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PLASMA DYNAMICS

Axisymmetric Plasma Equilibrium in Gravitational and Magnetic Fields[1]

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Abstract—Plasma equilibria in gravitational and open-ended magnetic fields are considered for the case of topologically disconnected regions of the magnetic flux surfaces where plasma occupies just one of these regions. Special dependences of the plasma temperature and density on the magnetic flux are used which allow the solution of the Grad–Shafranov equation in a separable form permitting analytic treatment. It is found that plasma pressure tends to play the dominant role in the setting the shape of magnetic field equilibrium, while a strong gravitational force localizes the plasma density to a thin disc centered at the equatorial plane.

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1. INTRODUCTION

Magnetohydrodynamic (MHD) equilibria of plasma subject to a gravitational field were considered originally by Chandrasekhar [1]. In recent years, other MHD plasma equilibria studies retaining gravity effects appeared in the literature [2–8]. Without discussing the validity of the approximations used in those papers, we notice that two of them [2, 8] are focused on disc-like plasma equilibria resembling the plasma shape of accretion discs. The goal of this work is to further study disc-like plasma equilibria in magnetic and gravity fields. However, in the contrast to [2, 8], where the whole space is topologically connected by the magnetic field lines, we consider an axisymmetric disc-like plasma equilibrium with open magnetic field lines for the case where the space is divided into regions which are topologically disconnected by magnetic flux surfaces and plasma occupies just one of these regions or lobes (see Fig. 1). Somewhat similar topology of the magnetic field and plasma was considered in [9] for the case of closed magnetic field lines corresponding to magnetic multipoles.

To demonstrate the possibility of the existence of such equilibria, we consider a model problem and find axisymmetric magnetic flux surfaces ψ, generated by the azimuthally symmetric equatorial plane current density

$$\mathbf{j}(R,z) = \mathbf{e}_{\varphi}\hat{j}_0 R^{-(\nu+1)}\delta(z), \tag{1}$$

[1] The article is published in the original.

where R and z are the cylindrical coordinates, $\mathbf{e}_{\varphi}$ is a unit vector along the azimuthal direction, $\hat{j}_0$ is a normalization constant, $\delta(z)$ is a delta-function, and ν is an adjustable parameter. Then, using the expression for the magnetic field $\mathbf{B} = (\nabla\psi\times\mathbf{e}_{\varphi})/R$, from Ampere's law we find

$$R\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = -\frac{4\pi\hat{j}_0}{c}R^{-(\nu+1)}\delta(z). \tag{2}$$

Looking for the solution of Eq. (2) in a separable form by letting $\psi(r,\mu) = -(4\pi\hat{j}_0/c)r^{-\nu}h(\mu)$, where r and

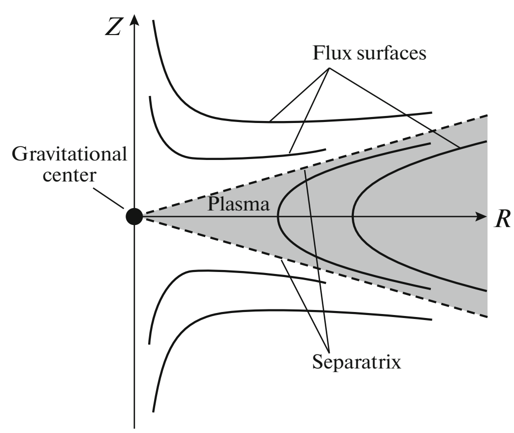

Fig. 1. Schematic view of plasma equilibrium in gravitational and open-ended magnetic fields with topologically disconnected regions of magnetic flux surfaces.

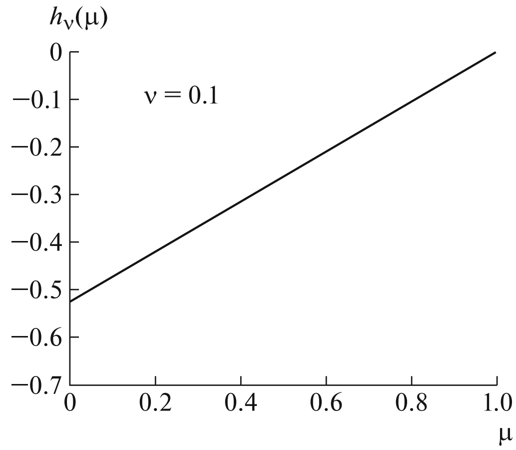

Fig. 2. The function $h_\nu(\mu)$ for $\nu = 0.1$.

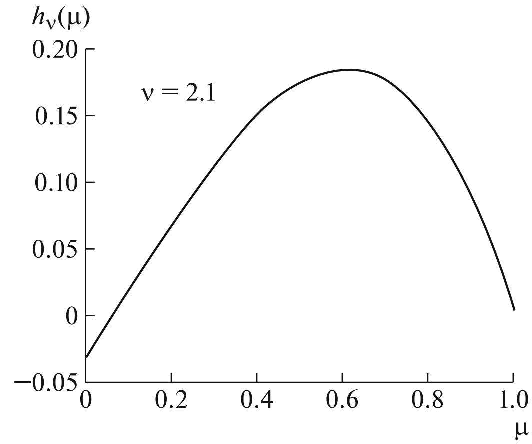

Fig. 3. The function $h_\nu(\mu)$ for $\nu = 2.1$.

$\mu = \cos(\vartheta)$ $(-1 \le \mu \le 1)$ are spherical coordinates, we find the following equation for $h(\mu)$:

$$\frac{d^2h}{d\mu^2} + \nu(\nu+1)\frac{h}{1-\mu^2} = \delta(\mu). \tag{3}$$

We notice that Eq. (3) has the same solution if we let $\nu \to -(\nu+1)$. To have the axial magnetic field dominate over poloidal field at the axis of symmetry, the solution of Eq. (3) should satisfy the boundary condition $h(\mu)/\left(\frac{dh}{d\mu}\sqrt{1-\mu^2}\right)\Big|_{\mu\to\pm1} \to 0$. Taking this into account and assuming that $h(\mu)$ is a symmetric function of μ, the solution of Eq. (3) for a noninteger ν can be written as

$$h(\mu) = h_\nu(\mu) - \frac{1}{2}\frac{P_{-\nu}(|\mu|) - |\mu|\,P_\nu(|\mu|)}{(1-\nu)P_{1-\nu}(0) + P_\nu(0)}, \tag{4}$$

where $P_\nu(\mu) = P_{-\nu-1}(\mu)$ is a Legendre function of the first kind and a homogeneous solution of Eq. (3) is $P_{-\nu}(\mu) - \mu P_\nu(\mu) = \sqrt{1-\mu^2}P_\nu^1(\mu)/(1+\nu)$. The jump condition, $dh/d\mu|_{\mu=0} = 1/2$, associated with Eq. (3), is used to determine the coefficient along with $dP_{-\nu}/d\mu|_{\mu=0} = -(1-\nu)P_{1-\nu}(0)$. For positive even integers ν, we can use Eq. (4), while for odd integer ν, we should replace $P_\nu(\mu)$ in Eq. (4) with $Q_\nu(\mu)$, which is a Legendre function of the second kind.

The functions $h_\nu(\mu)$ for $\nu = 0.1$ and $\nu = 2.1$ are shown in Figs. 2 and 3, respectively. As we see, for $\nu = 0.1$, the function $h_\nu(\mu)$ does not change its sign, which means that the whole space is topologically connected by magnetic field lines. For $\nu = 2.1$, however, $h_\nu(\mu)$ changes its sign at $\mu = \mu_{\text{sep}} \approx 0.064$, which means that $\psi(r, \mu = \mu_{\text{sep}}) = 0$ and the space is separated in three distinct magnetically disconnected regions or lobes with $\psi > 0$ and $\psi < 0$. In the next section, we will exploit this three lobe solution for a self-consistent current satisfying the Shafranov–Grad equation.

The paper is organized as follows, in Section 2, we describe our model of plasma equilibrium in gravitational and magnetic fields and formulate the governing equation; in Section 3, we present analytic solutions of this equation for some extreme conditions; and in Section 4, we summarize our findings.

2. GOVERNING EQUATION

We consider plasma equilibrium in a gravitational potential $G(r) = -G_0M_0/r$ and an axisymmetric magnetic field $\mathbf{B} = (\nabla\psi \times \mathbf{e}_\varphi)/R$, where G_0 and M_0 are the gravitational constant and the mass of the gravitational center located at $r = 0$, respectively. In the derivation of the equation governing our plasma equilibrium, we will follow [8] and assume that both electron, T_e, and ion, T_i, temperatures are functions of ψ, as required for a drifting Maxwellian solution of the drift-kinetic or gyrophase-averaged Fokker–Planck equation in an axisymmetric system [10–12]. Then, expressing the plasma density in the form $n(r,\mu) = \eta(\psi)\exp[\kappa(\psi,\mu)]$, from the force balance along the magnetic field lines, we find $\kappa(\psi,\mu) = -G(r)/C_s^2(\psi)$, where $C_S^2(\psi) = [Z_iT_e(\psi) + T_i(\psi)]/M$ and M is the mass of plasma ions having the charge Z_i. From the force balance equation across the magnetic field lines and Ampere's law, we obtain the Grad–Shafranov equation

$$R\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = -\frac{4\pi Mn}{R^2B^2}\nabla\psi \times \left\{C_S^2\nabla\ln(\eta) + \nabla C_S^2 + C_S^2\nabla\kappa + \nabla G\right\}. \tag{5}$$

Following [13], we will seek a solution of Eq. (5) in a separable form by taking

$$\psi(r,\mu) = \psi_0 h(\mu)(r_0/r)^{\alpha}, \tag{6}$$

where α is an adjustable parameter which plays the role of an eigenvalue to find a solution of Eq. (5) (see, e.g., [13]), while ψ_0 and r_0 are normalization constants such that $h(0) = 1$. One can show that ansatz (6) is compatible with Eq. (5) for the following dependences of $C_s^2(\psi)$ and $\eta(\psi)$:

$$\begin{aligned} C_s^2(\psi) &= C_0^2(\psi/\psi_0)^{1/\alpha}, \\ \eta(\psi) &= n_0(\psi/\psi_0)^{2+3/\alpha}, \end{aligned} \tag{7}$$

where C_0^2 and n_0 are normalization constants chosen in a such a way that $\kappa(\psi,\mu = 0) = 0$. Substituting expressions (6) and (7) into Eq. (5), we find the governing equation for $h(\mu)$ to be

$$\begin{aligned} &\frac{d^2h}{d\mu^2} + \frac{\alpha(\alpha+1)}{1-\mu^2}h \\ &= \alpha\beta\left\{\frac{g}{2}h^{-1/\alpha} - (\alpha+2)\right\}h^{1+4/\alpha}\exp\left[g(h^{-1/\alpha}-1)\right], \end{aligned} \tag{8}$$

where

$$\begin{aligned} \beta &= 8\pi M n_0\left(\frac{C_0 r_0}{\alpha\psi_0}\right)^2 \equiv \left.\frac{8\pi M n C_S^2}{B^2}\right|_{\mu=0}, \\ g &= \frac{G_0 M_0}{C_0^2 r_0} \equiv \left.\frac{G_0 M_0}{C_S^2 r}\right|_{\mu=0}. \end{aligned} \tag{9}$$

Equation (8) agrees with the different cases analyzed in [4, 8, 13–15]. The solution of Eq. (8) with the boundary conditions $h(0) = 1$ and $h(\mu)/\sqrt{1-\mu^2}\big|_{\mu\to\pm1} \to 0$ can only be satisfied for some particular values of α.

3. SOLUTIONS OF THE GOVERNING EQUATION

The plasma equilibria in gravitational and magnetic fields considered by Krasheninnikov and Catto [4, 8], where within the framework of separable solutions (6) for the case where the whole space was topologically connected by the magnetic field lines. Here, however, we follow [9] and consider equilibria with open magnetic field lines and topologically disconnected regions of the magnetic flux surfaces where plasma occupies just one of these regions (recall Fig. 1). Based on the results in Section 2, it is easy to see that the topology of the magnetic flux surfaces is determined by the magnitude of α (see [9]). Indeed, symmetric ($h(\mu) = h(-\mu)$) vacuum ($\beta = 0$) solutions of Eq. (8), which in this case is identical to the left-hand side of Eq. (3), for $h(0) = 1$ are given by the following expression:

$$\begin{aligned} h(\mu) &= h_{\text{vac}}^{(\alpha)}(\mu) \\ &\equiv \frac{P_{-\alpha}(\mu) - \mu P_{-\alpha-1}(\mu)}{P_{-\alpha}(0)} = \frac{P_{\alpha-1}(\mu) - \mu P_{\alpha}(\mu)}{P_{\alpha-1}(0)}, \end{aligned} \tag{10}$$

where α a nonzero adjustable parameter ($h = 1 \pm \mu$ for $\alpha = 0$), with the first form useful for α an even negative integer, and the second form useful for α an odd positive integer. From Eq. (10) one can see that, for $\alpha = 1$ or -2, the function $h_{\text{vac}}^{(1)}(\mu) = h_{\text{vac}}^{(-2)}(\mu) = 1-\mu^2$ reaches zero only at the poles ($\mu = \pm1$). For other $\alpha \neq 0$, additional roots of $h_{\text{vac}}^{(\alpha)}(\mu) = 0$ appear within the range $-1 \le \mu \le 1$, giving rise to additional lobes as was illustrated in Section 1. These roots become the separatrices, $\psi(r,\mu = \mu_{\text{sep}}) = 0$, dividing the space into magnetically disconnected regions with different signs of ψ.

In [9] the Grad–Shafranov equation for plasma equilibrium in a multipolar magnetic field was considered for closed magnetic field lines, corresponding to positive α (with $2 < \alpha < 3$). Here, we consider the case of plasma equilibria in gravitational and open magnetic fields corresponding to negative α. In [4, 8], only $\alpha > -2$ were considered for $G \neq 0$ and the whole space was topologically connected by the magnetic field lines. The simplest symmetric ($h(\mu) = h(-\mu)$) case with topologically disconnected magnetic flux surfaces and open magnetic field lines can be found for $-4 < \alpha < -3$.

For $\beta \ll 1$ and $\beta g \ll 1$, the right-hand side of Eq. (8) is small and $h(\mu)$ is close to the vacuum solution corresponding to $\alpha = -4$,

$$h_{\text{vac}}^{(-4)}(\mu) = (1-\mu^2)(1-5\mu^2), \tag{11}$$

so that we have $\mu_{\text{sep}} \approx 1/\sqrt{5}$. The correction for the eigenvalue $\alpha = -4$ caused by the small, but finite right-hand side of Eq. (8), can be found by multiplying Eq. (8) by $h_{\text{vac}}^{(-4)}(\mu)$ and integrating from $\mu = 0$ to $\mu = 1$, to find

$$\begin{aligned} &(\alpha-3)(\alpha+4)\int_0^1 h(\mu)(1-5\mu^2)d\mu \\ &= \alpha\beta\int_0^{\mu_{\text{sep}}} h_{\text{vac}}^{(-4)}(\mu)\left\{\frac{g}{2}h^{-1/\alpha} - (\alpha+2)\right\}h^{1+4/\alpha} \\ &\quad\times \exp\left[g(h^{-1/\alpha}-1)\right]d\mu. \end{aligned} \tag{12}$$

Substituting into Eq. (12), as a first approximation, $h(\mu) = h_{\text{vac}}^{(-4)}(\mu)$ (we discuss later the limits of the applicability of this approximation), we find the correction to the eigenvalue α,

$$\alpha = -4 + \frac{3}{4}\beta \int_0^{\mu_{sep}} h_{vac}^{(-4)}(\mu)\left\{\frac{g}{2}\left(h_{vac}^{(-4)}(\mu)\right)^{1/4} + 2\right\} \times \exp\left\{g\left[\left(h_{vac}^{(-4)}(\mu)\right)^{1/4} - 1\right]\right\} d\mu. \quad (13)$$

From this form we can get some insight into what would happen if the right-hand side of Eq. (8) were to dominate. We would then expect the function $h(\mu)$ to have a large negative second derivative about $\mu \approx 0$, which quickly reduces $h(\mu)$ to zero, so that the region occupied by plasma shrinks and $\mu_{sep} \ll 1$. In addition, we expect α to become greater than -4 and, as the region becomes narrower, move toward -3. As a result, in the limit $\alpha \to -3$, $h(\mu)$ in vacuum regions is given by Eq. (10) with $\alpha < -3$. Expanding for small μ and $\alpha \to -3$, by considering vacuum solution (10), gives

$$h_{vac}^{(\alpha)}(\mu) \approx (\mu_{sep}^{(\alpha)} - \mu)\bar{h}, \quad (14)$$

where $\bar{h}$ is a new normalization constant and

$$\mu_{sep}^{(\alpha\to -3)} \approx -2(3+\alpha)/3, \quad (15)$$

which becomes small for α in the vicinity of -3.

When the magnitude of the right-hand side of Eq. (8) is large, we can neglect the second term on the left-hand side of Eq. (8) to obtain

$$\frac{d^2h}{d\mu^2} = -3\beta\left(\frac{g}{2}h^{1/3} + 1\right)h^{-1/3}\exp\left[g(h^{1/3} - 1)\right], \quad (16)$$

where we let $\alpha \to -3$ on the right-hand side of Eq. (8). Accounting for the boundary conditions $h(0) = 1$ and $dh/d\mu|_{\mu=0} = 0$, multiplying Eq. (16) by $dh/d\mu$, and integrating from $\mu = 0$ gives

$$\frac{dh}{d\mu} = -3\sqrt{\beta}\left\{\int_{h^{1/3}}^{1} (g\xi + 2)\xi\exp[g(\xi - 1)]d\xi\right\}^{1/2}. \quad (17)$$

Integrating Eq. (17) from $\mu = 0$ to μ_{sep}, where $h(\mu_{sep}) = 0$, we obtain $h(\mu)$ and find $\mu_{sep}^{(\alpha)}$,

$$\mu_{sep}^{(\alpha)} = \frac{1}{3\sqrt{\beta}}\int_0^1\left\{\int_{h^{1/3}}^{1} (g\xi + 2)\xi\exp[g(\xi - 1)]d\xi\right\}^{-1/2} dh. \quad (18)$$

Matching expressions (15) and (18), we find α to be

$$\alpha = -3 - \frac{1}{2\sqrt{\beta}}\int_0^1\left\{\int_{h^{1/3}}^{1} (g\xi + 2)\xi\exp[g(\xi - 1)]d\xi\right\}^{-1/2} dh. \quad (19)$$

We can estimate the integral expression on the right-hand side of Eq. (19). For small and large g, we find

$$\int_0^1\left\{\int_{h^{1/3}}^{1} (g\xi + 2)\xi\exp[g(\xi - 1)]d\xi\right\}^{-1/2} dh \approx \begin{cases} 3\pi/4, & g \ll 1, \\ 1, & g \gg 1. \end{cases} \quad (20)$$

Thus, from Eqs. (19) and (20) we see that α can only be close to -3 for $\beta \gg 1$, giving

$$\alpha \approx -3 \times \begin{cases} 1 + \dfrac{\pi}{8\sqrt{\beta}}, & g \ll 1, \\ 1 + \dfrac{1}{6\sqrt{\beta}}, & g \gg 1. \end{cases} \quad (21)$$

We also conclude that, for $\beta \ll 1$, α stays close to -4 and $h(\mu)$ is close to $h_{vac}^{(-4)}(\mu)$ for both small and large g (although at $g \gg 1/\beta \gg 1$, $h(\mu)$ deviates significantly from $h_{vac}^{(-4)}(\mu)$ in a small region in the vicinity of $\mu = 0$, so that Eq. (13) may not be applied). As a result, from Eq. (13) we find

$$\alpha = -4 + \beta \times \begin{cases} \dfrac{1}{\sqrt{5}}\dfrac{24}{25}, & g \ll 1, \\ \dfrac{1}{8}\sqrt{\dfrac{3\pi g}{2}}, & 1/\beta \gg g \gg 1, \end{cases} \quad (22)$$

with the limit $1/\beta \gg g \gg 1$ a plasma disc equilibrium based on the localization of the density.

4. CONCLUSIONS

We have considered plasma equilibria in gravitational and open-ended magnetic fields for the case of topologically disconnected regions of the magnetic flux surfaces where plasma occupies just the central lobe (see Fig. 1). Choosing the special dependences of both plasma temperature and density on the magnetic flux as given by Eq. (7) allows us to search for the solutions of the Grad–Shafranov equation in separable form (6) that permits tractable analytic analysis. We find that the plasma pressure sets the shape of magnetic equilibrium at high β, while a strong gravitational force ($g \gg 1$) localizes the plasma density to a disc about the equatorial plane, but does not alter the magnetic equilibrium critically. At low β, the gravitational field affects the magnetic equilibrium when $1/\beta \gg g \gg 1$, but not for $g \ll 1$, and again a thin plasma disc about the equatorial plane is obtained only when gravity is strong.

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