

Sixty years of stochastic linearization technique

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Abstract Stochastic linearization technique is a versatile method of solving nonlinear stochastic boundary value problems. It allows obtaining estimates of the response of the system when exact solution is unavailable; in contrast to the perturbation technique, its realization does not demand smallness of the parameter; on the other hand, unlike the Monte Carlo simulation it does not involve extensive computational cost. Although its accuracy may be not very high, this is remedied by the fact that the stochastic excitation itself need not be known quite precisely. Although it was advanced about six decades ago, during which several hundreds of papers were written, its foundations, as exposed in many monographs, appear to be still attracting investigators in stochastic dynamics. This study considers the methodological and pedagogical aspects of its exposition.

Keywords Stochasticity · Nonlinear problems · Linearization · Galerkin method

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1 Introduction

This paper follows two recent articles, namely by Villaggio [1] and Maugin [2]. The former reviewed 60 years of solid mechanics whereas the latter dealt with the configurational forces. Villaggio [1] writes: “The end of the second world war marked a turning point of the history of solid mechanics. The reasons for this abrupt change are due to two causes: the opening of national frontiers, and a wave of enthusiasm for applied science, motivated by the technical achievements obtained in the production of new weapons”. The method of stochastic linearization technique was proposed more or less simultaneously on both sides of the Atlantic: by Booton [3] and Caughey [4] in the U.S. and by Kazakov [5] in the former S.U.

Around the method’s thirtieth anniversary, in his review, Spanos [6] wrote: “It can be stated, with only minor reservations, that the method of stochastic or statistical or equivalent linearization, has proved, over the period of the last three decades, the most useful approximate method for probabilistic analysis of nonlinear structural dynamical systems”. Around method’s half-century, Crandall [7] noted: “The procedure has been very popular with investigators in the field of random vibration. In 1998 it was estimated [8] that there had been over 400 papers published on the subject of statistical linearization”.

The method’s essence can be demonstrated on the simple problem of a single-degree-of-freedom structure, governed by the following differential equation:

$$m\ddot{X} + f(X, \dot{X}) = P(t) \tag{1}$$

where m = mass, X = displacement, \dot{X} = velocity, \ddot{X} = acceleration, f = nonlinear function, $P(t)$ = stationary random process in time.

The autocorrelation function and hence the spectral density of $P(t)$ are given. The problem consists in finding the probabilistic characteristics of X and \dot{X} . The simplest characteristics would be mathematical expectations of response quantities, $E(\cdot)$ indicating operation of mathematical expectation, namely $E(X)$, $E(\dot{X})$, $E(X^2)$ and $E(\dot{X}^2)$. Were $f(X, \dot{X})$ a linear function

$$f(X, \dot{X}) = k_0X + c_0\dot{X} \tag{2}$$

with k_0 the stiffness coefficient and c_0 the damping coefficient, the solution would be straightforward. For the case of the correlation function of $P(t)$

$$K_p(t, t') = E[P(t)P(t')] = 2\pi S_0\delta(t_2 - t_1) \tag{3}$$

(S_0 being the intensity of the noise), one obtains

$$E(X) = E(\dot{X}) = 0 \tag{4}$$

$$E(X^2) = \pi S_0/c_0k_0 \tag{5}$$

$$E(\dot{X}^2) = \pi S_0/c_0m \tag{6}$$

We are not concerned with the linear case, however. The closed-form solution for arbitrary nonlinearity as well as arbitrary excitation of the nonlinear oscillator is an unsolved problem. The pioneers of the stochastic linearization technique posed a question on possible linearization of the nonlinear function in Eq. (1), i.e. replacing the nonlinear function $f(X, \dot{X})$ by

$$f(X, \dot{X}) = k_{eq}X + c_{eq}\dot{X} \tag{7}$$

and finding equivalent values of the stiffness coefficient k_{eq} and the damping coefficient c_{eq} , so that the solution of thus obtained linear system

$$m\ddot{X} + k_{eq}X + c_{eq}\dot{X} = P(t) \tag{8}$$

would produce sufficiently good approximations for the desired quantities. The question is: How to determine k_{eq} and c_{eq} ? There is a gallery of answers. We dispense with a historical overview of these answers and direct the interested reader to various reviews, old and new, most recent perhaps being that

by Crandall [6]; the reader may also consult with the earlier reviews by Spanos [7], Roberts [8], Socha and Soong [9], Socha [10, 11], Falsone and Ricciardi [12], Elishakoff [13] and Proppe et al. [14]. There are two special monographs written on this subject, that by Roberts and Spanos [15], and by Socha [16]. It must be noted that Crandall [6] writes that some explanations provided in the literature since 1967 were “confusing”. To deal with controversial topics is beyond this study, however. We concern ourselves with a suggested explanation of the technique which hopefully will be free of “confusion”, on one hand, and will lead to rigorous pedagogical explanation of it for the novice. It is hoped that two alternative expositions proposed in this study will be adopted in future stochastic dynamics and random vibration textbooks.

2 A system possessing a nonlinear stiffness

Consider first the simplest form of nonlinearity which is exhibited by the system through its stiffness. In other words, the special form of Eq. (1) is studied

$$m\ddot{X} + c\dot{X} + f(X) = P(t) \tag{9}$$

We replace Eq. (9) by its “equivalent” given in Eq. (8). Since the damping is linear in both Eqs. (8) and (9), $c_{eq} = c$. We are looking for the equivalent linear stiffness k_{eq} . We evaluate the difference between the original nonlinear stiffness $f(X)$ and its linear equivalent $k_{eq}X$. Since $f(X)$ is in general a nonlinear function (no one would linearize a linear function!) the difference $f(X) - k_{eq}X$ does not vanish. At this stage one forms the mean-square difference

$$E(D^2) = E\{[f(X) - k_{eq}X]^2\} \tag{10}$$

and demands it to attain minimum with respect to k_{eq} ; in Eq. (10) the operator $E(\cdot)$ is that of mathematical expectation. Thus,

$$E(D^2) = \int_{-\infty}^{\infty} [f(x) - k_{eq}x]^2 \varphi(x) dx \tag{11}$$

where $\varphi(x)$ is the probability density function of $X(t)$. It makes sense to recall that we do not know the probability density function of the solution; indeed, had we known it, we would not use the approximate

technique of stochastic linearization; rather the desired probabilistic characteristics $E(X)$ and $E(X^2)$ would be evaluated by straightforward integration

$$\begin{aligned}
 E(X) &= \int_{-\infty}^{\infty} x\varphi(x)dx \\
 E(X^2) &= \int_{-\infty}^{\infty} x^2\varphi(x)dx
 \end{aligned}
 \tag{12}$$

i.e. without resorting to linearization technique.

There are two possibilities to proceed at this juncture. One possibility is to recognize our lack of knowledge of the exact $\varphi(x)$ and to employ some approximate probability density function of the linearized system $\psi(x, k_{eq})$. To obtain a linearized system we do not simply drop the nonlinear term: we replace the entire expression of restoring force that may contain linear and nonlinear expressions by an equivalent linear force $k_{eq}x$.

Here it must be remarked that since the linearized system in Eq. (8) inevitably depends on k_{eq} , so does the probability density $\psi(x, k_{eq})$. Then Eq. (11) is replaced by the approximate mean-square deviation $E_a(D^2)$ defined as

$$E_a(D^2) = \int_{-\infty}^{\infty} [f(x) - k_{eq}x]^2\psi(x, k_{eq})dx
 \tag{13}$$

The demand this mean square deviation to attain minimum with respect to k_{eq} leads to

$$\begin{aligned}
 \frac{dE_a(D^2)}{dk_{eq}} &= -2 \int_{-\infty}^{\infty} [f(x) - k_{eq}x]x\psi(x, k_{eq})dx \\
 &+ \int_{-\infty}^{\infty} [f(x) - k_{eq}x]^2 \frac{d\psi}{dk_{eq}} dx \\
 &= 0
 \end{aligned}
 \tag{14}$$

Another possibility is to assume that we know the exact probability density function in Eq. (11); proceeding then with minimization leads to

$$\frac{dE_a(D^2)}{dk_{eq}} = 2 \int_{-\infty}^{\infty} [f(x) - k_{eq}x]x\varphi(x)dx = 0
 \tag{15}$$

This demand reduces to the following expression for the equivalent stiffness coefficient k_{eq} :

$$k_{eq} = \frac{\int_{-\infty}^{\infty} f(x)x\varphi(x)dx}{\int_{-\infty}^{\infty} x^2\varphi(x)dx}
 \tag{16}$$

Had we known the exact probability density $\varphi(x)$ Eq. (16) would be replaced by

$$k_{eq} = \frac{E[Xf(X)]}{E(X^2)}
 \tag{17}$$

At this juncture it's recommended to ask ourselves to comment on this equation. Some are realizing the seemingly paradoxical situation we find ourselves in: We are looking for $E[X^2]$, yet we know it the stochastic linearization leads us to determine the equivalent linear stiffness k_{eq} whose determination demands the knowledge of the above sought quantity for Eq. (17) contains $E[X^2]$ in the denominator. Thus, the Eq. (17) must appear to the initial reader as totally useless, and the method of stochastic linearization as a nonsensical, for it leads, as it were, to catch 22, not less!

Such a situation is not pertinent solely to the stochastic linearization technique. It occurs even in deterministic problems. For example, analogous situation takes place while using the Rayleigh quotient method for the natural frequency evaluation. Whereas the quotient is derived in view of knowledge of exact mode shape, the better is approximate to obtain the estimate for the natural frequency.

Recalling that the exact density is not known, we instead of $\varphi(x)$ in Eq. (16) thus utilize its approximation $\psi(x, k_{eq})$:

$$k_{eq} = \frac{\int_{-\infty}^{\infty} f(x)x\psi(x)dx}{\int_{-\infty}^{\infty} x^2\psi(x)dx}
 \tag{18}$$

As can be observed by comparing Eqs. (14) and (15), the former contains an additional term. Natural question arises: "Which version of the stochastic linearization technique should be preferred?" As a popular proverb maintains, proof of the pudding is in eating. Thus, the above question must be changed into the following: "Which technique performs better?" In a series of studies, Socha and Pawleta [17], Elishakoff and Colajanni [18, 19], Colajanni and Elishakoff [20, 21] utilized Eq. (14) to derive k_{eq} . In Refs. [18, 20, 21] it was shown that the mean-square values of the

responses of several oscillators, the approach based on Eq. (14) led to results that were farther from exact solution than those obtained by employing Eq. 15. Only in one oscillator, originally studied by Booton [3], Elishakoff and Colajanni [19] demonstrated that both techniques lead to coincident results. Thus, in balance one has to prefer, due to pragmatic reasons, Eq. (15) to Eq. (14).

3 Discussion of Eqs. (14) and (15)

It must be noted that the above derivation of the two possible approaches when one minimizes the mean square error is presented herein for the first time. The approach given by Eq. (14) was given in Refs. [15–19]. Crandall [22] calls it a SPEC alternative, acronym being associated with the first letters in last names of the authors of papers [17–21]. How can one explain, *post factum*, the success of the second approach? It appears that in the second approach we carry as much as possible the attributes of exact analysis.

Crandall [6] characterizes Eq. (17) as “the recipe for selecting k_{eq} ”. Indeed, according to Paul Valéry, a French poet, essayist and philosopher, “Science is a collection of successful recipes”. The approach based on Eq. (14) was proposed due to the absence in the literature of specific statement that the recipe in Eq. (17) is associated with the assumption that until its derivation it was assumed that the exact probability density as known. This led, according to Crandall [22] to the fact that “there has been some confusion concerning the standard (i.e. second, IE & SC) procedure”. He also noted:

It must be admitted that the literature on this point has been confusing. Many descriptions of the standard procedure fail to explain why the expectation ... are considered to be independent of k before the differentiation ..., but immediately afterward are taken to be k -dependent.

Likewise in the personal communication by late Professor Caughey [23] to one of us, he writes:

Thank you for the papers that you sent me, I found them very interesting and a little disturbing. After reading both apps carefully, I have the following comments:

- (a) It’s surprising that both techniques lead to exactly the same first order corrections, it should be noted that perturbation theory also leads to the same first order correction term. As far as I know nobody has carried out the perturbation technique to obtain the higher corrections.
- (b) It’s also surprising that the improved minimization technique (i. e, Eq. (14)—IE&SC) leads to poorer results than the naïve technique. One thinks of asymptotic series where the best approximation given by the first couple of terms.
- (c) If the naïve technique is applied to Duffing’s equation with *Sinusoidal Excitation* it predicts the same first order correction that is given by the *Harmonic Balance*. I have not repeated the problem using your minimization technique. Duffing’s Equation with white noise excitation appears to be the simplest example to illustrate your technique; all other examples appear to be much more complex.

Likewise, Li and Chen [24] in their book note:

Although the above analysis [derivation of Eq. (14)—IE&SHC] is reasonable, the effect is not as good as expected. First, deduction is much more difficult and might be impossible for complex or multidimensional problems. Second, even for simple problems, it was shown that the accuracy of the ‘error-free’ linearization is sometimes lower than that of standard linearization (Elishakoff and Colajanni [18]).

Crandall [6] stressed that “The SPEC alternative has some interesting features [22], but unfortunately it is more labor intensive and, almost always, less accurate than the standard procedure”.

Another question arises on the role that the papers by Socha and Pawleta [17] and Elishakoff and Colajanni [18–21] had played in elucidation of stochastic linearization technique. Crandall [6] gives the following credit to the above studies:

...the inconsistency of applying recipes based on nonlinear response statistics independent of k to linear system statistics which were functions of k was recognized and corrected by Socha and

Pawleta [17, 25] and by Elishakoff and Colajanni [18–21].

These papers also inspired investigations by Crandall [6, 22, 26–28], Socha [10, 11, 16], Socha and Pawleta [25], Proppe et al. [14], Elishakoff [13, 29] and possibly others.

Here the method of Gaussian closure [30, 31] should be mentioned. It is widely known classical stochastic linearization technique coalesces with Gaussian closure technique. On the other hand, as we assume $\psi(x, k_{eq})$ to be Gaussin (and this is mandatory since the system is linearized and the input is Gaussian) then Eq. 14 will return to Gaussian closure technique.

4 Stochastic linearization via Bubnov–Galerkin technique

We would like to start discussion on the title topic by a comment that appears to be instructive on derivation of Eq. (14), or classical recipe for k_{eq} . One resorts to stochastic linearization as an approximate technique, knowing a priori that the exact probability density of the response is unknown. Yet, in order to derive Eq. (17) one has to assume the knowledge of the exact probability density. Therefore, the derivation of Eq. (17) may appear inconsistent. Consistency, naturally, is a desirable attribute to any derivation. According to William James’s philosophy, truth is associated with the term ‘leading’ in the sense that true beliefs “lead to consistency, stability”. However, the importance of consistency should not be overestimated. In words of Aldous Huxley, an English writer,

Too much consistency is as bad for the mind as it is for the body. Consistency is contrary to nature, contrary to life. The only completely consistent people are dead.

It appears instructive to reproduce here the quote form Levinson [32], commenting on his and Bickford’s [33] theories and the associated issues of consistency:

It would seems that the Bickford’s work has relegated the earlier work of the present writer to the status of an intellectual artifact in the history of applied mechanics whose importance is limited to providing the motivation for the work

of Bickford; from a certain theoretical point of view this is clearly so. What is vexations, however, is that Bickford’s theory, in the two elastostatic and one elastodynamic problems he consider, provides inferior results in two cases and essentially the same results in the remaining case when compared to the results of the present writer’s theory; exact elasticity solutions being available for purposes of comparison in all three of the examples considered.

As is seen here too, the less consistent theory turned out to produce better results!

Let us turn now to recasting stochastic linearization technique via the Bubnov–Galerkin method. It is naturally not possible to replace the nonlinear force $f(X)$ by a linear counterpart $k_{eq} X$ in Eq. (9). There is a difference between $f(X)$ and $k_{eq} X$. We refer to this difference as error $\varepsilon(X)$. Whereas we do not posses a magic wand to make it zero, we can try to make it as small as possible in some sense. We demand the first moment of $E(\varepsilon X)$ of this error to vanish

$$E(\varepsilon X) = 0 \tag{19}$$

where $E(\cdot)$ denotes mathematical expectation.

The fact that we do not know the probability density of the response to evaluate Eq. (19), does not prevent us from realizing that our condition (19) in fact is orthogonality condition between the error ε and the system’s displacement X . Yes, the error ε is not zero as we wish it to be, but at least we cannot see it, as it were, in the “direction” of X . We could metaphorically refer to condition (19) as the overlooking of one’s “misbehavior”, exhibiting itself in absence of being error-free, by parents, grandparents and friends (as a proverb maintains, “Friend is one who tolerates our success and accepts us with our mistakes”); one usually has a better grasp of having condition (19) presented as a “friendship’s” attribute. Thus, the condition (19) becomes:

$$E[\{f(X) - k_{eq}X\}X] = 0$$

For the coefficient k_{eq} we get

$$k_{eq} = \frac{E[f(X)X]}{E[X^2]} \tag{20}$$

Thus, by the Bubnov–Galerkin method we arrive at the same expression as Eq. (17).

5 Conclusion

In this paper we present a methodology for simple exposition of the celebrated stochastic linearization technique. It is rather hoped that this study eliminates the confusion that has surrounded this method for 27 years, in terminology of one of us (S.H.C.). Moreover, this study presents two alternative derivations of the classical scheme of stochastic linearization technique that may prove useful in reinforcing its foundations. It should be stressed that Spanos, Ghanem, Zeldin, Di Paola and Failla [34–39] extensively applied Bubnov–Galerkin method to various problems of nonlinear stochastic dynamics.

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