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Symmetric and nonsymmetric Koornwinder polynomials in the $q \to 0$ limit

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Abstract Koornwinder polynomials are a 6-parameter BC_n-symmetric family of Laurent polynomials indexed by partitions, from which Macdonald polynomials can be recovered in suitable limits of the parameters. As in the Macdonald polynomial case, standard constructions via difference operators do not allow one to directly control these polynomials at $q = 0$. In the first part of this paper, we provide an explicit construction for these polynomials in this limit, using the defining properties of Koornwinder polynomials. Our formula is a first step in developing the analogy between Hall–Littlewood polynomials and Koornwinder polynomials at $q = 0$. In the second part of the paper, we provide a construction for the nonsymmetric Koornwinder polynomials in the same limiting case; this parallels work by Descouens–Lascoux in type A. As an application, we prove an integral identity for Koornwinder polynomials at $q = 0$.

Keywords Koornwinder polynomials · Orthogonal polynomials · Symmetric functions · Hecke algebras

Mathematics Subject Classification 33D52 · 33D45

1 Introduction

In [9], Macdonald introduced a very important family of multivariate q-orthogonal polynomials associated with a root system. These polynomials, and their connec-

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tions to representation theory, combinatorics, and algebra, have been well studied and are an active area of research. For the type A root system, Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ contain many well-known families of symmetric functions as special cases: for example, the Schur, Hall–Littlewood, and Jack polynomials occur at $q = t$, $q = 0$, and $t = q^\alpha$, $q \to 1$, respectively. The existence of the top-level Macdonald polynomials (i.e., those that have parameters q and t) was proved by exhibiting a suitable operator, which has these polynomials as its eigenfunctions. A particularly important degeneration of the Macdonald polynomials is obtained in the $q = 0$ limit where one obtains zonal spherical functions on semisimple p-adic groups. In fact, Macdonald provides an explicit formula for the spherical functions of the Chevalley group $G(\mathbb{Q}_p)$ in terms of the root data for the group G [7]. In type A, one obtains the symmetrization formula for the Hall–Littlewood polynomials [8, Ch. III], which arise as zonal spherical functions for $GL_n(\mathbb{Q}_p)$.

The Askey–Wilson polynomials $p_n(y; a, b, c, d|q)$ for $n \in \mathbb{Z}_+$ are a fundamental class of orthogonal polynomials associated with the root system BC_1 [1]. They are q-polynomials in one variable, with parameters a, b, c, d. In [6], Koornwinder introduced a multivariate generalization of the Askey–Wilson polynomials associated with the nonreduced root system BC_n. These polynomials depend on six parameters q, t, a, b, c, d, and at $n = 1$, the t-dependence drops out and one recovers the Askey–Wilson polynomials. Koornwinder polynomials contain the three-parameter families of Macdonald polynomials associated with the pairs (BC_n, B_n) (obtained by replacing a, b, c, d, t by $q^{1/2}, -q^{1/2}, ab^{1/2}, -b^{1/2}, t$, respectively) and (BC_n, C_n) (obtained by replacing a, b, c, d, t, q by $ab^{1/2}, qab^{1/2}, -b^{1/2}, -qb^{1/2}, t, q^2$, respectively). As in the Macdonald polynomial case, the existence of these polynomials was proved by using q-difference operators; however, these behave badly as $q \to 0$. Given the relationship between Macdonald and Koornwinder polynomials, a natural question one can ask is whether there exists an explicit construction for the latter polynomials at $q = 0$, thereby providing an analog of the construction of Hall–Littlewood polynomials for this family; we will address this in the first part of the paper.

As in the case of Macdonald polynomials, Koornwinder polynomials are associated with the double affine Hecke algebra. From this viewpoint, the parameters above are expressed in terms of the geometry of the affine root system. These parameters are denoted by $t_0, \check{t}_0, t_n, \check{t}_n, t$ in the literature (in addition to parameter q) [14,15]. When one uses these parameters, the orthogonality density is a q-deformation of the Plancherel measure on the space of zonal spherical functions on p-adic groups, mentioned in the first paragraph. There is a reparametrization that allows one to translate between the two sets of parameters [see Eq. (2)]. In this paper, we will work with the Askey–Wilson parametrization a, b, c, d, t.

In the first part of this work, we use the defining properties (i.e., orthogonality and triangularity) of Koornwinder polynomials to provide a closed formula in the $q = 0$ limit:

Theorem 1.1 *Let λ be a partition with $l(\lambda) \leq n$ and $|a'|, |b'|, |t|, |a|, |b|, |c|, |d| < 1$. Then, $K_\lambda(z_1, \ldots, z_n; 0, t; a', b'; a, b, c, d)$ is given by*

$$\frac{1}{v_\lambda(t;a',b';a,b,c,d)}\sum_{w\in B_n} w\left(\prod_{1\le i\le n} u_\lambda(z_i)\prod_{1\le i<j\le n}\frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j}\frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\right), \tag{1}$$

where

$$u_\lambda(z_i)=\begin{cases}\dfrac{\left(1-a'z_i^{-1}\right)\left(1-b'z_i^{-1}\right)}{1-z_i^{-2}} & \text{if } \lambda_i=0,\\ z_i^{\lambda_i}\dfrac{\left(1-az_i^{-1}\right)\left(1-bz_i^{-1}\right)\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right)}{1-z_i^{-2}} & \text{if } \lambda_i>0.\end{cases}$$

Here, the sum is over the hyperoctahedral group B_n and a formula for $v_\lambda(t;a',b';a,b,c,d)$ is given in the first section. Verifying that the objects defined above satisfy the right type of symmetry is immediate; proving that they are indeed Laurent polynomials of correct leading degree which form an orthogonal family with respect to the Koornwinder density at $q=0$ is the difficult part. We note the difference in the univariate terms $u_\lambda(z_i)$ corresponding to zero parts versus nonzero parts (i.e., $\lambda_i=0$ versus $\lambda_i>0$), which makes the formula more complicated than existing symmetrization-type formulas. We also mention that our methods allow us to obtain families with two extra parameters a',b', as indicated above. To obtain Macdonald's two-parameter family of type B,C polynomials, one sets $a=a',b=b',c=0,d=0$; this eliminates the difference between univariate terms corresponding to zero and nonzero parts.

The second part of the paper deals with the nonsymmetric theory for Koornwinder polynomials in the limit $q\to 0$, in the Askey–Wilson parametrization. The $q\to 0$ limit was first studied by Ion in [4,5], but using parameters $(t_0,\check{t}_0,t_n,\check{t}_n)$ mentioned above, to preserve the representation-theoretic connection. In terms of Askey–Wilson parameters, this amounts to allowing $q,c,d\to 0$. Thus, we investigate a deformation, in which one allows nonzero parameters c,d as well. As in the symmetric case, our motivation will be from the special functions point of view—we will use the definition of these polynomials as orthogonal functions. We provide an explicit formula when these polynomials are indexed by partitions:

Theorem 1.2 *Let λ be a partition with $l(\lambda)\le n$, and $|c|,|d|<1$. Then $U_\lambda(z_1,\ldots,z_n;0,t;a,b,c,d)$ is given by*

$$\prod_{\lambda_i>0} z_i^{\lambda_i}\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right).$$

Verifying that this is a Laurent polynomial of correct degree is straightforward; showing orthogonality is the involved part. We then use elements of the affine Hecke algebra of type BC to recursively obtain *all* nonsymmetric Koornwinder polynomials in this limit. Our approach is similar in spirit to that of Descouens and Lascoux in the type A setting [2]. However, in that case, the polynomial at a dominant weight is a monomial; as indicated above, this is not the case in type BC. The authors then use the Yang-Baxter elements to recursively obtain formulas for all weights. Our results on

nonsymmetric Koornwinder polynomials in this paper may be viewed as a parallel of that paper, for type BC.

An important use of the previous two theorems is the application of such formulas to prove integral and summation identities. Indeed, in a previous paper [18], we used the explicit symmetrization formula for Hall–Littlewood polynomials of type A to strengthen and generalize results of Rains–Vazirani at $q = 0$ [11]. Explicit formulas of these polynomials may also be used to prove branching rules and Pieri identities, as well as to deduce information about change-of-basis coefficients. In this paper, we use the first theorem above to prove the following result:

Theorem 1.3 *Let λ be a partition with $l(\lambda) \leq n$, and $|a|, |b| < 1$. Then, the integral*

$$\int_T K_\lambda(z_1, \ldots, z_n; t^2; a, b; a, b, ta, tb)\tilde{\Delta}_K^{(n)}(z; t; \pm\sqrt{t}, a, b)dT$$

vanishes if λ is not an even partition. Moreover, if λ is an even partition, the integral is equal to

$$\frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \frac{N_\lambda(t; \pm\sqrt{t}, a, b)v_{\lambda+}(t; \pm\sqrt{t}, a, b)}{v_{\lambda+}(t^2; a, b, ta, tb)}.$$

We note that Rains–Vazirani [11] proved the vanishing part of the theorem at the top level, but were unable to calculate the evaluation when the integral does not vanish. Moreover, their methods do not directly carry through to $q = 0$, since the operators they use are ill-defined in this limit. Our technique provides a direct proof in this limit and allows us to compute this rational function explicitly in the nonvanishing case, as given above. We also provide self-contained proofs of the constant term evaluations and norm evaluations in both the symmetric and nonsymmetric cases (Theorems 2.8, 3.5 and Theorems 2.9, 3.7). We mention that, in the symmetric case, the constant term evaluation at the top level is a famous result of Gustafson [3]; this in turn is a multivariate generalization of a result of Askey and Wilson [1] and a q-generalization of Selberg's beta integral [13]. We note that Gustafson's approach requires $q \neq 0$, so one cannot directly apply that argument in this limiting case (although one can still obtain the evaluation by taking limits of his q-level formula). We give a self-contained proof of Gustafson's result in the $q = 0$ limit (Theorem 2.8 within the paper). Such proofs are useful, since the technique may be applied to formulate and prove other identities.

As mentioned above, this paper has two main components: the first part deals with the symmetric theory at $q = 0$, while the second deals with the nonsymmetric theory in the same limit. The first section of each part sets up the relevant notation, reviews some background material, and defines the polynomials in question. The second section of each part consists of the main theorems and proofs, in particular we prove that these are indeed the symmetric and nonsymmetric Koornwinder polynomials at $q = 0$, respectively. The third section of the first part contains an application of our formula to this work: We use the construction of the Koornwinder polynomials at $q = 0$ to prove an integral identity.

2 Symmetric Hall–Littlewood polynomials of type BC

2.1 Background and notation

We will first review some relevant notations before introducing the polynomials that are these subject of this paper; a good reference is [8, Ch. 1].

We first mention that, in our formulas, we will use the Askey–Wilson parameters (a, b, c, d) in addition to q, t. We note that, in the literature, Hecke parameters $(t_0, \check{t}_0, t_n, \check{t}_n)$ are also used (see [6,14,15], for example). There is a translation between these two sets of parameters according to the following reparametrization:

$$\{a, b, c, d, t\} \leftrightarrow \left\{t_n\check{t}_n, -t_n\check{t}_n^{-1}, t_0\check{t}_0 q^{1/2}, -t_0\check{t}_0^{-1} q^{1/2}, t^2\right\}. \tag{2}$$

Recall that a partition λ is a weakly decreasing string of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, in which some of the λ_i may be zero. We will denote the set of partitions by Λ^+. We call the λ_i the "parts" of λ. We write $l(\lambda) = \max\{k \geq 0 | \lambda_k \neq 0\}$ (the "length") and $|\lambda| = \sum_{i=1}^n \lambda_i$ (the "weight"). A string $\mu = (\mu_1, \ldots, \mu_n)$ of integers (not necessarily nonincreasing or positive) is called a composition of $|\mu| = \sum_{i=1}^n |\mu_i|$. We will say λ is an "even partition" if all parts of λ are even; in this case, we use the notation $\lambda = 2\mu$ where $\mu_i = \lambda_i/2$ for all i. We will also say λ has "all parts occurring with even multiplicity" if the conjugate partition λ' is an even partition. A composition λ is an element of $\mathbb{Z}^n$ for some $n \geq 1$; we will denote this set by Λ.

We briefly recall some orderings on compositions.

Definition 2.1 Let $\leq$ denote the dominance partial ordering on compositions, i.e., $\mu \leq \lambda$ if and only if

$$\sum_{1 \leq i \leq k} \mu_i \leq \sum_{1 \leq i \leq k} \lambda_i$$

for all $k \geq 1$ (and $\mu < \lambda$ if $\mu \leq \lambda$ and $\mu \neq \lambda$). Let $\overset{\text{lex}}{\leq}$ denote the reverse lexicographic ordering: $\mu \overset{\text{lex}}{\leq} \lambda$ if and only if $\lambda = \mu$ or the first nonvanishing difference $\lambda_i - \mu_i$ is positive.

Note that $\overset{\text{lex}}{\leq}$ is a total ordering.

Lemma 2.2 *Let $\mu, \lambda \in \mathbb{Z}^n$ such that $\mu \leq \lambda$. Then, $\mu \overset{lex}{\leq} \lambda$.*

Proof The claim is clearly true if $\mu = \lambda$, so suppose $\mu < \lambda$. If $\mu_1 < \lambda_1$, we are done; otherwise, $\mu_1 = \lambda_1$ and $\mu_2 \leq \lambda_2$ since $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$. Iterating this argument produces an integer i in $\{1, \ldots, n\}$ such that $\mu_1 = \lambda_1, \ldots, \mu_{i-1} = \lambda_{i-1}$ and $\mu_i < \lambda_i$. Thus, $\mu \overset{\text{lex}}{<} \lambda$ as desired. □

Definition 2.3 Let μ and λ be two elements of $\mathbb{Z}^n$. We will write μ^+ for the unique dominant weight in the BC_n orbit of μ (that is, the partition obtained by rearranging

the absolute values of the parts of μ in nonincreasing order). Then, we write $\mu \prec \lambda$ if and only if either 1) $\mu^+ < \lambda^+$ or if 2) $\mu^+ = \lambda^+$ and $\mu \leq \lambda$, and in either case $\mu \neq \lambda$.

Remarks This part will mostly deal with partitions and the dominance and reverse lexicographic orderings. Compositions, and the extended dominance ordering appearing in Definition 2.3, will become relevant in the following part that deals with the nonsymmetric theory.

Let λ be a partition. Let $m_i(\lambda)$ be the number of λ_j equal to i for each $i \geq 0$. Then, we define:

$$\begin{aligned} &v_\lambda(t; a', b'; a, b, c, d) \\ &= \left(\prod_{i\geq 0}\prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t}\right)\left(\prod_{i=1}^{m_1(\lambda)}\left(1-abcdt^{i-1+2m_0(\lambda)}\right)\right)\left(\prod_{i=1}^{m_0(\lambda)}\left(1-a'b't^{i-1}\right)\right), \end{aligned} \tag{3}$$

and

$$v_{\lambda+}(t; a, b, c, d) = \left(\prod_{i\geq 1}\prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t}\right)\left(\prod_{i=1}^{m_1(\lambda)}\left(1-abcdt^{i-1+2m_0(\lambda)}\right)\right). \tag{4}$$

Note the comparison with the factors making the Hall–Littlewood polynomials monic in [8, Ch. III]. Also note that

$$v_\lambda(t; a', b'; a, b, c, d) = v_{\lambda+}(t; a, b, c, d) v_{0^{m_0(\lambda)}}(t; a', b'; a, b, c, d).$$

For simplicity of notation, we will write v_λ, $v_{\lambda+}$, or $v_\lambda(t)$, $v_{\lambda+}(t)$, when the other parameters are clear from the context.

Throughout this paper, we will use

$$\begin{aligned} T = T_n &= \{(z_1, \ldots, z_n) : |z_1| = \cdots = |z_n| = 1\}, \\ dT = dT_n &= \prod_{1\leq j\leq n} \frac{dz_j}{2\pi\sqrt{-1}z_j} \end{aligned}$$

to denote the n-torus and Haar measure, respectively. When the number of variables may be unclear from context, we will use T_n, dT_n; otherwise, we will use T, dT. Since many of the objects we will be dealing with are functions of n variables, we will often use the superscript (n) with z in the argument, instead of $(z_1, \ldots, z_n)$. We define the (infinite) q-Pochhammer symbol

$$(a; q) = \prod_{k\geq 0}(1 - aq^k)$$

and let $(a_1, a_2, \ldots, a_l; q)$ denote $(a_1; q)(a_2; q)\cdots(a_l; q)$.

We recall the symmetric Koornwinder density:

$$\tilde{\Delta}_K^{(n)}(z;q,t;a,b,c,d) = \frac{(q;q)^n}{2^n n!} \prod_{1\le i\le n} \frac{\left(z_i^{\pm 2};q\right)}{\left(az_i^{\pm 1}, bz_i^{\pm 1}, cz_i^{\pm 1}, dz_i^{\pm 1};q\right)} \prod_{1\le i<j\le n} \frac{\left(z_i^{\pm 1}z_j^{\pm 1};q\right)}{\left(tz_i^{\pm 1}z_j^{\pm 1};q\right)}.$$

and the Koornwinder polynomials in six parameters (q,t,a,b,c,d), denoted by $K_\lambda^{(n)}(z;q,t;a,b,c,d)$ [6]. Since we are concerned with $q=0$ degenerations of Koornwinder polynomials, we will be interested in the symmetric Koornwinder density in the same limiting case:

$$\tilde{\Delta}_K^{(n)}(z;0,t;a,b,c,d) = \frac{1}{2^n n!} \prod_{1\le i\le n} \frac{\left(1-z_i^{\pm 2}\right)}{\left(1-az_i^{\pm 1}\right)\left(1-bz_i^{\pm 1}\right)\left(1-cz_i^{\pm 1}\right)\left(1-dz_i^{\pm 1}\right)} \prod_{1\le i<j\le n} \frac{\left(1-z_i^{\pm 1}z_j^{\pm 1}\right)}{\left(1-tz_i^{\pm 1}z_j^{\pm 1}\right)}, \tag{5}$$

where we write $(1-z_i^{\pm 2})$ for the product $(1-z_i^2)(1-z_i^{-2})$ and $(1-z_i^{\pm 1}z_j^{\pm 1})$ for $(1-z_iz_j)(1-z_i^{-1}z_j^{-1})(1-z_i^{-1}z_j)(1-z_iz_j^{-1})$, etc. We will write $\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)$ to denote this density.

Using the above density, we let

$$N_\lambda(t;a,b,c,d) = \frac{1}{v_{\lambda+}(t)} \int_T \tilde{\Delta}_K^{(m_0(\lambda))}(z;t;a,b,c,d)dT. \tag{6}$$

We note that, at the top level, the explicit evaluation of the integral above is a famous result of Gustafson [3]. The arguments do not directly apply at $q=0$, although one can obtain the evaluation via an appropriate limit. In keeping with the theme of this work, we will provide a self-contained proof of the evaluation of this integral in Theorem 2.8. This will provide an explicit formula for the quantity $N_\lambda(t;a,b,c,d)$. Our method, using induction and residue calculation to obtain a recursion, is very similar to Gustafson's argument at the top level.

For simplicity of notation, we will use the shorthand notation N_λ, $\tilde{\Delta}_K^{(n)}$, etc., when the parameters are clear from the context.

Finally, we explain some notation involving elements of the hyperoctahedral group, B_n. An element in B_n is determined by specifying a permutation $\rho \in S_n$ as well as a sign choice $\epsilon_\rho(i)$, for each $1 \le i \le n$. Thus, ρ acts on the subscripts of the variables, for example, by

$$\rho(z_1 \dots z_n) = z_{\rho(1)}^{\epsilon_\rho(1)} \dots z_{\rho(n)}^{\epsilon_\rho(n)}.$$

If $\rho(i) = 1$, we will say that z_1 occurs in position i of ρ. We also write

$$\text{“}z_i \prec_\rho z_j\text{”}$$

if $i = \rho(i')$ and $j = \rho(j')$ for some $i' < j'$, i.e., z_i appears to the left of z_j in the Laurent monomial $z_{\rho(1)}^{\epsilon_\rho(1)} \cdots z_{\rho(n)}^{\epsilon_\rho(n)}$. We also define $\epsilon_\rho(z_i)$ to be $\epsilon_\rho(i')$ if $i = \rho(i')$, i.e., it is the exponent (± 1) on z_i in $z_{\rho(1)}^{\epsilon_\rho(1)} \cdots z_{\rho(n)}^{\epsilon_\rho(n)}$.

We begin by establishing the existence of the $q \to 0$ limit of the symmetric Koornwinder polynomials.

Proposition 2.4 *Let $|t|, |a|, |b|, |c|, |d| < 1$. Then the $q \to 0$ limit of the Koornwinder polynomials $K_\lambda^{(n)}(z; q, t; a, b, c, d)$ exists.*

Proof We use induction on λ with respect to a total order extending $\leq$, for example, $\overset{\text{lex}}{\leq}$. The result is clear for $\lambda = 0$. Now suppose that the result holds for $\nu \leq \lambda$, we will show it holds for λ. We have the following Gram–Schmidt formula for the Koornwinder polynomials (see, e.g., [16, Eq 2.32]), which follows by expanding $m_\lambda(z) - K_\lambda^{(n)}(z; q, t; a, b, c, d)$ in terms of the orthogonal basis $K_\mu^{(n)}(z; q, t; a, b, c, d)$:

$$\begin{aligned} &K_\lambda^{(n)}(z; q, t; a, b, c, d) \\ &\quad = m_\lambda(z) - \sum_{\mu<\lambda} \frac{\langle m_\lambda(z), K_\mu^{(n)}(z; q, t; a, b, c, d)\rangle_q}{\langle K_\mu^{(n)}(z; q, t; a, b, c, d), K_\mu^{(n)}(z; q, t; a, b, c, d)\rangle_q} \\ &\qquad K_\mu^{(n)}(z; q, t; a, b, c, d), \end{aligned}$$

where

$$\langle f(z), g(z)\rangle_q = \int_{T_n} f(z_1, \ldots, z_n) g\left(z_1^{-1}, \ldots, z_n^{-1}\right) \tilde{\Delta}_K^{(n)}(z; q, t; a, b, c, d) dT_n.$$

By induction, we just need to check that the inner product in the numerators is well defined in the limit $q \to 0$ and that the norms in the denominator are well defined and nonzero. The former follows from the induction hypothesis and (5), and the latter follows immediately from the explicit formulas for the norms of the Koornwinder polynomials [17]. □

We finally define the main objects of this section, which we will eventually show agree with the $q \to 0$ limit of the previous proposition.

Definition 2.5 Let λ be a partition with $l(\lambda) \leq n$ and $|a'|, |b'|, |t|, |a|, |b|, |c|, |d| < 1$. Define $K_\lambda(z_1, \ldots, z_n; t; a', b'; a, b, c, d)$, indexed by λ by

$$\frac{1}{v_\lambda(t; a', b'; a, b, c, d)} \sum_{w \in B_n} w\left(\prod_{1 \leq i \leq n} u_\lambda(z_i) \prod_{1 \leq i < j \leq n} \frac{1 - t z_i^{-1} z_j}{1 - z_i^{-1} z_j} \frac{1 - t z_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}}\right), \tag{7}$$

where

$$u_\lambda(z_i) = \begin{cases} \frac{\left(1-a'z_i^{-1}\right)\left(1-b'z_i^{-1}\right)}{1-z_i^{-2}} & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i} \frac{\left(1-az_i^{-1}\right)\left(1-bz_i^{-1}\right)\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right)}{1-z_i^{-2}} & \text{if } \lambda_i > 0. \end{cases}$$

Remarks Note the slight abuse of notation: $K_\lambda^{(n)}(z; q, t; a, b, c, d)$ are the polynomials of [6], and above we are *defining* $K_\lambda^{(n)}(z; t; a', b'; a, b, c, d)$, which we will independently prove is the $q = 0$ limit of the former polynomials.

Remarks We note that the K_λ have two extra parameters a' and b'. In particular, the arguments below that prove that this is indeed the Koornwinder polynomial at $q = 0$ work for any choice of a', b'. However, we leave in arbitrary a', b' (as opposed to the choice ± 1) because the resulting form is useful in applications; an example that illustrates this appears in the last section.

We will also let

$$R_\lambda^{(n)}(z; t; a', b'; a, b, c, d) = v_\lambda(t; a', b'; a, b, c, d) K_\lambda^{(n)}(z; t; a', b'; a, b, c, d), \quad (8)$$

and for $w \in B_n$, we let

$$R_{\lambda,w}^{(n)}(z; t; a', b'; a, b, c, d) = w\left(\prod_{1\le i\le n} u_\lambda(z_i) \prod_{1\le i<j\le n} \frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j} \frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\right) \quad (9)$$

be the associated term in the summand. As usual, we will write $K_\lambda^{(n)}$, $R_\lambda^{(n)}$ and $R_{\lambda,w}^{(n)}$ when the parameters are clear from context.

Remarks When $(a, b, c, d) = (a', b', 0, 0)$, we obtain

$$K_\lambda(z_1, \ldots, z_n; t; a, b; a, b, 0, 0)$$
$$= \frac{1}{v_\lambda(t)} \sum_{w\in B_n} w\left(\prod_{1\le i\le n} z_i^{\lambda_i} \frac{\left(1-az_i^{-1}\right)\left(1-bz_i^{-1}\right)}{1-z_i^{-2}} \prod_{1\le i<j\le n} \frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j} \frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\right).$$

In particular, this is Macdonald's 2-parameter family $(BC_n, B_n) = (BC_n, C_n)$ polynomials at $q = 0$. We will write $K_\lambda^{(n)}(z; t; a, b, 0, 0)$ in this case.

2.2 Main results

In this section, we will prove Theorem 1.1: We will show that the $K_\lambda^{(n)}(z; t; a', b'; a, b, c, d)$, for λ a partition, are indeed the Koornwinder polynomials at $q = 0$. We note that while the BC-symmetry of these polynomials is fairly straightforward, proofs of polynomiality and orthogonality are more involved.

Theorem 2.6 *The function $K_\lambda^{(n)}(z;t;a',b';a,b,c,d)$ is a BC_n-symmetric Laurent polynomial (i.e., invariant under permuting variables $z_1,\dots,z_n$ and inverting variables $z_i \to z_i^{-1}$).*

Proof Recall the fully BC_n-antisymmetric Laurent polynomials:

$$\begin{aligned}\Delta_{BC} &= \prod_{1\le i\le n} z_i - z_i^{-1} \prod_{1\le i<j\le n} z_i^{-1} - z_j - z_j^{-1} + z_i \\ &= \prod_{1\le i\le n} \frac{z_i^2-1}{z_i} \prod_{1\le i<j\le n} \frac{1-z_iz_j}{z_iz_j}(z_j - z_i). \end{aligned} \tag{10}$$

Then, we have

$$\begin{aligned}&K_\lambda^{(n)}(z;a',b';a,b,c,d;t)\cdot\Delta_{BC} \\ &\quad= \frac{1}{v_\lambda(t)}\sum_{w\in B_n}\epsilon(w)w\left(\prod_{1\le i\le n}u'_\lambda(z_i)\prod_{1\le i<j\le n}\left(1-tz_i^{-1}z_j^{-1}\right)(z_i-tz_j)\right),\end{aligned} \tag{11}$$

where

$$u'_\lambda(z_i) = \begin{cases} z_i\left(1-a'z_i^{-1}\right)\left(1-b'z_i^{-1}\right) & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i+1}\left(1-az_i^{-1}\right)\left(1-bz_i^{-1}\right)\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right) & \text{if } \lambda_i > 0. \end{cases}$$

Notice that $K_\lambda^{(n)}\cdot\Delta_{BC}$ is a BC_n-antisymmetric Laurent polynomial, so in particular Δ_{BC} divides $K_\lambda^{(n)}\cdot\Delta_{BC}$ as polynomials. Consequently, $K_\lambda^{(n)}$ is a BC_n-symmetric Laurent polynomial, as desired. □

Theorem 2.7 *The functions $K_\lambda^{(n)}(z;t;a',b';a,b,c,d)$ are triangular with respect to dominance ordering:*

$$K_\lambda^{(n)}(z;t;a',b';a,b,c,d) = m_\lambda + \sum_{\mu<\lambda} c_\mu^\lambda m_\mu.$$

Remarks Here, $\{m_\lambda\}_\lambda$ is the monomial basis with respect to Weyl group of type BC:

$$m_\lambda = \frac{1}{|\mathrm{Stab}(\lambda)|}\sum_{w\in B_n} w\left(z_1^{\lambda_1}\dots z_n^{\lambda_n}\right),$$

where $\mathrm{Stab}(\lambda)$ is the stabilizer of λ in B_n.

Proof We show that when $K_\lambda^{(n)}$ is expressed in the monomial basis, the top degree term is m_λ; moreover, it is monic. First note that from (10) in the previous proof, we have

$$\Delta_{BC} = z^\rho + (\text{dominated terms}),$$

where $\rho = (n, n-1, \ldots, 2, 1)$. We compute the dominating monomial in $K_\lambda^{(n)} \cdot \Delta_{BC}$; see (11) in the previous proof for the formula. We look at the terms in the sum (corresponding to $w \in B_n$) and find the maximum total degree across these; it suffices to maximize the degree of the univariate terms and cross terms separately for a given $w \in B_n$. Note that if $\lambda_i = 0$, we have highest degree $\lambda_i + 1$ in $u'_\lambda(z_i)$. Similarly, if $\lambda_i > 0$, we note that $\lambda_i + 1 \geq -\lambda_i + 3$ (with equality if and only if $\lambda_i = 1$) so we have highest degree $\lambda_i + 1$ in $u'_\lambda(z_i)$. Moreover,

$$\prod_{1 \leq i < j \leq n} \left(1 - t z_i^{-1} z_j^{-1}\right)(z_i - t z_j) = \prod_{1 \leq i < j \leq n} \left(z_i - t z_j^{-1} - t z_j + t^2 z_i^{-1}\right) \tag{12}$$

has highest degree term $z^{\rho-1} = z_1^{n-1} z_2^{n-2} \cdots z_{n-1}$. Thus, the dominating monomial in $K_\lambda^{(n)} \cdot \Delta_{BC}$ is $z^{\lambda+\rho}$, so that the dominating monomial in $K_\lambda^{(n)}$ is z^λ.

We now show that the coefficient on $z^{\lambda+\rho}$ in $R_\lambda^{(n)} \cdot \Delta_{BC}$ (see (8) for the definition of $R_\lambda^{(n)}$) is $v_\lambda(t)$, so that $K_\lambda^{(n)}$ is indeed monic. Note first that by the above argument the only contributing w are those in the stabilizer of z^λ, i.e., those w such that (1) $z_1^{\lambda_1} \cdots z_n^{\lambda_n} = z_{w(1)}^{\lambda_1} \cdots z_{w(n)}^{\lambda_n}$ and (2) $\epsilon_w(z_i) = 1$ for all $1 \leq i \leq n - m_0(\lambda) - m_1(\lambda)$; let the set of these special permutations be denoted by $P_{\lambda,n}$. Now fix $w \in P_{\lambda,n}$, we will collect the coefficients (i.e., scalars t, a', b', a, b, c, d) on $z_1^{\lambda_1+n}, z_2^{\lambda_2+(n-1)}, \ldots$ from $u'_\lambda(z_i)$ and terms of (12). We will do this first for $z_1^{\lambda_1+n}$, then use an iterative argument to obtain the rest of the factors, and finally take the product of these to obtain the overall coefficient on $z^{\lambda+\rho}$. Using (11) and the arguments of the previous paragraph, one can check that the scalar factor on $z_1^{\lambda_1+n}$ is as follows:

(i) If $\lambda_1 > 1$:

$$t^{\#\{z_i \prec_w z_1\}}$$

(ii) If $\lambda_1 = 1$:

$$\begin{cases} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = 1 \\ -abcd(t^2)^{\#\{z_1 \prec_w z_i\}} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = -1 \end{cases}$$

(iii) If $\lambda_1 = 0$:

$$\begin{cases} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = 1 \\ -a'b'(t^2)^{\#\{z_1 \prec_w z_i\}} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = -1 \end{cases}$$

(note that we have used the contribution of (-1) factors from $\epsilon(w)$ in $K_\lambda^{(n)} \cdot \Delta_{BC}$).

Now define the following subsets of the variables $z_1, \ldots, z_n$:

$$N_{w,\lambda}^1 = \{z_i : n - m_0(\lambda) - m_1(\lambda) < i \leq n - m_0(\lambda) \text{ and } \epsilon_w(z_i) = -1\}$$
$$N_{w,\lambda}^0 = \{z_i : n - m_0(\lambda) < i \leq n \text{ and } \epsilon_w(z_i) = -1\}$$

$$N_{w,\lambda} = N^1_{w,\lambda} \cup N^0_{w,\lambda}.$$

Finally, define the following statistics of w:

$$n(w) = |\{(i,j) : 1 \le i < j \le n \text{ and } z_j \prec_w z_i\}|$$
$$c_\lambda(w) = |\{(i,j) : 1 \le i < j \le n \text{ and } z_i \prec_w z_j \text{ and } z_i \in N_{w,\lambda}\}|.$$

Then, by iterating the coefficient argument above, we get that the coefficient on $z^{\lambda+\rho}$ is given by

$$\sum_{w \in P_{\lambda,n}} t^{n(w)} t^{2c_\lambda(w)} (-abcd)^{|N^1_{w,\lambda}|} (-a'b')^{|N^0_{w,\lambda}|}.$$

Since $P_{\lambda,n} = B_{m_0(\lambda)} \times B_{m_1(\lambda)} \times \prod_{i \ge 2} S_{m_i(\lambda)}$, it is enough to show the following three cases:

$$\sum_{w \in S_m} t^{n(w)} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} \tag{13}$$

$$\sum_{w \in B_m} t^{n(w)} t^{2c_{(1^m)}(w) + 2m_0(\lambda)\cdot|N^1_{w,(1^m)}|} (-abcd)^{\left|N^1_{w,(1^m)}\right|}$$
$$= \prod_{j=1}^{m} \frac{1-t^j}{1-t} \left(1 - abcdt^{j-1+2m_0(\lambda)}\right) \tag{14}$$

$$\sum_{w \in B_m} t^{n(w)} t^{2c_{(0^m)}(w)} (-a'b')^{\left|N^0_{w,(0^m)}\right|} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} (1 - a'b't^{j-1}). \tag{15}$$

Equation (13) is well known, for example, refer to [8, Ch. III, proof of (1.2) and (1.3)]. We now show (14); (15) is analogous. One can verify that the LHS of (14) is exactly enumerated by the terms of

$$\prod_{k=1}^{m} \left[\sum_{i=1}^{k} \left(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-abcd) \right) \right]. \tag{16}$$

But we also have

$$\sum_{i=1}^{k} \left(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-abcd) \right) = \sum_{i=1}^{k} \left(t^{i-1} - abcdt^{k+2m_0(\lambda)-1} t^{k-i} \right)$$
$$= \left(1 - abcdt^{k+2m_0(\lambda)-1}\right) \left(1 + t + \cdots t^{k-1}\right) = \left(1 - abcdt^{k+2m_0(\lambda)-1}\right) \frac{1-t^k}{1-t};$$

substituting this into (16) gives the RHS of (14) as desired.

Multiplying these functions together for each distinct part i of λ (put $m = m_i(\lambda)$ in (13), (14), and (15), depending on whether $i \geq 2, i = 1,$ or $i = 0$, respectively), and using (3) shows that the coefficient on $z^{\lambda+\rho}$ in $R_\lambda^{(n)} \cdot \Delta_{BC}$ is indeed $v_\lambda(t)$, as desired. □

We will now provide a direct proof of Gustafson's formula [3] in the limit $q = 0$.

Theorem 2.8 *We have the following constant term evaluation in the symmetric case*

$$\begin{aligned}
&\int_T \tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT \\
&= \prod_{i=0}^{n-1} \frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)(1-t^i ab)} \\
&\quad \times \prod_{j=0}^{n-1}(1-t^{2n-2-j}abcd)\prod_{j=1}^{n}\frac{1-t}{1-t^j}.
\end{aligned}$$

Proof Note first that by Theorem 2.7, $K_{(0^n)}^{(n)}(z;t;a,b,0,0) = 1$. So in particular, we have

$$\begin{aligned}
\int_T \tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT &= \int_T K_{(0^n)}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT \\
&= \frac{1}{v_{(0^n)}(t;a,b,0,0)} \sum_{w\in B_n} \int_T R_{(0^n),w}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT \\
&= \frac{2^n n!}{v_{(0^n)}(t;a,b,0,0)} \int_T R_{(0^n),\mathrm{id}}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT,
\end{aligned}$$

where the last equality follows by symmetry of the integrand. But now using (9), one notes that

$$\begin{aligned}
&2^n n! R_{(0^n),\mathrm{id}}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d) \\
&= \prod_{1\leq i\leq n} \frac{\left(1-z_i^2\right)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right)} \\
&\quad \prod_{1\leq i<j\leq n} \frac{\left(1-z_i z_j^{\pm 1}\right)}{\left(1-tz_i z_j^{\pm 1}\right)}.
\end{aligned}$$

We will denote the right-hand side of the above equation by $\Delta_K^{(n)}(z;t;a,b,c,d)$.

We will now prove that

$$\int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT$$
$$= \prod_{i=0}^{n-1} \frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)} \prod_{j=n-1}^{2n-2} (1-t^j abcd). \tag{17}$$

For facility of notation, we will put $I_n(z;t;a,b;c,d) = \int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT$. We will prove (17) through the following two claims.

Claim 1: We have

$$I_n(z;t;a,b;c,d) = \frac{c}{(1-ac)(1-bc)(1-dc)(c-d)} I_{n-1}(z;t;a,b;tc,d) + \frac{d}{(1-ad)(1-bd)(1-cd)(d-c)} I_{n-1}(z;t;a,b;c,td), \tag{18}$$

with initial conditions $I_0(z;t;a,b;c,d) = 1$ and

$$I_1(z;t;a,b;c,d) = \frac{1-abcd}{(1-ac)(1-bc)(1-cd)(1-ad)(1-bd)}.$$

To prove the first claim, we note that

$$I_n(z;t;a,b;c,d) = \int \prod_{1\le i\le n} \frac{z_i\left(1-z_i^2\right)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(z_i-c)(z_i-d)} \prod_{1\le i<j\le n} \frac{(z_j-z_i)(1-z_iz_j)}{(z_j-tz_i)(1-tz_iz_j)} \prod_{j=1}^{n} \frac{dz_j}{2\pi\sqrt{-1}}.$$

We may now hold the variables $z_2,\ldots,z_n$ fixed and integrate with respect to z_1. There are simple poles at $z_1 = c$ and $z_1 = d$ (note that $|1/a|, |1/b|, |1/c|, |1/d| > 1$, since $|a|, |b|, |c|, |d| < 1$, so these do not contribute), so by the Residue Theorem, it will be the sum of residues at these poles. Consider the residue at $z_1 = c$:

$$\int \prod_{2\le i\le n} \frac{z_i\left(1-z_i^2\right)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(z_i-c)(z_i-d)} \prod_{2\le i<j\le n} \frac{(z_j-z_i)(1-z_iz_j)}{(z_j-tz_i)(1-tz_iz_j)} \times \frac{c}{(1-ac)(1-bc)(1-cd)(c-d)} \prod_{1<j\le n} \frac{(z_j-c)(1-cz_j)}{(z_j-tc)(1-tcz_j)} \prod_{j=2}^{n} \frac{dz_j}{2\pi\sqrt{-1}}$$

$$= C_1 \int_{T_{n-1}} \prod_{2\le i\le n} \frac{z_i\left(1-z_i^2\right)}{(1-az_i)(1-bz_i)(1-tcz_i)(1-dz_i)(z_i-tc)(z_i-d)}$$
$$\prod_{2\le i<j\le n} \frac{(z_j-z_i)(1-z_iz_j)}{(z_j-tz_i)(1-tz_iz_j)}dT$$

where $C_1 = \frac{c}{(1-ac)(1-bc)(1-cd)(c-d)}$. By renumbering the variables $(z_2,\dots,z_n)$ by $(z_1,\dots,z_{n-1})$, one sees that this is exactly $C_1 I_{n-1}(z;t;a,b;tc,d)$. An analogous argument applies for the residue at $z_1 = d$; this produces the second term $C_2 I_{n-1}(z;t;a,b;c,td)$, where $C_2 = \frac{d}{(1-ad)(1-bd)(1-cd)(d-c)}$.

To obtain the result at $n = 1$, one uses the above argument in this special case along with some algebraic manipulation. In particular, the computation of the sum of residues is as follows:

$$\begin{aligned}
&\frac{(1-c^2)c}{(1-ac)(1-bc)(1-c^2)(1-dc)(c-d)}\\
&\quad+\frac{(1-d^2)d}{(1-ad)(1-bd)(1-cd)(1-d^2)(d-c)}\\
&\quad=\frac{1}{(1-ac)(1-bc)(1-cd)(1-ad)(1-bd)}\\
&\qquad\left[\frac{c(1-ad)(1-bd)}{c-d}+\frac{d(1-ac)(1-bc)}{d-c}\right]\\
&\quad=\frac{1-abcd}{(1-ac)(1-bc)(1-cd)(1-ad)(1-bd)},
\end{aligned}$$

as desired. This proves the first claim.

Claim 2: We have the following solution to (18)

$$I_n(z;t;a,b;c,d) = \prod_{i=0}^{n-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)}$$
$$\prod_{j=n-1}^{2n-2}(1-t^jabcd).$$

We prove the second claim. One can first check that $n = 0, 1$ satisfies the initial conditions of (18). Then, for $n \ge 2$, we have

$$\begin{aligned}
&\frac{c}{(1-ac)(1-bc)(1-dc)(c-d)}I_{n-1}(z;t;a,b;tc,d)\\
&\quad+\frac{d}{(1-ad)(1-bd)(1-cd)(d-c)}I_{n-1}(z;t;a,b;c,td)\\
&\quad=\frac{c\prod_{j=n-2}^{2n-4}1-t^{j+1}abcd}{(1-ac)(1-bc)(1-dc)(c-d)}
\end{aligned}$$

$$\prod_{i=0}^{n-2} \frac{1}{(1-t^{i+1}ac)(1-t^{i+1}bc)(1-t^{i+1}cd)(1-t^{i}ad)(1-t^{i}bd)}$$
$$+\frac{d\prod_{j=n-2}^{2n-4} 1-t^{j+1}abcd}{(1-ad)(1-bd)(1-cd)(d-c)}$$
$$\prod_{i=0}^{n-2} \frac{1}{(1-t^{i}ac)(1-t^{i}bc)(1-t^{i+1}cd)(1-t^{i+1}ad)(1-t^{i+1}bd)}$$
$$=\left[\frac{c(1-t^{n-1}ad)(1-t^{n-1}bd)}{c-d}+\frac{d(1-t^{n-1}ac)(1-t^{n-1}bc)}{d-c}\right]$$
$$\times\prod_{i=0}^{n-1} \frac{1}{(1-t^{i}ac)(1-t^{i}bc)(1-t^{i}cd)(1-t^{i}ad)(1-t^{i}bd)}\prod_{j=n-2}^{2n-4}(1-t^{j+1}abcd)$$

But now note the following identity for the sum inside the parentheses:

$$\begin{aligned}&\frac{c(1-t^{n-1}ad)(1-t^{n-1}bd)}{c-d}+\frac{d(1-t^{n-1}ac)(1-t^{n-1}bc)}{d-c}\\&=\frac{c(1-t^{n-1}ad)(1-t^{n-1}bd)-d(1-t^{n-1}ac)(1-t^{n-1}bc)}{c-d}\\&=\frac{c-d+t^{2(n-1)}abcd^2-t^{2(n-1)}abc^2d}{c-d}=1-t^{2(n-1)}abcd,\end{aligned}$$

so the above finally becomes

$$\prod_{i=0}^{n-1} \frac{1}{(1-t^{i}ac)(1-t^{i}bc)(1-t^{i}cd)(1-t^{i}ad)(1-t^{i}bd)}$$
$$\prod_{j=n-1}^{2(n-1)}(1-t^{j}abcd)=I_n(z;t;a,b;c,d),$$

which proves (17).

Thus, putting this together, we have

$$\int_T \tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT=\frac{1}{v_{(0^n)}(t;a,b,0,0)}\int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT$$
$$=\prod_{i=0}^{n-1} \frac{1}{(1-t^{i}ac)(1-t^{i}bc)(1-t^{i}cd)(1-t^{i}ad)(1-t^{i}bd)}\prod_{j=n-1}^{2n-2}(1-t^{j}abcd)$$
$$\times\prod_{j=1}^{n}\frac{1-t}{1-t^j}\prod_{i=1}^{n}\frac{1}{1-abt^{i-1}}$$

$$= \prod_{i=0}^{n-1} \frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)(1-t^i ab)}$$
$$\prod_{j=0}^{n-1}(1-t^{2n-2-j}abcd)\prod_{j=1}^{n}\frac{1-t}{1-t^j},$$

where we have used Theorem 3.5 and (3). □

We note that the quantity $\Delta_K^{(n)}(z;t;a,b,c,d)$ which appears in the proof of Theorem 2.8 is actually the $q=0$ limit of the nonsymmetric Koornwinder density (see [6], for example); the nonsymmetric theory is investigated in the next section.

Theorem 2.9 *The family of polynomials* $\{K_\lambda^{(n)}(z;t;a',b';a,b,c,d)\}_\lambda$ *satisfy the following orthogonality result:*

$$\int_T K_\lambda^{(n)}(z;t;a',b';a,b,c,d)K_\mu^{(n)}(z;t;a',b';a,b,c,d)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT$$
$$= N_\lambda(t;a,b,c,d)\delta_{\lambda\mu}$$

(refer to (5) *and* (6) *for the definitions of* $\tilde{\Delta}_K^{(n)}$ *and* N_λ*, respectively; also see Theorem* 2.8*).*

Remarks Note that if $\lambda=\mu=(0^n)$, the left- and right-hand sides are both equal to $\int_T \tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT$, which was computed independently in Theorem 2.8.

Proof By symmetry of λ,μ, we may restrict to the case where $\lambda \overset{\text{lex}}{\geq} \mu$. We assume $\lambda_1>0$, so we need not consider the case $\lambda=\mu=0^n$; these assumptions hold throughout the proof. By definition of $K_\lambda^{(n)}(z;t;a',b';a,b,c,d)$ as a sum over B_n, the above integral is equal to

$$\sum_{w,\rho\in B_n}\int_T K_{\lambda,w}^{(n)}(z;t;a',b';a,b,c,d)K_{\mu,\rho}^{(n)}(z;t;a',b';a,b,c,d)$$
$$\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT.$$

Consider an arbitrary term in this sum over $B_n\times B_n$ indexed by (w,ρ). Note that using a change of variables in the integral and inverting variables (which preserves the integral), we may assume w is the identity permutation, and all sign choices are 1 (and ρ is arbitrary). That is, we have:

$$\int_T K_\lambda(z_1,\ldots,z_n;t;a',b';a,b,c,d)K_\mu(z_1,\ldots,z_n;t;a',b';a,b,c,d)$$
$$\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT$$
$$= 2^n n!\sum_{\rho\in B_n}\int_T K_{\lambda,\mathrm{id}}^{(n)}(z;t;a',b';a,b,c,d)K_{\mu,\rho}^{(n)}(z;t;a',b';a,b,c,d)$$

$$\begin{aligned}
&\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT \\
&= 2^n n! \frac{1}{v_\lambda(t)v_\mu(t)} \sum_{\rho\in B_n} \int_T R^{(n)}_{\lambda,\mathrm{id}}(z;t;a',b';a,b,c,d) R^{(n)}_{\mu,\rho}(z;t;a',b';a,b,c,d) \\
&\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT,
\end{aligned}$$

where $R^{(n)}_{\lambda,\rho}$ is as defined in (9).

We study an arbitrary term in this sum. In particular, we give an iterative formula that shows that each of these terms vanishes unless $\lambda = \mu$.

Claim 2.10 *Fix an arbitrary $\rho \in B_n$ and let $\rho(i) = 1$ for some $1 \le i \le n$. Then we have the following formula:*

$$\begin{aligned}
&2^n n! \int_{T_n} R^{(n)}_{\lambda,id}(z;t;a',b';a,b,c,d) R^{(n)}_{\mu,\rho}(z;t;a',b';a,b,c,d)\tilde{\Delta}_K^{(n)} dT_n \\
&= \begin{cases}
t^{i-1}2^{n-1}(n-1)! \int_{T_{n-1}} R^{(n-1)}_{\widehat{\lambda},\widehat{id}} R^{(n-1)}_{\widehat{\mu},\widehat{\rho}} \tilde{\Delta}_K^{(n-1)} dT_{n-1} & \text{if } \mu_i = \lambda_1 \text{ and } \epsilon_\rho(z_1) = -1, \\
t^{i-1}(t^2)^{m_0(\mu)+m_1(\mu)-i}(-abcd)2^{n-1}(n-1)! & \\
\quad \int_{T_{n-1}} R^{(n-1)}_{\widehat{\lambda},\widehat{id}} R^{(n-1)}_{\widehat{\mu},\widehat{\rho}} \tilde{\Delta}_K^{(n-1)} dT_{n-1} & \text{if } \mu_i = \lambda_1 = 1 \\
 & \text{and } \epsilon_\rho(z_1) = 1, \\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

where $\widehat{\lambda}$ and $\widehat{\mu}$ are the partitions λ and μ with parts λ_1 and μ_i deleted (respectively), and $\widehat{id}$ and $\widehat{\rho}$ are the permutations id and ρ with z_1 deleted (respectively) and signs preserved.

To prove the claim, we integrate with respect to z_1 in the iterated integral, using the definition of $R^{(n)}_{\lambda,\mathrm{id}}$, $R^{(n)}_{\mu,\rho}$ and $\tilde{\Delta}_K^{(n)}$.

First suppose $\mu_i > 0$. The univariate terms in z_1 are as follows:

$$\begin{aligned}
&z_1^{\lambda_1} \frac{\left(1-az_1^{-1}\right)\cdots\left(1-dz_1^{-1}\right)}{\left(1-z_1^{-2}\right)} z_1^{\mu_i} \frac{\left(1-az_1^{-1}\right)\cdots\left(1-dz_1^{-1}\right)}{\left(1-z_1^{-2}\right)} \frac{\left(1-z_1^{\pm 2}\right)}{\left(1-az_1^{\pm 1}\right)\cdots\left(1-dz_1^{\pm 1}\right)} \\
&= z_1^{\lambda_1+\mu_i} \frac{\left(-z_1^2\right)\left(1-az_1^{-1}\right)\cdots\left(1-dz_1^{-1}\right)}{(1-az_1)\cdots(1-dz_1)}
\end{aligned}$$

if $\epsilon_\rho(z_1) = 1$, and

$$\begin{aligned}
&z_1^{\lambda_1} \frac{\left(1-az_1^{-1}\right)\cdots\left(1-dz_1^{-1}\right)}{\left(1-z_1^{-2}\right)} z_1^{-\mu_i} \frac{(1-az_1)\cdots(1-dz_1)}{\left(1-z_1^{2}\right)} \frac{\left(1-z_1^{\pm 2}\right)}{\left(1-az_1^{\pm 1}\right)\cdots\left(1-dz_1^{\pm 1}\right)} \\
&= z_1^{\lambda_1-\mu_i}
\end{aligned}$$

if $\epsilon_\rho(z_1) = -1$.

Now suppose $\mu_i = 0$. The univariate terms in z_1 are as follows:

$$z_1^{\lambda_1} \frac{\left(1 - az_1^{-1}\right) \cdots \left(1 - dz_1^{-1}\right)}{\left(1 - z_1^{-2}\right)} \frac{\left(1 - a'z_1^{-1}\right)\left(1 - b'z_1^{-1}\right)}{\left(1 - z_1^{-2}\right)} \frac{\left(1 - z_1^{\pm 2}\right)}{\left(1 - az_1^{\pm 1}\right) \cdots \left(1 - dz_1^{\pm 1}\right)}$$
$$= z_1^{\lambda_1} \frac{\left(-z_1^2\right)\left(1 - a'z_1^{-1}\right)\left(1 - b'z_1^{-1}\right)}{(1 - az_1) \cdots (1 - dz_1)}$$

if $\epsilon_\rho(z_1) = 1$, and

$$z_1^{\lambda_1} \frac{\left(1 - az_1^{-1}\right) \cdots \left(1 - dz_1^{-1}\right)}{\left(1 - z_1^{-2}\right)} \frac{\left(1 - a'z_1\right)\left(1 - b'z_1\right)}{\left(1 - z_1^2\right)} \frac{\left(1 - z_1^{\pm 2}\right)}{\left(1 - az_1^{\pm 1}\right) \cdots \left(1 - dz_1^{\pm 1}\right)}$$
$$= z_1^{\lambda_1} \frac{\left(1 - a'z_1\right)\left(1 - b'z_1\right)}{(1 - az_1) \cdots (1 - dz_1)}$$

if $\epsilon_\rho(z_1) = -1$.

Notice that for the cross terms in z_1 (those involving z_j for $j \neq 1$), we have

$$\prod_{j>1} \frac{1 - tz_1^{-1}z_j^{-1}}{1 - z_1^{-1}z_j^{-1}} \frac{1 - tz_1^{-1}z_j}{1 - z_1^{-1}z_j} \times \prod_{j>1} \frac{1 - z_1^{\pm 1}z_j^{\pm 1}}{1 - tz_1^{\pm 1}z_j^{\pm 1}}$$

from the corresponding terms in z_1 of $R_{\lambda,\mathrm{id}}$ and the density. Combining this with the cross terms of $R_{\mu,\rho}$ in z_1 (and taking into account the various sign possibilities for ρ), we obtain

$$\prod_{\substack{z_i \prec_\rho z_1 \\ \epsilon_\rho(z_i)=1}} \frac{t - z_1 z_i}{1 - tz_1 z_i} \prod_{\substack{z_i \prec_\rho z_1 \\ \epsilon_\rho(z_i)=-1}} \frac{t - z_1 z_i^{-1}}{1 - tz_1 z_i^{-1}} \prod_{z_1 \prec_\rho z_j} \frac{\left(t - z_1 z_j^{-1}\right)\left(t - z_1 z_j\right)}{\left(1 - tz_1 z_j^{-1}\right)\left(1 - tz_1 z_j\right)}$$

if $\epsilon_\rho(z_1) = 1$, and

$$\prod_{\substack{z_i \prec_\rho z_1 \\ \epsilon_\rho(z_i)=1}} \frac{t - z_1 z_i}{1 - tz_1 z_i} \prod_{\substack{z_i \prec_\rho z_1 \\ \epsilon_\rho(z_i)=-1}} \frac{t - z_1 z_i^{-1}}{1 - tz_1 z_i^{-1}}$$

if $\epsilon_\rho(z_1) = -1$.

Thus, combining these computations, the integral in z_1 is:

$$\begin{cases} \int_{T_1} z_1^{\lambda_1+\mu_i} \frac{(-z_1^2)\left(1-az_1^{-1}\right)\cdots\left(1-dz_1^{-1}\right)}{(1-az_1)\cdots(1-dz_1)} \cdot \\ \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} \prod_{z_1 \prec_\rho z_j} \frac{\left(t-z_1z_j^{-1}\right)\left(t-z_1z_j\right)}{\left(1-tz_1z_j^{-1}\right)\left(1-tz_1z_j\right)} dT_1 & \text{if } \mu_i > 0 \text{ and } \epsilon_\rho(z_1) = 1, \\ \int_{T_1} z_1^{\lambda_1} \frac{(-z_1^2)\left(1-a'z_1^{-1}\right)\left(1-b'z_1^{-1}\right)}{(1-az_1)\cdots(1-dz_1)} \cdot \\ \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} \prod_{z_1 \prec_\rho z_j} \frac{\left(t-z_1z_j^{-1}\right)\left(t-z_1z_j\right)}{\left(1-tz_1z_j^{-1}\right)\left(1-tz_1z_j\right)} dT_1 & \text{if } \mu_i = 0 \text{ and } \epsilon_\rho(z_1) = 1, \\ \int_{T_1} z_1^{\lambda_1-\mu_i} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} dT_1 & \text{if } \mu_i > 0 \text{ and } \epsilon_\rho(z_1) = -1, \\ \int_{T_1} z_1^{\lambda_1} \frac{(1-a'z_1)(1-b'z_1)}{(1-az_1)\cdots(1-dz_1)} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} dT_1 & \text{if } \mu_i = 0 \text{ and } \epsilon_\rho(z_1) = -1. \end{cases}$$

In particular, the first integral vanishes unless $\lambda_1 = \mu_i = 1$; the second integral always vanishes; the third integral vanishes unless $\lambda_1 = \mu_i$; the fourth integral always vanishes. Thus, we obtain the vanishing conditions of the claim. To obtain the nonzero values, one can use the residue theorem and evaluate at the simple pole $z_1 = 0$ in the cases $\lambda_1 = \mu_i = 1$ and $\lambda_1 = \mu_i$. Finally, we combine with the original integrand involving terms in $z_2, \ldots, z_n$ to obtain the result of the claim.

Note that in particular the claim implies that if $\lambda \neq \mu$, each term vanishes and consequently the total integral is zero. This proves the vanishing part of the orthogonality statement.

Next, we compute the norm when $\lambda = \mu$. The claim shows that only certain $\rho \in B_n$ give nonvanishing term integrals. Such permutations must satisfy

$$z_1^{\lambda_1} \cdots z_n^{\lambda_n} z_{\rho(1)}^{-\lambda_1} \cdots z_{\rho(n)}^{-\lambda_n} = 1$$

and $\epsilon_\rho(z_i) = -1$ for all $1 \leq i \leq n - m_0(\lambda) - m_1(\lambda)$. For simplicity of notation, define $B_{\lambda,n}$ to be the set of such permutations $\rho \in B_n$. Then, we have:

$$\begin{aligned} &\int_T K_\lambda^{(n)}(z;t;a',b';a,b,c,d) K_\lambda^{(n)}(z;t;a',b';a,b,c,d) \tilde{\Delta}_K^{(n)} dT \\ &= \frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho \in B_n} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT \\ &= \frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho \in B_{\lambda,n}} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT, \end{aligned}$$

since only these permutations give nonvanishing terms.

Then, using the formula of Claim 2.10, we have

$$
\begin{aligned}
&2^n n! \sum_{\rho\in B_{\lambda,n}} \int_T R^{(n)}_{\lambda,\mathrm{id}} R^{(n)}_{\lambda,\rho} \tilde{\Delta}^{(n)}_K dT \\
&= \begin{cases} C_1 \times 2^{n-m_{\lambda_1}(\lambda)}(n-m_{\lambda_1}(\lambda))! \displaystyle\sum_{\rho\in B_{\tilde{\lambda},n-m_{\lambda_1}(\lambda)}} \int_T R^{(n-m_{\lambda_1}(\lambda))}_{\tilde{\lambda},\mathrm{id}} R^{(n-m_{\lambda_1}(\lambda))}_{\tilde{\lambda},\rho} \tilde{\Delta}^{(n-m_{\lambda_1}(\lambda))}_K dT & \text{if } \lambda_1 > 1, \\ C_2 \times 2^{n-m_{\lambda_1}(\lambda)}(n-m_{\lambda_1}(\lambda))! \displaystyle\sum_{\rho\in B_{\tilde{\lambda},n-m_{\lambda_1}(\lambda)}} \int_T R^{(n-m_{\lambda_1}(\lambda))}_{\tilde{\lambda},\mathrm{id}} R^{(n-m_{\lambda_1}(\lambda))}_{\tilde{\lambda},\rho} \tilde{\Delta}^{(n-m_{\lambda_1}(\lambda))}_K dT & \text{if } \lambda_1 = 1, \end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \prod_{k=1}^{m_{\lambda_1}(\lambda)} \left(\sum_{i=1}^{k} t^{i-1} \right) \\
C_2 &= \prod_{k=1}^{m_1(\lambda)} \left[\sum_{i=1}^{k} \left(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-abcd) \right) \right]
\end{aligned}
$$

and $\tilde{\lambda}$ is the partition λ with all $m_{\lambda_1}(\lambda)$ occurrences of λ_1 deleted, and the integrations in the right-hand side are with respect to $n - m_{\lambda_1}(\lambda)$ many variables. Iterating this argument gives that

$$
\begin{aligned}
&2^n n! \sum_{\rho\in B_{\lambda,n}} \int_T R^{(n)}_{\lambda,\mathrm{id}} R^{(n)}_{\lambda,\rho} \tilde{\Delta}^{(n)}_K dT \\
&= \left(\prod_{j>1} \prod_{k=1}^{m_j(\lambda)} \left(\sum_{i=1}^{k} t^{i-1} \right) \right) \left(\prod_{k=1}^{m_1(\lambda)} \sum_{i=1}^{k} \left(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-abcd) \right) \right) \\
&\quad \times 2^{m_0(\lambda)} m_0(\lambda)! \sum_{\rho\in B_{m_0(\lambda)}} \int_T R^{(m_0(\lambda))}_{(0^{m_0(\lambda)}),\mathrm{id}} R^{(m_0(\lambda))}_{(0^{m_0(\lambda)}),\rho} \tilde{\Delta}^{(m_0(\lambda))}_K dT;
\end{aligned}
$$

note that the expression on the final line is exactly $\int_T \left(R^{(m_0(\lambda))}_{(0^{m_0(\lambda)})} \right)^2 \tilde{\Delta}^{(m_0(\lambda))}_K dT$.

Thus,

$$
\begin{aligned}
&\frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho\in B_{\lambda,n}} \int_T R^{(n)}_{\lambda,\mathrm{id}} R^{(n)}_{\lambda,\rho} \tilde{\Delta}^{(n)}_K dT \\
&= \frac{1}{v_{\lambda+}(t)^2} \left(\prod_{j>1} \prod_{k=1}^{m_j(\lambda)} \left(\sum_{i=1}^{k} t^{i-1} \right) \right) \left(\prod_{k=1}^{m_1(\lambda)} \sum_{i=1}^{k} \left(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0 \cdots t_3) \right) \right) \\
&\quad \times \frac{1}{v_{(0^{m_0(\lambda)})}(t)^2} \int_T \left(R^{(m_0(\lambda))}_{(0^{m_0(\lambda)})} \right)^2 \tilde{\Delta}^{(m_0(\lambda))}_K dT,
\end{aligned}
$$

since by (3) and (4) we have $v_{\lambda+}(t) \cdot v_{(0^{m_0(\lambda)})}(t) = v_\lambda(t)$. Now using

$$\prod_{k=1}^{m_j(\lambda)} \left(\sum_{i=1}^{k} t^{i-1} \right) = \prod_{k=1}^{m_j(\lambda)} \frac{1-t^k}{1-t}$$

and

$$\begin{aligned}\sum_{i=1}^{k} \left(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-abcd) \right) &= \sum_{i=1}^{k} \left(t^{i-1} - abcdt^{k+2m_0(\lambda)-1} t^{k-i} \right) \\ &= \left(1 - abcdt^{k+2m_0(\lambda)-1}\right) \left(1 + t + \cdots t^{k-1}\right) \\ &= \left(1 - abcdt^{k+2m_0(\lambda)-1}\right) \frac{1-t^k}{1-t},\end{aligned}$$

the above expression can be simplified to

$$\begin{aligned}&\frac{1}{v_{\lambda+}(t)^2} \left(\prod_{j\geq 1} \prod_{k=1}^{m_j(\lambda)} \frac{1-t^k}{1-t} \right) \prod_{k=1}^{m_1(\lambda)} (1 - abcdt^{k+2m_0(\lambda)-1}) \int_T \left(K^{(m_0(\lambda))}_{(0^{m_0(\lambda)})} \right)^2 \tilde{\Delta}_K^{(m_0(\lambda))} dT \\ &= \frac{1}{v_{\lambda+}(t)} \int_T \tilde{\Delta}_K^{(m_0(\lambda))} dT = N_\lambda(t; a, b, c, d)\end{aligned}$$

since $K^{(m_0(\lambda))}_{(0^{m_0(\lambda)})} = 1$, by Theorem 2.7. Note that, by Theorem 2.8, there is an explicit evaluation for this norm. □

Proof of Theorem 1.1 In the previous theorems, we have shown that the polynomials we defined are (1) a basis for the space of BC_n-symmetric Laurent polynomials, (2) triangular with respect to the dominance ordering, and (3) orthogonal with respect to the $q = 0$ Koornwinder density function. In Proposition 2.4, we showed that the $q \to 0$ limit of the Koornwinder polynomials exists, and it is obvious that these limiting polynomials satisfy properties (1)–(3). We have

$$\langle K_\lambda^{(n)}(z; t; a, b, c, d), K_\lambda^{(n)}(z; t; a, b, c, d) - \lim_{q\to 0} K_\lambda^{(n)}(z; q, t; a, b, c, d)\rangle_0 = 0,$$

since $K_\lambda^{(n)}(z; t; a, b, c, d) - \lim_{q\to 0} K_\lambda^{(n)}(z; q, t; a, b, c, d)$ is in $\mathrm{span}(m_\mu(z) : \mu < \lambda) = \mathrm{span}(K_\mu^{(n)}(z; t, a, b, c, d) : \mu < \lambda)$ by property (2). Similarly,

$$\langle \lim_{q\to 0} K_\lambda^{(n)}(z; q, t; a, b, c, d), K_\lambda^{(n)}(z; t; a, b, c, d) - \lim_{q\to 0} K_\lambda^{(n)}(z; q, t; a, b, c, d)\rangle_0 = 0.$$

Thus, $\| K_\lambda^{(n)}(z; t; a, b, c, d) - \lim_{q\to 0} K_\lambda^{(n)}(z; q, t; a, b, c, d)\|_0 = 0$. Since the norm is nondegenerate on the space of BC_n-symmetric polynomials (we have exhibited an orthogonal basis whose norms are strictly positive), the result follows. □

2.3 Application

In this section, we use the closed formula (7) for the Koornwinder polynomials at $q = 0$ to prove Theorem 1.3. The vanishing condition of this identity was obtained by [11, Theorem 4.10] for arbitrary q; however, the methods used there do not provide the nonzero values. The idea used here is a type BC adaptation of that used in [18]: We use the structure of $K_\lambda^{(n)}$ as a sum over the Weyl group and the symmetry of the integral to restrict to one particular term. We obtain an explicit formula for the integral of this particular term by integrating with respect to one variable (holding the others fixed) and then proceed by induction.

Proof of Theorem 1.3 We have

$$\begin{aligned}
&\int_T K_\lambda(z_1,\dots,z_n;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT.\\
&= \frac{1}{v_\lambda(t^2;a,b;a,b,ta,tb)}\sum_{w\in B_n}\int_T R_{\lambda,w}^{(n)}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT\\
&= \frac{2^n n!}{v_\lambda(t^2;a,b;a,b,ta,tb)}\int_T R_{\lambda,\mathrm{id}}^{(n)}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT,
\end{aligned}$$

where in the last equation, we have used the symmetry of the integral. We assume $\lambda_1 > 0$ so that $\lambda \neq 0^n$. Next, we restrict to terms involving z_1 in the integrand and integrate with respect to z_1. Doing this computation gives the following:

$$\begin{aligned}
&\int_{T_1} z_1^{\lambda_1}\frac{\left(1-az_1^{-1}\right)\left(1-bz_1^{-1}\right)\left(1-taz_1^{-1}\right)\left(1-tbz_1^{-1}\right)}{\left(1-z_1^{-2}\right)}\\
&\quad\frac{\left(1-z_1^{\pm 2}\right)}{\left(1+\sqrt{t}z_1^{\pm 1}\right)\left(1-\sqrt{t}z_1^{\pm 1}\right)\left(1-az_1^{\pm 1}\right)\left(1-bz_1^{\pm 1}\right)}\\
&\quad\times\prod_{j>1}\frac{\left(1-t^2z_1^{-1}z_j\right)\left(1-t^2z_1^{-1}z_j^{-1}\right)}{\left(1-z_1^{-1}z_j\right)\left(1-z_1^{-1}z_j^{-1}\right)}\prod_{j>1}\frac{\left(1-z_1^{\pm 1}z_j^{\pm 1}\right)}{\left(1-tz_1^{\pm 1}z_j^{\pm 1}\right)}dT_1\\
&= \frac{1}{2\pi i}\int_C z_1^{\lambda_1-1}\frac{(z_1-ta)(z_1-tb)\left(1-z_1^2\right)}{\left(1-tz_1^2\right)\left(z_1+\sqrt{t}\right)\left(z_1-\sqrt{t}\right)(1-az_1)(1-bz_1)}\\
&\quad\times\prod_{j>1}\frac{\left(z_1-t^2z_j\right)\left(z_1-t^2z_j^{-1}\right)\left(1-z_1z_j\right)\left(1-z_1z_j^{-1}\right)}{\left(z_1-tz_j\right)\left(z_1-tz_j^{-1}\right)\left(1-tz_1z_j\right)\left(1-tz_1z_j^{-1}\right)}dz_1
\end{aligned}$$

Note that this integral has poles at $z_1 = \pm\sqrt{t}$ (note that $z = 1/b$, etc., do not contribute since they have norm larger than one) and $z_1 = tz_j, tz_j^{-1}$ for each $j > 1$.

We first compute the residue at $z_1 = \sqrt{t}$:

$$\begin{aligned}
&\left(\sqrt{t}\right)^{\lambda_1 - 1} \frac{\left(\sqrt{t} - ta\right)\left(\sqrt{t} - tb\right)(1-t)}{\left(1-t^2\right) 2\sqrt{t}\left(1 - a\sqrt{t}\right)\left(1 - b\sqrt{t}\right)} \\
&\prod_{j>1} \frac{\left(\sqrt{t} - t^2 z_j\right)\left(\sqrt{t} - t^2 z_j^{-1}\right)\left(1 - \sqrt{t} z_j\right)\left(1 - \sqrt{t} z_j^{-1}\right)}{\left(\sqrt{t} - t z_j\right)\left(\sqrt{t} - t z_j^{-1}\right)\left(1 - t\sqrt{t} z_j\right)\left(1 - t\sqrt{t} z_j^{-1}\right)} \\
&= \left(\sqrt{t}\right)^{\lambda_1} \frac{1}{2(1+t)} \prod_{j>1} \frac{\left(1 - t\sqrt{t} z_j\right)\left(1 - t\sqrt{t} z_j^{-1}\right)\left(1 - \sqrt{t} z_j\right)\left(1 - \sqrt{t} z_j^{-1}\right)}{\left(1 - \sqrt{t} z_j\right)\left(1 - \sqrt{t} z_j^{-1}\right)\left(1 - t\sqrt{t} z_j\right)\left(1 - t\sqrt{t} z_j^{-1}\right)} \\
&= \frac{\left(\sqrt{t}\right)^{\lambda_1}}{2(1+t)}
\end{aligned}$$

Similarly, we can compute the residue at $z_1 = -\sqrt{t}$:

$$\begin{aligned}
&\left(-\sqrt{t}\right)^{\lambda_1 - 1} \frac{\left(-\sqrt{t} - ta\right)\left(-\sqrt{t} - tb\right)(1-t)}{\left(1-t^2\right)\left(-2\sqrt{t}\right)\left(1 + a\sqrt{t}\right)\left(1 + b\sqrt{t}\right)} \\
&\prod_{j>1} \frac{\left(-\sqrt{t} - t^2 z_j\right)\left(-\sqrt{t} - t^2 z_j^{-1}\right)\left(1 + \sqrt{t} z_j\right)\left(1 + \sqrt{t} z_j^{-1}\right)}{\left(-\sqrt{t} - t z_j\right)\left(-\sqrt{t} - t z_j^{-1}\right)\left(1 + t\sqrt{t} z_j\right)\left(1 + t\sqrt{t} z_j^{-1}\right)} \\
&= \left(-\sqrt{t}\right)^{\lambda_1} \frac{1}{2(1+t)} \prod_{j>1} \frac{\left(1 + t\sqrt{t} z_j\right)\left(1 + t\sqrt{t} z_j^{-1}\right)\left(1 + \sqrt{t} z_j\right)\left(1 + \sqrt{t} z_j^{-1}\right)}{\left(1 + \sqrt{t} z_j\right)\left(1 + \sqrt{t} z_j^{-1}\right)\left(1 + t\sqrt{t} z_j\right)\left(1 + t\sqrt{t} z_j^{-1}\right)} \\
&= \frac{\left(-\sqrt{t}\right)^{\lambda_1}}{2(1+t)}
\end{aligned}$$

The residues at tz_j, tz_j^{-1} can be computed in a similar manner. One can then combine these residues (at tz_j, tz_j^{-1}) with the terms from the original integrand and integrate with respect to z_j. Some computations show the resulting integral is zero; the argument is similar that used in [18, Theorem 23].

Finally, we add the residues at $z_1 = \pm\sqrt{t}$ to get

$$\frac{(\sqrt{t})^{\lambda_1}}{2(1+t)} + \frac{(-\sqrt{t})^{\lambda_1}}{2(1+t)} = \begin{cases} \frac{(\sqrt{t})^{\lambda_1}}{(1+t)}, & \text{if } \lambda_1 \text{ is even} \\ 0, & \text{if } \lambda_1 \text{ is odd.} \end{cases}$$

Thus,

$$
\begin{aligned}
&2^n n! \int_T R^{(n)}_{\lambda,\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n)}_K(z;t;\pm\sqrt{t},a,b)dT \\
&= \begin{cases} \frac{(\sqrt{t})^{\lambda_1}}{(1+t)} 2^{n-1}(n-1)! \int_{T_{n-1}} R^{(n-1)}_{\widehat{\lambda},\widehat{\mathrm{id}}}(z;t^2;a,b;a,b,ta,tb) \\ \quad \tilde{\Delta}^{(n-1)}_K(z;t;\pm\sqrt{t},a,b)dT_{n-1}, & \text{if } \lambda_1 \text{ is even} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

where $\widehat{\lambda}$ is the partition λ with the part λ_1 deleted, and $\widehat{\mathrm{id}}$ is the permutation id with z_1 deleted and signs preserved.

Consequently, the entire integral vanishes if any part of λ is odd and if λ is even, it is equal to

$$
\begin{aligned}
&\frac{2^n n!}{v_\lambda(t^2;a,b;a,b,ta,tb)} \int_T R^{(n)}_{\lambda,\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n)}_K(z;t;\pm\sqrt{t},a,b)dT \\
&= \frac{2^{n-l(\lambda)}(n-l(\lambda)!}{v_{\lambda+}(t^2;a,b,ta,tb)v_{0^{n-l(\lambda)}}(t^2;a,b;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \\
&\quad \times \int_{T_{n-l(\lambda)}} R^{(n-l(\lambda))}_{0^{n-l(\lambda)},\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT_{n-l(\lambda)},
\end{aligned}
$$

where by a slight abuse of notation in the last line, we use id to denote the identity element in $B_{n-l(\lambda)}$. By (8), the last line is equal to

$$
\begin{aligned}
&\frac{2^{n-l(\lambda)}(n-l(\lambda)!}{v_{\lambda+}(t^2;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \\
&\quad \int_{T_{n-l(\lambda)}} K^{(n-l(\lambda))}_{0^{n-l(\lambda)},\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT_{n-l(\lambda)} \\
&= \frac{1}{v_{\lambda+}(t^2;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \\
&\quad \int_{T_{n-l(\lambda)}} K^{(n-l(\lambda))}_{0^{n-l(\lambda)}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT_{n-l(\lambda)} \\
&= \frac{1}{v_{\lambda+}(t^2;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \int_{T_{n-l(\lambda)}} \tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT_{n-l(\lambda)} \\
&= \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \frac{N_\lambda(t;\pm\sqrt{t},a,b)v_{\lambda+}(t;\pm\sqrt{t},a,b)}{v_{\lambda+}(t^2;a,b,ta,tb)},
\end{aligned}
$$

since $K^{(l)}_{0^l}(z;t;a',b';a,b,c,d) = 1$ by Theorem 2.7 and $n - l(\lambda) = m_0(\lambda)$. □

3 Nonsymmetric Hall–Littlewood polynomials of type BC

3.1 Background and notation

We first introduce the affine Hecke algebra of type BC, a crucial object in the study of nonsymmetric Koornwinder polynomials. We retain the notation on partitions, compositions, and orderings of Sect. 2.1.

Definition 3.1 (see [14,15]) The *affine Hecke algebra H of type C* is defined to be the $\mathbb{C}(q,t,a,b,c,d)$ algebra with generators $T_0, T_1, \ldots, T_n$ $(n > 1)$, subject to the following braid relations

$$\begin{aligned} T_iT_j &= T_jT_i, \quad |i-j| \geq 2, \\ T_iT_jT_i &= T_jT_iT_j, \quad |i-j| = 1, i, j \neq 0, n, \\ T_iT_{i+1}T_iT_{i+1} &= T_{i+1}T_iT_{i+1}T_i \quad (i = 0, i = n-1) \end{aligned}$$

and the quadratic relations

$$\begin{aligned} (T_0+1)(T_0+cd/q) &= 0, \\ (T_i+1)(T_i-t) &= 0, \quad i \neq 0, n, \\ (T_n+1)(T_n+ab) &= 0. \end{aligned}$$

Recall that, by the Noumi representation (see [12,14]), there is an action of H on the vector space of Laurent polynomials $\mathbb{C}(q^{1/2}, t, a, b, c, d)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ (here $x_1, \ldots, x_n$ are n independent indeterminates) as follows:

$$\begin{aligned} T_0 f &= -(cd/q)f + \frac{(1-c/x_1)(1-d/x_1)}{1-q/x_1^2}(f^{s_0} - f) \\ T_i f &= tf + \frac{x_{i+1} - tx_i}{x_{i+1} - x_i}(f^{s_i} - f) \quad (\text{for } 0 < i < n) \\ T_n f &= -abf + \frac{(1-ax_n)(1-bx_n)}{1-x_n^2}(f^{s_n} - f), \end{aligned}$$

where $f^{s_0}(x_1, \ldots, x_n) = f(q/x_1, x_2, \ldots, x_n)$, $f^{s_i}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$ for $0 < i < n$, and $f^{s_n}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}, 1/x_n)$. Note that, for $0 < i \leq n$, the action of T_i on polynomials is independent of q; this will be crucial for the rest of the paper.

We will denote the nonsymmetric Koornwinder polynomials in n variables by $U_\lambda^{(n)}(x; q, t; a, b, c, d)$ [12,14,15]. These polynomials are indexed by $\lambda \in \Lambda$ (recall from Sect. 2.1, Λ is the set of compositions). We will remind the reader how these polynomials are defined, but we must first set up some relevant notations. Let $^\iota$ be the involution defined by

$$^\iota : q \to q^{-1}, t \to t^{-1}, a \to a^{-1}, b \to b^{-1}, c \to c^{-1}, d \to d^{-1}, z^\mu \to z^\mu$$

and let $^{-}$ be the involution defined by

$$^{-}: q \to q, t \to t, a \to a, b \to b, c \to c, d \to d, z^{\mu} \to z^{-\mu}.$$

Define the weight

$$\Delta_K^{(n)}(z; q, t; a, b, c, d) = \prod_{1 \le i \le n} \frac{\left(z_i^2, q z_i^{-2}; q\right)}{\left(a z_i, b z_i, c z_i, d z_i, a q z_i^{-1}, b q z_i^{-1}, c z_i^{-1}, d z_i^{-1}; q\right)} \times \prod_{1 \le i < j \le n} \frac{\left(z_i z_j^{\pm 1}, q z_i^{-1} z_j^{\pm 1}; q\right)}{\left(t z_i z_j^{\pm 1}, q t z_i^{-1} z_j^{\pm 1}; q\right)}, \tag{19}$$

i.e., the full nonsymmetric density, see [11]. Note that $\Delta_K^{(n)}(z; q, 1; 1, -1, 0, 0) = 1$; this specialization is independent of q. As in the symmetric case, when the parameters are clear from context, we will suppress them to make the notation easier. Note the following formula for the nonsymmetric density at the specialization $q = 0$:

$$\prod_{1 \le i \le n} \frac{\left(1 - z_i^2\right)}{(1 - a z_i)(1 - b z_i)(1 - c z_i)(1 - d z_i)\left(1 - c z_i^{-1}\right)\left(1 - d z_i^{-1}\right)} \prod_{1 \le i < j \le n} \frac{1 - z_i z_j^{\pm 1}}{1 - t z_i z_j^{\pm 1}}. \tag{20}$$

We will write $\Delta_K^{(n)}(z; t; a, b, c, d)$ to indicate this particular limiting case.

With this terminology, consider the following inner product on functions of n variables with parameters q, t, a, b, c, d (see [10]):

$$\langle f, g \rangle_q = \int_{T_n} f \bar{g}^{\iota} \Delta_K^{(n)}(z; q, t; a, b, c, d) dT_n.$$

The integral above is the constant term of $f \bar{g}^{\iota} \Delta_K^{(n)}$. We also let

$$\begin{aligned}(f, g)_q &= \frac{1}{\int_{T_n} \Delta_K^{(n)}(z; q, t; a, b, c, d) dT_n} \int_{T_n} f \bar{g}^{\iota} \Delta_K^{(n)}(z; q, t; a, b, c, d) dT_n \\ &= \frac{\langle f, g \rangle_q}{\int_{T_n} \Delta_K^{(n)}(z; q, t; a, b, c, d) dT_n}.\end{aligned}$$

We have $(g, f)_q = \overline{(f, g)_q}^{\iota}$.

Also, denote by $\langle \cdot, \cdot \rangle_0$, the following inner product:

$$\langle f, g \rangle_0 = \int_{T_n} f \bar{g}^{\iota} \Delta_K^{(n)}(z; t; a, b, c, d) dT_n, \tag{21}$$

involving the $q = 0$ degeneration of the full nonsymmetric Koornwinder weight as in (20) and similarly

$$(f, g)_0 = \frac{1}{\int_{T_n} \Delta_K^{(n)}(z; t; a, b, c, d) dT_n} \int_{T_n} f \bar{g}^{\iota} \Delta_K^{(n)}(z; t; a, b, c, d) dT_n.$$

Recall that the polynomials $\{U_\mu^{(n)}(x; q, t; a, b, c, d)\}_{\mu \in \mathbb{Z}^n}$ are uniquely defined by the following conditions:

$$(i) \quad U_\mu = x^\mu + \sum_{\nu \prec \mu} w_{\mu\nu} x^\nu$$
$$(ii) \quad (U_\mu, x^\nu)_q = 0 \text{ if } \nu \prec \mu,$$

where as usual we write x^μ for the monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$.

Proposition 3.2 *Let $|t|, |a|, |b|, |c|, |d| < 1$. Then, the $q \to 0$ limit of the nonsymmetric Koornwinder polynomials $U_\mu^{(n)}(x; q, t; a, b, c, d)$ exists.*

Proof The argument is analogous to Proposition 2.4 in the symmetric case, here the relevant norms at the q-level are computed in [15]. □

Definition 3.3 For a partition λ with $l(\lambda) \leq n$, define

$$E_\lambda^{(n)}(z; c, d) = \prod_{\lambda_i > 0} z_i^{\lambda_i} \left(1 - c z_i^{-1}\right) \left(1 - d z_i^{-1}\right).$$

3.2 Main results

We will first prove Theorem 1.2: we will show that, under the assumption that λ is a partition with $l(\lambda) \leq n$, $E_\lambda^{(n)}(z; c, d)$ is the $q = 0$ limiting case of the nonsymmetric Koornwinder polynomial $U_\lambda^{(n)}(x; q, t; a, b, c, d)$.

Theorem 3.4 (Triangularity) *The polynomials $E_\lambda^{(n)}(z; c, d)$ are triangular with respect to dominance ordering, i.e.,*

$$E_\lambda^{(n)}(z; c, d) = z^\lambda + \sum_{\mu \prec \lambda} c_\mu z^\mu$$

for all partitions λ.

Proof It is clear that $E_\lambda^{(n)}(z; c, d) = z^\lambda +$ (dominated terms), since the term inside the product definition of $E_\lambda^{(n)}(z; c, d)$ is just $z_i^{\lambda_i} - (c + d) z_i^{\lambda_i - 1} + c d z_i^{\lambda_i - 2}$. □

Theorem 3.5 *We have the following constant term evaluation in the nonsymmetric case [with respect to $q = 0$ limit of the nonsymmetric density as in (20)]*

$$\int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT$$
$$= \prod_{i=0}^{n-1} \frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)} \prod_{j=n-1}^{2n-2} (1-t^j abcd).$$

Proof This follows from the proof of Theorem 2.8, in particular recall (17). □

Theorem 3.6 (Orthogonality) *Let λ be a partition with $l(\lambda) \leq n$ and $\mu \in \mathbb{Z}^n$ a composition, such that $\mu \prec \lambda$. Then, we have $\langle E_\lambda^{(n)}(z;c,d), z^\mu\rangle_0 = 0$.*

Proof Fix λ a partition. First note that, by definition of the inner product $\langle\cdot,\cdot\rangle_0$ in (21) we have

$$\langle E_\lambda^{(n)}(z;c,d), z^\mu\rangle_0 = \int_T E_\lambda^{(n)}(z;c,d) z^{-\mu}$$
$$\prod_{1\leq i\leq n} \frac{\left(1-z_i^2\right)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right)}$$
$$\prod_{1\leq i<j\leq n} \frac{1-z_i z_j^{\pm 1}}{1-tz_i z_j^{\pm 1}} dT.$$

We will first show $\langle E_\lambda^{(n)}(z;c,d), z^\mu\rangle_0 = 0$ for all compositions μ satisfying the following two properties:

Condition (i) $\mu \overset{\text{lex}}{<} \lambda$, so in particular there exists $1 \leq i \leq n$ such that $\mu_1 = \lambda_1, \ldots, \mu_{i-1} = \lambda_{i-1}$ and $\mu_i < \lambda_i$.

Condition (ii) $\lambda_i \neq 0$ (where i is as in (i)).

We mention that condition (ii) is necessary because of the difference between nonzero and zero parts of λ in Definition 3.3; in particular, if $\lambda_i = 0$, then one does not have the term $z_i^{\lambda_i}(1-cz_i^{-1})(1-dz_i^{-1})$ in $E_\lambda^{(n)}(z;c,d)$ (so that one still has the terms $1/(1-cz_i^{-1})(1-dz_i^{-1})$ in the product $E_\lambda^{(n)}(z;c,d)\Delta_K^{(n)}$). We give a proof by induction on n, the number of variables. Note first that condition (ii) implies that $\lambda_1, \ldots, \lambda_{i-1} \neq 0$. Consider the case $n = 1$. Then, in particular $i = 1$ and conditions (i) and (ii) give $\mu_1 < \lambda_1 \neq 0$. One can then compute

$$\langle E_\lambda^{(n)}, z^\mu\rangle_0 = \int_{T_1} z_1^{\lambda_1-\mu_1} \frac{\left(1-z_1^2\right)}{(1-az_1)(1-bz_1)(1-cz_1)(1-dz_1)} dT_1,$$

since $\lambda_1 - \mu_1 > 0$, this is necessarily zero. Now suppose the claim holds for $n-1$, we show it holds for n.

We may restrict the n-dimensional integral $\langle E_\lambda(z), z^\mu\rangle_0$ to the contribution involving z_1, one computes it to be

$$\int_{T_1} z_1^{\lambda_1-\mu_1} \frac{\left(1-z_1^2\right)}{(1-az_1)(1-bz_1)(1-cz_1)(1-dz_1)} \prod_{j>1} \frac{1-z_1 z_j^{\pm 1}}{1-tz_1 z_j^{\pm 1}} dT_1.$$

If $i=1$, then $\lambda_1 > \mu_1$ and this integral (and consequently the n-dimensional integral) is zero. If $i>1$, then $\lambda_1=\mu_1$ and this integral is 1. In this case, one notes that the resulting $n-1$ dimensional integral is exactly:

$$\int_{T_{n-1}} E_{\widehat{\lambda}}^{(n-1)}(z_2,\ldots,z_n) z^{-\widehat{\mu}} \Delta_K^{(n-1)} dT_{n-1},$$

where $\widehat{\lambda}=(\lambda_2,\ldots,\lambda_n)$ and $\widehat{\mu}=(\mu_2,\ldots,\mu_n)$. Note that conditions (i) and (ii) hold for $\widehat{\mu}$ and $\widehat{\lambda}$, and since this is the $n-1$ variable case, we may appeal to the induction hypothesis. Thus, the above integral is zero; consequently, $\langle E_\lambda^{(n)}, z^\mu\rangle_0 = 0$ as desired.

Finally, it remains to show that $\mu \prec \lambda$ implies conditions (i) and (ii). Recall that there are two cases for $\mu \prec \lambda$. In case 1), note that we have $\mu \le \mu^+ < \lambda$ with respect to the dominance ordering, so $\mu < \lambda$. This implies $\mu \overset{\text{lex}}{<} \lambda$ by Lemma 2.2. In case 2), it is clear. Now we show condition (ii). Suppose for contradiction that $\lambda_i = 0$, so that $\mu_1=\lambda_1 \ge 0,\ldots,\mu_{i-1}=\lambda_{i-1}\ge 0$ and $\mu_i<\lambda_i=0$ and $\lambda_k=0$ for all $i<k\le n$. Then, note that $\sum_{k=1}^i (\mu^+)_k > \sum_{k=1}^i \lambda_k$, which contradicts $\mu^+ \le \lambda$. Thus, we must have $\lambda_i \ne 0$ as desired. □

Theorem 3.7 *Let λ be a partition with $l(\lambda)\le n$, then*

$$\begin{aligned}
&\langle E_\lambda^{(n)}(z;c,d), E_\lambda^{(n)}(z;c,d)\rangle_0 = \langle E_\lambda^{(n)}(z;c,d), z^\lambda\rangle_0 \\
&= \prod_{i=0}^{m_0(\lambda)-1} \frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)} \\
&\quad \prod_{j=m_0(\lambda)-1}^{2m_0(\lambda)-2} (1-t^j abcd)
\end{aligned}$$

Proof The first equality follows from Theorems 3.4 and 3.6. For the second equality, we use arguments similar to those used in the proof of Theorem 3.6. We first note that

$$\begin{aligned}
&\langle E_\lambda^{(n)}(z;c,d), z^\lambda\rangle_0 = \int_{T_n} E_\lambda^{(n)}(z;c,d) z^{-\lambda} \\
&\prod_{1\le i\le n} \frac{\left(1-z_i^2\right)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)\left(1-cz_i^{-1}\right)\left(1-dz_i^{-1}\right)} \\
&\prod_{1\le i<j\le n} \frac{1-z_i z_j^{\pm 1}}{1-tz_i z_j^{\pm 1}} dT_n.
\end{aligned}$$

One can integrate with respect to z_1, holding the remaining variables fixed: after cancellations, the integral in z_1 is equal to

$$\int_{T_1} \frac{(1-z_1^2)}{(1-az_1)(1-bz_1)(1-cz_1)(1-dz_1)} \prod_{1<j\leq n} \frac{(1-z_1 z_j^{\pm 1})}{(1-tz_1 z_j^{\pm 1})} dT_1$$

The integrand has no poles in the unit disk, so to evaluate the integral set $z_1 = 0$, which gives 1. Thus,

$$\langle E_\lambda^{(n)}(z;c,d), z^\lambda \rangle_0 = \langle E_{\widehat{\lambda}}^{(n-1)}(z;c,d), z^{\widehat{\lambda}} \rangle_0$$

where $\widehat{\lambda} = (\lambda_2, \ldots, \lambda_n)$. Iterating this argument shows that

$$\langle E_\lambda^{(n)}(z;c,d), z^\lambda \rangle_0 = \int_{T_{m_0(\lambda)}} \Delta_K^{(m_0(\lambda))} dT_{m_0(\lambda)}.$$

By Theorem 3.5, this is equal to

$$\prod_{i=0}^{m_0(\lambda)-1} \frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)} \prod_{j=m_0(\lambda)-1}^{2m_0(\lambda)-2} (1-t^j abcd),$$

as desired. □

We will now prove that these polynomials $E_\lambda^{(n)}(z;c,d)$ are indeed the nonsymmetric Koornwinder polynomials indexed by a partition in the limit $q \to 0$. We have shown in Proposition 3.2 that the limit is well defined, so we just need to check that polynomials satisfying the above triangularity and orthogonality conditions are uniquely determined.

Proof of Theorem 1.2 Arguing along similar lines as the proof of Theorem 1.1, we find that

$$\| E_\lambda^{(n)}(z;c,d) - \lim_{q\to 0} U_\lambda^{(n)}(z;q,t;a,b,c,d) \|_0 = 0.$$

Now since the limits $\lim_{q\to 0} U_\lambda^{(n)}(z;q,t;a,b,c,d)$ form an orthogonal basis with positive norms [15], the norm $\|\cdot\|_0$ is nondegenerate and the result follows. □

Finally, we will use Definition 3.3 to extend to the case where $\lambda \in \Lambda$ via a particular recursion. We will first need to define some relevant rational functions in t, a, b, c, d.

Definition 3.8 Define

$$n_\lambda^i = -|\{l < i : \lambda_l = -1 \text{ or } 0\}| - 2|\{l > i+1 : \lambda_l = 0\}| - 1$$

and

$$r_\lambda = m_{-1}(\lambda) + m_0(\lambda) - 1;$$

they are statistics of the composition λ. We use this to define rational functions $\{p_i(\lambda)\}_{1\le i\le n}$ and $\{q_i(\lambda)\}_{1\le i\le n}$ as follows

$$p_n(\lambda) = \begin{cases} -ab-1, & \text{if } \lambda_n < -1 \\ -ab-1+abcdt^{2r_\lambda}, & \text{if } \lambda_n = -1 \\ 0, & \text{if } \lambda_n > 1 \\ -abcdt^{2r_\lambda}, & \text{if } \lambda_n = 1 \\ -ab, & \text{if } \lambda_n = 0 \end{cases}$$

and for $1 \le i \le n-1$

$$p_i(\lambda) = \begin{cases} t-1, & \text{if } \lambda_i < \lambda_{i+1} \text{ and } (\lambda_i, \lambda_{i+1}) \neq (-1, 0) \\ \frac{(1-t)t^{n_\lambda^i}}{abcd-t^{n_\lambda^i}}, & \text{if } (\lambda_i, \lambda_{i+1}) = (-1, 0) \\ 0, & \text{if } \lambda_i > \lambda_{i+1} \text{ and } (\lambda_i, \lambda_{i+1}) \neq (0, -1) \\ \frac{(t-1)abcd}{abcd-t^{n_\lambda^i}}, & \text{if } (\lambda_i, \lambda_{i+1}) = (0, -1) \\ t, & \text{if } \lambda_i = \lambda_{i+1}. \end{cases}$$

Similarly, define

$$q_n(\lambda) = \begin{cases} -ab, & \text{if } \lambda_n < 0 \\ 0, & \text{if } \lambda_n = 0 \\ 1, & \text{if } \lambda_n > 1 \\ 1 + cdt^{2r_\lambda}(-ab-1+abcdt^{2r_\lambda}), & \text{if } \lambda_n = 1 \end{cases}$$

and for $1 \le i \le n-1$

$$q_i(\lambda) = \begin{cases} t, & \text{if } \lambda_i < \lambda_{i+1} \\ 0, & \text{if } \lambda_i = \lambda_{i+1} \\ 1, & \text{if } \lambda_i > \lambda_{i+1} \text{ and } (\lambda_i, \lambda_{i+1}) \neq (0, -1) \\ 1 - \frac{(1-t)^2 abcdt^{n_\lambda^i - 1}}{(abcd-t^{n_\lambda^i})^2}, & \text{if } (\lambda_i, \lambda_{i+1}) = (0, -1). \end{cases}$$

Definition 3.9 For $\lambda \in \Lambda$ with $l(\lambda) \le n$, define $E^{(n)}_{s_i\lambda}(z; t; a, b, c, d)$ (for $1 \le i \le n$) by the following recursion

$$T_i E_\lambda = p_i(\lambda) E_\lambda + q_i(\lambda) E_{s_i\lambda} \tag{22}$$

where $p_i(\lambda), q_i(\lambda)$ are the rational functions of the previous definition, and for λ a partition $E_\lambda^{(n)}$ is given by Definition 3.3.

One can check that this action is well defined, i.e., it obeys the quadratic and braid relations of Definition 3.1.

Theorem 3.10 *For any $\lambda \in \Lambda$, $E_\lambda^{(n)}(z; t; a, b, c, d)$ is well defined, and we have*

$$\lim_{q\to 0} U_\lambda^{(n)}(z; q, t; a, b, c, d) = E_\lambda^{(n)}(z; t; a, b, c, d).$$

Proof The case when λ is a partition has been established by Theorem 1.2. The rest of the result is obtained by showing that [15] Proposition 6.1 admits the limit $q \to 0$ and that recursion in fact becomes (22) in this limit. As mentioned above, it is crucial that the operators T_i for $1 \le i \le n-1$ are independent of q; these are the operators that appear here. We note that the parameters must be translated according to the following reparametrization:

$$\{a, b, c, d, t\} \leftrightarrow \left\{t_n\check{t}_n, -t_n\check{t}_n^{-1}, t_0\check{t}_0 q^{1/2}, -t_0\check{t}_0^{-1} q^{1/2}, t^2\right\};$$

in particular, this reparametrization yields $T_0 = t_0 Y_0, T_n = t_n Y_n, T_i = tY_i$ (for $1 \le i \le n-1$), where $\{Y_i\}_{0\le i\le n}$ are the Hecke operators of [15].

It is a computation to directly verify that the limits exist in the cases $\lambda_n \le 0$ and $\lambda_i \le \lambda_{i+1} (1 \le i \le n-1)$. We then apply T_n and $T_i (1 \le i \le n-1)$ to the resulting recursions and use the quadratic relations

$$T_n^2 = -ab - T_n - abT_n$$

and

$$T_i^2 = t - T_i + tT_i$$

and simplify to obtain the recursion in the remaining cases $\lambda_n > 0$ and $\lambda_i > \lambda_{i+1}$. □

As a by-product of orthogonality for the $\{U_\lambda^{(n)}(z; q, t; a, b, c, d)\}_{\lambda\in\Lambda}$, we obtain the complete orthogonality for the $q = 0$ limiting case.

Corollary 3.11 *Let $\lambda, \mu \in \mathbb{Z}^n$ be compositions. If $\lambda \ne \mu$, we have $\langle E_\lambda^{(n)}, E_\mu^{(n)}\rangle_0 = 0$. If $\mu \prec \lambda$, then we have $\langle E_\lambda^{(n)}, z^\mu\rangle_0 = 0$.*

Proof This follows from the orthogonality for the q-nonsymmetric Koornwinder polynomials and Theorem 3.10. □

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