



Valid Orderings of Real Hyperplane Arrangements

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Abstract Given a real finite hyperplane arrangement \mathcal{A} and a point p not on any of the hyperplanes, we define an arrangement $\text{vo}(\mathcal{A}, p)$, called the *valid order arrangement*, whose regions correspond to the different orders in which a line through p can cross the hyperplanes in \mathcal{A} . If \mathcal{A} is the set of affine spans of the facets of a convex polytope \mathcal{P} and p lies in the interior of \mathcal{P} , then the valid orderings with respect to p are just the line shellings of \mathcal{P} where the shelling line contains p . When p is sufficiently generic, the intersection lattice of $\text{vo}(\mathcal{A}, p)$ is the *Dilworth truncation* of the semicone of \mathcal{A} . Various applications and examples are given. For instance, we determine the maximum number of line shellings of a d -polytope with m facets when the shelling line contains a fixed point p . If \mathcal{P} is the order polytope of a poset, then the sets of facets visible from a point involve a generalization of chromatic polynomials related to list colorings.

Keywords Hyperplane arrangement · Matroid · Dilworth truncation · Line shelling · Order polytope · Chromatic polynomial

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1 Introduction

Let \mathcal{A} be a (finite) real hyperplane arrangement, i.e., a finite set of affine hyperplanes in some d -dimensional real affine space $V \cong \mathbb{R}^d$. Since we consider only hyperplane

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arrangements in this paper, we call \mathcal{A} simply a real arrangement, always assumed to be finite. Basic information on arrangements may be found in Orlik and Terao [9] and Stanley [13].

The main question that will concern us is the following. Let L be a directed line in V . If L is sufficiently generic, then it will cross the hyperplanes $H \in \mathcal{A}$ in a certain order. What can we say about the possible orders of the hyperplanes? We can say more when we fix a point $p \in V$ not lying on any of the hyperplanes in \mathcal{A} and assume that L passes through p . The different orders then correspond in a simple way to regions of another arrangement, which we call the *valid order arrangement* $\text{vo}(\mathcal{A}, p)$.

A special situation occurs when \mathcal{A} consists of the affine spans of the facets of a d -dimensional convex polytope \mathcal{P} in \mathbb{R}^d . We then call \mathcal{A} the *visibility arrangement* $\text{vis}(\mathcal{P})$ of \mathcal{P} , since its regions correspond to sets of facets of \mathcal{P} visible from some point. If p lies in the interior of \mathcal{P} , then the regions of the valid order arrangement $\text{vo}(\mathcal{A})$ correspond to the line shellings of \mathcal{P} , where the line defining the shelling (which we call the *shelling line*) passes through p . In this case, we call $\text{vo}(\mathcal{A}, p)$ the *line shelling arrangement* of \mathcal{P} (with respect to p).

We will discuss a number of results concerning visibility and valid order arrangements. Most notably, when p is sufficiently generic, then the matroid corresponding to the semicone (defined below) of $\text{vo}(\mathcal{A}, p)$ is the Dilworth truncation of the matroid corresponding to \mathcal{A} . This observation enables us (Theorem 7) to answer the following question: given $n \geq d + 1$, what is the most number of line shellings that a convex d -polytope with n facets can have, where the shelling line passes through a fixed point p ? Another result (Theorem 4) is a connection between the visibility arrangement of the order polytope of a poset and a generalization of chromatic polynomials. Some of our work overlaps (in a dual setting) that of Edelman [5]. We will point out these instances in the appropriate places below. Edelman mentions that his paper provides a framework in which to extend the work of Ungar [15] to higher dimensions. Ungar's work was put in a general two-dimensional context by Goodman and Pollack [6], stated in terms of *allowable sequences*. Thus Edelman's paper and the present paper may be regarded as a higher dimensional generalization of the theory of allowable sequences.

2 The Valid Order Arrangement

Let \mathcal{A} be a hyperplane arrangement in a real affine space V , and let p be a point in V not lying on any hyperplane $H \in \mathcal{A}$. The following definition is equivalent to the (matroid) dual of Edelman's difference set $\mathcal{D}(\mathcal{C})$ [5, p. 147].

Definition 1 The *valid order arrangement* $\text{vo}(\mathcal{A}, p)$ consists of all hyperplanes of the following two types:

- The affine span of p and $H \cap H'$, where H and H' are two non-parallel hyperplanes in \mathcal{A} . We denote this affine span as $\text{aff}(p, H \cap H')$.
- The hyperplane through p parallel to two parallel hyperplanes $H, H' \in \mathcal{A}$, denoted by $\text{par}(p, H)$ or $\text{par}(p, H')$.

Note that $\text{vo}(\mathcal{A}, p)$ is a *central* arrangement, i.e., all the hyperplanes in $\text{vo}(\mathcal{A}, p)$ intersect, since every hyperplane in $\text{vo}(\mathcal{A}, p)$ contains p .

Consider a directed line L through p that is not parallel to any hyperplane $H \in \mathcal{A}$ and that does not intersect two distinct hyperplanes of \mathcal{A} in the same point. Thus L intersects the hyperplanes in \mathcal{A} in some order H_1, H_2, \dots, H_m as we come in from ∞ along L in the direction of L . We call the sequence H_1, \dots, H_m a *valid ordering* of \mathcal{A} with respect to p . Note that if we reverse the direction of L , then we get a new valid ordering H_m, \dots, H_1 .

Suppose that H and H' are two non-parallel hyperplanes of \mathcal{A} . The question of whether L intersects H before H' depends on the side of the hyperplane $\text{aff}(p, H \cap H')$ in which a point q lies, where q is a point of L near p in the positive direction (the direction of L) from p . Similarly, if H and H' are parallel hyperplanes of \mathcal{A} , then either q lies on the same side of both (i.e., not between them), in which case the order in which L intersects H and H' is independent of L , or else q lies between H and H' , in which case the order in which L intersects H and H' depends on the side of the hyperplane $\text{par}(p, H)$ in which the point q lies. It follows that the valid ordering corresponding to L is determined by the region of $\text{vo}(\mathcal{A}, p)$ in which the point q lies. In particular, we have the following result, which is equivalent to [5, Cor. 2.6].

Proposition 1 *The number of valid orderings of \mathcal{A} with respect to p is equal to the number $r(\text{vo}(\mathcal{A}, p))$ of regions of the valid order arrangement $\text{vo}(\mathcal{A}, p)$.*

We now wish to explain the connection between the valid order arrangement and a matroidal construction known as “Dilworth truncation.” Recall that a *matroid* on a set E may be defined as a collection \mathcal{I} of subsets of E , called *independent sets*, satisfying the following condition: for any subset $F \subseteq E$, the maximal (under inclusion) sets in \mathcal{I} that are contained in F all have the same number of elements. The prototypical example of a matroid consists of a finite subset E of a vector space, where a set $F \subseteq E$ is independent if it is linearly independent. For further information on matroid theory, see for instance [10, 16, 17].

We first define a matroid $M_{\mathcal{A}}$ associated with an arrangement. Given a real arrangement \mathcal{A} in a vector space V , which we identify with \mathbb{R}^d , let H be a hyperplane in \mathcal{A} defined by the equation $x \cdot \alpha = c$, where $0 \neq \alpha \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Associate with H the vector $v_H = (\alpha, -c) \in \mathbb{R}^{d+1}$. Let $M_{\mathcal{A}}$ be the matroid corresponding to the set $E_{\mathcal{A}} = \{v_H : H \in \mathcal{A}\}$. That is, the points of $M_{\mathcal{A}}$ are the vectors in $E_{\mathcal{A}}$, with independence in $M_{\mathcal{A}}$ given by linearly independence of vectors. Note that $E_{\mathcal{A}}$ is a *linear arrangement*, that is, all its hyperplanes pass through the origin.

Denote the coordinates in \mathbb{R}^{d+1} by x_1, \dots, x_d, y . Preserving the notation from above, let $\text{sc}(\mathcal{A})$ denote the set of all hyperplanes $\alpha \cdot x = cy$ in \mathbb{R}^{n+1} . We call $\text{sc}(\mathcal{A})$ the *semicone* of \mathcal{A} . If we add the additional hyperplane $y = 0$, then we obtain the *cone* $c(\mathcal{A})$, as defined e.g. in [13, §1.1]. Note that $\text{sc}(\mathcal{A})$ is a linear arrangement satisfying $M_{\mathcal{A}} \cong M_{\text{sc}(\mathcal{A})}$. Moreover, the “lines,” i.e., the rank two flats (or codimension two intersection subspaces) in the matroid $M_{\text{sc}(\mathcal{A})}$ correspond bijectively to the pair $\{H, H'\}$ of hyperplanes in \mathcal{A} . In particular, the parallel pairs $\{H, H'\}$ correspond to the codimension two intersections of $\text{sc}(\mathcal{A})$ lying within the hyperplane at infinity $y = 0$ (from $c(\mathcal{A}) \setminus \text{sc}(\mathcal{A})$).

Now let M be a matroid on a set E , and let $L = L_M$ denote the lattice of flats of M . If we remove the top k levels from L below the maximum element $\hat{1}$, then we obtain the k th *truncation* $T^k L$ of L . It is easy to see that $T^k L$ is a geometric lattice and hence the lattice of flats of a matroid. What if, however, we remove the *bottom* k levels from L above the minimum element $\hat{0}$? In general, we do not obtain a geometric lattice. We would like to “fill in” this lower truncation as generically as possible to obtain a geometric lattice, without adding any new atoms (elements of rank $k + 1$ of L) and without increasing the rank. This rather vague description was formalized by Dilworth [4]. Three other references are Brylawski [1, 2] and Mason [7]. We will give the definition at the level of matroids. Define the k th *Dilworth truncation* $D_k M$ to be the matroid on the set $\binom{E}{k+1}$ of $(k + 1)$ -element subsets of E , with independent sets

$$\mathcal{I} = \left\{ I \subseteq \binom{E}{k+1} : \text{rank}_M \left(\bigcup_{p \in I'} p \right) \geq \#I' + k, \forall \emptyset \neq I' \subseteq I \right\}.$$

Thus the flats of rank one of $D_k M$ are just the flats of rank $k + 1$ of M . In particular, the flats of $D_1 M$ are the lines (flats of rank two) of M . We carry over the notation D_k to geometric lattices. In other words, if L is a geometric lattice, so $L = L_M$ for some matroid M , then we define $D_k L = L_{D_k M}$.

NOTE. Various other notations are used for D_k , including D_{k+1} and T_{k+1} .

In general, $D_1 L$ seems to be an intractable object. For the boolean algebra B_m , we have [4, Thm. 3.2] and [7, p. 163]

$$D_1 B_m \cong \Pi_m, \tag{1}$$

the lattice of partitions of an m -set (or the intersection lattice of the braid arrangement B_m), but for more complicated geometric lattices L it is difficult to describe $D_1 L$ in a reasonable way. If L has rank two, then clearly $D_1 L$ consists of just two points $\hat{0}$ and $\hat{1}$. If L has rank three, then when we remove the atoms from L we still have a geometric lattice, so $D_1 L$ consists just of L with the atoms removed. When L has rank four, to obtain $D_1 L$ first remove the atoms from L to obtain a lattice L' of rank three. For any two atoms s, t of L' whose join in L' is the top element $\hat{1}$ of L' , adjoin a new element x_{st} covering s and t and covered by $\hat{1}$. The resulting poset is $D_1 L$. This construction allows us to give a formula for the characteristic polynomial (e.g., [13, §1.3] and [14, §3.11.2]) of $D_1 L$ when $\text{rank}(L) = 4$. Let ρ_2 be the number of elements of L of rank two, let L_3 be the set of elements of L of rank three, and let $c(t)$ be the number of elements u covering $t \in L$, i.e., $u > t$, and no element v satisfies $u > v > t$. Then

$$\begin{aligned} \chi_{D_1 L}(q) &= q^3 - \rho_2 q^2 + \left[\binom{\rho_2}{2} - \sum_{t \in L_3} \binom{c(t) - 1}{2} \right] q \\ &\quad + \sum_{t \in L_3} \binom{c(t) - 1}{2} - \binom{\rho_2 - 1}{2}. \end{aligned}$$

When $\text{rank}(L) = 5$ the situation becomes much more complicated.

We now come to our main result on the valid order arrangement.

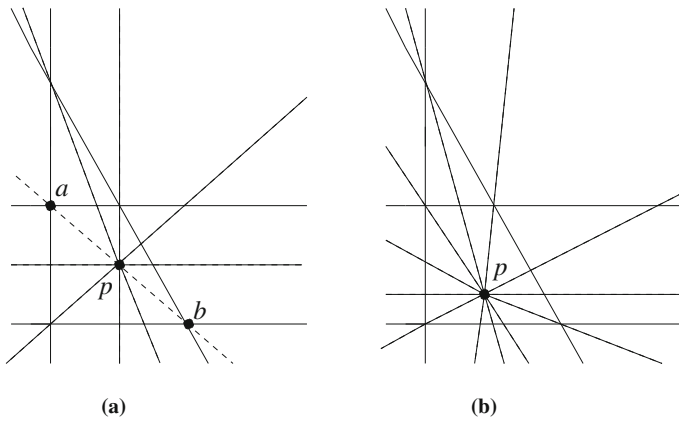


Fig. 1 Two valid order arrangements

Theorem 1 *Let \mathcal{A} be an arrangement in the real vector space V , and let p be a generic point of V . Then $L_{\text{vo}(\mathcal{A}, p)} \cong L_{D_1(\mathcal{A})}$.*

Proof Brylawski [1, p. 62] and [2, p. 197] and Mason [7, pp. 161–162] note that the Dilworth truncation of a geometry (simple matroid) M embedded in a vector space V of the same dimension (over a sufficiently large field if the field characteristic is nonzero) is obtained as the set of intersections of the lines of M with a generic hyperplane in V . This is precisely dual to the statement of our theorem. \square

As an example illustrating Theorem 1, Fig. 1a shows an arrangement \mathcal{A} of four hyperplanes (solid lines) in \mathbb{R}^2 and a nongeneric point p . The dashed lines are the hyperplanes in $\text{vo}(\mathcal{A})$. The point p is not generic since the same hyperplane of $\text{vo}(\mathcal{A})$ passes through the two intersections marked a and b . The arrangement $\text{vo}(\mathcal{A})$ has ten regions, so there are ten valid orderings of the four hyperplanes of \mathcal{A} with respect to p . Figure 1b shows the same situation with a generic point p . There are now 12 valid orderings with respect to p . In this case, the lattice $L_{\mathcal{A}}$ is an (upper) truncated boolean algebra $T^1 B_4$, with four atoms and six elements of rank two. Since $\text{rank}(L_{\mathcal{A}}) = 3$, the Dilworth truncation $D_1(L_{\mathcal{A}})$ is obtained simply by removing the atoms from $L_{\mathcal{A}}$.

3 Examples

As mentioned in Sect. 1, a special situation of interest occurs when \mathcal{A} consists of the affine spans $\text{aff}(F)$ of the facets F of a d -dimensional convex polytope \mathcal{P} in \mathbb{R}^d , in which case we call \mathcal{A} the *visibility arrangement* $\text{vis}(\mathcal{P})$ of \mathcal{P} . The regions of $\text{vis}(\mathcal{P})$ correspond to the sets of facets that are visible (on the outside) from some point in \mathbb{R}^d . In particular, the interior of \mathcal{P} is a region from which *no* facets are visible. Let $v(\mathcal{P}) = r(\text{vis}(\mathcal{P}))$, the number of regions of $\text{vis}(\mathcal{P})$ or visibility sets of facets of \mathcal{P} . If p is a point inside \mathcal{P} , then the valid orderings $(\text{aff}(F_1), \dots, \text{aff}(F_r))$ with respect to p correspond to the line shellings (F_1, \dots, F_r) where the shelling line passes through p . For basic information on line shellings, see Ziegler [20, Lecture 8].

For an arrangement \mathcal{A} in \mathbb{R}^d , let $\chi_{\mathcal{A}}(q)$ denote the *characteristic polynomial* of \mathcal{A} (e.g., [13, §1.3] and [14, §3.11.2]). A well-known theorem of Zaslavsky [13, Thm. 2.5] and [14, Thm. 3.11.7] states that the number $r(\mathcal{A})$ of regions of \mathcal{A} is given by

$$r(\mathcal{A}) = (-1)^d \chi_{\mathcal{A}}(-1). \tag{2}$$

Suppose that \mathcal{A} is defined over \mathbb{Z} , that is, the equations defining the hyperplanes in \mathcal{A} have integer coefficients. By taking these coefficients modulo a prime p , we get an arrangement \mathcal{A}_p defined over the finite field \mathbb{F}_p . It is also well known [13, Thm. 5.15] and [14, Thm. 3.11.10] that for p sufficiently large,

$$\chi_{\mathcal{A}}(p) = \# \left(\mathbb{F}_p^d - \bigcup_{H \in \mathcal{A}_p} H \right). \tag{3}$$

This result will be a useful tool below in computing some characteristic polynomials.

We now discuss two examples: the n -cube and the order polytope of a finite poset. Let \mathcal{C}_n denote the standard n -dimensional cube, given by the inequalities $0 \leq x_i \leq 1$, for $1 \leq i \leq n$. It is easy to see, e.g., by (3), that the visibility arrangement $\text{vis}(\mathcal{C}_n)$ satisfies

$$\chi_{\text{vis}(\mathcal{C}_n)}(q) = (q - 2)^n.$$

In particular, $r(\text{vis}(\mathcal{C}_n)) = 3^n$. Drawing a picture for $n = 2$ will make it geometrically clear why $\text{vis}(\mathcal{C}_n)$ has 3^n regions. In fact, the facets of \mathcal{C}_n come in n antipodal pairs F and \bar{F} . The sets of facets visible from some point are obtained by choosing for each pair $\{F, \bar{F}\}$ either F , \bar{F} , or neither. There are three choices for each pair and hence 3^n visibility sets in all.

More interesting are the line shellings of cubes. We summarize some information in the following result.

Theorem 2 (a) Let $p = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, the center of the cube \mathcal{C}_n . Then

$$\chi_{\text{vo}(\text{vis}(\mathcal{C}_n), p)}(q) = (q - 1)(q - 3) \cdots (q - (2n - 1)),$$

so the number of line shellings with respect to p is $2^n n!$.

- (b) The total number of line shellings of \mathcal{C}_n is $2^n n!^2$.
- (c) Let $f(n)$ denote the total number of shellings of \mathcal{C}_n . Then

$$\sum_{n \geq 1} f(n) \frac{x^n}{n!} = 1 - \frac{1}{\sum_{n \geq 0} (2n)! \frac{x^n}{n!}}. \tag{4}$$

- (d) Every shelling of \mathcal{C}_n can be realized as a corresponding line shelling of a polytope combinatorially equivalent to \mathcal{C}_n .

Proof (a) The hyperplanes of $\text{vo}(\text{vis}(\mathcal{C}_n))$ are given by $x_i = 0$, $1 \leq i \leq n$, and $x_i \pm x_j = 0$, $1 \leq i < j \leq n$. The characteristic polynomial can now easily

be computed from (3). Alternatively, $\text{vo}(\text{vis}(\mathcal{C}_n))$ is the *Coxeter arrangement* of type B_n , whose characteristic polynomial is well known ([13, p. 451] and [14, Exercise 3.115(d)]).

- (b) If we stand at a generic point far away from \mathcal{C}_n , we will see n facets of \mathcal{C}_n —all with a common vertex v . By symmetry, there are 2^n choices for v , and then $n!$ orderings of the n facets containing v that can begin a line shelling σ . Hence it remains to prove that the remaining n facets can come in any order in σ .

Let the parametric equation of the line L defining the shelling be $(a_1, a_2, \dots, a_n) + t(\alpha_1, \alpha_2, \dots, \alpha_n)$, where $t \in \mathbb{R}$. Making a small perturbation if necessary, we may assume that each $\alpha_i \neq 0$. We may also assume by symmetry that the facet F_i of the shelling, for $1 \leq i \leq n$, has the equation $x_i = 0$. The line L intersects the hyperplane $x_i = 0$ when $t = -a_i/\alpha_i$, so

$$\frac{a_1}{\alpha_1} > \frac{a_2}{\alpha_2} > \dots > \frac{a_n}{\alpha_n}.$$

The line L intersects the hyperplane $x_i = 1$ when $t = (1 - a_i)/\alpha_i$. Write

$$\frac{1 - a_i}{\alpha_i} = \frac{1}{\alpha_i} + b_i,$$

so $b_1 < b_2 < \dots < b_n$. Thus we can first choose $b_1 < b_2 < \dots < b_n$. Then choose $\alpha_1, \alpha_2, \dots, \alpha_n$ so that the numbers $\frac{1}{\alpha_i} + b_i$ come in any desired order. This then determines a_1, \dots, a_n uniquely, completing the proof.

- (c) This result is stated without proof in [14, Exercise 1.131]. To prove it, note that F_1, F_2, \dots, F_{2n} is a shelling if and only if for no $1 \leq j < n$ is it true that $\{F_1, F_2, \dots, F_{2j}\}$ consists of j pairs of antipodal facets. There follows the recurrence

$$(2n)! = \sum_{j=0}^n f(j) \binom{n}{j} (2n - 2j)!,$$

from which (4) is immediate.

- (d) See Develin [3, Cor. 2.12]. □

Conspicuously absent from Theorem 2 is the characteristic polynomial or number of regions of the line shelling arrangement $\text{vo}(\text{vis}(\mathcal{C}_n), p)$ when p is *generic*, the situation of Theorem 1. Suppose for instance that $n = 3$. Let $\mathcal{A}(p) = \text{vo}(\text{vis}(\mathcal{C}_3), p)$. If $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then by Theorem 2(a) we have

$$\chi_{\mathcal{A}(p)}(q) = (q - 1)(q - 3)(q - 5), \quad r(\mathcal{A}) = 48.$$

For $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$ we have

$$\chi_{\mathcal{A}(p)}(q) = (q - 1)(q - 5)(q - 7), \quad r(\mathcal{A}) = 96.$$

For generic p we have

$$\chi_{\mathcal{A}(p)}(q) = (q - 1)(q^2 - 14q + 53), \quad r(\mathcal{A}) = 136 = 2^3 \cdot 17.$$

The total number of line shellings of \mathcal{C}_3 is 288, and the total number of shellings is 480. While the Dilworth truncation $D_1(\text{vis}(\mathcal{C}_n))$ seems quite complicated, it might not be hopeless to compute its characteristic polynomial or number of regions. We leave this as an open problem.

We next consider the order polytope $\mathcal{O}(P)$ of a finite poset P , first defined explicitly in [11]. By definition, $\mathcal{O}(P)$ is the set of all order-preserving maps $\tau : P \rightarrow [0, 1]$ and is hence a convex polytope in the space \mathbb{R}^P of all maps $P \rightarrow \mathbb{R}$. Our main result will be a connection between the number of regions of $\text{vis}(\mathcal{O}(P))$, i.e., the number of visibility sets of facets of $\mathcal{O}(P)$, and a certain generalization of the chromatic polynomial of a graph.

Let G be a finite simple (i.e., no loops or multiple edges) graph with vertex set V . Recall that a *proper coloring* of G with colors from the set \mathbb{P} of positive integers is a map $f : V \rightarrow \mathbb{P}$ such that if u and v are adjacent in G then $f(u) \neq f(v)$. The *chromatic polynomial* $\chi_G(q)$ is defined when $q \in \mathbb{P}$ to be the number of proper colorings $f : V \rightarrow \{1, 2, \dots, q\}$. It is a standard result that $\chi_G(q)$ is a polynomial in q . Moreover, if $V = \{v_1, \dots, v_p\}$, then define the *graphical arrangement* \mathcal{A}_G to be the arrangement in \mathbb{R}^p with hyperplanes $x_i = x_j$, where v_i and v_j are adjacent vertices of G . Then $\chi_{\mathcal{A}_G}(q) = \mathcal{A}_G(q)$ ([13, Thm. 2.7] and [14, Exercise 3.108]).

We will generalize the definition of $\chi_G(q)$ by imposing finitely many disallowed colors at each vertex. More precisely, let $2^{\mathbb{P}}$ denote the set of all subsets of \mathbb{P} , and let $\psi : V \rightarrow 2^{\mathbb{P}}$ satisfy $\#\psi(v) < \infty$ for all $v \in V$. For $q \in \mathbb{P}$, define $\chi_{G,\psi}(q)$ to be the number of proper colorings $f : V \rightarrow \{1, 2, \dots, q\}$ such that $f(v) \notin \psi(v)$ for all $v \in V$. Thus for each vertex v , there is a finite set $\psi(v)$ of “disallowed colors.” We call such a coloring a ψ -coloring. The idea of permitting only certain colors of each vertex in a proper coloring of G has received much attention in the context of *list colorings* [19], but the function $\chi_{G,\psi}(q)$ seems to be new.

It is easy to see that $\chi_{G,\psi}(q)$ is a monic polynomial in q of degree p with integer coefficients. We call it the ψ -chromatic polynomial of G . Define the ψ -graphical arrangement $\mathcal{A}_{G,\psi}$ to be the arrangement in \mathbb{R}^p with hyperplanes $x_i = x_j$ whenever v_i and v_j are adjacent in V , together with $x_i = \alpha_j$ if $\alpha_j \in \psi(v_i)$.

Theorem 3 *We have $\chi_{\mathcal{A}_{G,\psi}}(q) = \chi_{G,\psi}(q)$, that is, the ψ -chromatic polynomial of G coincides with the characteristic polynomial of the ψ -graphical arrangement $\mathcal{A}_{G,\psi}$.*

Proof The proof is an immediate consequence of (3). □

Because $\chi_{G,\psi}$ is the characteristic polynomial of a hyperplane arrangement, it satisfies all the properties of such polynomials. For instance, there is a deletion–contraction recurrence, a broken circuit theorem, an extension to the Tutte polynomial, etc. We now give the connection between $\text{vis}(\mathcal{O}(P))$ and ψ -graphical arrangements.

Theorem 4 *Let P be a finite poset, and let H denote the Hasse diagram of P , considered as a graph with vertex set V . Define $\psi : V \rightarrow \mathbb{P}$ by*

$$\psi(v) = \begin{cases} \{1, 2\} & \text{if } v \text{ is an isolated point,} \\ \{1\} & \text{if } v \text{ is minimal but not maximal,} \\ \{2\} & \text{if } v \text{ is maximal but not minimal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\text{vis}(\mathcal{O}(P)) + (1, 1, \dots, 1) = \mathcal{A}_{H,\psi}$, where $\text{vis}(\mathcal{O}(P)) + (1, 1, \dots, 1)$ denotes the translation of $\text{vis}(\mathcal{O}(P))$ by the vector $(1, 1, \dots, 1)$.

Proof The result is an immediate consequence of the relevant definitions. Namely, if $V = \{v_1, \dots, v_p\}$, then the facets of $\mathcal{O}(P)$ are given by

$$\begin{aligned} x_i &= x_j && \text{if } v_j \text{ covers } v_i \text{ in } P, \\ x_i &= 0 && \text{if } x_i \text{ is a minimal element of } P, \\ x_i &= 1 && \text{if } x_i \text{ is a maximal element of } P, \end{aligned}$$

and the proof follows. □

NOTE. We could have avoided the translation by $(1, 1, \dots, 1)$ by allowing 0 to be a color, but it is more natural in many situations to let the set of colors be \mathbb{P} .

A curious result arises when P is graded of rank one, i.e., every maximal chain of P has two elements. For $W \subseteq V$, let H_W be the restriction of H to W , or in other words, the induced subgraph on the vertex set W .

Theorem 5 *Suppose that P is graded of rank one. Then*

$$\chi_{\text{vis}(\mathcal{O}(P))}(q) = \sum_{W \subseteq V} \chi_{H_W}(q - 2), \tag{5}$$

$$v(\mathcal{O}(P)) = (-1)^{\#P} \sum_{W \subseteq V} \chi_{H_W}(-3). \tag{6}$$

Proof Let $q \geq 2$. Choose a subset $W \subseteq V$. Color each minimal element of P not in W with the color 2, and color each maximal element of P not in W with the color 1. Color the remaining elements with the colors $\{3, 4, \dots, q\}$ in $\chi_{H_W}(q - 2)$ ways. This produces each ψ -coloring of H , so the proof of (5) follows. To obtain (6), put $q = -1$ in (5).

As an example, let P_{mn} denote the poset of rank one with m minimal elements, n maximal elements, and $u < v$ for every minimal element u and maximal element v . Hence H is the complete bipartite graph K_{mn} . It is known [12, Exercise 5.6] that

$$\sum_{m \geq 0} \sum_{n \geq 0} \chi_{K_{mn}}(q) \frac{x^m}{m!} \frac{x^n}{n!} = (e^x + e^y - 1)^q.$$

By simple properties of exponential generating functions, we get

$$\sum_{m \geq 0} \sum_{n \geq 0} \chi_{\text{vis}(\mathcal{O}(P_{mn}))}(q) \frac{x^m}{m!} \frac{x^n}{n!} = e^{x+y} (e^x + e^y - 1)^{q-2}$$

and

$$\begin{aligned} \sum_{m \geq 0} \sum_{n \geq 0} v(\mathcal{O}(P_{mn})) \frac{x^m}{m!} \frac{x^n}{n!} &= e^{-x-y} (e^{-x} + e^{-y} - 1)^{-3} \\ &= 1 + 2(x + y) + 7xy + 4 \frac{x^2 + y^2}{2!} + 23 \frac{x^2y + xy^2}{2!} \\ &\quad + 115 \frac{x^2y^2}{2!^2} + 8 \frac{x^3 + y^3}{3!} + 73 \frac{x^3y + xy^3}{3!} \\ &\quad + 533 \frac{x^3y^2 + x^2y^3}{2!3!} + 3451 \frac{x^3y^3}{3!^2} + \dots \end{aligned}$$

For instance, the order polytope of P_{22} has 8 facets and 115 visibility sets of facets.

We now pose the question of extending some results on graphical arrangements to ψ -graphical arrangements. An arrangement \mathcal{A} is *supersolvable* if the intersection lattice $L_{c(\mathcal{A})}$ of the cone $c(\mathcal{A})$ contains a maximal chain of modular elements. See for instance [13] for further details. If \mathcal{A} is supersolvable, then every zero of $\chi_{\mathcal{A}}(q)$ is a nonnegative integer. A graphical arrangement \mathcal{A}_G is supersolvable if and only if G is a *chordal* graph (also called a *triangulated* graph or *rigid circuit* graph) [13, Cor. 4.10]. It is natural to ask for an extension of this result to ψ -graphical arrangements. The proof of the following result is straightforward and will be omitted.

Theorem 6 *Let (G, ψ) be as above. Suppose that we can order the vertices of G as v_1, \dots, v_p such that*

- v_{i+1} connects to previous vertices along a clique (so G is chordal).
- If $i < j$ and v_i is adjacent to v_j , then $\psi(v_j) \subseteq \psi(v_i)$.

Then $\mathcal{A}_{G,\psi}$ is supersolvable.

The converse to Theorem 6 is proved in [8]. There are numerous characterizations of chordal graphs [18]. It would be interesting to investigate which of these characterizations have analogs for the pairs (G, ψ) satisfying the conditions of Theorem 6.

A profound generalization of supersolvable arrangements is due to Saito and Terao (e.g., [9, Chap. 4] and [13, Thm. 4.14]), called *free* arrangements. Freeness was defined originally for central arrangements, but we can define a noncentral arrangement \mathcal{A} to be free if the cone $c(\mathcal{A})$ is free. The “factorization theorem” of Terao asserts that if \mathcal{A} is free, then the zeros of $\chi_{\mathcal{A}}(q)$ are nonnegative integers (with an algebraic interpretation). Every supersolvable arrangement is free, and every free graphical arrangement is supersolvable. This leads to the following conjecture.

Conjecture 1 If $\mathcal{A}_{G,\psi}$ is a free ψ -graphical arrangement, then $\mathcal{A}_{G,\psi}$ is supersolvable.

Some partial results related to Conjecture 1 appear in [8].

4 Applications

One immediate application of Theorem 1 follows from the matroidal definition of Dilworth truncation.

Corollary 1 *The characteristic polynomial $\chi_{\text{vo}(\mathcal{A},p)}(q)$, where p is generic, is a matroidal invariant, that is, it depends only on $L_{\mathcal{A}}$. In particular, the number $v(\mathcal{A}, p)$ of valid orderings with respect to a generic point p is a matroidal invariant and hence is independent of the region in which p lies.*

Proof The Dilworth truncation $D_k L$ of a geometric lattice L is defined as $L_{D_k M}$, where M is the matroid associated to L . The proof that $L_{\text{vo}(\mathcal{A},p)}(q)$ is a matroidal invariant follows from Theorem 1. The statement for $v(\mathcal{A}, p)$ then follows from Zaslavsky’s Theorem (2). □

For our second application, let $c(n, k)$ denote the signless Stirling number of the first kind, i.e., the number of permutations $w \in \mathfrak{S}_n$ with k cycles. The following result is equivalent to [5, Cor. 3.2].

Theorem 7 *Let \mathcal{A} be an arrangement in \mathbb{R}^d with m hyperplanes, and let p be a point in \mathbb{R}^d not lying on any $H \in \mathcal{A}$. Then*

$$v(\mathcal{A}, p) \leq 2(c(m, m - d + 1) + c(m, m - d + 3) + c(m, m - d + 5) + \dots),$$

and this inequality is best possible. (The sum on the right is finite since $c(m, k) = 0$ for $k > m$.)

Proof It is not hard to see that $v(\mathcal{A}, p)$ will be maximized when the hyperplanes $H \in \mathcal{A}$ are as “generic as possible,” i.e., the intersection poset $L_{\mathcal{A}}$ is a boolean algebra B_m with all elements of rank greater than d (including the top element) removed, and when p is also generic. (Consider the effect of small perturbations of hyperplanes not in general position.) Assume then that $L_{\mathcal{A}}$ is such a truncated boolean algebra. Since $L_{\mathcal{A}}$ becomes a geometric lattice $\hat{L}_{\mathcal{A}}$ when we add a top element, it follows that the semicone $\text{sc}(\mathcal{A})$ satisfies $L_{\text{sc}(\mathcal{A})} \cong \hat{L}_{\mathcal{A}}$. Now ordinary truncation T^i and Dilworth truncation D_j commute (for $i + j < d$, the ambient dimension). By (1) we have $D_1 \hat{L}_{\mathcal{A}} \cong T^{m-d-1} \Pi_m$. Now [14, Examples 3.11.11]

$$\chi_{\Pi_m}(q) = (q - 1) \cdots (q - m + 1) = \sum_{j=0}^{m-1} (-1)^j c(m, m - j) q^{m-j-1}.$$

Thus

$$\chi_{T^{m-d-1} \Pi_m}(q) = \sum_{j=0}^{d-1} (-1)^j c(m, m - j) q^{d-j} + C$$

for some $C \in \mathbb{Z}$. Since $\chi_{\mathcal{B}}(1) = 0$ for any central arrangement \mathcal{B} , we get

$$C = - \sum_{j=0}^{d-1} (-1)^j c(m, m - j).$$

Therefore

$$\begin{aligned} v(\mathcal{A}) &= (-1)^d \chi_{T^{m-d-1}\Pi_m}(-1) \\ &= \sum_{j=0}^{d-1} c(m, m - j) - (-1)^d \sum_{j=0}^{d-1} (-1)^j c(m, m - j) \\ &= 2(c(m, m - d + 1) + c(m, m - d + 3) + c(m, m - d + 5) + \dots), \end{aligned}$$

and the proof follows. □

For fixed k , we have that $c(m, m - k)$ is a polynomial in m . Hence for fixed d , the bound in Theorem 7 is a polynomial $P_d(m)$ in m . For instance,

$$\begin{aligned} P_1(m) &= 2, \\ P_2(m) &= m(m - 1), \\ P_3(m) &= \frac{1}{12}(2m^4 - 10m^3 + 9m^2 - 2m + 4), \\ P_4(m) &= \frac{1}{24}m(m - 1)(m^4 - 6m^3 + 11m^2 - 6m + 24), \\ P_5(m) &= \frac{1}{2880}(15m^8 - 180m^7 + 830m^6 - 1848m^5 + 2735m^4 - 3300m^3 \\ &\quad + 2180m^2 - 432m + 5760). \end{aligned}$$

Clearly given $m > d$, we can find a convex d -polytope with m facets, where the affine spans of the facets are as “generic as possible,” as defined at the beginning of the proof of Theorem 7. Thus we obtain the following corollary to Theorem 7.

Corollary 2 *Let \mathcal{P} be a convex polytope in \mathbb{R}^d with m facets, and let p be a point in the interior of \mathcal{P} . Then the number $ls(\mathcal{P}, p)$ of line shellings of \mathcal{P} whose shelling line passes through p satisfies*

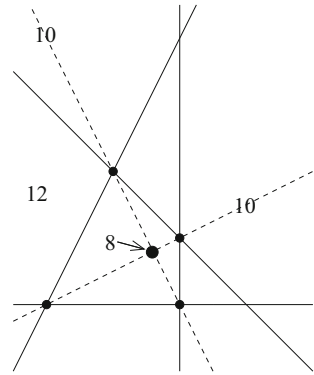
$$ls(\mathcal{P}, p) \leq 2(c(m, m - d + 1) + c(m, m - d + 3) + c(m, m - d + 5) + \dots),$$

and this inequality is best possible.

5 Further Vistas

We have considered the intersection of a line L through a point p with the hyperplanes of an arrangement \mathcal{A} . We will sketchily describe an extension. Namely, what if we replace L with an m -dimensional plane (or m -plane for short) P through m affinely

Fig. 2 An example of a polyhedral decomposition Γ associated to valid orderings



independent points p_1, \dots, p_m not lying on any $H \in \mathcal{A}$? We will obtain an induced arrangement

$$\mathcal{A}_P = \{H \cap P : H \in \mathcal{A}\}$$

in the ambient space P . Define the generalized valid order arrangement $\text{vo}(\mathcal{A}; p_1, \dots, p_m)$ to consist of all the hyperplanes passing through p_1, \dots, p_m and every intersection of $m + 1$ hyperplanes of \mathcal{A} , including “intersections at ∞ .” The regions of $\text{vo}(\mathcal{A}; p_1, \dots, p_m)$ correspond to the different equivalence classes of arrangements \mathcal{A}_P , where \mathcal{A}_P and \mathcal{A}_Q are considered equivalent if they correspond to the same *oriented* matroid. We then have the following analog of Theorem 1.

Theorem 8 *Let \mathcal{A} be an arrangement in the real vector space V , and let p_1, \dots, p_m be “sufficiently generic” points of V . Then $L_{\text{vo}(\mathcal{A}; p_1, \dots, p_m)} \cong L_{D_m(\mathcal{A})}$.*

Theorem 1 deals with $\text{vo}(\mathcal{A}, p)$ when p is generic. What about nongeneric p ? Define two points p, q not lying on any hyperplane of \mathcal{A} to be *equivalent* if there is a canonical bijection $\varphi : \text{vo}(\mathcal{A}, p) \rightarrow \text{vo}(\mathcal{A}, q)$. By *canonical*, we mean that if H is a hyperplane of $\text{vo}(\mathcal{A}, p)$ which is the affine span with p and the intersection $H_1 \cap H_2$ of two hyperplanes in \mathcal{A} (including an intersection at ∞ , i.e., H is parallel to H_1 and H_2), then $\varphi(H)$ is the affine span of q and $H_1 \cap H_2$. The equivalence classes of this equivalence relation form a polyhedral decomposition of \mathbb{R}^d . Figure 2 shows an example. The arrangement \mathcal{A} is given by solid lines and the lines (1-faces) of the polyhedral decomposition Γ by broken lines. Certain faces F of Γ are marked with the number $v_F(\mathcal{A}, p)$ of valid orderings for $p \in F$. If a face $F' \subseteq \text{aff}(F)$ satisfies $\dim F = \dim F'$, then $v_F(\mathcal{A}, p) = v_{F'}(\mathcal{A}, p)$.

What can be said about the polyhedral complex Γ ? The two-dimensional case illustrated in Fig. 2 is somewhat misleading. Let \mathcal{A} be an arrangement in \mathbb{R}^d , and let $p \in \mathbb{R}^d - \bigcup_{H \in \mathcal{A}} H$. Suppose that $H_1, \dots, H_4 \in \mathcal{A}$ with $H_1 \neq H_2$ and $H_3 \neq H_4$. If $\text{aff}(p, H_1 \cap H_2) = \text{aff}(p, H_3 \cap H_4)$, then the two $(d - 2)$ -dimensional subspaces $H_1 \cap H_2$ and $H_3 \cap H_4$ must both lie on an affine hyperplane K . If $d = 2$, then this condition always holds, but for $d > 2$ it does not hold for “generic” \mathcal{A} . Thus for generic \mathcal{A} and $d > 4$, the valid order arrangements $\text{vo}(\mathcal{A}, p)$ have the same number of

hyperplanes for any p . However, they may still differ in how the hyperplanes intersect. It may be interesting to further investigate the properties of Γ .

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