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## Linear growth for semigroups which are disjoint unions of finitely many copies of the free monogenic semigroup

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**Abstract.** We show that every semigroup which is a finite disjoint union of copies of the free monogenic semigroup (natural numbers under addition) has linear growth. This implies that the corresponding semigroup algebra is a PI algebra.

**Mathematics Subject Classification.** 20M05.

**Keywords.** Monogenic semigroup, Unions, Linear growth, PI algebra.

**1. Introduction.** It is well known that a semigroup may sometimes be decomposed into a disjoint union of subsemigroups which is unlike the structures of classical algebra such as groups and rings. For instance, the Rees theorem states that every completely simple semigroup is a Rees matrix semigroup over a group  $G$ , and is thus a disjoint union of copies of  $G$ , see [6, Theorem 3.3.1]; every Clifford semigroup is a strong semilattice of groups and as such it is a disjoint union of its maximal subgroups, see [6, Theorem 3.3.1]; every commutative semigroup is a semilattice of archimedean semigroups, see [5, Theorem 3.3.1].

If  $S$  is a semigroup which can be decomposed into a disjoint union of subsemigroups, then it is natural to ask how the properties of  $S$  depend on these subsemigroups. For example, if the subsemigroups are finitely generated, then so is  $S$ . Arajo et al. [3] consider the finite presentability of semigroups which are disjoint unions of finitely presented subsemigroups; Golubov [4] showed that a semigroup which is a disjoint union of residually finite subsemigroups is residually finite.

For semigroups  $S$  which are disjoint unions of finitely many copies of the free monogenic semigroup, the authors in [1] proved that  $S$  is finitely presented and residually finite; in [2] the authors proved that, up to isomorphism and anti-isomorphism, there are only two types of semigroups which are unions of two copies of the free monogenic semigroup. Similarly, they showed that there

are only nine types of semigroups which are unions of three copies of the free monogenic semigroup and provided finite presentations for semigroups of each of these types.

In this paper we continue investigating semigroups which are disjoint unions of finitely many copies of the free monogenic semigroup. Specifically, our main result is that every semigroup which is a finite disjoint union of copies of the free monogenic semigroup has linear growth. This implies that the corresponding semigroup algebra is a PI algebra.

**2. The linear growth theorem.** Let  $S$  be a semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup:

$$S = \bigcup_{a \in A} N_a,$$

where  $A$  is a finite set and  $N_a = \langle a \rangle$  for  $a \in A$ . Define the functions  $f: S \times A \times \mathbb{Z}_{>0} \rightarrow A$  and  $n: S \times A \times \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$  by the formula

$$xy^i = f(x, y, i)^{n(x, y, i)}.$$

For  $y, z \in A, x \in S$ , let  $E(x, y, z)$  be the set of  $i \in \mathbb{Z}_{>0}$  such that  $f(x, y, i) = z$  (i.e.,  $xy^i$  is a power of  $z$ ).

**Definition 2.1.** We say that  $z$  is  $(x, y)$ -persistent if  $E(x, y, z)$  is infinite.

**Lemma 2.2.** (i) *Let  $z$  be  $(x, y)$ -persistent. Then  $E(x, y, z) = i_0 + E_0(x, y, z)$ , where  $i_0 \in \mathbb{Z}_{>0}$  and  $E_0(x, y, z)$  is a nonzero submonoid of  $\mathbb{Z}_{\geq 0}$  (an additively closed subset containing 0). Moreover, there exists a rational number  $M_x(y, z) \geq 0$ , such that for any  $i \in E(x, y, z)$ ,*

$$n(x, y, i) = n_0 + M_x(y, z)(i - i_0).$$

(ii)  $M_x(y, z)$  is independent on  $x$ .

(iii) The function  $f(x, y, i)$  is periodic in  $i$  starting from some place.

*Proof.* First note that if  $z^t y^q = z^s$  for some  $t, q, s > 0$ , then for  $N \geq t$  we have  $z^N y^q = z^{N+s-t}$ , and hence

$$z^N y^{qm} = z^{N+(s-t)m} \tag{1}$$

for all  $m > 0$  and  $N \geq mt$ .

Now let  $i_0$  be the smallest number in  $E(x, y, z)$ , and  $xy^{i_0} = z^{n_0}$ . By our assumption, there exists  $r > 0$  such that  $xy^{i_0+r} = z^n$ ; let us take the smallest such  $r$ . Thus  $z^{n_0} y^r = z^n$ . Let  $M = M_x(y, z) := \frac{n-n_0}{r}$ . Then by (1), for any  $p > 0$  and  $N \geq pm_0$ , we have

$$z^N y^{pr} = z^{N+Mpr}. \tag{2}$$

Suppose now that  $z^t y^q = z^s$  for some  $t, q, s > 0$ . Then setting  $m = r$  in (1),  $p = q$  in (2), and comparing these equations, we get

$$s - t = Mq. \tag{3}$$

Now, let  $E_0(x, y, z) = E(x, y, z) - i_0$ . By (3), for all  $j \in E_0(x, y, z)$ , we have

$$z^{n_0} y^j = z^{Mj+n_0}. \tag{4}$$

Since this holds for infinitely many  $j$ , we have  $Mj + n_0 > 0$  for infinitely many  $j$ , hence  $M \geq 0$ .

Let us show that  $E_0(x, y, z)$  is closed under addition. Suppose  $j, k \in E_0(x, y, z)$ . Multiplying (4) by  $y^k$ , we get

$$z^{n_0}y^{j+k} = z^{Mj+n_0}y^k = z^{Mj}z^{Mk+n_0} = z^{M(j+k)+n_0}.$$

Thus,  $j + k \in E_0(x, y, z)$ . This proves (i).

To prove (ii), let  $x' \in S$  be another element such that  $z$  is  $(x', y)$ -persistent. Then by (2), we have

$$z^N y^{pr} = z^{N+Mpr}, \quad z^N y^{pr'} = z^{N+M'pr'} \tag{5}$$

for all  $p > 0$  and sufficiently large  $N$ , where  $M' = M_{x'}(y, z)$ . Thus,

$$z^N y^{rr'} = z^{N+Mrr'} = z^{N+M'rr'}$$

for large enough  $N$ , implying  $M = M'$ , as desired.

Finally, note that it follows from (i) that if  $f(x, y, i) = z$  for some  $(x, y)$ -persistent  $z$ , and if  $j \in E_0(x, y, z)$ , then  $f(x, y, i + j) = z$ . So statement (iii) follows from the fact that for sufficiently large  $j$ ,  $xy^j = z^m$  for an  $(x, y)$ -persistent  $z$ . Namely, the period  $T$  for  $f(x, y, j)$  can be taken to be any positive element of the intersection

$$\bigcap_{z \text{ is } (x,y)\text{-persistent}} E_0(x, y, z).$$

□

**Remark 2.3.** Lemma 2.2 in [1] states that if  $x^i y^j = x^k$  holds in  $S$  for some  $x, y \in A$  and  $i, j, k \in \mathbb{N}$  then  $i \leq k$ . By this lemma and Lemma 2.2, if the set  $E(x, y, z)$  contains more than one element then it contains an arithmetic progression, and hence is infinite.

In view of Lemma 2.2(ii), we will say that  $z$  is  $y$ -persistent if there is  $x \in S$  such that  $z$  is  $(x, y)$ -persistent, and will denote  $M_x(y, z)$  simply by  $M(y, z)$ . Notice that  $y$  is obviously  $y$ -persistent, with  $M(y, y) = 1$ .

**Lemma 2.4.** *Let  $x, y, z \in A$ . If  $z$  is  $y$ -persistent and  $y$  is  $x$ -persistent then  $z$  is  $x$ -persistent and  $M(x, z) = M(x, y)M(y, z)$ .*

*Proof.* Suppose  $z$  is  $(v, y)$ -persistent, and  $y$  is  $(u, x)$ -persistent. Then, by Lemma 2.2,

$$ux^{i_0+rm} = y^{n_0+M(x,y)rm}, \quad vy^{i'_0+r'm'} = z^{n'_0+M(y,z)r'm'}$$

for appropriate  $i_0, i'_0, n_0, n'_0, r, r' \in \mathbb{Z}_{>0}$  and any  $m, m' \in \mathbb{Z}_{\geq 0}$ . Let  $m = sr'$  with  $s \in \mathbb{Z}_{\geq 0}$ , and  $\ell \geq 0$  be such that  $\ell + n_0 - i'_0 = pr'$  for some  $p \in \mathbb{Z}_{\geq 0}$ . Then

$$\begin{aligned} (vy^\ell u)x^{i_0+rm} &= vy^{\ell+n_0+M(x,y)rm} = vy^{i'_0+r'(p+M(x,y)rs)} \\ &= z^{n'_0+M(y,z)r'(p+M(x,y)rs)}. \end{aligned}$$

Thus,

$$(vy^\ell u)x^{i_0+rr's} = z^{n'_0+M(y,z)pr'+M(x,y)M(y,z)rr's}.$$

By Lemma 2.2, this means that  $z$  is  $(vy^\ell u, x)$ -persistent, and  $M(x, z) = M(x, y)M(y, z)$ , as desired.  $\square$

**Lemma 2.5.** *Let  $A$  be a finite set,  $P \subset A \times A$  be a reflexive transitive relation, and  $M: P \rightarrow \mathbb{Z}_{\geq 0}$  be a nonnegative integer function such that  $M(y, y) = 1$ , and  $M(x, z) = M(x, y)M(y, z)$  whenever  $(x, y), (y, z) \in P$ . Then there exists a positive integer function  $d$  on  $A$  such that*

$$d(y) \geq d(z)M(y, z)$$

whenever  $(y, z) \in P$ .

*Proof.* We say that  $z$  is reachable from  $y$  (denote by  $z \geq y$ ) if  $(y, z) \in P$ . We define a relation  $\rho$  on  $A$  as follows:

$$\rho = \{(y, z) \subseteq A \times A \text{ such that } y \text{ is reachable from } z \text{ and } z \text{ is reachable from } y\}.$$

It is clear that the relation  $\rho$  is reflexive, symmetric, and transitive, so it is an equivalence relation, splitting  $A$  into equivalence classes  $A_1, A_2, \dots, A_n$ . Now we show that  $d$  exists if  $n = 1$ , which means that  $A$  is a single equivalence class. In this case,  $M(x, y)$  is defined for all  $x, y$ , and  $M(x, y)M(y, x) = M(y, y) = 1$ , so  $M(x, y) > 0$ . Let us pick  $a \in A$  and define  $d$  as follows:

$$d(b) = M(b, a) \quad \forall b \in A.$$

We can multiply this function by an integer to make it integer-valued. By our assumption, this function satisfies the required condition [in fact we have a stronger condition  $d(y) = d(z)M(y, z)$ ].

Now consider the case  $n > 1$ . We say that an equivalence class  $C$  is a sink if each element reachable from  $C$  is in  $C$ . We claim that there is a sink class  $A_i$  among the equivalence classes  $A_1, A_2, \dots, A_n$ . To prove this, assume the contrary, i.e., that there is no sink class. Pick an arbitrary equivalence class  $A_{i_1}$ . It is not a sink, so we can reach some  $A_{i_2}$  from it with  $i_2 \neq i_1$ , and we can reach  $A_{i_3}$  from  $A_{i_2}$ , with  $i_2 \neq i_3$ , and so on. For some  $k < l$ , we must have  $i_k = i_{l+1}$ , which is a contradiction, since  $A_{i_k} \neq A_{i_{k+1}}$ , and yet they are reachable from each other.

Now we continue the proof of the lemma by induction on  $|A| = N$ . It is clear that the statement is true when  $|A| = 1$ . Suppose that the statement is true when  $|A| < N$ , and let us prove it for  $|A| = N$ . Let  $C$  be a sink class, and consider  $A' = A \setminus C$ . By the induction assumption,  $d$  exists on  $A'$  and  $C$ . Since  $C$  is a sink, if  $y \in A'$  and  $z \in C$  then  $y$  is not reachable from  $z$ . So we may multiply the function  $d$  on  $A'$  by a large integer (without changing  $d$  on  $C$ ) so that  $d(y) \geq d(z)M(y, z)$  for all  $y \in A', z \in C$  such that  $z$  is reachable from  $y$ . Then  $d$  satisfies the required condition.  $\square$

**Corollary 2.6.** *For the semigroup  $S$ , there exists a positive integer function  $d$  on  $A$  such that*

$$d(y) \geq d(z)M(y, z) \tag{6}$$

whenever  $z$  is  $y$ -persistent.

*Proof.* This follows from Lemmas 2.4 and 2.5, applying the latter to the set  $P$  of pairs  $(y, z)$  such that  $z$  is  $y$ -persistent (and  $A, M$  as above).  $\square$

Now pick a function  $d$  as in Corollary 2.6. We extend this function from  $A$  to  $S$  by setting  $d(x^i) = d(x)i, i \in \mathbb{Z}_{>0}$ .

**Lemma 2.7.** *There exists a constant  $K > 0$  such that  $d(x) \leq K$  and  $d(xu) \leq K + d(u)$  for every  $u \in S$  and  $x \in A$ .*

*Proof.* Let  $y \in A$ . It suffices to prove that there exists  $K_y > 0$  such that

$$d(xy^i) \leq K_y + d(y^i) = K_y + d(y)i$$

for all  $i$ . Then we can take  $K' = \max_y K_y$ , and then pick  $K \geq K'$  so that  $d(x) \leq K$  for all  $x \in A$ . Further, it is enough to show that for each  $(x, y)$ -persistent  $z$ , there exists  $K_{y,z} > 0$  such that  $d(xy^i) \leq K_{y,z} + d(y)i$  whenever  $f(x, y, i) = z$ . Indeed, taking  $K'_y := \max_z K_{y,z}$ , we see that the inequality  $d(xy^i) \leq K'_y + d(y)i$  holds for almost all  $i$  [namely, for all  $i$  such that  $f(x, y, i)$  is  $(x, y)$ -persistent], so there exists  $K_y \geq K'_y$  such that  $d(xy^i) \leq K_y + d(y)i$  for all  $i$ .

By Lemma 2.2, if  $f(x, y, i) = z$  then  $xy^i = z^{n_0 + M(y,z)(i-i_0)}$ , so by (6),

$$\begin{aligned} d(xy^i) &= d(z^{n_0 + M(y,z)(i-i_0)}) = d(z)(n_0 + M(y, z)(i - i_0)) \\ &\leq d(z)n_0 + d(y)(i - i_0) = d(z)n_0 - d(y)i_0 + d(y)i. \end{aligned}$$

So we may take  $K_{y,z}$  to be any positive number such that  $K_{y,z} \geq d(z)n_0 - d(y)i_0$ .  $\square$

**Corollary 2.8.** *If  $u$  is a word of length  $n$ , then  $d(u) \leq Kn$ .*

*Proof.* This clearly follows from Lemma 2.7 by induction in  $n$ .  $\square$

**Proposition 2.9.** *For  $r > 0$ , let  $I(r) = \{w \in S \text{ such that } d(w) \leq r\}$ . Then there exists  $L > 0$  such that  $|I(r)| \leq Lr$  for all  $r$ .*

*Proof.* Let  $a \in A$  and suppose  $a^m \in I(r)$ . Since  $d(a^m) = d(a)m$ , we get that  $m \leq \frac{r}{d(a)}$ . So the number of elements of  $I(r)$  of the form  $a^m$  is  $\lfloor \frac{r}{d(a)} \rfloor$ . Hence we may take  $L = \sum_{a \in A} \frac{1}{d(a)}$ .  $\square$

**Theorem 2.10.** *The semigroup  $S$  has linear growth.*

*Proof.* Let  $J(m)$  be the set of  $w \in S$  which can be represented by a word of length  $\leq m$ . Then by Corollary 2.8  $J(m) \subset I(Km)$ . Hence  $|J(m)| \leq |I(Km)| \leq LKm$  by Proposition 2.9. This implies the theorem.  $\square$

**Corollary 2.11.** *Let  $\bar{S} := 1 \cup S$  be the monoid obtained by adding a unit to  $S$ , and  $A = \mathbb{F}\bar{S}$  be the algebra spanned by  $\bar{S}$  over any field  $\mathbb{F}$ . Then  $A$  is a PI algebra.*

*Proof.* By a theorem of Small et al. [7] any finitely generated algebra of linear growth (i.e., GK dimension 1) is PI. Thus the result follows from Theorem 2.10.  $\square$

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