{z_{i+1}}\right)=\frac{z_i-qz_{i+1}}{qz_iz_{i+1}}.$$

In both cases, the factor z_i-qz_{i+1} cancels one of the denominators in $\prod_{1\leq A<B\leq k}\frac{z_A-z_B}{z_A-qz_B}$. This allows us to deform (without encountering any poles) the z_i and z_{i+1} contours so that they coincide. Since $n_i=n_{i+1}$, this means that we may write both $\left([\nabla_{\mathbf{a}}]_i - q\cdot[\nabla_{\mathbf{a}}]_{i+1}\right)m(t;\mathbf{n})$ and $\left([\nabla_{\mathbf{a}}^{-1}]_i - q^{-1}\cdot[\nabla_{\mathbf{a}}^{-1}]_{i+1}\right)m(t;\mathbf{n})$ in the form

$$\int\int(z_i-z_{i+1})G(z_i,z_{i+1})dz_idz_{i+1}$$

for a suitable function $G(z_i,z_{i+1})$ involving integration in all variables except z_i and z_{i+1}. The function G (in both cases) is symmetric in z_i,z_{i+1}, which implies that the above integral is identically zero. Thus, the second condition in Definition 3.1 is also checked.

(3) To check the third condition, observe that if $n_k=0$, then there is no pole $z_k=1$ in the integral over z_k in (1.3). Thus, the nested integral vanishes.

(4) Because for the step initial condition $x_i(0)=-i$, the left-hand side of (1.3) is identically one. We thus need to show that $m(0;\mathbf{n})\equiv 1$. This follows from the residue calculus. Expanding the z_1 contour to infinity, one encounters only the pole at $z_1=0$ (clearly, $z_1=\infty$ is not a pole because of the factors $a_i/(a_i-z_1)$). The residue at $z_1=0$ is equal to $-q^{-(k-1)}$. After having expanded the z_1 contour, the remaining integral is the same as in (1.3) but in $k-1$ variables. Thus, repeating this proceedure, we see that the fourth condition is also verified.

By virtue of Corollary 3.3, this completes the proof of Theorem 1.2. □

4.2 Discussion of the Fredholm Determinantal Formula (Conjecture 1.4)

Assume that $\mathsf{L}>0$. Let us first discuss the growth of the moments of the q-PushASEP.

Lemma 4.2 *For any $k\geq 1$ and $\mathbf{n}\in\mathbb{W}_{\geq 0}^{k,N}$,*

$$\mathbb{E}\left(\prod_{i=1}^{k}q^{x_{n_i}(t)+n_i}\right)\geq \text{const}\cdot e^{\mathsf{L}a_1^{-1}t\cdot e^{k\ln(1/q)}}.$$

Here const *is some positive constant, and* $\mathbf{x}(t)$ *is the* q*-PushASEP started from an arbitrary (non-random) initial configuration* $\mathbf{x}(0) = \mathbf{x}$.

Proof Clearly,

$$\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i} \geq q^{k(x_1(t)+1)} = \text{const} \cdot q^{k(\xi-\eta)},$$

where ξ and η are independent Poisson random variables with parameters $\mathsf{R}a_1 t$ and $\mathsf{L}a_1^{-1} t$, respectively (cf. the end of Sect. 1.1). The constant in front accounts for the initial condition $x_1(0)$. We have

$$\mathbb{E}\, q^{k(\xi-\eta)} = e^{\mathsf{R}a_1 t(q^k-1)+\mathsf{L}a_1^{-1} t(q^{-k}-1)},$$

which yields the claim. □

Let us now explain how one could *formally* obtain the Fredholm determinant (Conjecture 1.4) from the moment formulas of Theorem 1.2 that were proved in Sects. 2 and 3. Using the q-Binomial theorem, write the q-Laplace transform as

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n}; q)_\infty}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{\zeta^k q^{k(x_n(t)+n)}}{(q; q)_k}\right].$$

This identity is rigorous. Next, let us interchange the expectation and the summation. By Lemma 4.2, we get a *divergent* series (of course this is because the interchange of expectation and summation is not justifiable):

$$\sum_{k=0}^{\infty} \frac{\zeta^k\, \mathbb{E}(q^{k(x_n(t)+n)})}{(q; q)_k}. \tag{4.3}$$

However, plugging nested contour integral expressions for the moments $\mathbb{E}(q^{k(x_n(t)+n)})$ afforded by Theorem 1.2, it is possible to formally write (4.3) as a Fredholm determinant. A general scheme for doing this is explained in Sect. 3.1 of [10] and was initially developed in Sect. 3.2 of [5]. It amounts to deforming (and accounting for residues coming from this deformation) the nested contours in (1.3) so that they all become a small circle around $z = 1$. This is a rigorous operation, see [10, Prop. 3.2]. Then one should reorder summands in (4.3), and also use the Mellin-Barnes summation formula. These two latter operations may not be done in a rigorous way in our situation.[3] However, applied to the divergent series (4.3), these steps yield a valid Fredholm determinantal expression of Conjecture 1.4.

It is possible that Conjecture 1.4 (which we have formally argued for above) can be rigorously proved with the help of the algebraic framework of Macdonald processes [5]. Namely, one may be able to show (in a manner similar to [7,8]) that identity (1.7) is a specialization of an algebraic identity which in turn can be established without running into convergence issues. Then (1.7) arises for certain particular values of parameters. Another possible way of resolving the convergence issues is to pass to a suitable discrete-time regularization, cf. the discussion in Sect. 1.7.

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[3] When $\mathsf{L} = 0$, i.e., for the q-TASEP, all operations are valid, see [5] and [10].

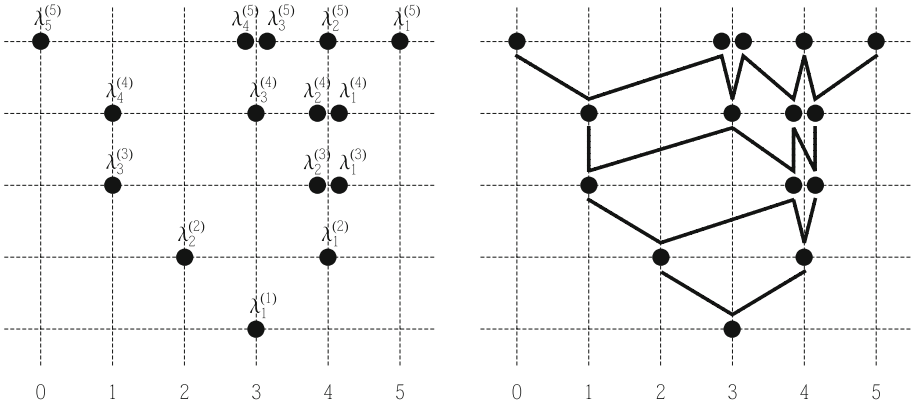

Fig. 4 Particle configuration $\boldsymbol{\lambda}$ and a visualization of the interlacing property

Appendix 1: Dynamics on Two-Dimensional Interlacing Arrays

Here we briefly explain how the q-PushASEP arises as a one-dimensional marginal of a certain two-dimensional stochastic Markov dynamics on interlacing arrays of particles. This two-dimensional dynamics may be constructed as an interpolation between the "push-block" dynamics of [5, Sect. 2.3.3] (see also Dynamics 1 in [12, Sect. 5.5]), and the q-version of the dynamics driven by row insertion RSK algorithm (Dynamics 8 in [12, Sect. 8.2.1]).[4] Note that the latter dynamics has to be reflected, i.e., the particles under this dynamics must jump to the left instead of jumping to the right. Let us now proceed to the definition of the two-dimensional dynamics.

The state space of the two-dimensional dynamics is the set of triangular arrays of interlacing particles which have integer coordinates (see Fig. 4 for an example):

$$\boldsymbol{\lambda} = \{\lambda_j^{(k)} \in \mathbb{Z},\ 1 \le j \le k \le N \colon \lambda_j^{(k)} \le \lambda_{j-1}^{(k-1)} \le \lambda_{j-1}^{(k)}\}.$$

Each particle $\lambda_j^{(k)}$ can jump either to the right or to the left by one.

The right jumps are described as follows. Each particle $\lambda_j^{(k)}$ has an independent exponential clock with rate

$$\mathsf{R} a_k \frac{\left(1 - q^{\lambda_{j-1}^{(k-1)} - \lambda_j^{(k)}}\right)\left(1 - q^{\lambda_j^{(k)} - \lambda_{j+1}^{(k)} + 1}\right)}{1 - q^{\lambda_j^{(k)} - \lambda_j^{(k-1)} + 1}}.$$

When the clock of $\lambda_j^{(k)}$ rings, the particle jumps to the right by one. If this jump of $\lambda_j^{(k)}$ would break the interlacing with upper particles, i.e., if $\lambda_j^{(k)} = \lambda_j^{(k+1)} = \cdots = \lambda_j^{(k+m)}$ (for some

[4] There is no unique way of defining a dynamics on two-dimensional interlacing arrays with these properties. For instance, the "push-block" dynamics may be replaced by the dynamics coming from the q-version of the Robinson-Schensted column insertion algorithm introduced in [22]. See also [12] for more examples and a general discussion.

$m \geq 1$), then all the particles $\lambda_j^{(k+1)}, \ldots, \lambda_j^{(k+m)}$ are instantaneously pushed to the right by one.[5]

The left jumps are different. Only the leftmost particles $\lambda_k^{(k)}$ can independently jump to the left by one. At level k of the array the independent jumps of left particles happen at rate $\mathsf{L} a_k^{-1}$. When any particle $\lambda_j^{(k-1)}$ moves to the left by one (independently or due to a push), it instantaneously forces one of its two immediate upper neighbors, $\lambda_{j+1}^{(k)}$ or $\lambda_j^{(k)}$, to move to the left by one with probabilities ℓ and $1-\ell$, respectively, where

$$\ell = q^{\lambda_j^{(k-1)} - \lambda_{j+1}^{(k)}} \frac{1 - q^{\lambda_{j+1}^{(k)} - \lambda_{j+1}^{(k-1)}}}{1 - q^{\lambda_j^{(k-1)} - \lambda_{j+1}^{(k-1)}}}$$

(here $\lambda_j^{(k-1)}$ denotes the position of the particle *before* the move).

In the description of the dynamics, all factors of the form $(1 - q^{\cdots})$ having nonexistent indices are set to be equal to one. One can readily see that the leftmost particles under this two-sided dynamics on two-dimensional interlacing arrays marginally evolve as a *Markov* process. In the shifted coordinates $x_n(t) := \lambda_n^{(n)}(t) - n$, where $n = 1, \ldots, N$, the evolution of the particles is governed by our q-PushASEP.

The fixed-time distributions of the two-dimensional dynamics $\boldsymbol{\lambda}(t)$ described above are probability measures on interlacing arrays. Let the initial configuration be the *densely packed* one, i.e., $\lambda_j^{(k)}(0) = 0$ for all $1 \leq j \leq k \leq N$. This configuration corresponds to the step initial condition for the q-PushASEP.

After time $t \geq 0$, the distribution of $\boldsymbol{\lambda}(t)$ generalizes the (one-sided) Macdonald processes of [5], [8]. The second Macdonald parameter which is usually denoted by t is set to zero (so that there is no notational conflict with the time parameter); such Macdonald processes are also referred to as the q-Whittaker processes.

Put $a_i \equiv 1$ for simplicity. If L is zero, then $\boldsymbol{\lambda}(t)$ is distributed according to

$$\operatorname{Prob}\big(\boldsymbol{\lambda}(t)\big) = \frac{1}{Z} P_{\lambda^{(1)}}(1) P_{\lambda^{(2)}/\lambda^{(1)}}(1) \cdots P_{\lambda^{(N)}/\lambda^{(N-1)}}(1) Q_{\lambda^{(N)}}(\rho_{\mathsf{R}t}), \tag{5.1}$$

where each $\lambda^{(k)} = (\lambda_1^{(k)} \geq \cdots \geq \lambda_k^{(k)}) \in \mathbb{Z}^k$ is an ordered collection of nonnegative integers, P and Q are the (ordinary and skew) Macdonald symmetric functions [19], and $\rho_{\mathsf{R}t}$ is the so-call Plancherel specialization of $Q_{\lambda^{(N)}}$, e.g., see [5, Sect. 2.2.1]. The Plancherel specialization may be defined, e.g., in terms of the generating function for the *one-row* Macdonald Q functions (i.e., functions indexed by ordered k-tuples of integers with $k = 1$):

$$\sum_{n \geq 0} Q_{(n)}(\rho_t) u^n = e^{tu}. \tag{5.2}$$

On the other hand, for $\mathsf{R} = 0$, the distribution of $-\boldsymbol{\lambda}(t)$ (this simply means negating all components of the interlacing array) is described by the Macdonald process (5.1) (with $\mathsf{R}t$ replaced by $\mathsf{L}t$ in the Plancherel specialization of $Q_{-\lambda^{(N)}}$).

In the general case when L and R are both positive, we expect that the distribution of $\boldsymbol{\lambda}(t)$ (started from the packed initial configuration) is given by a certain *two-sided* version of a

[5] This mechanism of instantaneous pushes is built into the jump rates. Indeed, if the interlacing is broken, then the higher particles have infinite jump rates due to vanishing denominator. Moreover, if the jump of some $\lambda_j^{(k)}$ would break the interlacing with lower particles, then the rate assigned to this jump is equal to zero.

Macdonald process. This two-sided version should necessarily have the form

$$\mathrm{Prob}\left(\boldsymbol{\lambda}(t)\right) = \frac{1}{Z} P_{\lambda^{(1)}}(1) P_{\lambda^{(2)}/\lambda^{(1)}}(1) \cdots P_{\lambda^{(N)}/\lambda^{(N-1)}}(1) \mathcal{M}_N^{(\mathsf{R}t;\mathsf{L}t)}(\lambda^{(N)}) \tag{5.3}$$

for a suitable nonnegative function $\mathcal{M}_N^{(\mathsf{R}t;\mathsf{L}t)}$ on the Nth floor (cf. (5.1)). Note that here the coordinates $\lambda_j^{(k)}$ can be positive or negative (but still must interlace).

Indeed, the product of the P functions, $P_{\lambda^{(1)}}(1) P_{\lambda^{(2)}/\lambda^{(1)}}(1) \dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(1)$, corresponds to a certain *Gibbs* property of Macdonald processes (see [12] for more detail) which is preserved by both the dynamics with $\mathsf{L} = 0$ or $\mathsf{R} = 0$, and thus also by the dynamics with general positive R and L (this is because the Markov generator of the latter process is a linear combination of the two "pure" right and left generators).

When $N = 1$, the measure (5.3) is simply the convolution of the two "pure" one-sided measures (note that $P_{\lambda^{(1)}}(1) = 1$), and so the generating function for $\mathcal{M}_1^{(\mathsf{R}t;\mathsf{L}t)}$ takes the form (cf. (1.4))

$$\sum_{n \in \mathbb{Z}} \mathcal{M}_1^{(\mathsf{R}t;\mathsf{L}t)}(n) u^n = e^{t(\mathsf{R}u + \mathsf{L}u^{-1})}. \tag{5.4}$$

Note that in the one-sided case, the one-row functions $Q_{(n)}$ generate the algebra of symmetric functions to which all the Q_λ's (with λ having nonnegative parts) belong. Thus, identity (5.2) defines $Q_\lambda(\rho_t)$ for all λ, and one can proceed to the definition of the one-sided Macdonald processes. In the two-sided case, it is not clear what algebraic structures are responsible for the passage from $\mathcal{M}_1^{(\mathsf{R}t;\mathsf{L}t)}(n)$ (viewed as one-row functions $Q_{(n)}(\rho^{\text{two-sided}}_{\mathsf{R}t;\,\mathsf{L}t})$) to the functions Q_λ with λ general. Therefore, at this point we are left to view (5.3) as a *defn* of the two-sided Plancherel specialization of the general Macdonald symmetric functions $Q_{\lambda^{(N)}}(\rho^{\text{two-sided}}_{\mathsf{R}t;\,\mathsf{L}t}) := \mathcal{M}_N^{(\mathsf{R}t;\mathsf{L}t)}(\lambda^{(N)})$ corresponding to not necessarily one-row λ's. We do not further develop the theory of two-sided Macdonald processes in the present paper, but note that the desire to understand the distribution of the two-sided dynamics on two-dimensional interlacing integer arrays (5.3), as well as the question of proving Conjecture 1.4, provide some motivation for these objects.[6]

Appendix 2: Formal Scaling Limit as $q \nearrow 1$

Consider the scaling of the two-dimensional dynamics described by [5, Thm. 4.1.21]:

$$q = e^{-\varepsilon}, \qquad t = \varepsilon^{-2}\tau, \qquad a_k = e^{-\varepsilon \mathsf{a}_k}, \qquad k = 1, \dots, N;$$
$$\lambda_j^{(k)} = C(\varepsilon;\tau) - (k + 1 - 2j)\varepsilon^{-1} \log \varepsilon + G_j^{(k)} \varepsilon^{-1}, \; k = 1, \dots, N, \; j = 1, \dots, k.$$

Here $\tau > 0$ is the scaled time, $C(\varepsilon;\tau)$ represents the global shift of the coordinate system, and $(\mathsf{a}_1, \dots, \mathsf{a}_N)$ are the scaled values of the a_j's. In the one-sided setting, the Macdonald processes (5.1) converge under this scaling with $C(\varepsilon;\tau) = \varepsilon^{-2}\tau$ to Whittaker processes introduced in [21], see also [5, Ch. 4].

As explained in [5, Sects. 4.1 and 5.2] and [12, Sect. 8.4], the q-TASEP and the q-PushTASEP (i.e., the "pure" dynamics corresponding to $\mathsf{L} = 0$ or $\mathsf{R} = 0$) under this scaling with $C(\varepsilon;\tau) = +\varepsilon^{-2}\tau$ or $C(\varepsilon;\tau) = -\varepsilon^{-2}\tau$, respectively, correspond to stochastic differential equations (SDEs) which describe evolution of the hierarchy of the free energies of the

[6] The $q = 0$ version of the two-sided Macdonald processes (i.e., the two-sided Schur processes) was introduced and investigated in [4].

O'Connell–Yor semi-discrete directed polymer [21,23].[7] These free energies may also be represented as logarithms of solutions to the semi-discrete stochastic heat equation

$$du_j(t) = u_{j-1}(t) - u_j(t) + u_j(t)dB_j(t), \qquad j = 1, \dots, N; \qquad u(0, N) = \delta_{1N}, \tag{6.1}$$

where $B_1, \dots, B_N$ are independent standard Brownian motions (possibly with linear drifts).

Let us now discuss the formal scaling limit of the two-sided (q-PushASEP) evolution, i.e., with $\mathsf{R}, \mathsf{L} > 0$. Let us scale the R and L parameters around 1:

$$\mathsf{R} = e^{-\varepsilon \mathsf{r}}, \qquad \mathsf{L} = e^{-\varepsilon \mathsf{l}},$$

where $\mathsf{r}, \mathsf{l} \in \mathbb{R}$ are the scaled values. Moreover, one should take the global shift $C(\varepsilon; \tau)$ to be zero (one should think that the shifts $\pm\varepsilon^{-2}\tau$ corresponding to the "pure" right and left dynamics compensate each other).

We will focus only on the leftmost particles $\lambda_k^{(k)}$, the whole array can be considered in a similar way. The limiting SDEs for the quantities $G_k^{(k)}$ look as (with the agreement that $G_0^{(0)} \equiv 0$)

$$dG_k^{(k)} = \sqrt{2} \cdot dW_k + \left(-2\mathsf{a}_k + \mathsf{l} - \mathsf{r} - e^{G_k^{(k)} - G_{k-1}^{(k-1)}}\right) d\tau, \qquad k = 1, \dots, N. \tag{6.2}$$

Here $W_1, \dots, W_N$ are independent standard driftless Brownian motions.

Remark 6.1 The $G_k^{(k)}$'s satisfying (6.2) can also be formally interpreted as logarithms of solutions to the semi-discrete stochastic heat equation (6.1). The terms $(-2\mathsf{a}_k + \mathsf{l} - \mathsf{r})$ are absorbed into drifts of the Brownian motions $B_1, \dots, B_N$ in (6.1).

Calculations leading to (6.2) are analogous to what is done in [5, Sect. 5.4.4] and [12, Sect. 8.4.4]. First, note that our scaling dictates

$$G_k^{(k)}(\tau + d\tau) - G_k^{(k)}(\tau) = \frac{\lambda_k^{(k)}(\tau + \varepsilon^{-2}d\tau) - \lambda_k^{(k)}(\tau)}{\varepsilon^{-1}}. \tag{6.3}$$

Right jumps of the particle $\lambda_k^{(k)}$ occur with probability

$$\mathsf{R}a_k(1 - q^{\lambda_{k-1}^{(k-1)} - \lambda_k^{(k)}}) = 1 - \varepsilon(\mathsf{a}_k + \mathsf{r} + e^{G_k^{(k)} - G_{k-1}^{(k-1)}}) + O(\varepsilon^2). \tag{6.4}$$

Left jumps happen at rate

$$\mathsf{L}a_k^{-1} = e^{-\varepsilon(\mathsf{l} - \mathsf{a}_k)} = 1 - (\mathsf{l} - \mathsf{a}_k)\varepsilon + O(\varepsilon^2), \tag{6.5}$$

and, moreover, the particle $\lambda_k^{(k)}$ is pushed to the left by $\lambda_{k-1}^{(k-1)}$ with probability $\varepsilon e^{G_k^{(k)} - G_{k-1}^{(k-1)}} + O(\varepsilon^2)$. One should multiply this probability by the change in the position of $\lambda_{k-1}^{(k-1)}$ during time interval $t = \varepsilon^{-2}\tau$, this yields

$$\left(\varepsilon e^{G_k^{(k)} - G_{k-1}^{(k-1)}} + O(\varepsilon^2)\right)\left(\varepsilon^{-1}\left(G_{k-1}^{(k-1)}(\tau + d\tau) - G_{k-1}^{(k-1)}(\tau)\right)\right) = O(1). \tag{6.6}$$

The constant factors in (6.4) and (6.5) give rise to the change in the position of $\lambda_k^{(k)}$ (during time interval $\varepsilon^{-2}d\tau$) equal to the difference of two independent Poisson random variables with mean $\varepsilon^{-2}d\tau$. In view of (6.3), these summands correspond to the differential of the Brownian motion $\sqrt{2} \cdot dW_k(\tau)$. The summands of order ε in (6.4)–(6.5) give rise to constant terms. The constant term (6.6) is multiplied by $\frac{1}{\varepsilon^{-1}}$, and thus vanishes.

[7] These are our quantities $G_k^{(k)}$ in the description of the scaling.

References

1. Alimohammadi, M., Karimipour, V., Khorrami, M.: A two-parametric family of asymmetric exclusion processes and its exact solution. J. Stat. Phys. **97**(1–2), 373–394 (1999). arXiv:cond-mat/9805155
2. Balász, M., Komjáthy, J., Seppäläinen, T.: Microscopic concavity and fluctuation bounds in a class of deposition processes. Ann. Inst. H. Poincaré B **48**, 151–187 (2012)
3. Bethe, H.: Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette. (On the theory of metals. I. Eigenvalues and eigenfunctions of the linear atom chain). Zeitschrift fur Physik **71**, 205–226 (1931)
4. Borodin, A.: Schur dynamics of the Schur processes. Adv. Math. **228**(4), 2268–2291 (2011). arXiv:1001.3442 [math.CO]
5. Borodin, A., Corwin, I.: Macdonald processes. Probab. Theor. Relat. Fields **158**(1–2), 225–400 (2014). arXiv:1111.4408 [math.PR]
6. Borodin, A., Corwin, I.: Discrete time q-TASEPs. Int. Math. Res. Not. **2**, 499–537 (2015). doi:10.1093/imrn/rnt206, arXiv:1305.2972 [math.PR]
7. Borodin, A., Corwin, I., Ferrari, P., Veto, B.: Height fluctuations for the stationary KPZ equation (2013, preprint). arXiv:1308.3475 [math-ph]
8. Borodin, A., Corwin, I., Gorin, V., Shakirov, S.: Observables of Macdonald processes. Trans. Am. Math. Soc. (2013, to appear). arXiv:1306.0659 [math.PR]
9. Borodin, A., Corwin, I., Petrov, L., Sasamoto, T.: Spectral theory for the q-Boson particle system. Compos. Math. **151**(1), 1–67 (2015). arXiv:1308.3475 [math-ph]
10. Borodin, A., Corwin, I., Sasamoto, T.: From duality to determinants for q-TASEP and ASEP. Ann. Probab. **42**(6), 2314–2382 (2014). arXiv:1207.5035 [math.PR]
11. Borodin, A., Ferrari, P.: Large time asymptotics of growth models on space-like paths I: PushASEP. Electron. J. Probab. **13**, 1380–1418 (2008). arXiv:0707.2813 [math-ph]
12. Borodin, A., Petrov, L.: Nearest neighbor Markov dynamics on Macdonald processes. Adv. Math. (2013, to appear). arXiv:1305.5501 [math.PR]
13. Calabrese, P., Le Doussal, P., Rosso, A.: Free-energy distribution of the directed polymer at high temperature. Eur. Phys. Lett. **90**(2), 20002 (2010)
14. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equation. McGraw Hill, New York (1955)
15. Dotsenko, V.: Replica Bethe ansatz derivation of the Tracy-Widom distribution of the free energy fluctuations in one-dimensional directed polymers. J. Stat. Mech. (07), P07010 (2010). arXiv:1004.4455 [cond-mat.dis-nn]
16. Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence. Wiley-Interscience, New York (1986)
17. Liggett, T.: Interacting Particle Systems. Springer, New York (1985)
18. Liggett, T.: Stochastic Interacting Systems: Contact, Voter and Exclusion Processes, Grundlehren de mathematischen Wissenschaften, vol. 324. Springer, New York (1999)
19. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, New York (1995)
20. Matveev, K., Petrov, L.: In preparation
21. O'Connell, N.: Directed polymers and the quantum Toda lattice. Ann. Probab. **40**(2), 437–458 (2012). arXiv:0910.0069 [math.PR]
22. O'Connell, N., Pei, Y.: A q-weighted version of the Robinson–Schensted algorithm. Electron. J. Probab. **18**(95), 1–25 (2013). arXiv:1212.6716 [math.CO]
23. O'Connell, N., Yor, M.: Brownian analogues of Burke's theorem. Stoch. Process. Appl. **96**(2), 285–304 (2001)
24. Povolotsky, A.: On integrability of zero-range chipping models with factorized steady state. J. Phys. A **46**(465205) (2013). arXiv:1308.3250 [math-ph]
25. Povolotsky, A., Mendes, J.F.F.: Bethe ansatz solution of discrete time stochastic processes with fully parallel update. J. Stat. Phys. **123**(1), 125–166 (2006). arXiv:cond-mat/0411558 [cond-mat.stat-mech]
26. Sasamoto, T., Wadati, M.: Exact results for one-dimensional totally asymmetric diffusion models. J. Phys. A **31**, 6057–6071 (1998)
27. Spitzer, F.: Interaction of Markov processes. Adv. Math. **5**(2), 246–290 (1970)