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SYMPOSIUM

The nature of tournaments

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Abstract This paper characterizes the optimal way for a principal to structure a rank-order tournament in a moral hazard setting (as in Lazear and Rosen in J Polit Econ 89:841–864, 1981). We find that it is often optimal to give rewards to top performers that are smaller in magnitude than corresponding punishments to poor performers. The paper identifies four reasons why the principal might prefer to give larger rewards than punishments: (1) R is small relative to P (where R is risk aversion and P is absolute prudence); (2) the distribution of shocks to output is asymmetric and the asymmetry takes a particular form; (3) the principal faces a limited liability constraint; and (4) there is agent heterogeneity of a particular form.

Keywords Prizes · Tournaments

JEL Classification L22

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1 Introduction

Lazear and Rosen (1981) argue that rank-order tournaments help to solve a moral hazard problem faced by firms.[1] Such tournaments have been interpreted as explaining many features of firms, such as within-firm job promotions, wage increases, bonuses, and CEO compensation; as well as "punishments " such as firings and up-or-out policies (Lazear 1991; Prendergast 1999).

In assessing this claim, it is important to understand what optimal prize structures look like in such tournaments (where abilities are identical and common knowledge, agents are risk-averse,[2] and there are both common and idiosyncratic shocks to output). This paper provides a characterization of the optimal prizes in tournaments of the type first analyzed by Lazear and Rosen (1981) and Green and Stokey (1983) (LRGS tournaments). Our results have considerable practical significance. They allow us to test whether aspects of employee compensation arise because of or in spite of the moral hazard theory of tournaments.

We analyze a LRGS-style model with and without binding limited liability constraints for the agents. We identify conditions under which the optimal prize structure has the property that the reward for placing ith in the tournament rather than $(i+1)$th is smaller than the optimal punishment for placing $(n-i+1)$th rather than $(n-i)$th (where n is the number of agents in the tournament) when $i \leq \frac{n-1}{2}$. In particular, this means that the punishments for the worst performers are greater in magnitude than the rewards for the best performers.

The particular shape of the optimal prize schedule depends crucially upon the distribution of the shocks to agents' output. We find that a set of weights, $\{\beta_i\}_{i=1}^{n}$, which can be calculated solely based upon the shock distribution, encapsulates the effect of the shock distribution on the optimal prize schedule. The weight β_i is equal to the marginal change in the probability of placing ith in the tournament from a marginal change in effort. In fact, when agents' utility for wealth is logarithmic, the optimal prize schedule is simply an affine transformation of the weight schedule.[3]

Many common noise distributions, such as the normal distribution and uniform distribution, yield weight schedules that spike at the top and bottom. When the weight schedule spikes at the top and bottom, and the limited liability constraint does not bind, the optimal prize schedule gives special rewards to a few of the best performers, special punishments to a few of the worst performers, and somewhat smaller rewards/punishments for those whose performance is neither at the top nor bottom of the distribution.

While, often, optimal tournaments punish more than they reward, there are four factors that lead the rewards to be large relative to the punishments. We find that the amount of punishment relative to reward depends upon the size of R relative to P,

[1] Green and Stokey (1983) provide a general treatment of the problem with risk-averse agents.

[2] Lazear and Rosen (1981) have risk-neutral agents but Green and Stokey extend this, inter alia, to risk-averse agents.

[3] When utility for wealth is logarithmic and the shock distribution is symmetric (in the sense that $F(-x) = 1 - F(x)$), we find that the rewards for the best performers are exactly equal to the punishments for the worst performers.

where R is Arrow–Pratt risk aversion and P is the coefficient of absolute prudence. When R is sufficiently low relative to P, it may be optimal for the principal to give larger rewards than punishments.[4] If there is a limited liability constraint, this may limit the principal's ability to punish and lead the principal to rely more heavily upon rewards to incentivize agents. The optimal size of rewards relative to punishments also depends upon the distribution of the shocks to agents' output. If the shock distribution is asymmetric ($F(-x) \neq 1 - F(x)$) in a manner to be defined below, it may be optimal to give large rewards relative to punishments. Finally, if the agents participating in the tournament are heterogeneous in a manner to be defined below, the principal may wish to give large rewards. These results speak to the importance of punishment as a tool to the principal and in what settings it might be expected to arise.

The paper will proceed as follows. Section 2 provides a brief review of the existing literature. Section 3 gives the basic setup of the model and states the problem of the principal designing the tournament. Section 4 establishes the main results of the paper (in four corollaries to Proposition 1), giving a partial characterization of the optimal prize schedule. Intuition for the results is provided in Section 4.1. Section 5 considers the case where the principal may offer only two prizes, providing further intuition and applications. Section 6 contains some concluding remarks.

2 Brief literature review

Since the seminal contributions of Lazear and Rosen (1981), Green and Stokey (1983) and Nalebuff and Stiglitz (1983) there has been a vast amount of research on labor market tournaments, as well as tournaments between firms such as R&D tournaments. For excellent overviews see Lazear (1991) and Prendergast (1999).

Our paper analyzes the optimal prize structure and the relative importance of rewards versus punishments in a framework which is essentially identical to Green and Stokey (1983). We are certainly not the first to consider optimal prize structures in tournaments. As long ago as 1902, Francis Galton addressed this question in two prize tournaments.[5] The most important and recent paper relating to ours is Moldovanu and Sela (2001). They consider a contest with multiple prizes where the players are privately informed about their ability and analyze optimal prize structures within the framework of private value all-pay auctions. This is formally similar to models analyzed by Weber (1985), Glazer and Hassin (1988), Hillman and Riley (1989), Baye et al. (1996), Krishna and Morgan (1997), Clark and Riis (1998a), and Barut and Kovenock (1998). Moldovanu and Sela (2001) analyze a model where risk-neutral players have different costs of exerting effort, which is private information. The contest designed seeks to maximize the sum of the efforts by determining the allocation of a fixed purse among the contestants. They show that if the contestants have linear or concave cost of effort functions then the optimal prize structure involves allocating

[4] The concept of absolute prudence is due to Kimball (1990) who analyzes its role on precautionary saving in a dynamic model. The relationship between risk aversion and absolute prudence has been explored in a variety of settings different from ours [see, for example, Carroll and Kimball (1996) and Caplin and Nalebuff (1991)].

[5] This is cited in Moldovanu and Sela (2001).

the entire prize to the first-place getter. With convex costs, entry fees, or minimum effort requirements, more prizes can be optimal.[6]

The central distinguishing feature of our approach is the focus on tournaments in a moral hazard context (LRGS tournaments), where the risk faced by agents arises from noise between effort and output.[7] In Moldovanu and Sela (2001), risk arises from an agent's uncertainty about her relative productivity. Krishna and Morgan (1998) also examine the LRGS context, but under somewhat restrictive assumptions: in particular, they assume a limited liability constraint but no participation constraint (which is equivalent in our framework to a limited liability constraint sufficiently strong that it causes the participation constraint to be non-binding.) They also restrict attention to tournaments with four or fewer players and assume that the total purse is fixed.

An early paper on prize structures in tournaments is O'Keeffe et al. (1984) which focuses on how to get contestants of unequal ability to compete in the "correct" tournament, and what prizes to use. Two other notable papers that relate to the appropriate use of tournaments and optimal design are Levin (2002), and Jaramillo (2004).

3 The model

3.1 Statement of the problem

Suppose there are n agents available to compete in a rank-order tournament. This tournament is set up by a principal whose goal is to maximize her expected profits. The principal pays a prize w_i to the agent who places ith in the tournament. The profits which accrue to the principal are equal to the sum of the outputs of the participating agents minus the amount she pays out: $\pi = \sum_{i=1}^{n}(q_i - w_i)$. We assume that the principal is risk-neutral. For now, we will assume that agents are homogeneous in ability. If agent j exerts effort e_j, her output is given by $q_j = e_j + \varepsilon_j + \eta$, where ε_j and η are random variables with mean zero and distributed according to distributions F and G, respectively. We assume that the ε_j's are independent of one another and η. We will refer to η as the "common shock" to output and ε_j as the "idiosyncratic shock" to output. Since rank-order tournaments filter out the noise created by common shocks but individual contracts do not, rank-order tournaments are considered most advantageous when common shocks are large.[8]

We will assume that agents have utility that is additively separable in wealth and effort. If agent j places ith in the tournament, her utility is given by: $u(w_i) - c(e_j)$ where $u' \geq 0, u'' \leq 0, c' \geq 0, c'' \geq 0$. Agents have an outside option which guaran-

[6] In a setting with identical players, Schottner (2008) shows that when entry fees cannot be charged a fixed-price tournament may dominate a first-price auction.

[7] A different strand of the literature analyzes tournaments through so-called "contest success functions", which specifies the probability of each player winning as a function of the vector of efforts of all players. See, for instance, Skaperdas (1996) and Clark and Riis (1998b).

[8] See Holmström (1982) for a definitive treatment of relative performance evaluation individual contracts. He shows that an appropriately structured individual contract with a relative performance component dominates a rank-order tournament for n finite. Green and Stokey (1983) prove convergence of optimal tournaments to the individual contract second-best as $n \to \infty$ when there are no common shocks.

tees them $\bar{U}$, so unless the expected utility from participation is at least equal to $\bar{U}$, agents will not be willing to participate. We also assume that agents must receive a wage of at least $\bar{w}$ (which we can think of as a limited liability constraint).

The timing of events is as follows. Time 1: the principal commits to a prize schedule $\{w_i\}_{i=1}^n$. Time 2: agents decide whether or not to participate. Time 3: individuals choose how much effort to exert. Time 4: output is realized and prizes are awarded according to the prize schedule set at time 1.

3.2 Solving the model

We will restrict attention to symmetric pure strategy equilibria (as do Green and Stokey 1983; Krishna and Morgan 1998). In a symmetric equilibrium, every agent will exert effort e^*. Furthermore, every agent has an equal chance of winning any prize. Thus, an agent's expected utility is

$$\frac{1}{n}\sum_i u(w_i) - c(e^*)$$

In order for it to be worthwhile for an agent to participate in the tournament, it is necessary that

$$\frac{1}{n}\sum_i u(w_i) - c(e^*) \geq \bar{U}$$

An agent who exerts effort e while everyone else exerts effort e^* receives expected utility

$$U(e, e^*) = \sum_i \varphi_i(e, e^*)u(w_i) - c(e)$$
$$\text{where } \varphi_i(e, e^*) = \Pr(i\text{th place}|e, e^*),$$

The problem faced by an agent is to choose e to maximize $U(e, e^*)$. The first-order condition for this problem is

$$c'(e) = \sum_i \frac{\partial}{\partial e}\varphi_i(e, e^*)u(w_i)$$

By assumption, the solution to the agent's maximization problem is $e = e^*$. If the first-order condition gives the solution to the agent's maximization problem, it follows that

$$c'(e^*) = \sum_i \beta_i u(w_i)$$
$$\text{where } \beta_i = \left.\frac{\partial}{\partial e}\varphi_i(e, e^*)\right|_{e=e^*}$$

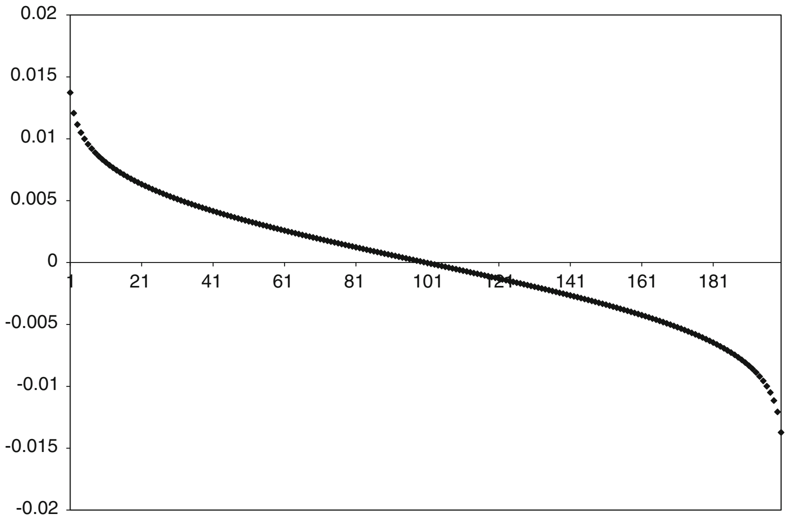

Fig. 1 Weights for the normal

We will often refer to the β_i's as "weights." The β_i's do not depend upon e^* but simply upon the noise distribution function F. Lemma 1 gives a formula for β_i and some additional properties.

Lemma 1 1. *The following is a formula for β_i as a function of F and the corresponding pdf, f:*

$$\beta_i = \binom{n-1}{i-1} \int\limits_{\mathbb{R}} F(x)^{n-i-1}(1-F(x))^{i-2}\left((n-i)-(n-1)F(x)\right) f(x)^2 \mathrm{d}x.$$

2. *For all F, $\sum_i \beta_i = 0$, $\beta_1 \geq 0$, and $\beta_n \leq 0$. If F is symmetric ($F(-x) = 1 - F(x)$), $\beta_i = -\beta_{n-i+1}$ for all i. 3. If F is a uniform distribution on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$, $\beta_1 = -\beta_n = \frac{1}{\sigma}$ and $\beta_i = 0$ for $1 < i < n$.*

Under special conditions that we will see in Sect. 4, the optimal prize schedule will be an affine transformation of the weight schedule. More generally, when there is no limited liability constraint, the optimal prize schedule will have a shape similar to the weight schedule.

Lemma 1 shows that the weight schedule for the uniform distribution is completely flat in the middle and spikes at the top and bottom. We find that many other distributions have weight schedules that are relatively flat in the middle and spike at the top and bottom. The normal distribution has this pattern. Figure 1 gives a plot of the weights for a normal distribution with standard deviation (SD) of 1 and $n = 200$.

While the weights associated with uniformly distributed and normally distributed noise are always decreasing in i, the weights need not be monotonic. When the noise distribution is not single-peaked, non-monotonicities tend to arise. It should be noted that, while the weights can be increasing in i over some range, the weights cannot be increasing over the entire range. As Lemma 1 shows, $\beta_1 > \beta_n$ unless $\beta_1 = \beta_n = 0$. As we will see in the next section, non-monotonicities in the weights lead to non-monotonicities in the optimal prize schedule.

In general, the agents' first-order condition may or may not give the solution to the agents' maximization problem.[9] In order for the first-order condition to give the solution, the second-order condition must be satisfied. Lemma 2 gives conditions under which the second-order condition will be satisfied at $e = e^*$.

Lemma 2 *Suppose that F is symmetric $(F(-x) = 1 - F(x))$, $u(w_i) - u(w_j) \leq u(w_{n-j+1}) - u(w_{n-i+1})$ for all $i \leq j \leq \frac{n+1}{2}$, and $\sum_{i=1}^{j} \gamma_i \geq 0$ for all $j \leq \frac{n}{2}$, where $\gamma_i = \frac{\partial^2}{\partial e^2}\varphi_i(e, e^*)\Big|_{e=e^*}$. Then, the agents' second-order condition is satisfied at $e = e^*$.*

The condition on the γ_i's holds when F is a uniform, normal, double exponential, or Cauchy distribution. In the next section, we will give conditions under which the principal will choose a prize schedule for which $u(w_i) - u(w_j) \leq u(w_{n-j+1}) - u(w_{n-i+1})$ for all $i \leq j \leq \frac{n+1}{2}$ when agents act according to the first-order condition.

Now that we have elaborated the agents' problem, we turn to the principal's problem. We have assumed that the principal is risk neutral. This implies that the principal's objective is to maximize expected profits

$$E(\pi) = \sum_j e_j - \sum_i w_i = n\left(e^* - \frac{1}{n}\sum_i w_i\right).$$

When the agents' first-order condition is equivalent to the agents' incentive compatibility constraint, the problem of the principal can be stated as follows:

$$\max_{w_i}\left(e^* - \frac{1}{n}\sum_i w_i\right)$$

subject to

$$\frac{1}{n}\sum_i u(w_i) - c(e^*) \geq \bar{U} \tag{IR}$$

$$c'(e^*) = \sum_i \beta_i u(w_i) \tag{IC}$$

$$w_i \geq \bar{w} \quad \text{for all } i \tag{LL}$$

Substituting $(c')^{-1}\left(\sum_i \beta_i u(w_i)\right)$ for e^*, and $u^{-1}(u_i)$ for w_i, we can rewrite the principal's problem as:

[9] One way to ensure that the first-order condition and the IC constraint are equivalent is to make a particular assumption on the agent's utility function and assume that the parameterized distributions of output are: (a) quasiconvex and (b) have the Monotone Likelihood Ratio Property (Jewitt 1988, Theorem 3). The assumption on the utility function requires that u is a concave transformation of $1/u'$.

$$\max_{u_i}\left((c')^{-1}\left(\sum_i \beta_i u(w_i)\right)-\frac{1}{n}\sum_i u^{-1}(u_i)\right)$$
subject to
$$\bar{U}-\frac{1}{n}\sum_i u_i+(c')^{-1}\left(\sum_i \beta_i u(w_i)\right)\leq 0 \quad \text{and} \quad u^{-1}(u_i)\geq u^{-1}(\bar{w})$$

The Lagrangian associated with this maximization problem is:

$$\begin{aligned}\mathcal{L}=&\left((c')^{-1}\left(\sum_i \beta_i u_i\right)-\frac{1}{n}\sum_i u^{-1}(u_i)\right)\\ &-\lambda\left(\bar{U}-\frac{1}{n}\sum_i u_i+c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)\right)\\ &-\sum_i \mu_i\left(u^{-1}(u_i)-u^{-1}(\bar{w})\right)\end{aligned}$$

Just as the agents' first-order condition does not necessarily solve the agents' maximization problem, the first-order conditions of the Lagrangian may not solve the principal's maximization problem. The following Lemma gives a condition under which the principal will act according to the first-order conditions of the Lagrangian.

Lemma 3 *If* $c''' \leq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, *and* $(u_1, \ldots, u_n, \lambda, \mu_1, \ldots, \mu_n)$ *satisfies the Kuhn–Tucker conditions of* $\mathcal{L}$, $(u_1, \ldots, u_n)$ *solves the principal's problem.*

These conditions on the cost of effort function are somewhat restrictive, but they do hold for all functions of the form $c(e) = de^{\alpha}$ for which $\alpha \geq 2$.

4 The optimal prize schedule

We will now give a partial characterization of the principal's optimal prize schedule. We will identify three important determinants of the optimal prize schedule: (1) the size of R relative to P (R is risk aversion and P is absolute prudence), (2) the size of $\bar{w}$ (the minimum prize that can be awarded), and (3) the shape of the noise distribution F. In what we will think of as a base case, in which $R \geq \frac{P}{2}$, F is symmetric, and the limited liability constraint is non-binding, the rewards given at the top of the prize schedule are smaller than the punishments given at the bottom of the prize schedule. It might be optimal to give larger rewards than punishments if R is low relative to P, F is asymmetric, or the limited liability constraint is binding. We will develop an intuition for these results below.

The main results of this section follow from Proposition 1. However, it may not be immediately clear to readers what the implications of the proposition are. Corollaries 1–4 develop the main implications of the proposition.

The first-order conditions of the Lagrangian lead to the following lemma, which tells us a great deal about the optimal prize schedule.

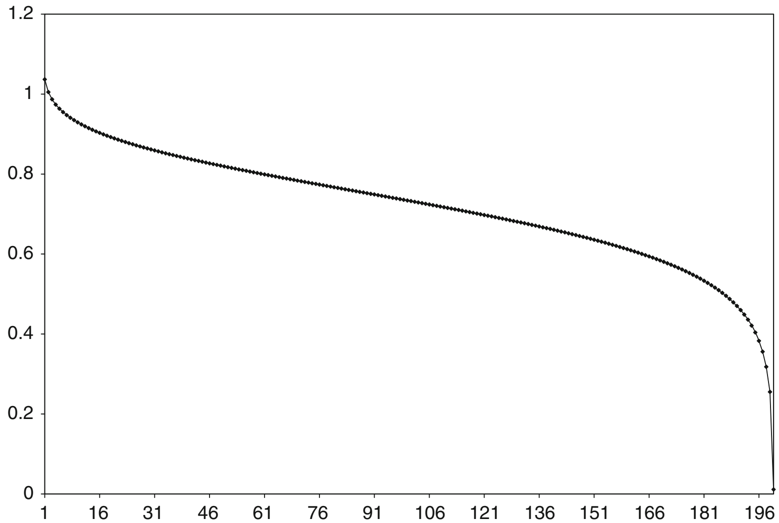

Fig. 2 Optimal prize schedule

Lemma 4 *Suppose* $w^* = (w_1^*, \ldots w_n^*)$ *is the optimal prize schedule and let* $v_i = u'(w_i^*)$. *If the agents act according to their first-order condition,* $c''' \geq 0$, *and* $\frac{c''}{c'} \geq \frac{c'''}{c''}$, *then* $\frac{\frac{1}{v_i} - \frac{1}{v_{i+k}}}{\frac{1}{v_j} - \frac{1}{v_{j+l}}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$ *whenever* $w_i, w_j, w_k, w_l > \bar{w}$.

Proposition 1 follows directly from Lemma 4, and relates the slope of the prize schedule to the slope of the weight schedule. What we will find is that, under the special condition that u is logarithmic and the limited liability constraint is non-binding, Lemma 4 implies that the optimal prize schedule is simply an affine transformation of the weight schedule.

What we find more generally is that the optimal prize schedule tends to look similar to an affine transformation of the weight schedule when the limited liability constraint is non-binding. When R is large relative to P, the optimal prize schedule differs from an affine transformation of the weight schedule in that the prizes at the top are revised in the direction of the median prize while the prizes at the bottom are revised in the opposite direction from the median prize. When R is small relative to P , the optimal prize schedule differs from an affine transformation of the weight schedule in that the prizes at the bottom are revised in the direction of the median prize while the prizes at the bottom are revised in the opposite direction from the median prize.

We see this in comparing the prize schedule in Fig. 2 (a case where R is large relative to P) to the corresponding weights shown in Fig. 1. Figure 2 shows the prize schedule in money (as opposed to utils) in the case where $n = 200$, F is a normal distribution with SD 1, $c(e) = \frac{e^2}{2}$, the utility function is CRRA with $\theta = 2$, and there is no limited liability constraint. We observe that the shape of the prize schedule is similar to the shape of the weight schedule in Fig. 1 but the prizes for the best performers are revised in the direction of the median prize and the prizes for the worst performers are revised in the opposite direction.

Proposition 1 is as follows.

Proposition 1 *Suppose* $\min(w_i^*, w_{i+k}^*, w_j^*, w_{j+l}^*) > \bar{w}$ *and* $\min(i, i+k) \geq \max(j, j+l)$ *(k and l can be positive or negative). Suppose further that* $\beta_i - \beta_{i+k} \geq 0$ *and* $\beta_j - \beta_{j+l} \geq 0$. *Let* $R = -\frac{u''}{u'}$ *denote the Arrow-Pratt measure of risk aversion. Let* $P = -\frac{u'''}{u''}$ *denote the coefficient of absolute prudence. Suppose* $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, *and the agents act according to their first-order condition.*

(i) *If* $R \geq \frac{P}{2}$:

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*}$$
$$\leq \left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

(ii) *If* $R \leq \frac{P}{2}$:

$$\left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*}$$
$$\leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

(iii) *Let* $u_i^* = u(w_i^*)$. *If* $R \geq \frac{P}{3}$:

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u_i^* - u_{i+k}^*}{u_j^* - u_{j+l}^*}$$
$$\leq \left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

(iv) *If* $R \leq \frac{P}{3}$:

$$\left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u_i^* - u_{i+k}^*}{u_j^* - u_{j+l}^*}$$
$$\leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

When u is logarithmic, $R = \frac{P}{2}$. Proposition 1 implies that $\frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$, which means the optimal prize schedule is an affine transformation of the weight schedule. Corollary 1 states this precisely.

Corollary 1 *(1) If $u(w) = \log(w)$ (in which case $R = \frac{P}{2}$), $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and the agents act according to their first-order condition, then $\frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$ whenever $w_i, w_j, w_k, w_l > \bar{w}$. If the limited liability constraint does not bind, the vector $w^* = (w_1^*, \ldots, w_n^*)$ is an affine transformation of the vector $\beta = (\beta_1, \ldots, \beta_n)$. (2) If $u(w) = w^{1/2}$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and the agents act according to their first-order condition, then $\frac{u_i^* - u_{i+k}^*}{u_j^* - u_{j+l}^*} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$ whenever $w_i, w_j, w_k, w_l > \bar{w}$ where $u_i^* = u(w_i^*)$. If the limited liability constraint does not bind, the vector $u^* = (u_1^*, \ldots, u_n^*)$ is an affine transformation of the vector $\beta = (\beta_1, \ldots, \beta_n)$.*

Proposition 1 allows us to compare the size of rewards at the top of the optimal prize distribution to the size of punishments at the bottom of the optimal prize distribution (therefore, the size of $w_i^* - w_{i+1}^*$ relative to $w_{n-i}^* - w_{n-i+1}^*$, $i \geq \frac{n+1}{2}$). In particular, when F is symmetric, $R \geq \frac{P}{2}$, and there is no limited liability constraint the size of punishments inflicted at the bottom of the prize schedule ($w_i^* - w_{i+1}^*$, $i \geq \frac{n+1}{2}$) will be larger than corresponding rewards at the top of the prize schedule ($w_{n-i}^* - w_{n-i+1}^*$). When F is symmetric, $R \leq \frac{P}{2}$, and there is no limited liability constraint, the size of punishments inflicted at the bottom of the prize schedule ($w_i^* - w_{i+1}^*$, $i \geq \frac{n+1}{2}$) will be smaller than corresponding rewards at the top of the prize schedule ($w_{n-i}^* - w_{n-i+1}^*$). In Fig. 2, for example, (a case where $R \geq \frac{P}{2}$ and F is symmetric) we see that the rewards at the top of the prize schedule are small compared to the punishments at the bottom.

Corollary 2 states this point more formally, giving conditions when $r_i = \frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*}$ will be greater than or less than 1. Observe that $r_i \geq 1$ for all $i \geq \frac{n+1}{2}$ means that punishments are larger than corresponding rewards and $r_i \leq 1$ for all $i \geq \frac{n+1}{2}$ means that punishments are smaller than corresponding rewards. Corollary 2 also makes conclusions about how $r_i = \frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*}$ changes as a function of i.

Corollary 2 *Let $r_i = \frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*}$ and $q_i = \frac{u_i^* - u_{i+1}^*}{u_{n-i}^* - u_{n-i+1}^*}$. Suppose F is symmetric, $\{\beta_i\}$ is decreasing in i, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and agents act according to their first-order condition. Let $m = \max\{j : w_j^* > \bar{w}\} \cup \{0\}$ (m is the highest integer for which $w_j^* > \bar{w}$ or 0 if $w_1^* = \bar{w}$).*

(i) If $R \geq \frac{P}{2}$: $r_i \geq 1$ for $m > i \geq \frac{n+1}{2}$, and $r_{i+1} \geq r_i$ for $m > i \geq n - m + 1$. (ii) If $R \leq \frac{P}{2}$: $r_i \leq 1$ for $m > i \geq \frac{n+1}{2}$, and $r_{i+1} \leq r_i$ for $m > i \geq n - m + 1$. (iii) If $R \geq \frac{P}{3}$: $q_i \geq 1$ for $m > i \geq \frac{n+1}{2}$, and $q_{i+1} \geq q_i$ for $m > i \geq n - m + 1$. (iv) If $R \leq \frac{P}{3}$: $q_i \leq 1$ for $m > i \geq \frac{n+1}{2}$, and $q_{i+1} \leq q_i$ for $m > i \geq n - m + 1$.

It follows from Corollary 2 that when $R \geq \frac{P}{2}$, F is symmetric, and there is no limited liability constraint, $r_i \geq 1$ for $i \geq \frac{n+1}{2}$ and r_i is increasing in i. These conditions

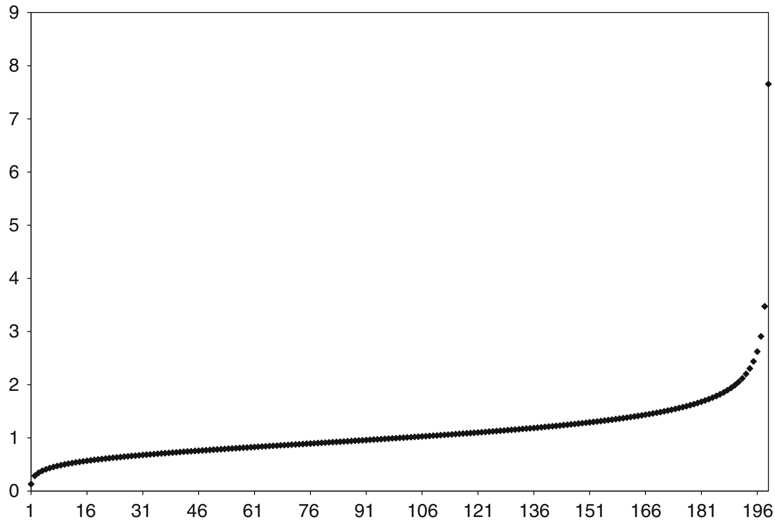

Fig. 3 r_i

hold for the prize schedule in Fig. 2. Figure 3 plots the ratios r_i corresponding the prize schedule in Fig. 2.

$R \geq \frac{P}{2}$ for a large class of utility functions. For this reason, we think of this as the "base case." $R \geq \frac{P}{2}$ for all CARA utility functions and CRRA utility functions with $\theta \geq 1. R \geq \frac{P}{3}$ for all CARA utility functions and CRRA utility functions with $\theta \geq \frac{1}{2}$. $R \leq \frac{P}{3}$ for CRRA utility functions with $\theta \leq \frac{1}{2}$, and $R \leq \frac{P}{2}$ for CRRA utility functions with $\theta \leq 1$.

Another conclusion that can be drawn from Proposition 1 is that, when the limited liability constraint does not bind, the optimal prize schedule will be relatively flat in the middle and spike at the top and bottom when the weight schedule has this shape. Many common distributions, such as the normal distribution, result in weight schedules with this shape. In particular, when noise is uniformly distributed, we find that there is a special prize for first place, a special punishment for last place, and a single prize for everyone else (the prize schedule is perfectly flat in the middle).

Corollary 3 *If F is uniformly distributed, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and agents act according to their first-order condition:*

$$w_i^* = w_j^*, \quad 1 < i, j < n$$

Limited liability

Proposition 1 suggests that a binding limited liability constraint reduces the size of the punishment at the bottom of the prize schedule relative to the reward at the top.

Consider an example. Suppose $u(w)$ is logarithmic and F is uniform. Corollary 1 tells us that if there is no limited liability constraint, the optimal prize schedule has the form: $w_i = w$ for $1 < i < n$, $w_1 = w + \phi$, and $w_n = w - \phi$. In this case, the punishment at the bottom is the same as the reward at the top.

But, suppose the limited liability constraint binds: $w - \phi < \bar{w}$. In this case, the optimal prize schedule may give a larger reward at the top than punishment at the bottom: $w_i = w'$ for $1 < i < n$, $w_1 = w' + \phi_1$, and $w_n = w' - \phi_2$ with $\phi_1 > \phi_2$.

The agents' second-order condition

In the previous section, Lemma 2 gave a condition on the prize schedule under which the agents' second-order condition will hold for certain F at $e = e^*$. The following corollary to Proposition 1 gives us conditions under which the principal will choose a prize schedule that meets the condition of Lemma 2.

Corollary 4 *Suppose the limited liability constraint is non-binding. If F is symmetric, $\{\beta_i\}$ is decreasing in i, $R \geq \frac{P}{3}$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and agents act according to their first-order condition:*

$$u(w_i^*) - u(w_j^*) \leq u(w_{n-j+1}^*) - u(w_{n-i+1}^*) \quad \textit{if } i \leq j \leq \frac{n+1}{2}$$

Therefore, when the principal assumes that agents act according to the first-order condition, F is symmetric, $\{\beta_i\}$ is decreasing in i, $\sum_{i=1}^{j} \gamma_i \geq 0$ for $j \leq \frac{n}{2}$, $R \geq \frac{P}{3}$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and the limited liability constraint is non-binding, the principal will choose a prize schedule that satisfies the agent's second-order condition at $e = e^*$.

4.1 Some intuition for the results

We have identified three key determinants of the optimal prize schedule: the size of R relative to P, the size of the minimum prize $\bar{w}$, and the noise distribution F. Let us consider the reasons why these are important determinants.

(1) The size of R relative to P

The size of R relative to P determines how the principal balances two considerations in the choice of the optimal prize schedule.

One consideration is how risk aversion, R, changes with wealth. If agents become less risk averse as they become wealthier, they are less averse to upside risk than they are to downside risk. This is a reason to give larger rewards to top performers than punishments to poor performers.

A second consideration is how quickly the marginal utility of wealth declines. When the marginal utility of wealth declines quickly (u'' low), it is necessary to give much larger monetary rewards to top performers to induce effort than punishments to poor performers. This inclines the principal to give larger punishments than rewards.

The larger R is relative to P, the more important the second consideration is to the principal relative to the first.

(2) Limited liability

A limited liability constraint decreases a principal's ability to punish poor performers. As the limited liability constraint becomes more severe (w increases), the principal becomes increasingly inclined to rely on rewards rather than punishments as a means of incentivizing agents. In some instances, a limited liability constraint might yield a winner-take-all tournament in which $w_i = \bar{w}$ for $1 \leq i < n$ and $w_n > \bar{w}$.

If the limited liability constraint makes the ex-ante participation constraint non-binding, this is equivalent to the Krishna and Morgan case. They find that, in this case, it is generally optimal for the principal to implement a winner-take-all tournament.

It should be noted that, in the absence of a limited liability constraint, in cases where $R \geq \frac{P}{2}$ and F is symmetric, punishments for losers are typically not exorbitant. Thus it is possible to imagine cases in which a limited liability constraint might be non-binding.

(3) The noise distribution

Corollary 3 shows that, when u is logarithmic, the optimal prize schedule is an affine transformation of the weights, β_i. When F is symmetric, $\beta_i = -\beta_{n-i+1}$. This means that, when F is symmetric, the optimal prize schedule rewards winners and punishes losers equally.

But, when F is asymmetric, β_i may be larger or smaller than $-\beta_{n-i+1}$. There are distributions F for which the weight schedule, and hence the prize schedule, is steep for low i and flat for high i. The prize schedule in this case clearly rewards winners more than it punishes losers.

Why does a weight schedule that is flat at the bottom lead to a prize schedule that is flat at the bottom? Suppose, for the sake of argument, that $\beta_{n-1} = \beta_n$. What this says is that a marginal change in agent effort does not affect the probability of placing $(n-1)$th relative to nth. Therefore, placing nth rather than $(n-1)$th is a matter of luck rather than effort. In punishing agents for placing nth rather than $(n-1)$th, the principal gives a reward for luck without giving a reward for effort. Since agents are risk averse, it is costly to the principal to reward luck. Therefore, it does not make sense for the principal to reward agents for placing nth rather than $(n-1)$th. So, $w^*_{n-1} = w^*_n$. If, in contrast, $\beta_{n-1} > \beta_n$, punishing nth place relative to $(n-1)$th place rewards effort as well as luck. So, it makes sense for the principal to punish nth place in this case.

It should be noted that there are asymmetric F that produce weight schedules that are steeper for high i than for low i. Such F lead to prize schedules that reward winners less than they punish losers. Therefore, asymmetry of the noise distribution can lead to more or less reward for winners depending upon the particular type of asymmetry.

4.2 The effect of agent heterogeneity

Agent heterogeneity can have an effect on the optimal size of rewards relative to punishments. Whether heterogeneity increases rewards relative to punishments, decreases rewards relative to punishments, or is neutral depends, however, on the exact type of heterogeneity that exists. There are two leading cases: *additive heterogeneity* where agent i's output is given by $q_i = e_i + \theta_i + \varepsilon_i + \eta$, where θ_i is agent i's type, ε_i is idiosyncratic noise, and η is a common shock to output, and *multiplicative heterogeneity* where agent i's output is given by $q_i = \theta_i e_i + \varepsilon_i + \eta$. In supplementary online material, we explore this issue in detail,[10] but particularly in the case of multiplicative heterogeneity large rewards can be optimal. In that setting, since high θ agents are more productive than low θ agents, the principal cares more about inducing effort among high θ agents than low θ agents and high θ agents have a low probability of placing at the bottom of the tournament. Therefore, high θ agents are given a greater incentive to exert effort by rewards than by punishments and low θ agents are given a greater incentive to exert effort by punishments than by rewards. Moreover, the effort of high θ types is more valuable to the principal than the effort of low θ types. This gives the principal a strong reason to rely more upon rewarding winners than punishing losers in the multiplicative case.

5 Two-prize tournaments

In the previous section, we found that when R is large relative to P, F is symmetric, and there is no limited liability constraint, the principal relies more heavily on punishment than on reward. To examine how important punishments are relative to rewards, we will consider what happens when the principal is limited to using just two prizes. That is, suppose she can only give a prize w_1 to the top j performers and a prize w_2 to the bottom $n-j$ performers. When the principal is restricted in this way, where would she like to set j? One possibility would be to set $j=\frac{n}{2}$, so that the top half earns one prize and the bottom half earns another. Another possibility would be to set $j=1$, which gives a special prize to the best performer. The opposite would be to set $j=n-1$, so that there is a special punishment in store for the worst performer.

We will find that, when $R \geq \frac{P}{3}$ and F is symmetric, it is always optimal to set $j \geq \frac{n}{2}$. We also identify conditions for which it is optimal to set $j=n-1$, giving a special punishment to the worst performer. This is somewhat indicative of the importance of punishments to the principal relative to the importance or rewards.

Definition 1 We will call a tournament a "j tournament" when the principal pays a prize w_1 to the top j performers and a prize w_2 to the bottom $n-j$ performers. Let $u_1=u(w_1)$ and $u_2=u(w_2)$. We will call a tournament a "winner-prize tournament" if $j \leq \frac{n}{2}$ and a "strict winner-prize tournament" if $j=1$. We will call a tournament a "loser-prize tournament" is $j \geq \frac{n}{2}$ and a "strict loser-prize tournament" if $j=n-1$.

We will consider when the principal prefers to implement a loser-prize tournament rather than a winner-prize tournament. To answer this question, we will compare a j tournament and an $n-j$ tournament that induce the same level of effort and both meet the individual rationality constraint. It will be shown that, when R is large relative to P, F is symmetric, and $j \leq \frac{n}{2}$, the payment made to agents by the principal is greater when she uses the j tournament. When R is small relative to P, F is symmetric, and $j \leq \frac{n}{2}$, the payment made to agents by the principal is smaller when she uses the j tournament.

[10] See http://www.mit.edu/~rholden/Papers.htm.

First, we must know when a j tournament and an $n - j$ tournament induce the same effort. The following corollary of Lemma 1 provides the answer.

Corollary 5 *If F is symmetric and agents act according to the first-order condition, a j tournament and an $n - j$ tournament for which $u_1 - u_2$ is the same induce the same level of effort. This level of effort is given by*

$$c'(e) = \left(\sum_{i=1}^{j} \beta_i \right) (u_1 - u_2)$$

Using this corollary, we will now establish the main result of this section.

Proposition 2 *Suppose the principal is restricted to use a j tournament (but has a choice over w_1 and w_2), that the principal is restricted to implementing a tournament that induces effort level e, and that there is no limited liability constraint. Let π_j denote the expected profits from the optimal choice of w_1 and w_2. Suppose further that F is symmetric and agents act according to the first-order condition.*

(i) *If $R \geq \frac{P}{3}$,*

$$\pi_j \leq \pi_{n-j} \quad \text{for } j \leq \frac{n}{2}$$

(ii) *If $R \leq \frac{P}{3}$,*

$$\pi_j \geq \pi_{n-j} \quad \text{for } j \leq \frac{n}{2}$$

The following is an immediate corollary.

Corollary 6 *Suppose the principal is restricted to implementing a j tournament, but can choose whatever j she likes. Suppose F is symmetric, agents act according to the first-order condition, and there is no limited liability constraint. If u satisfies $R \geq \frac{P}{3}$, then the optimal j tournament is a loser-prize tournament (a tournament with $j \geq \frac{n}{2}$). If u satisfies $R \leq \frac{P}{3}$, then the optimal j tournament is a winner-prize tournament (a tournament with $j \leq \frac{n}{2}$).*

So far, we have given conditions under which the optimal two-prize tournament is a loser-prize tournament. We can go further and make comparisons between loser-prize tournaments when we assume that the idiosyncratic noise distribution is uniform.

Proposition 3 *Suppose the principal is restricted to use a j tournament, F is a symmetric uniform distribution, agents act according to the first-order condition, and there is no limited liability constraint. If u satisfies $R \geq \frac{P}{3}$, the optimal j tournament is the strict loser-prize tournament. If u satisfies $R \leq \frac{P}{3}$, the optimal j tournament is the strict winner-prize tournament.*

When the noise distribution is not uniform, the optimal j depends upon the utility function as well as the distributional weights. However, as mentioned above, many distributions (including the normal distribution) have weight schedules that are similar to the uniform distribution: they are relatively flat for $1 < i < n$ and spike at the top and bottom. The strict loser-prize tournament tends to be optimal when $R > \frac{P}{3}$ and the noise distribution has weights that look similar to those of a uniform distribution. In the numerical examples that we have considered, we have generally found $j = n - 1$ to be the optimal two-prize tournament when F is normal and $R > \frac{P}{3}$.[11]

6 Concluding remarks

This paper gives a framework and an intuition for thinking about how prizes should be structured in rank-order tournaments created to deal with moral hazard.

We identify four key determinants of the optimal tournament prize structure: the size of R relative to P, limited liability, the noise distribution, and agent heterogeneity. We find, in particular, that rewards for the best performers tend to be smaller than punishments for the worst performers when R is large relative to P, there is no limited liability constraint, the noise distribution is symmetric, and agents are homogeneous. Larger rewards for the best performers might be optimal when R is small relative to P, there is limited liability, the noise distribution is asymmetric, or agents are heterogeneous.

These results allow us to test whether aspects of employee compensation are explained by the moral hazard theory of tournaments or arise for other reasons. Within-firm job promotions, wage increases, bonuses, and CEO compensation have often been interpreted as prizes for top performers in Lazear–Rosen rank-order tournaments. Our results, for example, cast some doubt on the idea that tournaments that reward winners without punishing losers exist *purely* to solve a moral hazard problem.

The key determinants of the optimal tournament prize structure identified in this paper (the size of R relative to P, limited liability, the noise distribution, and agent heterogeneity) are also key determinants of the optimal individual contract. Indeed, by Green and Stokey (1983, Theorem 3), as the number of players in the tournament grows large, the two reward schedules converge. This is a topic we address in other work.

[11] Our results in this section do not give a sense of how much the choice of j matters to the principal's profits. In a case where $j = n - 1$ is optimal, we would like to know how much worse off the principal would be if she chose $j = 1$ instead. We have looked at numerical examples in order to get a sense of the magnitude of the loss. The numerical examples we have considered suggest that the profits from the optimal j tournament are generally close to the profits from the optimal n prize tournament. The induced effort level is also similar. However, we find that the choice of j matters a great deal. When j is not chosen optimally, the principal's profit may be quite far from the profit from the optimal j tournament and the profit from the optimal n-prize tournament. Since $j = n - 1$ is often the optimal j when $R > \frac{P}{3}$, we find that there are many cases where the optimal $j = n - 1$ tournament closely approximates the optimal n prize tournament while the optimal $j = 1$ tournament returns a profit that is markedly worse. Therefore, in many cases, punishing the worst performer is the most important incentive the principal has at her disposal.

7 Appendix

Proof of Lemma 1

$$\varphi_i(e,e^*) = \Pr(i\text{th place}|e,e^*) = \int_{\mathbb{R}} \binom{n-1}{i-1} \left(F(e-e^*+x)\right)^{n-i} \times \left(1-F(e-e^*+x)\right)^{i-1} f(x)\mathrm{d}x$$

$$\frac{\partial}{\partial e}\varphi_i(e,e^*) = \int_{\mathbb{R}} \binom{n-1}{i-1} \left(F(e-e^*+x)\right)^{n-i-1} \left(1-F(e-e^*+x)\right)^{i-2} \times \left[(n-i)-(n-1)\left(F(e-e^*+x)\right)\right] f(x)f(e-e^*+x)\mathrm{d}x$$

$$\beta_i = \frac{\partial}{\partial e}\varphi_i(e,e^*)\Big|_{e=e^*} = \binom{n-1}{i-1}\int_{\mathbb{R}} F(x)^{n-i-1}(1-F(x))^{i-2} \times ((n-i)-(n-1)F(x))\, f(x)^2\mathrm{d}x$$

Since $\sum_{i=1}^n \varphi_i(e,e^*) = 1, \sum_{i=1}^n \frac{\partial}{\partial e}\varphi_i(e,e^*) = 0$. Hence, $\sum_{i=1}^n \beta_i = \sum_{i=1}^n \frac{\partial}{\partial e}\varphi_i(e,e^*)\big|_{e=e^*} = 0$. $\beta_1 = (n-1)\int_{\mathbb{R}} F(x)^{n-2} f(x)^2\mathrm{d}x \geq 0$ and $\beta_n = -(n-1)\int_{\mathbb{R}}(1-F(x))^{n-2}f(x)^2\mathrm{d}x \leq 0$. If F is symmetric:

$$\begin{aligned}\beta_{n-i+1} &= \binom{n-1}{n-i}\int_{\mathbb{R}} F(x)^{i-2}(1-F(x))^{n-i-1}\left((i-1)-(n-1)F(x)\right) f(x)^2\mathrm{d}x\\ &= -\binom{n-1}{i-1}\int_{\mathbb{R}}(1-F(x))^{i-2}F(x)^{n-i-1}\left((n-i)-(n-1)F(x)\right) f(x)^2\mathrm{d}x\\ &= -\beta_i\end{aligned}$$

Hence, $\beta_{n-i+1} = -\beta_i$ for F symmetric. Suppose F is uniform on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$. It follows from the formula for β_i that $\beta_1 = -\beta_n = \frac{1}{\sigma}$ and $\beta_i = 0, 1 < i < n$. □

Proof of Lemma 2 Suppose that F is symmetric and $u(w_i) - u(w_j) \leq u(w_{n-j+1}) - u(w_{n-i+1})$ for all $i \leq j \leq \frac{n+1}{2}$. The second-order condition of the agent's problem is:

$$\sum_{i=1}^n \frac{\partial^2}{\partial e^2}\varphi_i(e,e^*)u(w_i) - c''(e) \leq 0.$$

Since $c'' \leq 0$ by assumption, the second-order condition will hold at $e = e^*$ if: $\sum_{i=1}^n \gamma_i u(w_i) \leq 0$, where $\gamma_i = \frac{\partial^2}{\partial e^2}\varphi_i(e,e^*)\Big|_{e=e^*}$. In the proof of Lemma 1, a formula was given for $\frac{\partial}{\partial e}\varphi_i(e,e^*)$. Differentiating this w.r.t. e yields

$$\begin{aligned}\gamma_i &= \frac{\partial^2}{\partial e^2}\varphi_i(e,e^*)\Big|_{e=e^*}\\ &= \binom{n-1}{i-1}\int_{\mathbb{R}} (F(x))^{n-i-2}\,(1-F(x))^{i-3}\\ &\quad\times\left[\begin{array}{c}(n-i)(n-i-1)\\ -2(n-i)(n-2)F(x)+(n-1)(n-2)F^2(x)\end{array}\right] f^3(x)\mathrm{d}x\\ &\quad+\binom{n-1}{i-1}\int_{\mathbb{R}} F(x)^{n-i-1}\,(1-F(x))^{i-2}\,[(n-i)-(n-1)F(x)]\,f(x)f'(x)\mathrm{d}x\end{aligned}$$

Since $\sum_{i=1}^{n}\varphi_i(e,e^*)=1$, $\sum_{i=1}^{n}\frac{\partial^2}{\partial e^2}\varphi_i(e,e^*)\Big|_{e=e^*}=0$ and $\sum_{i=1}^{n}\gamma_i=0$. The formula implies that when F is symmetric, $\gamma_{n-i+1}=\gamma_i$.

Let us define γ_i' as follows. $\gamma_i'=\gamma_i$ for $i\neq\frac{n+1}{2}$. If $i=\frac{n+1}{2}$, $\gamma_i'=\frac{1}{2}\gamma_i$. Notice that, since $\gamma_i=\gamma_{n-i+1}$, $\sum_{i=1}^{\lceil n/2\rceil}\gamma_i'=\frac{1}{2}\sum_{i=1}^{n}\gamma_i=0$.

Let $P=\{i\le\lceil n/2\rceil\,|\gamma_i'\ge 0\}$ and $N=\{i\le\lceil n/2\rceil\,|\gamma_i'<0\}$. By assumption $\sum_{i=1}^{j}\gamma_i\ge 0$ for all $j\le\frac{n}{2}$. Furthermore, $\sum_{i=1}^{\lceil n/2\rceil}\gamma_i'=0$. As a result, for $i\in P$ it is possible to write γ_i' as $\gamma_i'=-\sum_{k\in N}\delta_{ik}\gamma_k'$, where $\delta_{ik}\ge 0$ for $i>k$, $\delta_{ik}=0$ for $k<i$, and $\sum_{i\in P}\delta_{ik}=1$ for all $k\in\mathbb{N}$.

$$\begin{aligned}\sum_{i=1}^{n}\gamma_i u(w_i) &= \sum_{i=1}^{\lceil n/2\rceil}\gamma_i'(u(w_i)+u(w_{n-i+1}))\\ &= \sum_{i\in P}\gamma_i'(u(w_i)+u(w_{n-i+1}))+\sum_{k\in N}\gamma_k'(u(w_k)+u(w_{n-k+1}))\\ &= -\sum_{i\in P}\sum_{k\in N}\delta_{ik}\gamma_k'(u(w_i)+u(w_{n-i+1}))+\sum_{i\in N}\gamma_i'(u(w_i)+u(w_{n-i+1}))\\ &= \sum_{k\in N}(-\gamma_k')\left(\sum_{i\in P}\delta_{ik}(u(w_i)+u(w_{n-i+1}))-(u(w_k)+u(w_{n-k+1})\right)\end{aligned}$$

For $i\le k\le\frac{n+1}{2}$, $u(w_i)-u(w_k)\le u(w_{n-k+1})-u(w_{n-i+1})$ and $u(w_i)+u(w_{n-i+1})\le u(w_k)+u(w_{n-k+1})$. Since $i\le k\le\frac{n+1}{2}$ when $\delta_{ik}>0$, it follows that

$$\sum_{k\in N}(-\gamma_k)\left(\sum_{i\in P}\delta_{ik}(u(w_i)+u(w_{n-i+1}))-(u(w_k)+u(w_{n-k+1})\right)\le 0.$$

Therefore, $\sum_{i=1}^{n}\gamma_i u(w_i)\le 0$. Hence, the second-order condition holds at $e=e^*$. □

Proof of Lemma 3 Let $s(u_1,\ldots,u_n)=(c')^{-1}\left(\sum_i \beta_i u_i\right)-\frac{1}{n}\sum_i u^{-1}(u_i)$

Since $u''\le 0$ and $u'\ge 0$, $(u^{-1})''(x)=\frac{-u''(u^{-1}(x))}{(u'(u^{-1}(x)))^3}\ge 0$. Therefore, $-\frac{1}{n}\sum_i u^{-1}(u_i)$ is concave. Since $\sum_i \beta_i u_i$ is linear in u_i, $(c')^{-1}\left(\sum_i \beta_i u_i\right)$ is concave if $(c')^{-1}$ is concave. For simplicity of notation, let $z_1(x)=(c')^{-1}(x)$. $z_1''(x)=\frac{-c'''((c')^{-1}(x))}{\left(c''((c')^{-1}(x))\right)^3}$. Therefore, $(c')^{-1}$ is concave if and only if $\frac{c'''}{c''}\ge 0$. Since $c'', c'''\ge 0$, $(c')^{-1}$ is indeed concave. Since $(c')^{-1}\left(\sum_i \beta_i u_i\right)$ and $-\frac{1}{n}\sum_i u^{-1}(u_i)$ are both concave, s is a concave function.

Let $q(u_1,\ldots,u_n)=-\frac{1}{n}\sum_i u_i+c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)+\bar{U}-\frac{1}{n}\sum_i u_i+\bar{U}$ is linear, and therefore both concave and convex. $c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$ will be convex if $c\left((c')^{-1}(x)\right)$ is convex since $\sum_i \beta_i u_i$ is linear. For simplicity of notation, let $z_2(x)=c\left((c')^{-1}(x)\right)$. $z_2''(x)=\frac{\left(c''((c')^{-1}(x))\right)^2-\left(c'((c')^{-1}(x))\right)\left(c'''((c')^{-1}(x))\right)}{\left(c''((c')^{-1}(x))\right)^3}$. Since $-\frac{c'''}{c''}\ge-\frac{c''}{c'}$, it follows that $z_2''\ge 0$. Hence, $c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$ is convex. Since $-\frac{1}{n}\sum_i u_i+\bar{U}$ and $c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$ are both convex, q is convex.

Let $l_i(u_1,\ldots,u_n)=u^{-1}(u_i)-u^{-1}(\bar{w})$. From our previous analysis, it is immediately clear that l_i is convex. Since s is concave, and q and l_i are convex, the Kuhn-Tucker conditions are met. □

Proof of Lemma 4

$$\mathcal{L}=\left((c')^{-1}\left(\sum_i \beta_i u_i\right)-\frac{1}{n}\sum_i u^{-1}(u_i)\right)-\lambda\left(\bar{U}-\frac{1}{n}\sum_i u_i+c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)\right)$$
$$-\sum_i \mu_i\left(u^{-1}(u_i)-u^{-1}(\bar{w})\right)$$

Let $h(x)=(c')^{-1}(x)$, $v(x)=u'(x)$, and $v_i=u'(w_i)=u'(u^{-1}(u_i))$. When the limited liability constraint does not bind, $\mu_i=0$. The first order condition for u_i in such a case is as follows:

$$\beta_i n h'\left(\sum_i \beta_i u_i\right)\left(1-\lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right)+\lambda=\frac{1}{v_i}$$

It follows that, for any i and k, $\frac{1}{v_i}-\frac{1}{v_{i+k}}=(\beta_i-\beta_{i+k})nh'(\sum_i \beta_i u_i)(1-\lambda c'(h(\sum_i \beta_i u_i)))$. Similarly, for any j and l, $\frac{1}{v_j}-\frac{1}{v_{j+l}}=\frac{1}{v_i}-\frac{1}{v_{i+k}}=(\beta_j-\beta_{j+l})nh'(\sum_i \beta_i u_i)(1-\lambda c'(h(\sum_i \beta_i u_i)))$. Therefore,

$$\frac{\frac{1}{v_i}-\frac{1}{v_{i+k}}}{\frac{1}{v_j}-\frac{1}{v_{j+l}}}=\frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}.$$

□

Proof of Proposition 1 Let $r(w) = \frac{1}{v(w)} = \frac{1}{u'(w)}$. r is increasing since $r'(w) = \frac{-u''}{(u')^2} \geq 0$. $r'' = \left(\frac{-u''}{(u')^2}\right)(2R - P)$, so $r'' \geq 0$ if $R \geq \frac{P}{2}$ and $r'' \leq 0$ if $R \leq \frac{P}{2}$. Let us consider two cases.

Case 1 $R \geq \frac{P}{2}$

Since $\beta_i \geq \beta_{i+k}$ and r is increasing, it follows that $w_i^* \geq w_{i+k}^*$. Because $r'' \geq 0$, it follows that:

$$r'(w_{i+k}^*)(w_i^* - w_{i+k}^*) \leq r(w_i^*) - r(w_{i+k}^*) \leq r'(w_i^*)(w_i^* - w_{i+k}^*)$$

Similarly, since $\beta_j \geq \beta_{j+l}$, $w_j^* \geq w_{j+l}^*$ and:

$$r'(w_{j+l}^*)(w_j^* - w_{j+l}^*) \leq r(w_j^*) - r(w_{j+l}^*) \leq r'(w_j^*)(w_j^* - w_{j+l}^*)$$

Hence,

$$\left(\frac{r'(w_{i+k}^*)}{r'(w_j^*)}\right)\frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)} \leq \left(\frac{r'(w_i^*)}{r'(w_{j+l}^*)}\right)\frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*}$$

And,

$$\left(\frac{r'(w_{j+l}^*)}{r'(w_i^*)}\right)\frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \left(\frac{r'(w_j^*)}{r'(w_{i+k}^*)}\right)\frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)}$$

By Lemma 4, $\frac{r(w_i^*)-r(w_{i+k}^*)}{r(w_j^*)-r(w_{j+l}^*)} = \frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}$. Therefore,

$$\left(\frac{r'(w_{j+l}^*)}{r'(w_i^*)}\right)\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \left(\frac{r'(w_j^*)}{r'(w_{i+k}^*)}\right)\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

$r'' \geq 0$ and $\min(i, i+k) \geq \max(j, j+l)$ implies that:

$$1 \leq \frac{r'(w_{j+l}^*)}{r'(w_i^*)} \leq \frac{r'(w_j^*)}{r'(w_{i+k}^*)}$$

And,

$$\frac{r'(w_{j+l}^*)}{r'(w_i^*)} = \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^2$$

So,

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w^*_i - w^*_{i+k}}{w^*_j - w^*_{j+l}}$$
$$\leq \left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

Case 2 $R \leq \frac{P}{2}$
Following a similar logic, when $R \leq \frac{P}{2}$ and $\min(i, i+k) \geq \max(j, j+l)$:
Since $w^*_i \geq w^*_{i+k}$ and $r'' \leq 0$ (since $R \leq \frac{P}{2}$), it follows that:

$$\left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w^*_i - w^*_{i+k}}{w^*_j - w^*_{j+l}}$$
$$\leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

Let $z(x) = \frac{1}{u'(u^{-1}(x))}$. z is increasing since $z' = \frac{-u''(u^{-1}(x))}{(u'(u^{-1}(x)))^3} \geq 0$. $z'' = \left(\frac{-u''}{(u')^4}\right)$ $(3R - P)$, so $z'' \geq 0$ if $R \geq \frac{P}{3}$ and $z'' \leq 0$ if $R \leq \frac{P}{3}$. Following the same logic for $z(x)$ as for $r(x)$, we find the following. $R \geq \frac{P}{3}$ implies that:

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}}$$
$$\leq \left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

and $R \leq \frac{P}{3}$ implies that

$$\left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}}$$
$$\leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}.$$

□

Proof of Proposition 2 We will begin by considering the case where $R \geq \frac{P}{3}$. We will compare a j tournament ($j \leq \frac{n}{2}$) and an $n - j$ tournament that both meet the IR constraint and lead to the same exertion of effort, e, by players in the IC constraint. We will show that the sum of prizes paid by the principal in the j tournament exceeds the sum of prizes paid by the principal in the $n-j$ tournament. Given this result, we know that we can obtain the same effort with an $n-j$ tournament as a j tournament while meeting the IR constraint and paying out less in prizes. This shows that the optimal j tournament is dominated by the optimal $n-j$ tournament.

Following this argument, we will now consider a j tournament and an $n-j$ tournament that both meet the IR constraint and lead to the same effort exertion. Let w_1 and w_2 denote the prizes paid in the j tournament and let $u_i = u(w_i)$. Similarly, let $\tilde{w}_1$ and $\tilde{w}_2$ denote the prizes paid in the $n-j$ tournament and let $\tilde{u}_i = u(\tilde{w}_i)$. Further, let $\alpha = \frac{j}{n}$. The IR constraints for the j and $n-j$ tournaments imply that $\alpha u_1 + (1-\alpha)u_2 = \bar{u}$ and $(1-\alpha)\tilde{u}_1 + \alpha\tilde{u}_2 = \bar{u}$ where $\bar{u} = \bar{U} + c(e)$. Lemma 1 tells us that effort is the same in the j and $n-j$ tournaments when $u_1 - u_2 = \tilde{u}_1 - \tilde{u}_2$. These three equations tell us that $\tilde{u}_1 = \frac{\alpha}{1-\alpha}u_1 + \frac{1-2\alpha}{1-\alpha}\bar{u}$, $\tilde{u}_2 = 2\bar{u} - u_1$, and $u_2 = \frac{-\alpha}{1-\alpha}u_1 + \frac{1}{1-\alpha}\bar{u}$. Let W denote the sum of prizes in the j tournament and $\tilde{W}$ denote the sum of prizes in the $n-j$ tournament. Also, let $h = u^{-1}$. Then

$$W = \alpha w_1 + (1-\alpha)w_2 = \alpha h(u_1) + (1-\alpha)h\left(\frac{-\alpha}{1-\alpha}u_1 + \frac{1}{1-\alpha}\bar{u}\right)$$

$$\tilde{W} = \alpha\tilde{w}_2 + (1-\alpha)\tilde{w}_1 = \alpha h(2\bar{u} - u_1) + (1-\alpha)h\left(\frac{\alpha}{1-\alpha}u_1 + \frac{1-2\alpha}{1-\alpha}\bar{u}\right)$$

Let $g(x) = \alpha h(x) + (1-\alpha)h(\frac{\bar{u}-\alpha x}{1-\alpha})$ and $\Delta = u_1 - \bar{u} \geq 0$. We need to show that, for $\alpha \leq \frac{1}{2}$, $W \geq \tilde{W}$, or $g(\bar{u}+\Delta) - g(\bar{u}-\Delta) \geq 0$ (*). We see that $g'(x) = \alpha(h'(x) - h'(\frac{\bar{u}-\alpha x}{1-\alpha}))$. $h''(y) = \frac{-u''(h(y))}{[u'(h(y))]^3} \geq 0$ since u is concave. Observe that $g'(x) \geq 0$ for $x \geq \bar{u}$ and $g'(x) \leq 0$ for $x \leq \bar{u}$ since $h'' \geq 0$. Let $\varphi(\Delta) \equiv g(\bar{u}+\Delta) - g(\bar{u}-\Delta)$. A sufficient condition for (*) is that: $\varphi'(\Delta) \geq 0 \forall \Delta \geq 0$ since $\varphi(0) = 0$. We see that

$$\varphi'(\Delta) = \alpha\left(h'(\bar{u}+\Delta) - h'(\bar{u} + \frac{\alpha}{1-\alpha}\Delta)\right) - \alpha\left(h'(\bar{u} - \frac{\alpha}{1-\alpha}\Delta) - h'(\bar{u}-\Delta)\right)$$

Let $\omega(\theta, x, y) = \alpha\left[(h'(x+\theta) - h'(x)) - (h'(y+\theta) - h'(y))\right]$. Then, $\varphi'(\Delta) = \omega(\frac{1-2\alpha}{1-\alpha}, \bar{u} + \frac{\alpha}{1-\alpha}\Delta, \bar{u} - \Delta)$. Observe that $\frac{1-2\alpha}{1-\alpha} \geq 0$ since $\alpha \leq \frac{1}{2}$ and $\bar{u} + \frac{\alpha}{1-\alpha}\Delta \geq \bar{u} - \Delta$. Since, $\omega(0, x, y) = 0$, it is sufficient to show that $\frac{\partial\omega}{\partial\theta}(\theta, x, y) \geq 0$ when $x \geq y$. Because, $\frac{\partial\omega}{\partial\theta}(\theta, x, y) = \alpha(h''(x+\theta) - h''(y+\theta))$, a sufficient condition for $\frac{\partial\omega}{\partial\theta}(\theta, x, y) \geq 0$ is $h''' \geq 0$. $h'''(y) = \frac{-3u''}{(u')^4}\left(R - \frac{P}{3}\right) \geq 0$. This proves that $W \geq \tilde{W}$. Under the assumption that $R \leq \frac{P}{3}$, the argument can be replicated to show that $W \leq \tilde{W}$. □

Proof of Proposition 3 Suppose $R \geq \frac{P}{3}$. Let us consider a j tournament and a j' tournament with $j' > j \geq n/2$. Let $\alpha = \frac{j}{n}$ and $\alpha' = \frac{j'}{n}$. We will compare j and j' tournaments that lead to the same level of effort exertion, e, and consider the amounts paid out in prizes by the principal. Let w_1 and w_2 denote the prizes paid in the j tournament and $\tilde{w}_1$ and $\tilde{w}_2$ denote the prizes paid in the j' tournament. Let $u_i = u(w_i)$, $\tilde{u}_i = u(\tilde{w}_i)$ and let W and $\tilde{W}$ denote the sum of prizes in the j and

j' tournaments, respectively. Before we proceed, we need to define two functions: $\beta(x) = \beta_{\lceil nx \rceil}$ and $\gamma(x) = n\int_0^x \beta(x)\mathrm{d}x$. We see that $\gamma(\frac{j}{n}) = \sum_{i=1}^{j} \beta_i$. Thus, the incentive compatibility constraints for the j and j' tournaments can be written as $c'(e) = \gamma(\alpha)(u(w_1) - u(w_2))$ and $c'(e) = \gamma(\alpha')(u(\tilde{w}_1) - u(\tilde{w}_2))$, respectively. Individual rationality implies that $\alpha u_1 + (1-\alpha)u_2 = \bar{u}$ and $\alpha'\tilde{u}_1 + (1-\alpha')\tilde{u}_2 = \bar{u}$ where $\bar{u} = \bar{U} + c(e)$. Combining these four constraints, we can solve for W and $\tilde{W}$ in terms of u_1. Let us define a few functions: $\Phi(\alpha') = \frac{\gamma(\alpha')-\gamma(\alpha)}{\gamma(\alpha')} + \alpha' \frac{\gamma(\alpha)}{\gamma(\alpha')}$, $h = u^{-1}$, $g(x, \alpha') = \beta h(\alpha') + (1-\alpha')h(\frac{\bar{u}-\alpha' x}{1-\alpha'})$, and $\psi(\alpha') = g(\frac{1-\Phi(\alpha')}{1-\alpha}u_1 + \frac{\Phi(\alpha')-\alpha}{1-\alpha}\bar{u}, \beta)$. Then, we find that W and $\tilde{W}$ can be expressed as follows: $\tilde{W} = \psi(\alpha')$ and $W = \psi(\alpha)$. Let us consider $\psi'(x)$. If we find that $\psi'(x) \leq 0$ for $x \in [\alpha, \alpha']$, then it follows that $\tilde{W} \leq W$. This implies that the j' tournament dominates the j tournament.

$$\psi'(x) = (\bar{u} - u_1)\,\Phi'(x)\left(\frac{1}{1-x}\right)\left(xh'(u_1) + \left(\frac{1}{\Phi'(x)} - x\right)h'\left(\frac{\bar{u} - xu_1}{1-x}\right)\right) + \left(h(u_1) - h\left(\frac{\bar{u} - xu_1}{1-x}\right)\right)$$

Let us define

$$\Gamma(u) = (\bar{u} - u)\,\Phi'(x)\left(\frac{1}{1-x}\right)\left(xh'(u) + \left(\frac{1}{\Phi'(x)} - x\right)h'\left(\frac{\bar{u} - xu}{1-x}\right)\right) + \left(h(u) - h\left(\frac{\bar{u} - xu}{1-x}\right)\right)$$

We see that $\Gamma(\bar{u}) = 0$. Since $u_1 > \bar{u}$, $\psi'(x) = \Gamma(u_1) \leq 0$ if $\Gamma'(u) \leq 0$ for $u > \bar{u}$.

$$\Gamma'(u) = \left(\frac{\bar{u} - u}{1-x}\right)x\left(h''(u) - \frac{\frac{1}{\Phi'(x)} - x}{1-x}h''\left(\frac{\bar{u} - xu}{1-x}\right)\right) + \left(\frac{1 - x(1 + \Phi'(x))}{1-x}\right)\left(h'(u) - h'\left(\frac{\bar{u} - xu}{1-x}\right)\right)$$

Suppose it were the case that $\Phi'(x) = 1$. Then,

$$\Gamma'(u) = \left(\frac{\bar{u} - u}{1-x}\right)x\left(h''(u) - h''\left(\frac{\bar{u} - xu}{1-x}\right)\right) + \left(\frac{1-2x}{1-x}\right)\left(h'(u) - h'\left(\frac{\bar{u} - xu}{1-x}\right)\right)$$

Recall that we are assuming $1 > x \geq \frac{1}{2}$ and $u > \bar{u}$. Since $R \geq \frac{P}{3}$, it follows that $h'', h''' \geq 0$ (to see the argument, see the proof of Proposition 6). It therefore follows that the above expression is less than zero. Thus, if $\Phi'(x) = 1$, $\psi'(x) < 0$. From the definition of Φ, it follows that $\Phi'(x) = 1 + \frac{\gamma'(x)}{\gamma(x)}(1-x)$. Since $\gamma(x) = n\int_0^x \beta(x)\mathrm{d}x$, $\gamma'(x) = n\beta(x) = n\beta_{\lceil nx \rceil}$. Thus, $\Phi'(x) = 1 + \frac{n\beta_{\lceil nx \rceil}}{\gamma(x)}(1-x)$. Suppose that F is a symmetric and uniform distribution. It follows from Lemma 4 that $\beta_i = 0$ for $1 < i < n$. This implies that $\Phi'(x) = 1$ for $x \in [\frac{1}{n}, \frac{n-1}{n})$. Hence, when F is a symmetric uniform distribution, the j' tournament dominates the j tournament where $j' > j \geq n/2$. If follows from this and Corollary 3 that, for F a symmetric uniform distribution, the optimal j tournament is the strict loser prize tournament. The argument can be replicated for the case where $R \leq \frac{P}{3}$. □

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