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Examples of abelian surfaces with everywhere good reduction

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Abstract We describe several explicit examples of simple abelian surfaces over real quadratic fields with real multiplication and everywhere good reduction. These examples provide evidence for the Eichler–Shimura conjecture for Hilbert modular forms over a real quadratic field. Several of the examples also support a conjecture of Brumer and Kramer on abelian varieties associated to Siegel modular forms with paramodular level structures.

Mathematics Subject Classification 11F41 · 11F46 · 11F67 · 11G10

1 Introduction

A celebrated result of Fontaine [20] (see also Abrashkin [1]) asserts that there is no abelian scheme over **Z**. In other words, there is no abelian variety over **Q** with everywhere good reduction. However, long before this result, there were a few examples of elliptic curves of unit conductor over quadratic fields in the literature. For example,

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the curve

$$E : y^2 + xy + \epsilon^2 y = x^3,$$

where $\epsilon = \frac{5+\sqrt{29}}{2}$ is the fundamental unit in $F = \mathbf{Q}(\sqrt{29})$, was known to Tate, and to Serre who extensively studied it in [38]; it is also alluded to in [42]. Since then, there has been much work on finding elliptic curves with everywhere good reduction over number fields, with a particular emphasis on quadratic fields; see for example [8,12,27,36,40,48]. For real quadratic fields, the database of such curves has been considerably expanded by Elkies [16]. In [14], it is shown that this database is complete for all fundamental discriminants ≤ 1000 of narrow class number one, if one assumes modularity. A more systematic algorithm which, given a number field F and a finite set of primes S of its ring of integers, returns the set of all elliptic curves over F with good reduction outside of S is given in [11]. However, this algorithm relies on algorithms for S-integral points for elliptic curves, and has not yet been full implemented for this reason. An alternate (and perhaps more efficient) approach which uses S-unit equations is currently being explored by Cremona and Elkies [10]. In fact, a similar method has already been used by Smart [45] to find hyperelliptic curves of genus 2 with good reduction outside S when $F = \mathbf{Q}$.

In contrast, to the best of our knowledge there is not a single example of an abelian surface with everywhere good reduction in the literature (except in the case when the abelian surface has complex multiplication [13], or is a $\mathbf{Q}$-surface [6,41] or a product of elliptic curves). This could possibly be explained by the fact that all the algorithms we mentioned above do not readily generalize to the genus 2 situation. The goal of this paper is to remedy that situation by providing the first equations for such surfaces over real quadratic fields.

We note that the non-existence of abelian varieties over $\mathbf{Q}$ with good reduction everywhere is instrumental in the Khare–Wintenberger proof of the Serre conjecture for Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. As described in [28], the proof of the Serre conjecture in retrospect can be viewed as a method to exploit an accident which occurs in three different guises:

(a) (Fontaine, Abrashkin) There are no non-zero abelian varieties over $\mathbf{Z}$.
(b) (Serre, Tate) There are no irreducible representations

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}),$$

where $\overline{\mathbf{F}}$ is the algebraic closure of $\mathbf{F}_2$ or $\mathbf{F}_3$, that are unramified outside of 2 and 3 respectively.
(c) $S_2(\mathrm{SL}_2(\mathbf{Z})) = 0$, i.e., there are no cusp forms of level $\mathrm{SL}_2(\mathbf{Z})$ and weight 2.

The failure of this happy accident over general number fields, such as real quadratic fields, means that new techniques are needed for analogous modularity results.

Our approach to the construction of abelian surfaces with everywhere good reduction combines three key elements: (a) recent advances in the computation of Hecke eigenvalues of Hilbert modular forms, (b) new rational models of Hilbert modular surfaces, and (c) the Eichler–Shimura conjecture for Hilbert modular forms. As a result of our investigation, we produce further evidence for the Eichler–Shimura conjecture, as well as for a conjecture of Brumer and Kramer [5] associating abelian varieties to paramodular Siegel modular forms on Sp(4).

The outline of the paper is as follows: in Sect. 2, we briefly recall the basic facts regarding these three ingredients. In Sect. 3, we describe our strategy to predict and find examples of good reduction abelian surfaces, assuming the Eichler–Shimura conjecture. Section 4 provides several illustrative examples of our methods, giving explicit abelian surfaces with good reduction everywhere, and connecting them to appropriate Hilbert modular forms. We conclude with a list of all our examples in Sect. 5.

2 Background

2.1 Hilbert modular forms

Let F be a totally real field of narrow class number one and degree d. We let $\mathcal{O}_F$ be the ring of integers of F, $\mathfrak{d}_F$ the different of F. For each $i = 1, \ldots, d$, let $a \mapsto a^{(i)}$ denote the i-th embedding of F into $\mathbf{R}$, so that we have an identification $F \otimes \mathbf{R} \simeq \mathbf{R}^d$. We let F_+ be the set of totally positive elements in F, i.e. the inverse image of $(\mathbf{R}_+)^d$, and $\mathcal{O}_{F,+} = F_+ \cap \mathcal{O}_F$. We fix a totally positive generator δ of $\mathfrak{d}_F$. (Note that every ideal has such a generator since F has narrow class number one.)

Let $\mathcal{H}$ be the Poincaré upper half plane. The Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathcal{H}^d$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, \ldots, z_d) = \left(\frac{a^{(i)} z_i + b^{(i)}}{c^{(i)} z_i + d^{(i)}} \right)_{i=1,\ldots,d}.$$

Let $\mathfrak{N}$ be an integral ideal, and set

$$\Gamma_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : c \in \mathfrak{N} \right\}.$$

Let $k \geq 2$ be an even integer. A Hilbert modular form of weight[1] k and level $\mathfrak{N}$ is a holomorphic function $f : \mathcal{H}^d \to \mathbf{C}$ such that

$$f(\gamma z) = \left(\prod_{i=1}^{d} (c^{(i)} z_i + d^{(i)}) \right)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N}).$$

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. Then f is invariant under the matrices $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ for $\mu \in \mathcal{O}_F$, which act as $z \mapsto z + \mu$. Hence, by the Koecher principle [4], f admits a q-expansion of the form

[1] More precisely, this defines a Hilbert modular form of *parallel* weight k.

$$f(z) = a_0 + \sum_{\mu \in \mathcal{O}_{F,+}} a_\mu e^{2\pi i \mathrm{Tr}(\frac{\mu z}{\delta})},$$

where $\mathrm{Tr}(\nu z) = \nu^{(1)} z_1 + \cdots + \nu^{(d)} z_d$, for $\nu \in F_+$. We say that f is a cusp form if $a_0 = 0$. Since f is invariant under the action of the matrices $\mathrm{diag}(\epsilon, \epsilon^{-1})$ for $\epsilon \in \mathcal{O}_F^\times$ in $\mathrm{SL}_2(\mathcal{O}_F)$, which act as $z \mapsto \epsilon^2 z$, we have $a_{\epsilon^2\mu} = a_\mu$ for all $\mu \in \mathcal{O}_{F,+}$ and $\epsilon \in \mathcal{O}_F^\times$. Let f be a cusp form of weight k and level $\mathfrak{N}$. Then, for every ideal $\mathfrak{m} \subseteq \mathcal{O}_F$, the quantity $a_\mathfrak{m}(f) = a_\mu$, where μ is a totally positive generator of $\mathfrak{m}$, is well-defined and depends only on $\mathfrak{m}$. We call it the $\mathfrak{m}$-th Fourier coefficient of f. When f is a normalized eigenform for the Hecke operators (i.e. $a_{(1)}(f) = 1$), the eigenvalue of the Hecke operator $T_\mathfrak{m}$ is $a_\mathfrak{m}(f)$ for each $\mathfrak{m} \nmid \mathfrak{N}$. It is a theorem of Shimura [44] that in this situation, the $a_\mathfrak{m}(f)$ are algebraic integers and the $\mathbf{Z}$-subalgebra $\mathcal{O}_f = \mathbf{Z}[a_\mathfrak{m}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] \subset \mathbf{C}$ has finite rank and is therefore an order in some number field K_f (called the field of Fourier coefficients of f). For more background on Hilbert modular forms, see [4,14,19].

Here, we wish to point out some new techniques in the computation of Hilbert modular forms, which arise from the Eichler–Jacquet–Langlands–Shimizu correspondence between Hilbert modular forms and quaternionic modular forms. We will not go into details here, but instead refer the reader to [14] for a detailed description of these methods. The upshot is that it is possible to efficiently compute systems of Hecke eigenvalues for Hilbert modular cusp forms by instead computing modular forms on finite spaces or on Shimura curves. This will be crucial to our methods in this paper. The corresponding algorithms have been implemented in the Hilbert Modular Forms Package in `Magma` [3]).

2.2 Hilbert modular surfaces

Let K be a real quadratic field of discriminant D'. The Hilbert modular surface $Y_-(D')$ is a compactification of the coarse moduli space which parametrizes principally polarized abelian surfaces with real multiplication by the ring of integers $\mathcal{O}_K$ of K, i.e. pairs (A, ι), where $\iota : \mathcal{O}_K \to \mathrm{End}_{\overline{\mathbf{Q}}}(A)$ is a homomorphism. The complex points $Y_-(D')(\mathbf{C})$ of this space are obtained by compactifying $\mathrm{SL}_2(\mathcal{O}_K)\backslash(\mathcal{H}^+ \times \mathcal{H}^-)$, where $\mathcal{H}^+$ and $\mathcal{H}^-$ are the upper and lower half-planes respectively, by adding finitely many cusps and resolving the singularities of the resulting space. The Hilbert modular surface maps to the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces, by forgetting the action of $\mathcal{O}_K$. Its image is the Humbert surface $\mathcal{H}_{D'}$, and the map $Y_-(D') \to \mathcal{H}_{D'}$ is a double cover, ramified along a union of modular curves. The surfaces $Y_-(D')$ have models over the integers, with good reduction away from primes dividing D'.

Recently, Elkies and the second author [17] computed explicit birational models over $\mathbf{Q}$ for these Hilbert modular surfaces for all the fundamental discriminants D' less than 100, by identifying the Humbert surface $\mathcal{H}_{D'}$ with a moduli space of elliptic K3 surfaces, which may be computed explicitly. For the fundamental discriminants in the range $1 < D' < 100$, the Humbert surface is a rational surface, i.e. birational to $\mathbf{P}^2$ over $\overline{\mathbf{Q}}$ (and in fact, even over $\mathbf{Q}$). Therefore, they are able to exhibit $\mathcal{H}_{D'}$ as a double cover of $\mathbf{P}^2$, with equation $z^2 = f(r, s)$, where r, s are parameters on $\mathbf{P}^2$.

They also get the map to $\mathcal{A}_2$, which is birational to $\mathcal{M}_2$, the moduli space of genus 2 curves. It is given by producing the Igusa–Clebsch invariants of the image point as rational functions of r and s.

Remark 1 Hilbert modular surfaces have been an object of extensive study in number theory and arithmetic geometry, especially in the latter half of the twentieth century. In particular, their geometric classification was described by Hirzebruch, van de Ven, Zagier, van der Geer and others. In the comprehensive reference [50], arithmetic models for some of them are also described. However, for our work, we need explicit equations for these surfaces along with the map to $\mathcal{A}_2$, and this does not seem to be available for any discriminant other than 5 except in [17] (though it could be worked out in principle using Hilbert and Siegel modular forms). Consequently, we will use the equations from [17] throughout.

2.3 Eichler–Shimura conjecture

The following conjecture is instrumental in identifying the examples in this paper.

Conjecture 1 (Eichler–Shimura) *Let F be a totally real number field and $\mathfrak{N}$ an integral ideal of F. Let f be a Hilbert newform of weight 2 and level $\mathfrak{N}$. Let $\mathcal{O}_f = \mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F]$ be the order generated by the Fourier coefficients of f, and K_f its field of fractions. There exists an abelian variety A_f/F of dimension $[K_f : \mathbf{Q}]$ with good reduction outside of $\mathfrak{N}$ and with $\mathcal{O}_f \hookrightarrow \mathrm{End}_F(A_f)$, such that*

$$L(A_f, s) = \prod_{\tau \in \mathrm{Hom}(K_f, \mathbf{C})} L(f^\tau, s),$$

where

$$L(f^\tau, s) := \sum_{\mathfrak{m} \subseteq \mathcal{O}_F} \frac{a_{\mathfrak{m}}(f)^\tau}{\mathrm{N}\mathfrak{m}^s}.$$

When $F = \mathbf{Q}$, this conjecture is a theorem, due to Eichler for prime level and Shimura in the general case. The Eichler–Shimura construction can be summarized as follows. Let $N > 1$ be an integer, and let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$. This curve and its Jacobian $J_1(N)$ are defined over $\mathbf{Q}$. We recall that the space $S_2(\Gamma_1(N))$ of cusp forms of weight 2 and level $\Gamma_1(N)$ is a $\mathbf{T}$-module, where $\mathbf{T}$ is the Hecke algebra. Let $f \in S_2(\Gamma_1(N))$ be a newform, and let $I_f = \mathrm{Ann}_{\mathbf{T}}(f)$. Shimura [43] showed that the quotient

$$A_f := J_1(N)/I_f J_1(N)$$

is an abelian variety A_f of dimension $[K_f : \mathbf{Q}]$ defined over $\mathbf{Q}$ with endomorphisms by the order $\mathcal{O}_f = \mathbf{Z}[a_n(f) : n \geq 1]$ and that

$$L(A_f, s) = \prod_{g \in [f]} L(g, s),$$

where $[f]$ denotes the Galois orbit of f.

One of the main consequences of the proof of the Serre conjecture [39] by Khare–Wintenberger [29] is that the converse to Conjecture 1 is true when $F = \mathbf{Q}$. That is, an abelian variety of GL_2-type is isogenous to a $\mathbf{Q}$-simple factor of $J_1(N)$ for some N [30]. And so, this provides a theoretical construction of all abelian varieties of GL_2-type over $\mathbf{Q}$ with a prescribed conductor. In fact, one can make this explicit in many cases (see [9] for elliptic curves, and [23,25] for abelian surfaces).

For $[F : \mathbf{Q}] > 1$, the known cases of Conjecture 1 exploit the cohomology of Shimura curves. For instance, the conjecture is known when $[F : \mathbf{Q}]$ is odd, or when $\mathfrak{N}$ is exactly divisible by a prime $\mathfrak{p}$ of $\mathcal{O}_F$ [51]. The simplest case in which Conjecture 1 is still unknown is when f is a newform of level (1) and weight 2 over a real quadratic field. In that case, the conjecture predicts that the associated abelian variety A_f has everywhere good reduction.

3 The strategy

Let F be a number field of class number 1, and E an elliptic curve over F given by a (global minimal) Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_i \in \mathcal{O}_F$, the ring of integers of F. The invariants c_4 and c_6 of E satisfy the equation $c_4^3 - c_6^2 = 1728\Delta$, where Δ is the discriminant of E. In other words, the pair (c_4, c_6) is an $\mathcal{O}_F$-integral point on the curve

$$y^2 = x^3 - 1728\Delta. \tag{1}$$

Since E has everywhere good reduction if and only if Δ is a unit in $\mathcal{O}_F$, we can find all the elliptic curves over F with everywhere good reduction by solving (1) as Δ runs over a finite set of representatives of $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^{12}$. (Note that given a pair (c_4, c_6), we get a minimal model by using the Tate algorithm.) Most of the algorithms we mentioned earlier rely on this fact.

Unfortunately, the reduction of abelian varieties of higher dimension is not characterized by a nice single diophantine equation such as (1). For this reason, we need an additional ingredient which will guide our search. This extra input is provided by the Eichler–Shimura conjecture.

Suppose we have a Hilbert modular eigenform f of weight 2 over F, with Hecke eigenvalues $a_{\mathfrak{m}}(f)$ in a real quadratic field K_f of discriminant D'. The Eichler–Shimura conjecture predicts that there should be an abelian variety A over F of dimension $[K_f : \mathbf{Q}] = 2$, (up to isogeny) associated to this data, which has real multiplication by an order in K_f. Furthermore, the conductor of A should divide the level $\mathfrak{N}$ of f. In particular, if f has level (1), the conjectural abelian surface A has good reduction everywhere. This observation will be the source of our examples in this paper, for which the abelian surface turns out to be principally polarized, and also has real multiplication by the full ring of integers of K_f. Our strategy to produce such A is as follows:

(a) Find a Hilbert modular form of level (1) and weight 2 for a real quadratic field F, with coefficients in a real quadratic field K_f of discriminant D'.
(b) Find an F-rational point on the Hilbert modular surface $Y_-(D')$, for which the L-function of the associated abelian surface matches that of f at several Euler factors, up to twist.
(c) Compute the correct quadratic twist of the abelian surface, or the genus 2 curve.
(d) Check that the abelian surface has good reduction everywhere.
(e) Prove that the L-functions indeed match up.

Note that there is no reason one has to restrict to the case when the base field is a real quadratic field F. The next interesting case in which the Eichler–Shimura conjecture is not known is that of totally real quartic base fields L. So one could look for eigenforms of weight 2 for $\mathrm{SL}_2(\mathcal{O}_L)$ whose Fourier coefficients are in a real quadratic field K of discriminant D, and on the other hand try to find L-rational points on the Hilbert modular surface $Y_-(D)$. In this paper, we looked at quadratic base fields F for convenience. On the other hand, if we instead want examples for which the field K_f has larger degree, we might need explicit rational models for the appropriate Hilbert modular varieties, which are not currently available. Hence the choice of K is restricted.

For simplicity, we investigated only real quadratic fields F of narrow class number 1 and discriminant less than 1000. We found twenty-eight examples of Hilbert newforms, and corresponding abelian surfaces for most of these forms. We will say a few words later about the "missing" examples, which we hope will be found in future work.

4 The examples

From now on, F will denote a real quadratic field of narrow class number one. We let D be its fundamental discriminant. We will denote its ring of integers by $\mathcal{O}_F$. Let $w = \sqrt{D}$ or $(1+\sqrt{D})/2$ according as D is 0 or 1 mod 4, so that $\{1, w\}$ is a $\mathbf{Z}$-basis of $\mathcal{O}_F$. For a Hilbert newform f of weight 2 over F, we will let $\mathcal{O}_f = \mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F]$ and K_f be the order and the field generated by the Fourier coefficients, respectively. We will focus on forms such that $[K_f : \mathbf{Q}] = 2$, since we do not yet know how to write simple equations for general Hilbert modular varieties. We let D' be the discriminant of K_f and write $e = \sqrt{D'}$ or $(1+\sqrt{D'})/2$. We denote the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q})$ and $\mathrm{Gal}(K_f/\mathbf{Q})$ by σ and τ respectively. The L-series of the conjectural surface A_f attached to f is written as

$$L(A_f, s) = L(f, s)L(f^\tau, s) = \prod_{\mathfrak{p}} \frac{1}{Q_{\mathfrak{p}}(\mathrm{N}(\mathfrak{p})^{-s})},$$

where

$$\begin{aligned} Q_{\mathfrak{p}}(T) &:= (T^2 - a_{\mathfrak{p}}(f)T + \mathrm{N}(\mathfrak{p}))(T^2 - a_{\mathfrak{p}}(f)^\tau T + \mathrm{N}(\mathfrak{p})) \\ &= T^4 - s_{\mathfrak{p}}(f)T^3 + t_{\mathfrak{p}}(f)T^2 - \mathrm{N}(\mathfrak{p})s_{\mathfrak{p}}(f)T + \mathrm{N}(\mathfrak{p})^2. \end{aligned}$$

Table 1 A summary of the examples

D	D'	Case
53	8	I
61	12	I
73	5	I
193	17	II
233	17	II
277	29	II
349	21	II
353	5	III
373	93	II
389	8	II
397	24	II
409	13	II
421	5	I
421	5	III
433	12	II
461	29	II
613	21	II
677	13	II
677	29	II
677	85	II
709	5	II
797	8	II
797	29	II
809	5	II
821	44	II
853	21	II
929	13	II
997	13	II

Our examples (see Table 1) can be subdivided in the following cases, with the majority of examples coming from Case II.

I: The form f is $\mathrm{Gal}(F/\mathbf{Q})$-invariant.
II: The form f is not $\mathrm{Gal}(F/\mathbf{Q})$-invariant, but its $\mathrm{Gal}(K_f/\mathbf{Q})$-orbit $\{f, f^\tau\}$ is.
III: The $\mathrm{Gal}(K_f/\mathbf{Q})$-orbit $\{f, f^\tau\}$ is not $\mathrm{Gal}(F/\mathbf{Q})$-invariant.

We will see that Case I is somewhat special: it is frequently possible to produce the associated abelian surface through analytic methods for classical modular forms.

In [5], Brumer–Kramer proposed the following conjecture as a genus 2 analogue of the Eichler–Shimura construction for classical newforms of weight 2 (with integer coefficients).

Conjecture 2 (Brumer–Kramer) *Let g be a paramodular Siegel newform of genus* 2*, weight* 2 *and level N, with integer Hecke eigenvalues, which is not in the span of*

Gritsenko lifts. Then there exists an abelian surface B defined over $\mathbf{Q}$ of conductor N such that $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$ *and* $L(g, s) = L(B, s)$.

The examples in Case II show that there is a strong connection between this conjecture and Conjecture 1.

4.1 Case I

In this case, the Hecke eigenvalues of the Hilbert modular form f satisfy

$$a_{\mathfrak{p}}(f) = a_{\mathfrak{p}^\sigma}(f).$$

This implies that the form f is a base change from $\mathbf{Q}$. Let g be a newform in $S_2(\Gamma_1(D))$ whose base change is f. Since the level of f is (1), the form $g \in S_2(\Gamma_1(D), \chi_D)^{\mathrm{new}}$ by [33, Prop. 2, p. 263], where χ_D is the fundamental character of the quadratic field $F = \mathbf{Q}(\sqrt{D})$. Let L_g be the coefficient field of g. Then, L_g is a quartic CM field which contains K_f. The non-trivial element of $\mathrm{Gal}(L_g/K_f)$, which we denote by $(x \mapsto \overline{x}, x \in L_g)$, extends to complex conjugation. Let B_g be the abelian variety attached to the form g. Then B_g is a fourfold such that $\mathrm{End}_{\mathbf{Q}}(B_g) \otimes \mathbf{Q} \simeq L_g$. Let w_D be the Atkin–Lehner involution on $S_2(\Gamma_1(D), \chi_D)^{\mathrm{new}}$. This induces an involution on B_g, which we still denote by w_D. Shimura [41, § 7.7] shows the following:

(a) w_D is defined over F, and $w_D^\sigma = -w_D$;
(b) $w_D \cdot [x] = [\overline{x}] \cdot w_D$, where $[x]$ denotes the endomorphism induced on B_g by $x \in L_g$.
(c) The abelian surface $A_f := (1 + w_D)B_g$ is defined over F, and is isogenous to its Galois conjugate given by $A_f^\sigma := (1 - w_D)B_g$. Moreover, we have

$$B_g \otimes_{\mathbf{Q}} F \sim A_f \times A_f^\sigma.$$

So in this case, the existence of the surface A_f is a direct consequence of the classical Eichler–Shimura construction.

Although Conjecture 1 is known in this case, it would still be desirable to have an explicit equation for the surface A_f. We outline two methods to find it, the first of which is special to this case.

4.1.1 Method 1

This method is analytic, and has an obvious connection with the Oda conjecture [35, p. xii] for Hilbert modular forms that arise from base change. It assumes that both A_f and A_f^σ are principally polarizable. To describe it, we recall that by [7, Theorems 6.2.4 and 6.2.6], there exist newforms $g_1, g_2 \in S_2(\Gamma_1(D), \chi_D)^{\mathrm{new}}$ such that $g_1, \overline{g}_1, g_2$ and $\overline{g}_2$ form a basis of the Hecke constituent of g and

$$w_D(g_1) = \overline{\lambda}_D(g_1)\overline{g}_1, \quad w_D(g_2) = \overline{\lambda}_D(g_2)\overline{g}_2,$$

where $a_D(g)$ is the Hecke eigenvalue of g at D and $\lambda_D(g) = \frac{a_D(g)}{\sqrt{D}}$, the pseudo-eigenvalue of w_D. The matrix of w_D in the basis $\{g_1, \overline{g}_1, g_2, \overline{g}_2\}$ is given by

$$W_D := \begin{bmatrix} 0 & \lambda_D(g_1) & 0 & 0 \\ \overline{\lambda}_D(g_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_D(g_2) \\ 0 & 0 & \overline{\lambda}_D(g_2) & 0 \end{bmatrix}$$

From this, we see that $W_D^\sigma = -W_D$. The following lemma is a simple adaptation of Cremona's [8, Lemma 5.6.2].

Lemma 1 *The set of forms $h_i^\pm := \frac{1}{2}(g_i \pm w_D(g_i))$, $i = 1, 2$, are bases for the $\pm$-eigenspaces of W_D, acting on the Hecke constituent of g, which give a decomposition of the space of differential 1-forms $H^0(B_g \otimes_{\mathbf{Q}} F, \Omega^1_{B_g \otimes_{\mathbf{Q}} F/F})$ according to the action of* $\mathrm{Gal}(F/\mathbf{Q})$.

Let $H_1(B_g, \mathbf{Z})^\pm$ denote the $\pm$-eigenspaces of w_D. They are free Hecke submodules of $H_1(B_g, \mathbf{Z})$ of rank 4 over $\mathbf{Z}$, which are direct summands.

Lemma 2 *Let $\Lambda_g^\pm$ be the period lattices obtained by integrating the forms in Lemma 1 against $H_1(B_g, \mathbf{Z})^\pm$, and set $\Lambda_g = \Lambda_g^+ \oplus \Lambda_g^-$. Then, there exist an abelian fourfold B_g' defined over $\mathbf{Q}$, and an isogeny $\phi : B_g' \to B_g$ whose degree is a power of 2, such that $B_g'(\mathbf{C}) = \mathbf{C}^4/\Lambda_g$. Moreover, $B_g' = \mathrm{Res}_{F/\mathbf{Q}}(A_f)$ where A_f is an abelian surface defined over F.*

Proof We first note that the complex tori $\mathbf{C}^2/\Lambda_g^\pm$ and $\mathbf{C}^4/\Lambda_g$ have canonical Riemann forms obtained by restriction of the intersection pairing $\langle \cdot, \cdot \rangle$ on B_g. Therefore, they are the complex points of some abelian varieties. Since $h_1^+, h_2^+, h_1^-, h_2^-$ is a basis of the Hecke constituent of g, [41, Theorem 7.14 and Proposition 7.19] imply that there exist a fourfold B_g' defined over $\mathbf{Q}$, and an isogeny $\phi : B_g' \to B_g$, such that $B_g'(\mathbf{C}) = \mathbf{C}^4/\Lambda_g$.

Next, let $x \in H_1(B_g, \mathbf{Z})$, then we have

$$2x = (x + w_D x) + (x - w_D x) = y_+ + y_- \in H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^-.$$

Hence the exponent of $H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^-$ inside $H_1(B_g, \mathbf{Z})$ divides 2. This implies that the degree of ϕ is a power of 2.

Since w_D is defined over F and $w_D^\sigma = -w_D$, the bases $\{h_1^+, h_2^+\}$ and $\{h_1^-, h_2^-\}$ are $\mathrm{Gal}(F/\mathbf{Q})$-conjugate. Therefore $\mathbf{C}^2/\Lambda_g^+$ and $\mathbf{C}^2/\Lambda_g^-$ are the complex points of some abelian surfaces defined over F that are Galois conjugate. Let A_f be the surface such that $A_f(\mathbf{C}) = \mathbf{C}^2/\Lambda_g^+$. Then, we see that $B_g' = \mathrm{Res}_{F/\mathbf{Q}} A_f$ by construction.

In practice, we can replace B_g by B_g', and hence assume that

$$H_1(B_g, \mathbf{Z}) = H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^- = H_1(A_f, \mathbf{Z}) \oplus H_1(A_f^\sigma, \mathbf{Z}).$$

The above integration then gives the period lattice decomposition

$$\Omega_{B_g} = \Omega_{A_f} \times \Omega_{A_f^\sigma} = (\Omega_1 \mid \Omega_2) \times (\Omega_1^\sigma \mid \Omega_2^\sigma).$$

Provided that the intersection pairing restricted to $H_1(A_f, \mathbf{Z})$ and $H_1(A_f^\sigma, \mathbf{Z})$ induces principal polarizations, we can compute the surfaces A_f and A_f^σ as Jacobians of curves C_f and C_f^σ (defined over F).

We illustrate this with the following example. The smallest discriminant for which we obtain a surface which satisfies Case I is $D = 53$. The abelian surface A_f has real multiplication by (an order in) the field $\mathbf{Q}(\sqrt{2})$. In fact, we will see that it has real multiplication by the full ring of integers.

A symplectic basis for $H_1(B_g, \mathbf{Z})$ is given by the modular symbols [47]

$$\begin{aligned}
\gamma_1 &:= -\{-1/35, 0\} + \{-1/26, 0\},\\
\gamma_2 &:= -\{-1/47, 0\},\\
\gamma_3 &:= \{-1/37, 0\},\\
\gamma_4 &:= \{-1/47, 0\} - \{-1/15, 0\} + \{-1/13, 0\},\\
\gamma_5 &:= -\{-1/28, 0\},\\
\gamma_6 &:= -\{-1/44, 0\},\\
\gamma_7 &:= \{-1/15, 0\} - \{-1/44, 0\},\\
\gamma_8 &:= \{-1/28, 0\} + \{-1/21, 0\} - \{-1/26, 0\}.
\end{aligned}$$

Computing the matrix G of the intersection pairing in that basis, we see that B_g is principally polarized. We obtain the integral bases $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ and $\{\delta_1', \delta_2', \delta_3', \delta_4'\}$ for $H_1(B_g, \mathbf{Z})^+$ and $H_1(B_g, \mathbf{Z})^-$, respectively, where

$$\begin{aligned}
\delta_1 &:= -\{-1/35, 0\} + \{-1/26, 0\},\\
\delta_2 &:= \{-1/37, 0\} - \{-1/47, 0\} + \{-1/15, 0\} - \{-1/13, 0\},\\
\delta_3 &:= -\{-1/28, 0\},\\
\delta_4 &:= -\{-1/28, 0\} + \{-1/15, 0\} - \{-1/44, 0\} - \{-1/21, 0\} + \{-1/26, 0\},\\
\delta_1' &:= -\{-1/47, 0\},\\
\delta_2' &:= \{-1/37, 0\} + \{-1/47, 0\} - \{-1/15, 0\} + \{-1/13, 0\},\\
\delta_3' &:= -\{-1/44, 0\},\\
\delta_4' &:= \{-1/28, 0\} + \{-1/15, 0\} - \{-1/44, 0\} + \{-1/21, 0\} - \{-1/26, 0\}.
\end{aligned}$$

In this case, we verify that the index of $H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^-$ inside $H_1(B_g, \mathbf{Z})$ is 4, and that the restriction of the intersection pairing to each direct summand $H_1(B_g, \mathbf{Z})^\pm$ is of type $(1, 2)$. This means that A_f and A_f^σ are not principally polarized with respect to the Riemann form given by the restriction of the intersection pairing from B_g. Let $G^\pm$ be the corresponding matrices for these pairings. We remedy this situation by finding a suitable element of the Hecke algebra, as in [24, Section 4.2]. The element $u = -e - 2 \in \mathcal{O}_f$ has norm 2, and acts on $H_1(B_g, \mathbf{Z})^\pm$ as $T_7^\pm$ where T_7

is the Hecke operator at 7. Letting $G_u^{\pm} = T_7^{\pm} \cdot G^{\pm}$, we obtain principal polarizations on A_f and A_f^{σ} by [24, Proposition 3.11].

By integrating the bases of differential forms $\{h_1^+, h_2^+\}$ and $\{h_1^-, h_2^-\}$ from Lemma 1 against the Darboux bases

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix} := \begin{pmatrix}0 & 0 & 1 & 2\\ 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{pmatrix} \text{ and } \begin{pmatrix}\eta_1'\\ \eta_2'\\ \eta_3'\\ \eta_4'\end{pmatrix} := \begin{pmatrix}1 & 0 & 0 & 4\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\delta_1'\\ \delta_2'\\ \delta_3'\\ \delta_4'\end{pmatrix},$$

respectively, we obtain the Riemann period matrices Ω_{A_f} and $\Omega_{A_f^{\sigma}}$, where

$$\begin{aligned}
\Omega_1 &:= \begin{pmatrix} 2.53595... + 2.39271...i & -4.32914... - 4.08462...i\\ -66.45185... - 24.43147...i & 19.46329... + 7.15581...i\end{pmatrix},\\
\Omega_2 &:= \begin{pmatrix} 1.79318... - 1.69190...i & 6.12233... - 5.77653...i\\ 46.98855... - 17.27566...i & 27.52526... - 10.11984...i\end{pmatrix},\\
\Omega_1^{\sigma} &:= \begin{pmatrix} -2.44814... + 4.22343...i & 2.44814... + 4.22343...i\\ 0.78506... + 1.10501...i & -0.78506... + 1.10501...i\end{pmatrix},\\
\Omega_2^{\sigma} &:= \begin{pmatrix} 1.43409... + 2.47403...i & -8.35849... + 14.41970...i\\ -2.68038... + 3.77277...i & 0.45988... + 0.64730...i\end{pmatrix}.
\end{aligned}$$

This yields the normalized period matrices

$$\begin{aligned}
Z &:= \begin{pmatrix} -0.65878... + 0.69909...i & -0.40996... + 0.82303...i\\ -0.40996... + 0.82303...i & -0.32227... + 1.89394...i\end{pmatrix},\\
Z^{\sigma} &:= \begin{pmatrix} -0.14337... + 1.54762...i & 1.99999... - 0.64475...i\\ 2.00000... - 0.64475...i & 0.14337... + 1.54762...i\end{pmatrix}.
\end{aligned}$$

We compute the Igusa–Clebsch invariants I_2, I_4, I_6 and I_{10} to 200 decimal digits of precision using Z and Z^{σ}, and identify them as elements in F (due to Lemma 2). In the weighted projective space $\mathbf{P}^2_{(1:2:3:5)}$, this gives the point

$$\begin{aligned}
&(I_2 : I_4 : I_6 : I_{10})\\
&\quad = \left(1 : \frac{-21504b + 81889}{5973136} : \frac{-1241984b + 3114075}{1122949568} : \frac{1564843b + 21688699}{1362467130944816}\right),
\end{aligned}$$

where $b = \sqrt{53}$. By using Mestre's algorithm [34] which is implemented in `Magma`, we obtain a curve with above invariants. We reduce this curve using the algorithm in [2] implemented in `Sage` [37] to get the curve

$$\begin{aligned}
C_f' : y^2 = {} & (-6w + 25)x^6 + (-60w + 246)x^5 + (-242w + 1017)x^4\\
& + (-534w + 2160)x^3 + (-626w + 2688)x^2\\
& + (-440w + 1724)x - 127w + 567.
\end{aligned}$$

Table 2 The first few Hecke eigenvalues of a base change newform of level (1) and weight 2 over $\mathbf{Q}(\sqrt{53})$

$N\mathfrak{p}$	$\mathfrak{p}$	$a_\mathfrak{p}(f)$	$s_\mathfrak{p}(f)$	$t_\mathfrak{p}(f)$
4	2	$e+1$	2	7
7	$-w-2$	$-e-2$	-4	16
7	$-w+3$	$-e-2$	-4	16
9	3	$-3e+1$	2	1
11	$w-2$	$3e$	0	4
11	$w+1$	$3e$	0	4
13	$w-1$	$-2e+1$	2	19
13	$-w$	$-2e+1$	2	19
17	$-w-5$	-3	-6	43
17	$w-6$	-3	-6	43
25	5	$2e+4$	8	58
29	$-w-6$	$3e-3$	-6	49
29	$w-7$	$3e-3$	-6	49

Here $e=\sqrt{2}$

We have used floating point calculations to get the equation of the curve C'_f, but now we can directly check that the Frobenius data of its Jacobian matches that of the Hilbert modular form, up to quadratic twist.

Remark 2 We computed the curve C'_f by using the normalized period matrix Z. We could have instead applied the Jacobian nullwerte method [23,25] to the periods matrices Ω_{A_f} and $\Omega_{A_f^\sigma}$. This has the advantage of producing curves with small coefficients, needing no further reduction.

Remark 3 For the other Hilbert modular forms in Case I, we obtained the corresponding abelian surfaces using Method 1. The only exception is $D=61$, where the abelian surface has RM by $\mathbf{Z}[\sqrt{3}]$ and is naturally $(1,2)$-polarized, and is therefore not principally polarizable by [24, Corollary 2.12 and Proposition 3.11]; it is not treated in this paper (Table 2).

4.1.2 Method 2

An equation for the Hilbert modular surface $Y_-(8)$ is given in [17] (see 2.2 for a quick review of the results we need here). As a double-cover of $\mathbf{P}^2_{r,s}$, it is given by

$$z^2 = 2(16rs^2 + 32r^2s - 40rs - s + 16r^3 + 24r^2 + 12r + 2).$$

It is a rational surface (even over $\mathbf{Q}$) and therefore the rational points are dense. In particular, there is an abundance of rational points of small height. The Igusa–Clebsch invariants $(I_2 : I_4 : I_6 : I_{10}) \in \mathbf{P}^2_{(1:2:3:5)}$ are given by

$$\left(-\frac{24B_1}{A_1}, -12A, \frac{96AB_1 - 36A_1B}{A_1}, -4A_1B_2\right),$$

where

$$\begin{aligned}A_1 &= 2rs^2,\\ A &= -(9rs + 4r^2 + 4r + 1)/3,\\ B_1 &= (rs^2(3s + 8r - 2))/3,\\ B &= -(54r^2 s + 81rs - 16r^3 - 24r^2 - 12r - 2)/27,\\ B_2 &= r^2.\end{aligned}$$

Recall that we expect to find a point of $Y_-(8)$ over $F = \mathbf{Q}(\sqrt{53})$, corresponding to the principally polarized abelian surface A which should match the Hilbert modular form f. We first make a list of all F-rational points of height $\leq$200 on the Hilbert modular surface. Next, for each of these rational points, we try to construct the corresponding genus 2 curve C over F, whose Jacobian corresponds to the moduli point (r, s) we have chosen, and check whether the characteristic polynomial of Frobenius on its first étale cohomology group matches up the polynomial $Q_{\mathfrak{p}}(T)$ giving the corresponding Euler factor of surface A_f attached to the Hilbert modular form. If a candidate point (r, s) passes this test for say the first 50 primes (ordered by norm) of F of good reduction for f and $A = J(C)$, we can be reasonably convinced that it is the correct curve, and then try to prove that A is associated to f.

There are two subtleties in the search. First, since the Hilbert modular surface $Y_-(D')$ is only a coarse moduli space, the point (r, s) is not enough to recover the curve up to F-isomorphism. The Igusa–Clebsch invariants are rational functions in r and s, and they are only enough to pin down C up to quadratic twist. Therefore, when we match the quartic L-factors $L_{\mathfrak{p}}(A, T)$ and $Q_{\mathfrak{p}}(T)$, we need to allow for

$$L_{\mathfrak{p}}(A, \pm T) = Q_{\mathfrak{p}}(T)$$

rather than just the plus sign. Second, the Igusa–Clebsch invariants do not always allow us to define C over the base field F; there is often a Brauer obstruction. Even when C is definable over F (which is the case we are interested in), it can be computationally expensive to do so. Therefore, it is convenient to speed up the process of testing compatibility with f by first reducing (I_2, I_4, I_6, I_{10}) modulo $\mathfrak{p}$ (assuming good reduction) and then producing a curve $D_{\mathfrak{p}}$ over $\mathbf{F}_q$ from these reduced invariants, where $q = \mathrm{N}\mathfrak{p}$. If C exists over F, then its reduction $C_{\mathfrak{p}}$ will be the same as $D_{\mathfrak{p}}$ up to quadratic twist. The advantage is that the Brauer obstruction vanishes over the finite field $\mathbf{F}_q$, making it very easy to check compatibility at $\mathfrak{p}$.

In this particular example, a search of $Y_-(8)$ for all points of height $\leq$200 using [15] (implemented in `Sage`) gives the parameters

$$r = -\frac{24 + 10w}{11^2}, \quad s = \frac{136 - 24w}{11^2},$$

and the Igusa–Clebsch invariants

$$I_2 = 208 + 88w,$$

$$\begin{aligned} I_4 &= -1660 - 588w, \\ I_6 &= -428792 - 135456w, \\ I_{10} &= 643072 + 204800w. \end{aligned}$$

This leads to the same curve C'_f as above.

By further reducing the curve we obtained by either of Methods 1 or 2, we get the following.

Theorem 1 *Let $C = C_f : y^2 + Q(x)y = P(x)$ be the curve over F, where*

$$\begin{aligned} P &:= -4x^6 + (w-17)x^5 + (12w-27)x^4 + (5w-122)x^3 \\ &\quad +(45w-25)x^2 + (-9w-137)x + 14w + 9, \\ Q &:= wx^3 + wx^2 + w + 1. \end{aligned}$$

Then

(a) *The discriminant of this curve is $\Delta_C = -\epsilon^7$. Thus C has everywhere good reduction.*
(b) *The surface $A := J(C)$ is modular and corresponds to the unique Hecke constituent $[f]$ in $S_2(1)$, the space of Hilbert cusp forms of weight 2 and level (1) over $F = \mathbf{Q}(\sqrt{53})$.*

Proof A direct calculation shows that $\Delta_C = -\epsilon^7$. By construction, A has real multiplication by $\mathcal{O}_f = \mathbf{Z}[\sqrt{2}]$, where 7 is split. Let λ be one of the primes above 7, and consider the λ-adic representation

$$\rho = \rho_{A,\lambda} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(K_{f,\lambda}) \simeq \mathrm{GL}_2(\mathbf{Q}_7),$$

and its reduction $\overline{\rho}$ modulo λ. We will show that ρ is modular by using [46, Theorem A]. For this, it suffices to show that $\overline{\rho}$ is reducible or, equivalently, that A has a 7-torsion point defined over F. By definition, we have

$$A(F) \simeq \mathrm{Pic}^0(C)(F).$$

So it is enough to find a degree zero divisor D defined over F such that $7D$ is principal. To this end, we consider the field $L = F(\alpha)$, where α is a root of the polynomial $x^2 - wx + 3$. Let $\sigma' \in \mathrm{Gal}(L/F)$ be the non-trivial involution. Then, the point $P = (\alpha, (-6w-12)\alpha + 2w + 18) \in C(L)$, and the divisor $D := P + \sigma'(P) - 2\infty$ belongs to $\mathrm{Pic}^0(C)(F)$. An easy calculation shows that $7D \sim (0)$. Hence, D corresponds to a point of order 7 in $A(F)$.

Since $S_2(1)$ has dimension 2 and is spanned by $[f]$, A must correspond to this Hilbert newform.

Remark 4 Both C and A have everywhere good reduction. However, this is not true in some of the other examples. Indeed, it can happen that a curve C has bad reduction at a prime $\mathfrak{p}$ while $\mathrm{Jac}(C)$ does not. (See the example of Theorem 3.)

Remark 5 The modularity of the abelian surface $A = \mathrm{Jac}(C)$ we found means that it is isogenous to the surface A_f obtained from the Eichler–Shimura construction over $\mathbf{Q}$. Since A_f is a $\mathbf{Q}$-surface, so is A. In fact, the proof of the reducibility of $\overline{\rho}_{A,\lambda}$ implies that A and its Galois conjugate are related by a 7-isogeny.

4.2 Case II

The following result explains the connection between Conjectures 1 and 2.

Proposition 1 *Assume that Conjecture 2 is true. Let F be a real quadratic field. Let f be a Hilbert newform of weight 2 and level $\mathfrak{N}$ over F, which satisfies the hypotheses of Case II. Then f satisfies Conjecture 1.*

Proof Since f is a non-base change, [26, Main Theorem] implies that there is a paramodular Siegel newform g of genus 2, level ND^2 and weight 2 attached to f, where $N = \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{N})$. Moreover, since $\mathrm{Gal}(F/\mathbf{Q})$ preserves $\{f, f^\tau\}$, we must have

$$a_{\mathfrak{p}^\sigma}(f) = a_{\mathfrak{p}}(f)^\tau$$

for all primes $\mathfrak{p} \subseteq \mathcal{O}_F$. Therefore, the Hecke eigenvalues of the form g are integers. So by Conjecture 2, there is an abelian surface B_g defined over $\mathbf{Q}$ with $\mathrm{End}_{\mathbf{Q}}(B_g) = \mathbf{Z}$ such that $L(B_g, s) = L(g, s)$. Let A_f be the base change of B_g to F. Then, by construction, we have

$$L(A_f, s) = L(f, s)L(f^\tau, s).$$

Hence, A_f satisfies Conjecture 1. $\square$

Remark 6 Assume Conjecture 2. By Proposition 1, if A_f is an abelian surface attached to a Hilbert newform f satisfying Case II, then A_f is the base change to F of some surface B defined over $\mathbf{Q}$, which acquires extra endomorphisms. Therefore, we know that the Igusa–Clebsch invariants of A_f are in $\mathbf{Q}$, and we can use this fact in looking for A_f.

The first real quadratic field of narrow class number 1 where there is a form f of level (1) and weight 2, which satisfies Case II, is $F = \mathbf{Q}(\sqrt{193})$ (see Table 3). The coefficients of f generate the ring of integers $\mathcal{O}_f := \mathbf{Z}[\frac{1+\sqrt{17}}{2}]$ of the field $K_f = \mathbf{Q}(\sqrt{17})$.

Theorem 2 *Let $C : y^2 + Q(x)y = P(x)$ be the curve over F, where*

$$\begin{aligned} P(x) &:= 2x^6 + (-2w + 7)x^5 + (-5w + 47)x^4 + (-12w + 85)x^3 \\ &\quad +(-13w + 97)x^2 + (-8w + 56)x - 2w + 1, \\ Q(x) &:= -x - w. \end{aligned}$$

Then

Table 3 The first few Hecke eigenvalues of a non-base change newform of level (1) and weight 2 over $\mathbf{Q}(\sqrt{193})$

$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$s_{\mathfrak{p}}(f)$	$t_{\mathfrak{p}}(f)$
2	$9w-67$	e	1	0
2	$9w+58$	$-e+1$	1	0
3	$-2w+15$	e	1	2
3	$2w+13$	$-e+1$	1	2
7	$-186w-1199$	$-e+2$	3	12
7	$186w-1385$	$e+1$	3	12
23	$38w-283$	$-e-6$	-13	84
23	$-38w-245$	$e-7$	-13	84
25	5	1	2	51
31	$-16w-103$	$e-3$	-5	64
31	$-16w+119$	$-e-2$	-5	64
43	$4w+25$	$e+4$	9	102
43	$-4w+29$	$-e+5$	9	102

Here $e=(1+\sqrt{17})/2$

(a) *The discriminant* $\Delta_C=-1$, *hence* C *has everywhere good reduction.*
(b) *The surface* $J(C)$ *is modular and corresponds to the form* f *listed in Table 3.*

Remark 7 A theorem of Stroeker [48] implies[2] that if E is an elliptic curve defined over a real quadratic field F having good reduction everywhere, then $\Delta_E \notin \{-1,1\}$. However, this fails for curves of genus 2, by the above example.

Proof We show that $\Delta_C=-1$ as before, which implies that C and $J(C)$ both have everywhere good reduction. However, it is important to observe that we located the curve based on our heuristics which rely on Conjectures 1 and 2. Indeed, let $S_2(1)$ be the space of Hilbert cuspforms of level (1) and weight 2 over $F=\mathbf{Q}(\sqrt{193})$. Then $S_2(1)$ has dimension 9, and decomposes into two Hecke constituents of dimension 2 and 7 respectively. The form f in Table 3 is an eigenvector in the 2-dimensional constituent, and it is a non-base change whose Hecke constituent is Galois stable. So we can look for our surface A_f with the help of Proposition 1.

To find the curve C, we proceed as in Sect. 4.1.2, using the results from [17]. The surface $Y_-(17)$ is a double-cover of the (weighted) projective space $\mathbf{P}^2_{g,h}/\mathbf{Q}$ given by

$$
\begin{aligned}
z^2 = &-256\,h^3 + (192\,g^2 + 464\,g + 185)\,h^2 \\
&-2\,(2\,g+1)\,(12\,g^3 - 65\,g^2 - 54\,g - 9)\,h + (g+1)^4\,(2\,g+1)^2.
\end{aligned}
$$

A search for $\mathbf{Q}$-rational points of low height on this surface yields the following parameters, Igusa–Clebsch and G_2 invariants:

$$
\begin{aligned}
&g = 0, h = -1/4, \\
&I_2 = 40,\ I_4 = -56,\ I_6 = -669,\ I_{10} = -4,
\end{aligned}
$$

[2] Stroeker's result is stated for imaginary quadratic fields. Elkies [16] remarks that the argument implies the statement above for real quadratic fields.

$$j_1 = -3200000,\ j_2 = -208000,\ j_3 = -16400.$$

Over $\mathbf{Q}$, this gives the curve

$$C' : y^2 = -8x^6 + 220x^5 - 44x^4 - 14828x^3 - 4661x^2 - 21016x + 10028.$$

After finding a suitable twist and reducing the Weierstrass equation, we get the curve C displayed in the statement of the theorem.

To prove modularity, we note that 3 is inert in $K_f = \mathbf{Q}(\sqrt{17})$, and consider the 3-adic representation attached to A,

$$\rho_{A,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(K_{f,(3)}) \simeq \mathrm{GL}_2(\mathbf{Q}_9).$$

By computing the orders of Frobenius for the first few primes, we see that the mod3 representation

$$\overline{\rho}_{A,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\mathbf{F}_9)$$

is surjective, and absolutely irreducible. Hence $\rho_{A,3}$ is also absolutely irreducible. Since 3 and 5 are unramified in the quadratic field F, the ramification indices of $\overline{\rho}_{A,3}$ at the primes of F above them are odd. Also, since $\overline{\rho}_{A,3}$ is unramified at (5), the image of the inertia group at $I_{(5)}$ at 5 in $\mathrm{GL}_2(\mathbf{F}_9)$ is trivial. In particular, the image of $I_{(5)}$ has odd order and lies in $\mathrm{SL}_2(\mathbf{F}_9)$. By studying the Tate module of $A \times_F F(\zeta_3)$, we also see that $\overline{\rho}|_{G_{F(\zeta_3)}}$ is absolutely irreducible. Therefore, $\overline{\rho}_{A,3}$ is modular by [18, Theorem 3.2 and Proposition 3.4]. We then apply [21, Theorem 1.1 in Erratum], [22] to conclude that $\rho_{A,3}$ is modular. So, A is modular and corresponds to the unique newform $f \in S_2(1)$ with coefficients in $\mathcal{O}_f = \mathbf{Z}[\frac{1+\sqrt{17}}{2}]$.

Corollary 1 *Let B be the Jacobian of the curve $C'/\mathbf{Q}$ in the proof of Theorem 2. Then B is paramodular of level* 193^2.

Remark 8 In [5], the authors remarked that Conjecture 1.4 in their paper should be verifiable by current technology for paramodular abelian surfaces B over $\mathbf{Q}$ with $\mathrm{End}_{\overline{\mathbf{Q}}}(B) \supsetneq \mathbf{Z}$. The majority of the surfaces we found fall in Case II (see Sect. 5), and provide such evidence by Corollary 1.

In contrast to the curves in Theorems 1 and 2, we found a few curves whose Jacobians had everywhere good reduction while the curves themselves did not. We now discuss one such example, for the field $F = \mathbf{Q}(\sqrt{929})$, with Hecke eigenvalues in $\mathbf{Q}(\sqrt{13})$.

Theorem 3 *Let $C : y^2 + Q(x)y = P(x)$ be the curve over F, where*

$$\begin{aligned} P(x) &:= 23x^6 + (90w - 45)x^5 + 33601x^4 + (28707w - 14354)x^3 \\ &\quad +3192149x^2 + (811953w - 405977)x + 19904990, \\ Q(x) &:= x^3 + x + 1. \end{aligned}$$

Then

Table 4 The first few Hecke eigenvalues of a non-base change newform of level (1) and weight 2 over $\mathbf{Q}(\sqrt{929})$

N$\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$s_{\mathfrak{p}}(f)$	$t_{\mathfrak{p}}(f)$
2	$561w-8830$	$-e+1$	1	1
2	$561w+8269$	e	1	1
5	$-4w-59$	$-e+1$	1	7
5	$4w-63$	e	1	7
9	3	3	6	27
11	$-8342w+131301$	$2e-3$	-4	13
11	$8342w+122959$	$-2e-1$	-4	13
19	$-50w-737$	$e-2$	-3	37
19	$50w-787$	$-e-1$	-3	37
23	$-42832w+674165$	$4e-4$	-4	-2
23	$42832w+631333$	$-4e$	-4	-2
29	$-2w+31$	$-2e+6$	10	70
29	$2w+29$	$2e+4$	10	70

Here $e=(1+\sqrt{13})/2$

(a) *The discriminant $\Delta_C = 3^{22}$, hence C has bad reduction at* (3).
(b) *The surface $A := J(C)$ has everywhere good reduction. It is modular and corresponds to the form f listed in Table 4.*

Proof The curve C is a global minimal model for the base change to F of the curve $C'/\mathbf{Q}$ given by

$$C' : y^2 = 93x^6 - 14688x^5 + 549594x^4 + 2268918x^3 + 2259369x^2 - 1488402x \\ +13059345.$$

We compute the reduction $\widetilde{C}'$ of C' at 3 by combining [32, Theorem 1 and Proposition 2], and Liu's algorithm implemented in `Sage`. This returns the type (V), $[\mathrm{I}_0 - \mathrm{I}_0 - 1]$. So the reduction $\widetilde{A}'$ of the Jacobian A' of C' is a product of two elliptic curves whose j-invariants are $j_1 = j_2 = 0$ ([32, Proposition 2, (v)]). This implies that A' has non-ordinary good reduction at (3); and so does A since 3 is inert in F. (Note that this is consistent with the fact that $a_{(3)}(f) = 3$.) Since 3 is the only prime dividing Δ_C, we see that A has everywhere good reduction.

To prove modularity, we recall that by construction A has real multiplication by $\mathcal{O}_f = \mathbf{Z}[\frac{1+\sqrt{13}}{2}]$, where 3 splits. We choose a prime λ above 3, and consider the λ-adic representation

$$\rho_{A,\lambda} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(K_{f,\lambda}) \simeq \mathrm{GL}_2(\mathbf{Q}_3)$$

and its reduction $\overline{\rho}_{A,\lambda}$ modulo λ. By computing the first few Frobenii, we see that $\overline{\rho}_{A,\lambda}$ is surjective, hence irreducible. Since $\mathrm{GL}_2(\mathbf{F}_3)$ is solvable, $\overline{\rho}$ is modular by Langlands–Tunnell [31, Chap. I] and [49]. By looking at the Tate module of $A \times_F F(\zeta_3)$, we also

Table 5 Unresolved cases

Case	List of (D, D')
II	(433, 12), (613, 21), (677, 85), (821, 44), (853, 21)
III	(353, 5), (421, 5)

see that $\rho_{A,\lambda}$ is not induced from $F(\zeta_3)$. So, we conclude that $\rho_{A,\lambda}$ is modular by [21, Theorem 1.1 in Erratum], [22]. □

Remark 9 The example in Theorem 3 and other similar ones in Table 7 underscore the difficulty in producing effective algorithms for principally polarized abelian surfaces with good reduction outside a (finite) prescribed set of primes S of $\mathcal{O}_F$. Indeed, let A be such a surface so that $A = \mathrm{Jac}(C)$, where C is a curve defined over F with good reduction outside a finite set of primes $T \supseteq S$. Then, the set $T \backslash S$ is non-empty in general, depends *a priori* on A, and is hard to predict. When A has real multiplication by some quadratic field K and is attached to a modular form f, $T \backslash S$ is contained in the set of non-ordinary primes for f, which is possibly infinite.

Similar proofs apply for the other Hilbert modular forms in Case II for which we were able to find matching principally polarized abelian surfaces. However, there are five examples (listed in Table 5) for which we were unable as yet to find matching abelian surfaces. In each case, the Fourier coefficients of the form indicate that the missing surface would have real multiplication by the full ring of integers $\mathcal{O}_{D'}$. So, assuming the Eichler–Shimura conjecture holds, our difficulties in matching those forms could be due to one of the following reasons:

(a) Our height bound for the rational point search on the corresponding Hilbert modular surfaces is too small. We searched for parameters $r, s \in \mathbf{Q}$ of height up to 1000.
(b) The corresponding abelian surface is not principally polarized. Note that the criteria given in [24, Proposition 3.11] to convert an arbitrary polarization to a principal polarization fail for each of the missing discriminants D'. For $(D, D') = (677, 85)$, the field $\mathbf{Q}(\sqrt{D'})$ has class number 2, whereas for the other examples, there is no unit of negative norm.

There is also the possibility, since the models in [17] are birational to $Y_-(D')$ (rather than isomorphic), that we might have missed some curves or points in our search. However, this is unlikely to be the case, as the extra points should correspond to abelian surfaces with extra endomorphisms.

4.3 Case III

This is by far the trickiest case, since the Igusa–Clebsch invariants (and therefore r, s) are not in $\mathbf{Q}$. This leads to a much slower search for F-points on $Y_-(D')$, compared to searching for $\mathbf{Q}$-points. We searched for points of height up to 400 using the enumeration of points of small height developed in [15] (implemented in `Sage`), but were

Table 6 Case I examples

D	D'	Hyperelliptic polynomials
53	8	$Q = wx^3 + wx^2 + w + 1$ $P = -4x^6 + (w-17)x^5 + (12w-27)x^4 + (5w-122)x^3$ $+(45w-25)x^2 + (-9w-137)x + 14w + 9$
73	5	$Q = x^3 + x + 1$ $P = (w-5)x^6 + (3w-14)x^5 + (3w-19)x^4$ $+(4w-3)x^3 - (3w+16)x^2 + (3w+11)x - (w+4)$
421	5	$Q = w(x^3 + 1)$ $P = -2(4414w + 43089)x^6 + (31147w + 303963)x^5$ $-10(4522w + 44133)x^4 + 2(17290w + 168687)x^3$ $-18(816w + 7967)x^2 + 27(122w + 1189)x - (304w + 3003)$

Table 7 Case II examples

D	D'	Hyperellipticpolynomials
193	17	$Q = -x - w$ $P = 2x^6 + (-2w+7)x^5 + (-5w+47)x^4 + (-12w+85)x^3$ $+(-13w+97)x^2 + (-8w+56)x - 2w + 1$
233	17	$Q = x + 1$ $P = -2x^6 - (2w-1)x^5 - 45x^4 - 4(2w-1)x^3 - 31x^2 + (w-1)x + 9$
277	29	$Q = -1$ $P = -24x^6 + 31bx^5 - 4615x^4 + 1321bx^3 + 58837x^2 + 5039bx - 49745$
349	21	$Q = x^3$ $P = -2x^6 + 4bx^5 - 1328x^4 + 673bx^3 - 66879x^2 + 10145bx - 223536$
373	93	$P = (265b - 5118)(8x - b)(8960bx^5 - 2020471x^4 + 488608bx^3$ $-22037369x^2 + 1332394bx - 12019522)$
389	8	$Q = x^3 + x^2 + x + 1$ $P = -wx^5 + 159x^4 - (138w - 68)x^3 + 6429x^2 - (1619w - 809)x$ $+16260$
397	24	$P = -601x^6 + 748bx^5 - 154001x^4 + 42596bx^3 - 2631127x^2$ $+218342bx - 2997270$
409	13	$Q = x^3 + x^2 + 1$ $P = -2x^6 + (-3w+1)x^5 - 219x^4 + (-83w+41)x^3 - 1806x^2$ $+(-204w+102)x - 977$
461	29	$Q = x^3$ $P = -32x^6 - 34bx^5 - 6916x^4 - 1605bx^3 - 94873x^2 - 6335bx - 78584$

Table 7 continued

D	D'	Hyperellipticpolynomials
677	13	$Q = x^3 + x^2 + x + 1$ $P = -12x^6 + (61w - 31)x^5 - 22335x^4 + (25770w - 12886)x^3$ $-2830998x^2 + (980087w - 490044)x - 23929668$
677	29	$P = -4453x^6 - 5786bx^5 - 2120768x^4 - 612392bx^3 - 67342400x^2$ $-5834038bx - 142573513$
709	5	$P = 2x^6 + 2bx^5 + 560x^4 + 114bx^3 + 9040x^2 + 530bx + 9058$
797	8	$P = 1856x^6 - 3784bx^5 + 2561907x^4 - 1160668bx^3 + 235735797x^2$ $-32038746bx + 1445987770$
797	29	$Q = x^3 + x^2 + x + 1$ $P = x^6 + (3w - 2)x^5 + 594x^4 + (314w - 158)x^3 + 18483x^2$ $+(2897w - 1449)x + 37491$
809	5	$Q = x^3 + x + 1$ $P = -134x^6 - (146w - 73)x^5 - 13427x^4 - (3255w - 1627)x^3$ $-89746x^2 - (6523w - 3261)x - 39941$
929	13	$Q = x^3 + x + 1$ $P = 23x^6 + (90w - 45)x^5 + 33601x^4 + (28707w - 14354)x^3$ $+3192149x^2 + (811953w - 405977)x + 19904990$
997	13	$Q = x^3$ $P = x^6 + 3bx^5 + 2989x^4 + 1592bx^3 + 475212x^2 + 75831bx + 5023486$

unable to find either of the two examples predicted by the Eichler–Shimura conjecture, corresponding to the Hilbert modular forms of level 1 and weight 2 over $\mathbf{Q}(\sqrt{353})$ and $\mathbf{Q}(\sqrt{421})$, both with Fourier coefficients in $\mathbf{Q}(\sqrt{5})$. In addition to the reduced search height bound, another complicating factor is the fundamental unit of F, which might be quite large. In Case II, the discriminant of the genus 2 curve differed from $I_{10}(r, s)$ by only a few small (rational) primes. However, in Case III, one has to take into account the fact that a power of the fundamental unit might also appear in the discriminant. On the other hand, principal polarizability is not an obstruction, as $\mathbf{Q}(\sqrt{5})$ has a fundamental unit of negative norm.

We hope to address the missing examples using different techniques in future work.

5 The data

In Tables 6 and 7 we list genus 2 curves $y^2 + Q(x)y = P(x)$ matching the data. We always set $b = \sqrt{D}$ and $w = (b+1)/2$. We suppress $Q(x)$ when it is 0. (We recall that each of the curves listed has a modular Jacobian. In Case I, this is true as the Jacobian is a $\mathbf{Q}$-surface. While in Case II, we prove the modularity by the same technique as above.)

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Appendix

In Table 8 below we list Hilbert modular form data for all the examples considered in this paper.

Table 8 Hecke eigenvalues for the Hilbert modular forms in this paper

$D = 53,\ D' = 8$			$D = 61,\ D' = 12$			$D = 73,\ D' = 5$		
$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	$e+1$	3	$-w-3$	$e-1$	2	$-w-4$	$-e$
7	$-w-2$	$-e-2$	3	$-w+4$	$e-1$	2	$w-5$	$-e$
7	$-w+3$	$-e-2$	4	2	e	3	$-4w-15$	$-e+1$
9	3	$-3e+1$	5	$w-5$	$-e$	3	$-4w+19$	$-e+1$
11	$w-2$	$3e$	5	$-w-4$	$-e$	19	$6w-29$	$4e-1$
11	$w+1$	$3e$	13	$-w-1$	3	19	$-6w-23$	$4e-1$
13	$w-1$	$-2e+1$	13	$w-2$	3	23	$14w-67$	$-3e+4$
13	$-w$	$-2e+1$	19	$-3w-11$	$-e+3$	23	$-14w-53$	$-3e+4$
17	$-w-5$	-3	19	$3w-14$	$-e+3$	25	5	$-e+1$
17	$w-6$	-3	41	$w-8$	$-2e-3$	37	$-2w-5$	5
25	5	$2e+4$	41	$-w-7$	$-2e-3$	37	$2w-7$	5
29	$-w-6$	$3e-3$	47	$-3w-8$	$4e+6$	41	$30w-143$	$2e+4$
$D = 193,\ D' = 17$			$D = 233,\ D' = 17$			$D = 277,\ D' = 29$		
$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
2	$9w+58$	$-e+1$	2	$-w-7$	e	3	$w+8$	$-e+1$
2	$9w-67$	e	2	$-w+8$	$-e+1$	3	$-w+9$	e
3	$-2w+15$	e	7	$-8w+65$	$e-1$	4	2	-2
3	$2w+13$	$-e+1$	7	$8w+57$	$-e$	7	$6w-53$	$-e+3$
7	$186w-1385$	$e+1$	9	3	-2	7	$-6w-47$	$e+2$
7	$-186w-1199$	$-e+2$	13	$38w-309$	$-e+3$	13	$-w-7$	$-e-1$
23	$-38w+283$	$-e-6$	13	$-38w-271$	$e+2$	13	$w-8$	$e-2$
23	$-38w-245$	$e-7$	19	$-6w+49$	$-3e+3$	19	$4w+31$	$-2e+1$
25	5	1	19	$6w+43$	$3e$	19	$-4w+35$	$2e-1$
31	$-16w+119$	$-e-2$	23	$2w+15$	$-e+2$	23	$-3w+26$	3
31	$-16w-103$	$e-3$	23	$-2w+17$	$e+1$	23	$-3w-23$	3
43	$4w+25$	$e+4$	25	5	-3	25	5	-3

Table 8 continued

$D = 349,\ D' = 21$			$D = 353,\ D' = 5$			$D = 353,\ D' = 5$		
$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
3	$-w-9$	$-e+1$	2	$w+9$	$2e-1$	2	$w+9$	$-e+1$
3	$w-10$	e	2	$-w+10$	$-e+1$	2	$-w+10$	$2e-1$
4	2	-2	9	3	$-2e-2$	9	3	$-2e-2$
5	$-6w+59$	e	11	$10w+89$	$-2e+2$	11	$10w+89$	$2e+3$
5	$-6w-53$	$-e+1$	11	$-10w+99$	$2e+3$	11	$-10w+99$	$-2e+2$
17	$13w-128$	$-e+2$	17	$66w-653$	$-4e+2$	17	$66w-653$	3
17	$13w+115$	$e+1$	17	$-66w-587$	3	17	$-66w-587$	$-4e+2$
19	$-5w-44$	$2e$	19	$-28w+277$	2	19	$-28w+277$	$2e-3$
19	$5w-49$	$-2e+2$	19	$28w+249$	$2e-3$	19	$28w+249$	2
23	$-w+11$	$-2e+5$	23	$-8w-71$	$4e-2$	23	$-8w-71$	$2e+3$
23	$w+10$	$2e+3$	23	$8w-79$	$2e+3$	23	$8w-79$	$4e-2$
29	$-3w-26$	$-2e-1$	25	5	-3	25	5	-3

$D = 373,\ D' = 93$			$D = 389,\ D' = 8$			$D = 397,\ D' = 24$		
$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
3	$w-10$	-2	4	2	-2	3	$2w+19$	$-e$
3	$w+9$	-2	5	$-3w-28$	$2e-1$	3	$-2w+21$	e
4	2	3	5	$-3w+31$	$-2e-1$	4	2	-1
7	$-6w-55$	-2	7	$-w-9$	$-2e-1$	11	$w-11$	$-e+2$
7	$6w-61$	-2	7	$w-10$	$2e-1$	11	$-w-10$	$e+2$
13	$-7w+71$	$e+1$	9	3	-4	19	$-11w+115$	$2e-2$
13	$-7w-64$	$e+1$	11	$2w+19$	$-2e-2$	19	$-11w-104$	$-2e-2$
17	$-w-10$	$e-2$	11	$-2w+21$	$2e-2$	23	$-3w-28$	2
17	$-w+11$	$e-2$	13	$-w-10$	$2e+1$	23	$3w-31$	2
25	5	6	13	$w-11$	$-2e+1$	25	5	-4
29	$-4w+41$	$-e-1$	17	$-8w-75$	$2e-4$	29	$9w-94$	1
29	$-4w-37$	$-e-1$	17	$-8w+83$	$-2e-4$	29	$-9w-85$	1

$D = 409,\ D' = 13$			$D = 421,\ D' = 5$			$D = 421,\ D' = 5$		
$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
2	$219w+2105$	$e-1$	3	$4w-43$	$2e$	3	$4w-43$	$-2e+1$
2	$219w-2324$	$-e$	3	$4w+39$	$-2e+1$	3	$4w+39$	$2e$
3	$-11066w-106365$	$-e+2$	4	2	3	4	2	$e-2$
3	$11066w-117431$	$e+1$	5	$-w-10$	$e-2$	5	$-w-10$	3
5	$-18w+191$	$-e$	5	$w-11$	3	5	$w-11$	$e-2$
5	$-18w-173$	$e-1$	7	$54w+527$	$e-2$	7	$54w+527$	3
17	$8w+77$	4	7	$-54w+581$	$-e+5$	7	$-54w+581$	$e-2$
17	$8w-85$	4	11	$25w-269$	$e-2$	11	$25w-269$	$-e+5$
23	$286w-3035$	$-4e+3$	11	$-25w-244$	4	11	$-25w-244$	0
23	$286w+2749$	$4e-1$	17	$-3w+32$	0	17	$-3w+32$	4
41	$-1600w+16979$	$-e+5$	17	$-3w-29$	$-6e+3$	17	$-3w-29$	$-4e+5$
41	$-1600w-15379$	$e+4$	31	$9w-97$	$-4e+5$	31	$9w-97$	$-6e+3$

Table 8 continued

$D = 421,\ D' = 5$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
3	$4w - 43$	2
3	$4w + 39$	2
4	2	$2e - 1$
5	$-w - 10$	$e + 2$
5	$w - 11$	$e + 2$
7	$54w + 527$	$-2e + 2$
7	$-54w + 581$	$-2e + 2$
11	$25w - 269$	-4
11	$-25w - 244$	-4
17	$-3w + 32$	$-5e + 3$
17	$-3w - 29$	$-5e + 3$
31	$9w - 97$	$2e + 4$

$D = 433,\ D' = 12$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
2	$-w + 11$	$-e$
2	$w + 10$	e
3	$1202w - 13107$	$e - 1$
3	$-1202w - 11905$	$-e - 1$
11	$-324w - 3209$	$-e - 3$
11	$-324w + 3533$	$e - 3$
13	$94w + 931$	-3
13	$-94w + 1025$	-3
17	$17152w - 187031$	$-2e - 3$
17	$-17152w - 169879$	$2e - 3$
25	5	0
37	$-12w - 119$	-3

$D = 461,\ D' = 29$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	-2
5	$-w - 10$	e
5	$-w + 11$	$-e + 1$
9	3	-3
17	$w + 11$	$-e + 4$
17	$-w + 12$	$e + 3$
19	$3w - 34$	$-e + 3$
19	$-3w - 31$	$e + 2$
23	$-2w + 23$	$-e + 3$
23	$-2w - 21$	$e + 2$
41	$w - 13$	$-2e + 2$
41	$-w - 12$	$2e$

$D = 613,\ D' = 21$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
3	$w - 13$	e
3	$-w - 12$	$-e + 1$
4	2	0
7	$8w + 95$	2
7	$8w - 103$	2
17	$33w + 392$	$-e + 5$
17	$33w - 425$	$e + 4$
19	$-9w - 107$	3
19	$9w - 116$	3
25	5	-6
29	$-w + 14$	$-2e + 7$
29	$w + 13$	$2e + 5$

$D = 677,\ D' = 13$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	0
9	3	-4
13	$-w + 13$	$-e + 1$
13	$-w - 12$	e
25	5	-7
37	$-w - 11$	$-4e - 1$
37	$-w + 12$	$4e - 5$
41	$w - 15$	$-e + 9$
41	$w + 14$	$e + 8$
49	7	-3
59	$w - 11$	$-2e + 5$
59	$-w - 10$	$2e + 3$

$D = 677,\ D' = 29$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	-1
9	3	-3
13	$-w + 13$	$e + 2$
13	$-w - 12$	$-e + 3$
25	5	-3
37	$-w - 11$	$e - 3$
37	$-w + 12$	$-e - 2$
41	$w - 15$	$3e$
41	$w + 14$	$-3e + 3$
49	7	-10
59	$w - 11$	$-3e + 6$
59	$-w - 10$	$3e + 3$

$D = 677,\ D' = 85$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	-3
9	3	-1
13	$-w + 13$	e
13	$-w - 12$	$-e + 1$
25	5	-7
37	$-w - 11$	$e + 7$
37	$-w + 12$	$-e + 8$
41	$w - 15$	$e + 2$
41	$w + 14$	$-e + 3$
49	7	-6
59	$w - 11$	$-e - 6$
59	$-w - 10$	$e - 7$

$D = 709,\ D' = 5$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
3	$-59w - 756$	$2e - 1$
3	$59w - 815$	$-2e + 1$
4	2	0
5	$w - 14$	$2e + 1$
5	$-w - 13$	$-2e + 3$
7	$-16w + 221$	$-2e$
7	$-16w - 205$	$2e - 2$
11	$-547w - 7009$	$-4e + 1$
11	$547w - 7556$	$4e - 3$
19	$6w - 83$	$2e + 4$
19	$6w + 77$	$-2e + 6$
29	$75w - 1036$	$2e - 3$

$D = 797,\ D' = 8$

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	-3
9	3	-3
11	$-w + 15$	$3e$
11	$w + 14$	$-3e$
13	$2w - 29$	$-2e - 1$
13	$2w + 27$	$2e - 1$
17	$w + 13$	$-2e$
17	$w - 14$	$2e$
25	5	0
41	$-w - 15$	$2e - 5$
41	$-w + 16$	$-2e - 5$
43	$w - 13$	$-e - 4$

Table 8 continued

$D=797,\ D'=29$			$D=809,\ D'=5$			$D=821,\ D'=44$		
$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
4	2	0	2	$-219w+3224$	$-e+1$	4	2	-1
9	3	-3	2	$-219w-3005$	e	5	$w-15$	0
11	$-w+15$	3	5	$21796w-320869$	e	5	$-w-14$	0
11	$w+14$	3	5	$-21796w-299073$	$-e+1$	7	$-6w-83$	$e-1$
13	$2w-29$	$e+3$	7	$-18w-247$	$2e-2$	7	$6w-89$	$-e-1$
13	$2w+27$	$-e+4$	7	$18w-265$	$-2e$	9	3	-3
17	$w+13$	$e+1$	9	3	-4	19	$5w-74$	$e-5$
17	$w-14$	$-e+2$	13	$-4w-55$	$-3e+2$	19	$5w+69$	$-e-5$
25	5	-6	13	$4w-59$	$3e-1$	23	$-w-13$	$-e-3$
41	$-w-15$	$-e+6$	19	$140w+1921$	$e-5$	23	$-w+14$	$e-3$
41	$-w+16$	$e+5$	19	$-140w+2061$	$-e-4$	29	$11w-163$	$-2e-3$
43	$w-13$	$2e-5$	23	$2926w-43075$	$-3e+6$	29	$-11w-152$	$2e-3$

$D=853,\ D'=21$			$D=929,\ D'=13$			$D=997,\ D'=13$		
$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$
3	$-w+15$	$-e+1$	2	$561w-8830$	$-e+1$	3	$-7w-107$	e
3	$-w-14$	e	2	$561w+8269$	e	3	$-7w+114$	$-e+1$
4	2	0	5	$-4w-59$	$-e+1$	4	2	0
19	$9w+127$	5	5	$4w-63$	e	13	$3w+46$	$2e+2$
19	$-9w+136$	5	9	3	3	13	$-3w+49$	$-2e+4$
23	$19w-287$	$e+1$	11	$-8342w+131301$	$2e-3$	19	$4w-65$	$e+1$
23	$19w+268$	$-e+2$	11	$8342w+122959$	$-2e-1$	19	$4w+61$	$-e+2$
25	5	3	19	$-50w-737$	$e-2$	23	$-w-16$	6
31	$w-14$	-3	19	$50w-787$	$-e-1$	23	$-w+17$	6
31	$-w-13$	-3	23	$-42832w+674165$	$4e-4$	25	5	-4
41	$8w-121$	$-3e+3$	23	$42832w+631333$	$-4e$	31	$80w+1223$	$-2e+1$
41	$8w+113$	$3e$	29	$2w+29$	$-2e+6$	31	$80w-1303$	$2e-1$

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