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Price of Airline Frequency Competition

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Abstract

Frequency competition influences capacity allocation decisions in airline markets and has important implications to airline profitability and airport congestion. Market share of a competing airline is a function of its frequency share and the relationship between the two is pivotal for understanding the impacts of frequency competition on airline business. Based on the most commonly accepted form of this relationship, we propose a game-theoretic model of airline frequency competition. We characterize the conditions for existence and uniqueness of a Nash equilibrium for the 2-player case. We analyze two different myopic learning dynamics for the non-equilibrium situations and prove their convergence to Nash equilibrium under mild conditions. For the N-player game between identical players, we characterize all the pure strategy equilibria and identify the worst-case equilibrium, i.e. the equilibrium with maximum total cost. We provide an expression for the measure of inefficiency, similar to the price of anarchy, which is the ratio of the total cost of the worst-case equilibrium to the total cost of the cost minimizing solution and investigate its dependence on different parameters of the game.

1 Introduction

Since the deregulation of US domestic airline business in the 1970's, airlines have used fare and service frequency as the two most important instruments of competition. Passengers have greatly benefitted from fare competition, which has resulted in a substantial decrease in real (inflation adjusted) airfares over the years. On the other hand, frequency competition has resulted in availability of more options for air travel. The benefits of increased competition to the airlines themselves are not as obvious. Throughout the post-deregulation period, airline profits have been highly volatile. Several major US carriers have made substantial losses over the last decade with some of them filing for Chapter 11 bankruptcy and some others narrowly escaping bankruptcy. Provision of excess capacity is one of the reasons often cited for the poor

economic health of airlines. Due to the so called S-curve relationship [5] between market share and frequency share, an airline is expected to attract disproportionately more passengers by increasing its frequency share in a market. To increase their market share, airlines engage in frequency competition by providing more flights per day on competitive routes. As a result, they prefer operating many flights with small aircraft rather than operating fewer flights with larger aircraft. The average aircraft sizes in domestic US markets have been falling continuously over the last couple of decades in spite of increasing passenger demand [6]. Similarly, the average load factors, i.e. the ratio of the number of passengers to the number of seats, on some of the most competitive and high demand markets have been found to be lower than the industry average.

Apart from the chronic worries about the industry's financial health, worsening congestion and delays at the major US airports has become another cause of serious concern. Increases in passenger demand, coupled with decreases in average aircraft size have led to a great increase in the number of flights being operated, especially between the major airports, leading to congestion. The US Congress Joint Economic Committee has estimated that in calendar year 2007, delays cost around \$18 billion to the airlines and another \$12 billion to passengers [20].

Thus, frequency competition affects airlines' capacity allocation decisions, which in turn have a strong impact on airline profitability, as well as on airport congestion. In this paper, we propose a game theoretic framework, which is consistent with the most prevalent model of frequency competition. Section 2 provides background on airline schedule planning and reviews the literature on frequency competition. Section 3 presents the N-player game model. Best response curves are characterized in section 4. In section 5, we focus on the 2-player game. We provide the conditions for existence and uniqueness of a Nash equilibrium and discuss realistic parameter ranges. We then provide two different myopic learning models for the 2-player game and provide proof of their convergence to the Nash equilibrium. In section 6, we identify all possible equilibria in a N-player game with identical players and find the worst-case equilibrium. In section 7, we evaluate the price of anarchy and establish the dependence of airline profitability and airport congestion on airline frequency competition. We conclude with a summary of main results in section 8.

2 Frequency Planning under Competition

The airline planning process involves decisions ranging from long-term strategic decisions such as fleet planning and route planning, to medium-term decisions about schedule development [4]. Fleet planning is the process of determining the composition of a fleet of aircraft, and involves decisions about acquiring new aircraft and retiring existing aircraft in the fleet. Given a fleet, the second step in the airline planning process involves the choice of routes to be flown, and

is known as the route planning process. A route is a combination of origin and destination airports (occasionally with intermediate stops) between which flights are to be operated. Route planning decisions take into account the expected profitability of a route based on demand and revenue projections as well as the overall structure of the airline’s network. Given a set of selected routes, the next step in the planning process is airline schedule development, which in itself is a combination of decisions about frequency, departure times and aircraft sizes for each route, and aircraft rotations over the network.

Frequency planning is the part of the airline schedule development process that involves decisions about the number of flights to be operated on each route. By providing more frequency on a route, an airline can attract more passengers. Given an estimate of total demand on a route, the market share of each airline depends on its own frequency as well as on competitor frequency. The S-curve or sigmoidal relationship between the market share and frequency share is a widely accepted notion in the airline industry [17][5]. However, it is difficult to trace the origins and evolution of this S-shaped relationship in the airline literature [10]. Empirical evidence of the relationship was documented in some early studies and regression analysis was used to estimate the model parameters [23][22][21]. Over the years, there have been several references to the S-curve including Kahn [14] and Baseler [3]. A recent empirical study by Button and Drexler [10] suggests a somewhat different trend. While reporting some evidence of the S-curve relationship in the early 1990’s, they observe a much flatter linear relationship between market share and frequency share in the early 2000’s. In another recent study, Wei and Hansen [24] provide statistical support for the S-curve effect of airline frequency on market shares, based on a nested logit model for non-stop duopoly markets. In this paper, we use a more general model that is compatible with the linear, as well as the S-curve assumptions. The mathematical expression for the S-curve relationship [21][5] is given by:

$$MS_i = \frac{FS_i^\alpha}{\sum_{j=1}^n FS_j^\alpha} \quad (1)$$

for parameter α such that $\alpha > 1$, where MS_i = market share of airline i , FS_i = frequency share of airline i and n = number of competing airlines.

Empirical studies have shown that the typical values of α range between 1.3 and 1.7, with $\alpha = 1.5$ being a good representative value. This model is general enough to even accommodate the linear relationship suggested by Button and Drexler, by setting $\alpha = 1$.

Many of these studies go on to discuss the financial implications of the S-curve. Button and Drexler [10] associate it with provision of "excess capacity" and an "ever-expanding number of flights", while O’Connor [17] associates it with "an inherent tendency to overschedule". Kahn goes even further and raises the question of whether it is possible at all to have a financially strong and yet highly competitive airline industry at the same time [17].

Despite continuing interest in frequency competition based on the S-curve phenomenon, literature on game theoretic aspects of such competition is limited. Hansen [12] analyzed frequency competition in a hub-dominated environment using a strategic form game model. Dobson and Lederer [11] modeled schedule and fare competition as a strategic form game. Adler [1] used an extensive form game model to analyze airlines competing on fare, frequency and aircraft sizes. Each of these three studies adopted a successive optimizations approach to solve for a Nash equilibrium. Only Hansen [12] mentions some of the issues regarding convergence through discussion of different possible cases. But none of these three studies provides any conditions for convergence properties of the algorithm. Wei and Hansen [25] analyze three different models of airline competition and solve for equilibrium through explicit enumeration of the entire strategy space. Brander and Zhang [7] and Aguirregabiria and Ho [2] model airline competition as a dynamic game and estimate the model parameters using empirical data. Norman and Strandens [16] also calibrate model parameters using empirical data but for a strategic form game. None of the studies mentioned so far provides any guarantee or conditions for existence or uniqueness of a pure strategy equilibrium. Brueckner and Flores-Fillol [9] and Brueckner [8] obtain closed form expressions for equilibrium decisions analytically. They focus on symmetric equilibria while ignoring the possibility of any unsymmetric equilibria. Most of the previous studies involving game theoretic analysis of frequency competition, such as Adler [1], Pels et. al [18], Hansen [12], Wei and Hansen [25], Aguirregabiria and Ho [2], Dobson and Lederer [11], Hong and Harker [13], model market share using Logit or nested Logit type models, with utility typically being an affine function of the inverse of frequency. Such relationships can be substantially different from the S-shaped relationship between market share and frequency share, depending on the exact values of utility parameters.

All of these studies involve finding a Nash equilibrium or some refinement of it. But there isn't sufficient justification of the predictive power of the equilibrium concept. Hansen [12] provides some discussion of the shapes of best response curves and stability of equilibrium points. But none of the studies has focused on any learning dynamics through which less than perfectly rational players may eventually reach the equilibrium state. In this paper, we analyze a strategic form game among airlines with frequency of service being the only decision variable. We will only consider pure strategies of the players, i.e. we will assume that the frequency decisions made by the airlines are deterministic. We use the Nash equilibrium solution concept under pure strategy assumption. The research contributions of this study are threefold. First, we make use of the S-curve relationship between market share and frequency share and analyze its impact on the existence and uniqueness of pure strategy Nash equilibria. Second, we provide reasonable learning dynamics and provide theoretical proof for their convergence to the unique Nash equilibrium for the 2-player game. Third, we provide a measure of inefficiency, similar to the price of anarchy, of a system of competing profit-maximizing airlines in comparison to a system with centralized control. This measure can be used as a proxy to understand the effects

of frequency competition on airline profitability and airport congestion.

3 Model

Let M be the total market size i.e. the number of passengers wishing to travel from a particular origin to a particular destination on a non-stop flight. In general, an airline passenger may have more than one flight in his itinerary. Conversely, two passengers on the same flight may have different origins and/or destinations. But for our analysis, we will ignore these network effects and assume the origin and destination pair of airports to be isolated from the rest of the network. Let $I = \{1, 2, \dots, n\}$ be the set of airlines competing in a particular non-stop market. Although most of the major airlines today follow the practices of differential pricing and revenue management, we will assume that the airfare charged by each airline remains constant across all passengers. Let p_i be the fare charged by each airline i . Further, we will assume that the type and seating capacity of aircraft to be operated on this non-stop route are known. Let S_i be the seating capacities for airline i and C_i be the operating cost per flight for airline i . Let α be the parameter in the S-curve relationship. A typical value suggested by literature is around 1.5. To keep our analysis general, we assume that $1 < \alpha < 2$. Our results are applicable even in the case of a linear relationship between market share and frequency share by taking the limit as $\alpha \rightarrow 1^+$.

Assumption 1. $1 < \alpha < 2$

Let x_i be the frequency of airline i . As per the S-curve relationship between market share and frequency share, the i^{th} airline's share of the market (MS_i) is given by:

$$MS_i = \frac{x_i^\alpha}{\sum_{j=1}^n x_j^\alpha}.$$

This is obtained by multiplying the numerator and denominator of the right hand side of equation (1) by $\left(\sum_{j=1}^n x_j^\alpha\right)^\alpha$. The number of passengers (PAX_i) traveling on airline i is given by:

$$PAX_i = \min \left(M \frac{x_i^\alpha}{\sum_{j=1}^n x_j^\alpha}, S_i x_i \right).$$

Airline i 's profit (Π_i) is given by:

$$\Pi_i = p_i * \min \left(M \frac{x_i^\alpha}{n}, S_i x_i \right) - C_i x_i.$$

We will assume that for every i , $C_i < p_i S_i$. In other words, the total operating cost of a flight is lower than the total revenue generated when the flight is completely filled. This assumption is reasonable because if it is violated for some airline i , then there is a trivial optimal solution $x_i = 0$ for that airline.

Assumption 2. $C_i < p_i S_i \forall i \in I$

4 Best Response Curves

Let us define the effective competitor frequency, $y_i = \left(\sum_{j \in I, j \neq i} x_j^\alpha \right)^{1/\alpha}$, and

$$\begin{aligned} \Pi_i &= \min(\Pi'_i, \Pi''_i) \\ \text{where, } \Pi'_i &= M p_i \frac{x_i^\alpha}{x_i^\alpha + y_i^\alpha} - C_i x_i \text{ and } \Pi''_i = p_i S_i x_i - C_i x_i. \end{aligned}$$

Π'_i is a twice continuously differentiable function of x_i .

$$\begin{aligned} \frac{\partial \Pi'_i}{\partial x_i} &= \frac{M p_i \alpha x_i^{\alpha-1} y_i^\alpha}{(x_i^\alpha + y_i^\alpha)^2} - C_i \\ \text{and } \frac{\partial^2 \Pi'_i}{\partial x_i^2} &= \frac{M p_i \alpha x_i^{\alpha-2} y_i^\alpha}{(x_i^\alpha + y_i^\alpha)^3} ((\alpha - 1) y_i^\alpha - (\alpha + 1) x_i^\alpha). \end{aligned}$$

Π'_i has a single point of contraflexure at $x_i = y_i \left(\frac{\alpha-1}{\alpha+1} \right)^{1/\alpha}$ such that the function is strictly convex for all lower values of x_i and strictly concave for all higher values of x_i . Π'_i can have at most two points of zero slope (stationary points). If two such points exist, then the one with lower x_i will be a local minima in the convex region and the one with higher x_i will be a local maxima in the concave region. Therefore, Π'_i has at most one local maximum and exactly one boundary point at $x_i = 0$. Therefore, global maxima of Π'_i will be at either of these two points.

Π_i'' is a linear function of x_i with a positive slope. For a given combination of parameters α , M , p_i , C_i , S_i and a given effective competitor frequency y_i , the global maximum of Π_i can satisfy any one of the following three cases. These three cases are also illustrated in figures 1, 2 and 3 respectively.

Case A: $\Pi_i' \leq 0$ for all $x_i > 0$. Under this case, either a local maximum does not exist for Π_i' or it exists but value of the function Π_i' at that point is negative. In this case, a global maximum of $\Pi_i(x_i)$ is at $x_i = 0$. This describes a situation where the effective competitor frequency is so large that airline i cannot earn a positive profit at any frequency. Therefore, the best response of airline i is to have a zero frequency, i.e. not to operate any flights in that market.

Case B: Local maximum of Π_i' exists and the value of the function Π_i' at that local maximum is positive and less than or equal to $\Pi_i''(x_i)$. In this case, the unique global maximum of $\Pi_i(x_i)$ exists at the local maximum of $\Pi_i'(x_i)$. In this case, the optimum frequency is positive and at this frequency, airline i earns the maximum profit that it could have earned had the aircraft seating capacity been infinite.

Case C: A local maximum of Π_i' exists in the concave part and the value of the function $\Pi_i'(x_i)$ at this local maximum is greater than $\Pi_i''(x_i)$. In this case, $\Pi_i'(x_i)$ and $\Pi_i''(x_i)$ intersect at two distinct points (apart from $x_i = 0$). The unique global maximum of $\Pi_i(x_i)$ exists at the point of intersection with highest x_i value. This describes the case where optimum frequency is positive and greater than the optimum frequency under the assumption of infinite aircraft seating capacity. At this frequency, airline i earns lower profit than the maximum profit it could have earned had the aircraft seating capacity been infinite.

$\Pi_i'(0) = 0$ and for very low positive values of x_i , $\frac{\partial \Pi_i'}{\partial x_i}$ is negative. Therefore, at the first stationary point (the one with lower x_i value), the Π_i' function value will be negative. Moreover, as y_i tends to infinity, Π_i' is negative for any finite value of x_i . Therefore, $\Pi_i'(x_i) > 0$ for some x_i if and only if $\Pi_i'(x_i') > 0$ for some stationary point x_i' . For a given combination of parameters α , M , p_i , C_i and S_i , there exists a threshold value of effective competitor frequency y_i such that, for any y_i value above this threshold, $\Pi_i'(x_i) \leq 0$ for all $x_i > 0$ and therefore the best response of airline i is $x_i = 0$. Let us denote this threshold by y_{th} and the corresponding x_i value as x_{th} . At $x_i = x_{th}$ and $y_i = y_{th}$,

$$\begin{aligned} \Pi_i' &= 0, & \frac{\partial \Pi_i'}{\partial x_i} &= 0, & \frac{\partial^2 \Pi_i'}{\partial x_i^2} &\leq 0 \\ \Rightarrow x_{th} &= (\alpha - 1) \frac{Mp_i}{\alpha C_i} \text{ and } y_{th} &= (\alpha - 1)^{\frac{\alpha-1}{\alpha}} \frac{Mp_i}{\alpha C_i}. \end{aligned}$$

Of course, at $y_i = y_{th}$, $x_i = 0$ is also optimal. It turns out that it is the only y_i value at which there is more than one best response possible. This situation is unlikely to be observed in

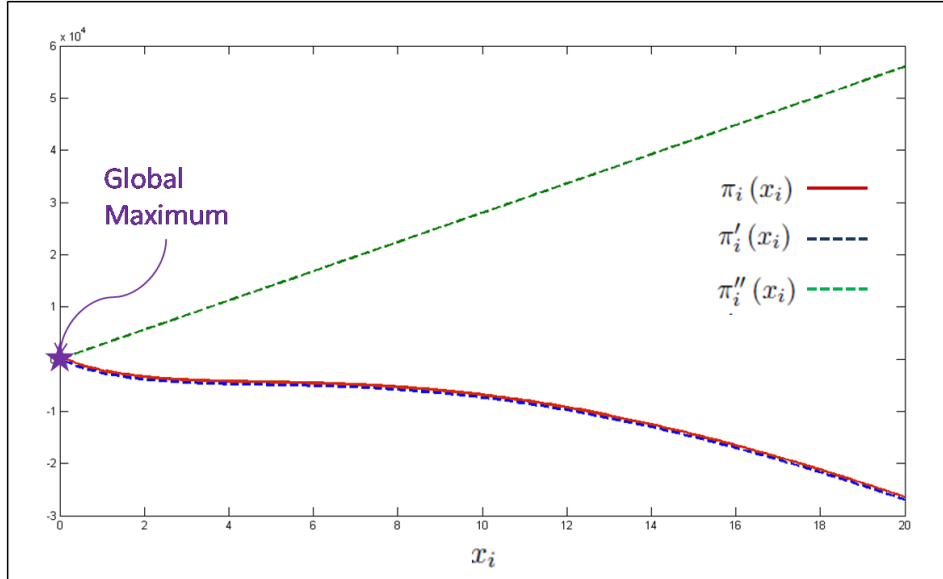


Figure 1: Case A

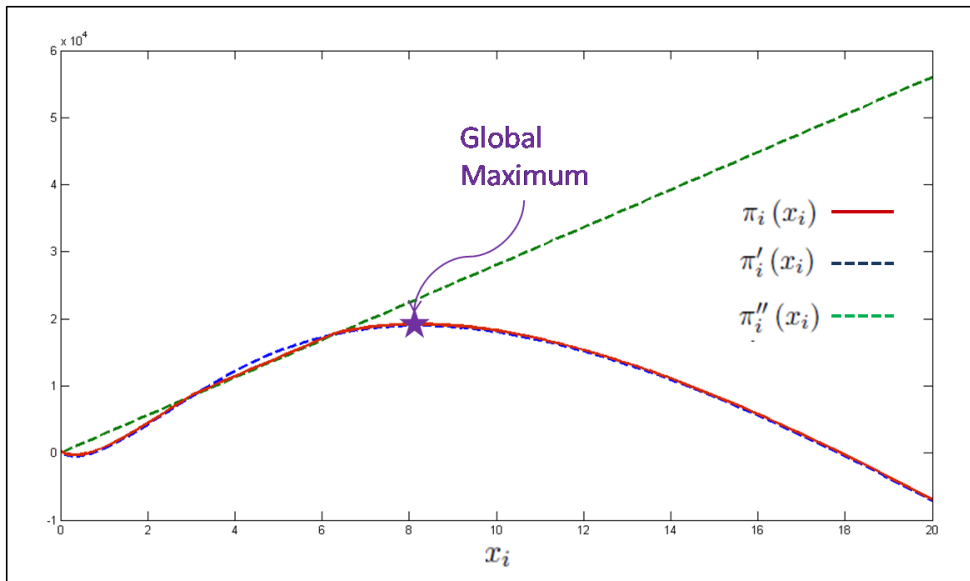


Figure 2: Case B

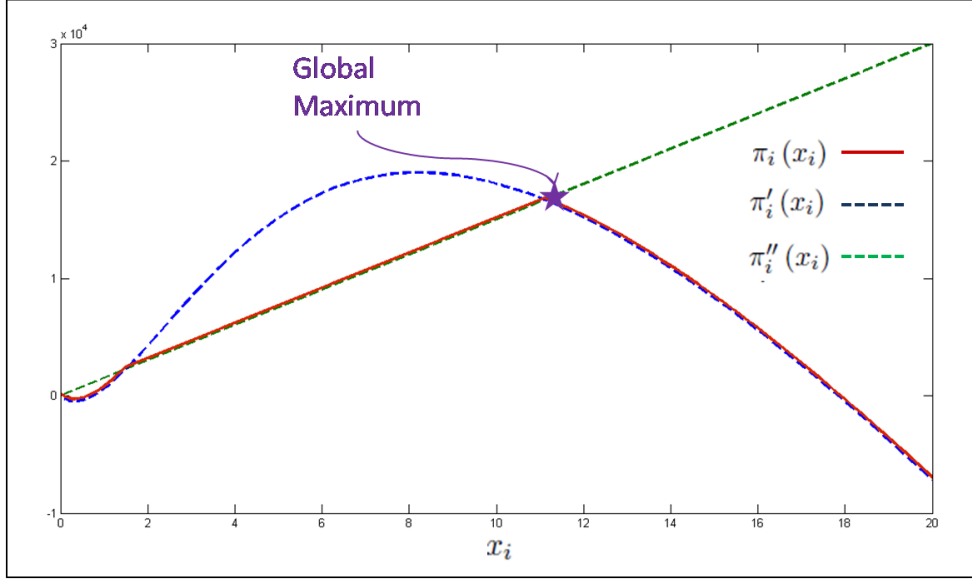


Figure 3: Case C

real world examples, because the parameters of the model are all real numbers with continuous distributions. So the probability of observing this exact idiosyncratic case is zero. If we arbitrarily assume that in the event of two optimal frequencies, an airline chooses the greater of the two values, then the best response correspondence reduces to a function which we will refer to as the best response function. The existence of two different maximum values at $y_i = y_{th}$ means that the best response correspondence is not always convex valued. Therefore, in general, a pure strategy Nash equilibrium may or may not exist for this game.

For y_i values slightly below y_{th} , the global maximum of Π_i corresponds to the stationary point of Π_i' in the concave part as described in case B above. Therefore, for y_i values slightly below y_{th} , at the stationary point of Π_i' in the concave part, $\Pi_i' < \Pi_i''$. However, as $y_i \rightarrow 0$, $\text{argmax}(\Pi_i'(x_i)) \rightarrow 0$. Therefore, the $\text{argmax}(\Pi_i(x_i))$ exists at the point of intersection of Π_i' and Π_i'' curves, as explained in case C above. For y_i values slightly above 0, at the stationary point of Π_i' in the concave part, $\Pi_i' > \Pi_i''$. Therefore by continuity, for some y_i such that $0 \leq y_i \leq y_{th}$, there exists x_i such that, $\Pi_i' = \Pi_i''$, $\frac{\partial \Pi_i'}{\partial x_i} = 0$ and $\frac{\partial^2 \Pi_i'}{\partial x_i^2} \leq 0$. It turns out that there is only one such y_i value that satisfies these conditions. Let us denote this y_i value by y_{cr} , since this is critical value of effective competitor frequency such that case B prevails for higher y_i values (as long as $y_i \leq y_{th}$) and case C prevails for all lower y_i values. The value of y_{cr} and the corresponding x_i value, x_{cr} , is given by,

$$x_{cr} = \frac{M}{S_i} \left(1 - \frac{C_i}{\alpha p_i S_i} \right) \text{ and } y_{cr} = \frac{\frac{M}{S_i} \left(1 - \frac{C_i}{\alpha p_i S_i} \right)}{\left(\frac{\alpha p_i S_i}{C_i} - 1 \right)^{\frac{1}{\alpha}}}.$$

For $y_i = 0$, as $x_i \rightarrow 0^+$, Π'_i keeps increasing and Π''_i keeps decreasing. However, $\Pi''_i < \Pi'_i$ for sufficiently low values of x_i . Therefore, Π_i is maximized when $\Pi'_i = \Pi''_i$. Let us denote this x_i value as x_0 . It is easy to see that $x_0 = \frac{M}{S_i}$. We will denote the range of y_i values with $y_i \geq y_{th}$ as region A, $y_{cr} \leq y_i < y_{th}$ as region B and $y_i < y_{cr}$ as range C.

In region C, Π_i is maximized for a unique x_i value such that $\Pi'_i = \Pi''_i$ and $\frac{\partial \Pi'_i}{\partial x_i} \leq 0$. The equality condition translates into,

$$\frac{M}{S_i} x_i^{\alpha-1} - x_i^\alpha = y_i^\alpha. \quad (2)$$

The left hand side (LHS) of equation (2) is strictly concave because $1 < \alpha < 2$. Further, the LHS is maximized at $x_i = \frac{\alpha-1}{\alpha} \frac{M}{S_i}$. The maximum value of LHS is at $y_i = (\alpha-1)^{\frac{\alpha-1}{\alpha}} \frac{M}{\alpha S_i}$. So for every y_i value, there are two corresponding x_i values satisfying equation (2) that correspond to the two points of intersection of the Π'_i and Π''_i curves. The one corresponding to the higher x_i value is of interest to us. That always corresponds to x_i values greater than $\frac{\alpha-1}{\alpha} \frac{M}{S_i}$. Differentiating both sides of equation (2) with respect to y_i ,

$$\frac{\partial x_i}{\partial y_i} = \alpha \frac{y_i^{\alpha-1}}{x_i^{\alpha-2}} \frac{1}{(\alpha-1) \frac{M}{S_i} - \alpha x_i} < 0.$$

So the best response of airline i in region C is strictly decreasing. Let us again differentiate with respect to y_i to obtain the second derivative of best response x_i ,

$$\frac{\partial^2 x_i}{\partial y_i^2} = \frac{\left(\frac{\partial x_i}{\partial y_i} \right)^2 (\alpha-1) x_i^{\alpha-3} \left(\alpha x_i + (2-\alpha) \frac{M}{S_i} \right) + \alpha (\alpha-1) y_i^{\alpha-2}}{\left((\alpha-1) \frac{M}{S_i} - \alpha x_i \right) x_i^{\alpha-2}} < 0. \quad (3)$$

Therefore, the best response curve is a strictly decreasing and concave function for all $0 \leq y_i < y_{cr}$.

In region B, Π_i is maximized for a unique x_i value such that $\frac{\partial \Pi'_i}{\partial x_i} = 0$ and $\frac{\partial^2 \Pi'_i}{\partial x_i^2} < 0$. The first order equality condition translates into,

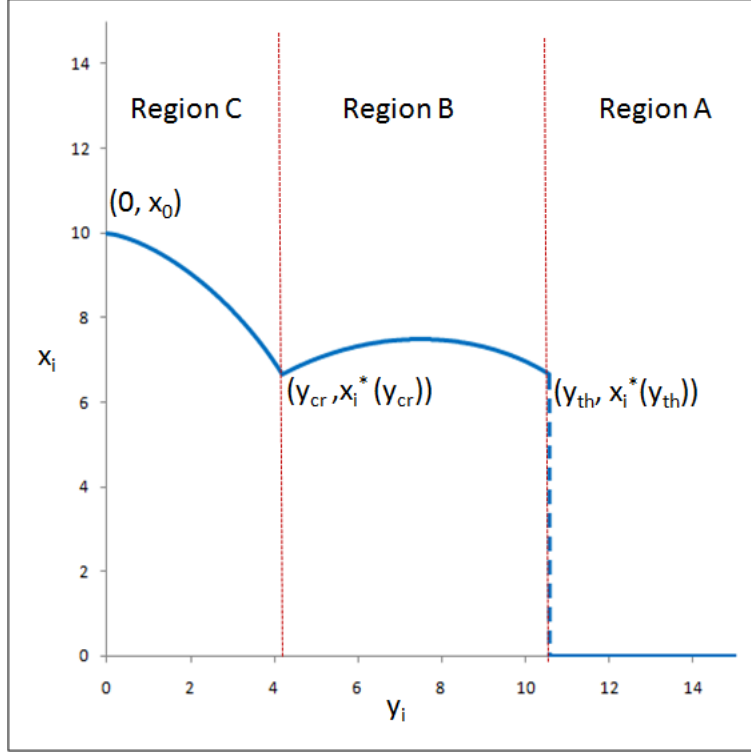


Figure 4: Best Response Curve

$$Mp_i \alpha \frac{x_i^{\alpha-1} y_i^\alpha}{(x_i^\alpha + y_i^\alpha)^2} = C_i. \quad (4)$$

Differentiating both sides of equation (4) with respect to y_i and again substituting equation (4) we get,

$$\frac{\partial x_i}{\partial y_i} = \frac{x_i}{y_i} \frac{x_i^\alpha - y_i^\alpha}{\left(1 + \frac{1}{\alpha}\right) x_i^\alpha - \left(1 - \frac{1}{\alpha}\right) y_i^\alpha}. \quad (5)$$

The second order inequality condition translates into,

$$\begin{aligned} & \frac{Mp_i \alpha x_i^{\alpha-2} y_i^\alpha}{(x_i^\alpha + y_i^\alpha)^3} ((\alpha - 1) y_i^\alpha - (\alpha + 1) x_i^\alpha) < 0 \\ \Rightarrow & \left(1 + \frac{1}{\alpha}\right) x_i^\alpha - \left(1 - \frac{1}{\alpha}\right) y_i^\alpha > 0. \end{aligned} \quad (6)$$

So the denominator of the right hand side of equation (5) is positive. Therefore, $\frac{\partial x_i}{\partial y_i} = 0$ if and only if $x_i = y_i$, $\frac{\partial x_i}{\partial y_i} > 0$ if and only if $x_i > y_i$ and $\frac{\partial x_i}{\partial y_i} < 0$ if and only if $x_i < y_i$. Therefore, the best response curve $x_i(y_i)$ in region B has zero slope at $x_i = y_i$, is strictly increasing for $x_i > y_i$ and strictly decreasing for $x_i < y_i$. Substituting $x_i = y_i$ in equation (4) we get, $x_i = y_i = \frac{\alpha M p_i}{4 C_i}$.

Figure 4 describes a typical best response curve as a function of effective competitor frequency. In region A, the effective competitor frequency is so small that airline i attracts a large market share even with a small frequency. Therefore, the optimal frequency ignoring seating capacity constraints is so low that, the number of seats is exceeded by the number of passengers wishing to travel with airline i . As a result, the optimal frequency and the maximum profit that can be earned by airline i are decided by the aircraft seating capacity constraint. In this region, the optimal number of flights scheduled by airline i is just sufficient to carry all the passengers that wish to travel on airline i . In this region, airline i has 100% load factor at the optimal frequency. With increasing effective competitor frequency, the market share attracted by airline i reduces and hence fewer flights are required to carry those passengers. Therefore, the best response curve is strictly decreasing in this region. Once the effective competitor frequency exceeds a critical value y_{cr} , the seating capacity constraint ceases to affect the optimal frequency decision.

In region B, the effective competitor frequency is sufficiently large due to which the number of passengers attracted by airline i does not exceed the seating capacity. Therefore, the aircraft seating capacity constraint becomes redundant in this region. The optimal frequency is equal to the frequency at which the marginal revenue equals marginal cost, which is a constant C_i . As the effective competitor frequency increases, the market share of airline i at the optimal frequency decreases and the load factor of airline i at optimal frequency also decreases. At a large value, y_{th} , of effective competitor frequency, the load factor of airline i at its optimal frequency reduces to a value $\frac{C_i}{p_i S_i}$ and the optimal profit drops to zero.

For all values of effective competitor frequency above y_{th} , i.e. in region C, there is no positive frequency for which the airline i can make positive profit. Therefore, the optimal frequency of airline i in region C is zero.

5 2-Player Game

Let x and y be the frequency of carrier 1 and 2 respectively. The effective competitor frequency for carrier 1 is y and that for carrier 2 is x . For any pure strategy Nash equilibrium (PSNE), the competitor frequency for each carrier can belong to any one of the three regions, A, B and C. So potentially there are 9 different combinations possible. We define the *type* of a PSNE as the combination of regions to which the competitor frequency belongs at equilibrium. We will denote each type by a pair of capital letters denoting the regions. For example, if carrier 1's

effective competitor frequency, i.e. y , belongs to region B and carrier 2's effective competitor frequency, i.e. x , belongs to region C, then that PSNE is said to be of type BC. Accordingly, there are 9 different types of PSNE possible for this game, namely AA, AB, AC, BA, BB, BC, CA, CB and CC.

Frequency competition among carriers is the primary focus of this research. However, it is important to realize that frequency planning is just one part of the entire airline planning process. Frequency planning decisions are not taken in isolation, the route planning phase precedes the frequency planning phase. Once the set of routes to be operated is decided, the airline proceeds to the decision of the operating frequency on that route. This implicitly means that once a route is deemed profitable in the route planning phase, frequency planning is the phase that decides the number of flights per day, which is supposed to be a positive number. However, in AA, AB, BA, AC or CA type equilibria, the equilibrium frequency of at least one of the carriers is zero, which is inconsistent with the actual airline planning process. Moreover, for ease of modeling, we have made a simplifying assumption that the seating capacity is constant. In reality, seating capacities are chosen considering the estimated demand in a market. If the demand for an airline in a market exceeds available seats on a regular basis, the airline would be inclined to use larger aircraft. Sustained presence of close to 100% load factors is a rarity. However type AC, BC, CA, CB and CC type equilibria involve one or both carriers having 100% load factors. Zero frequency and 100% load factors make all types of equilibria, apart from type BB equilibrium, suspect in terms of their portrayal of reality.

We will now investigate each of these possible types of pure strategy equilibria of this game and obtain the existence and uniqueness conditions for each of them.

5.1 Existence and Uniqueness

Proposition 1. *A type AA equilibrium cannot exist.*

Proof. If $x^* = 0$ then, $\Pi_2 = p_2 * \min(M, S_2y) - C_2y$, which is maximized at $y = \frac{M}{S_2}$ because $C_2 < S_2p_2$. So $y^* > 0$ whenever $x^* = 0$. So this type of equilibrium cannot exist. \square

Proposition 2. *A type AB (and type BA) equilibrium cannot exist.*

Proof. Type AB equilibrium exists if and only if $x^* = 0$, $y^* > 0$ and $PAX_2 < S_2y^*$. As shown before, if $x^* = 0$ then, Π_2 is maximized at $y = \frac{M}{S_2}$ as long as $C_2 < S_2p_2$. So $PAX_2 = M = S_2y^*$ whenever $x^* = 0$. So this type of equilibrium cannot exist. By symmetry, type BA equilibrium cannot exist either. \square

Proposition 3. *A type AC equilibrium exists if and only if $\frac{C_1}{S_1p_1} \geq \frac{S_2}{S_1} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$ and if it exists, then it is a unique type AC equilibrium.*

Proof. This type of equilibrium requires $x^* = 0$ and $y^* = \frac{M}{S_2}$. So if an equilibrium of this type exists, then it must be the unique type AC equilibrium. For this equilibrium to exist, the only condition we need to check is that $\frac{M}{S_2} = y \geq y_{th} = (\alpha - 1)^{\frac{\alpha-1}{\alpha}} \frac{Mp_1}{\alpha C_1}$. For all $y^* = \frac{M}{S_2}$, $x^* = 0$ is true if and only if $\Pi_1 \leq 0$, for all $x \geq 0$. So type AC equilibrium will exist if and only if $\frac{C_1}{S_1 p_1} \geq \frac{S_2}{S_1} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$. \square

By symmetry, a type CA equilibrium exists if and only if $\frac{C_2}{S_2 p_2} \geq \frac{S_1}{S_2} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$ and if it exists, then it is the unique type CA equilibrium.

Proposition 4. *A type BB equilibrium exists if and only if $k \leq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\alpha}}$, $\frac{1}{k} \leq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\alpha}}$, $\frac{C_1}{S_1 p_1} < \alpha \frac{k^\alpha}{1+k^\alpha}$ and $\frac{C_2}{S_2 p_2} < \alpha \frac{1}{1+k^\alpha}$, where $k = \frac{C_1 p_2}{C_2 p_1}$, and if it exists, then it is a unique type BB equilibrium.*

Proof. In type BB equilibrium, $x^* > 0$, $y^* > 0$, $PAX_1 < S_1 x$ and $PAX_2 < S_2 y$. Therefore, $\Pi_1(x^*, y^*) = \Pi'_1(x^*, y^*)$ and $\Pi_2(x^*, y^*) = \Pi'_2(x^*, y^*)$. So Π_1 and Π_2 are both twice continuously differentiable at (x^*, y^*) . So type BB equilibrium exists if and only if there exist x and y such that $\frac{\partial \Pi'_1}{\partial x} = 0$, $\frac{\partial \Pi'_2}{\partial y} = 0$, $\frac{\partial^2 \Pi'_1}{\partial x^2} \leq 0$, $\frac{\partial^2 \Pi'_2}{\partial y^2} \leq 0$, $\Pi'_1 \geq 0$, $\Pi'_2 \geq 0$, $M \frac{x^\alpha}{x^\alpha + y^\alpha} < S_1 x$ and $M \frac{y^\alpha}{x^\alpha + y^\alpha} < S_2 y$. Solving the two FOCs simultaneously, we get $x = \frac{\alpha M p_1}{C_1} \frac{k^\alpha}{(1+k^\alpha)^2}$ and $y = \frac{\alpha M p_1}{C_1} \frac{k^{\alpha+1}}{(1+k^\alpha)^2}$. So if this equilibrium exists, then it must be the unique type BB equilibrium.

The second order conditions (SOCs) can be simplified to $k \leq \left(\frac{\alpha+1}{\alpha-1}\right)^{\frac{1}{\alpha}}$ and $\frac{1}{k} \leq \left(\frac{\alpha+1}{\alpha-1}\right)^{\frac{1}{\alpha}}$. Also the $\Pi'_1 \geq 0$ and $\Pi'_2 \geq 0$ translate into,

$$k \leq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\alpha}} \quad (7)$$

$$\text{and } \frac{1}{k} \leq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\alpha}}. \quad (8)$$

Conditions (7) and (8) make the second order conditions redundant. Finally, the last two conditions translate into,

$$\frac{C_1}{S_1 p_1} < \alpha \frac{k^\alpha}{1+k^\alpha} \quad (9)$$

$$\text{and } \frac{C_2}{S_2 p_2} < \alpha \frac{1}{1+k^\alpha}. \quad (10)$$

Therefore, type BB equilibrium exists if and only if conditions (7), (8), (9) and (10) are satisfied. \square

Proposition 5. *A type BC equilibrium exists if and only if $\frac{C_1 S_1}{p_1 S_1 S_2} \leq (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$, $\frac{k^\alpha}{1+k^\alpha} \leq \frac{1}{\alpha}$, $\frac{1}{1+k^\alpha} \leq \frac{1}{\alpha} \frac{C_2}{p_2 S_2}$ and $\frac{1}{\alpha} \frac{C_1}{p_1 S_1} \geq \frac{1}{1 + \left(\frac{S_1}{S_2}\right)^{\frac{\alpha-1}{\alpha}}}$, where $k = \frac{C_1 p_2}{C_2 p_1}$, and if it exists, then it is a unique type BC equilibrium.*

Proof. In type BC equilibrium, $x^* > 0$, $y^* > 0$, $PAX_1 < S_1 x$ and $PAX_2 = S_2 y$. Therefore $\Pi_1(x^*, y^*) = \Pi'_1(x^*, y^*)$. So $\Pi_1(x)$ is twice continuously differentiable at (x^*, y^*) . For local maxima of Π_2 at (x^*, y^*) , we need $\Pi'_2 = \Pi''_2$ and $\frac{\partial \Pi'_2}{\partial y} \leq 0$.

A type BC equilibrium then exists if and only if there exists (x, y) such that $\frac{\partial \Pi'_1}{\partial x} = 0$, $\Pi'_2 = \Pi''_2$, $\frac{\partial^2 \Pi'_1}{\partial x^2} \leq 0$, $\frac{\partial \Pi'_2}{\partial y} \leq 0$, $\Pi'_1 \geq 0$ and $M \frac{x^\alpha}{x^\alpha + y^\alpha} < S_1 x$. The first two conditions translate into,

$$\frac{x^{\alpha-1} y^\alpha}{(x^\alpha + y^\alpha)^2} = \frac{C_1}{\alpha M p_1} \text{ and } \frac{y^\alpha}{x^\alpha + y^\alpha} = \frac{S_2}{M} y.$$

Solving these two equations simultaneously we get,

$$x = \left(\frac{M C_1}{\alpha p_1 S_2^2} \right)^{\frac{1}{\alpha-1}} y^{\frac{\alpha-2}{\alpha-1}} \quad (11)$$

$$\text{and } \left(\frac{y S_2}{M} \right)^{\frac{1}{\alpha-1}} - \left(\frac{y S_2}{M} \right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{C_1}{\alpha p_1 S_2} \right)^{\frac{\alpha}{\alpha-1}} = 0. \quad (12)$$

The nonnegativity condition on airline 1's profit implies that $M p_1 \frac{x^\alpha}{x^\alpha + y^\alpha} \geq C_1 x$. Substituting equation (11) and (12) we get,

$$\frac{y S_2}{M} \leq \frac{1}{\alpha}. \quad (13)$$

The LHS of equation (12) is a strictly increasing function of y for $\frac{y S_2}{M} < \frac{1}{\alpha}$. Therefore, there exists a y that satisfies equation (12) and inequality (13) if and only if $\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{C_1}{\alpha p_1 S_2}\right)^{\frac{\alpha}{\alpha-1}} \geq 0$, i.e. if and only if

$$\frac{C_1 S_1}{p_1 S_1 S_2} \leq (\alpha - 1)^{\frac{\alpha-1}{\alpha}}, \quad (14)$$

and if it exists, then it is unique. Therefore, if a type BC equilibrium exists, then it must be a unique type BC equilibrium.

Simplifying the second order condition and substituting equation (11) and equation (12), we get $\frac{yS_2}{M} \leq \frac{\alpha+1}{2\alpha}$. Therefore, condition (13) makes the second order condition redundant.

First order condition on $\Pi'_2(y)$ simplifies to $\frac{x}{y} \leq \frac{C_2 p_1}{C_1 p_2}$. Substituting equation (11) and equation (12) we get,

$$\frac{yS_2}{M} \geq \frac{k^\alpha}{1+k^\alpha}. \quad (15)$$

Therefore, there exists a y that satisfies equation (12), inequality (13) and inequality (15) if and only if

$$\begin{aligned} \frac{k^\alpha}{1+k^\alpha} \leq \frac{1}{\alpha} \text{ and } \left(\frac{C_1}{\alpha p_1 S_2}\right)^{\frac{\alpha}{\alpha-1}} &\geq \left(\frac{k^\alpha}{1+k^\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{k^\alpha}{1+k^\alpha}\right)^{\frac{\alpha}{\alpha-1}} \\ \iff \frac{k^\alpha}{1+k^\alpha} &\leq \frac{1}{\alpha} \end{aligned} \quad (16)$$

$$\text{and } \frac{1}{1+k^\alpha} \leq \frac{1}{\alpha} \frac{C_2}{p_2 S_2}. \quad (17)$$

Finally, the last condition, i.e. the condition that the seating capacity exceeds the number of passengers for airline 1, simplifies to $\frac{x^{\alpha-1}}{y^{\alpha-1}} \leq \frac{S_1}{S_2}$. Substituting equation (11) we get,

$$\frac{yS_2}{M} \geq \frac{C_1}{\alpha p_1 S_1}. \quad (18)$$

Combining with inequality (13) we get, $\frac{a}{\alpha} \geq \frac{yS_2}{M} \geq \frac{C_1}{\alpha p_1 S_1}$. Therefore, there exists a y that satisfies equation (12), inequality (13) and inequality (18) if and only if,

$$\begin{aligned} \left(\frac{C_1}{\alpha p_1 S_2}\right)^{\frac{\alpha}{\alpha-1}} &\geq \left(\frac{C_1}{\alpha p_1 S_1}\right)^{\frac{1}{\alpha-1}} - \left(\frac{C_1}{\alpha p_1 S_1}\right)^{\frac{\alpha}{\alpha-1}} \\ \iff \frac{1}{\alpha} \frac{C_1}{p_1 S_1} &\geq \frac{1}{1 + \left(\frac{S_1}{S_2}\right)^{\frac{\alpha}{\alpha-1}}}. \end{aligned} \quad (19)$$

Therefore, type BC equilibrium exists if and only if inequality conditions (14), (16), (17) and (19) are satisfied. \square

By symmetry, a type CB equilibrium exists if and only if $\frac{C_2 S_2}{p_2 S_2 S_1} \leq (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$, $\frac{1}{1+k^\alpha} \leq \frac{1}{\alpha}$, $\frac{k^\alpha}{1+k^\alpha} \leq \frac{1}{\alpha} \frac{C_1}{p_1 S_1}$, $\frac{1}{\alpha} \frac{C_2}{p_2 S_2} \geq \frac{1}{1+\left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}}$, and if it exists, then it is a unique CB type equilibrium.

Proposition 6. *A type CC equilibrium exists if and only if $\frac{\left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}}{1+\left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}} \leq \frac{1}{\alpha} \frac{C_1}{S_1 p_1}$ and $\frac{1}{1+\left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}} \leq \frac{1}{\alpha} \frac{C_2}{S_2 p_2}$, and if it exists, then it is a unique type CC equilibrium.*

Proof. For type CC equilibrium, $x > 0$, $y > 0$, $PAX_1 = S_1 x$ and $PAX_2 = S_2 y$. Existence of local maxima of Π_1 at $x = x^*$ requires that $\frac{\partial \Pi_1'}{\partial x} \leq 0$. Similarly existence of local maxima of Π_2 at $y = y^*$ requires that $\frac{\partial \Pi_2'}{\partial y} \leq 0$. So for a type CC equilibrium to exist at (x, y) , the necessary and sufficient conditions to be satisfied are $\frac{x^\alpha}{x^\alpha + y^\alpha} M = S_1 x$, $\frac{y^\alpha}{x^\alpha + y^\alpha} M = S_2 y$, $\frac{\partial \Pi_1'}{\partial x} \leq 0$ and $\frac{\partial \Pi_2'}{\partial y} \leq 0$. Solving the two equalities simultaneously we get, $x = \frac{M}{S_1 \left(1 + \left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}\right)}$ and $y = \frac{M}{S_2 \left(1 + \left(\frac{S_1}{S_2}\right)^{\frac{\alpha}{\alpha-1}}\right)}$.

Therefore, if a type CC equilibrium exists, then it must be the unique type CC equilibrium. The two inequality conditions translate into,

$$\frac{\left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}}{1 + \left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}} \leq \frac{1}{\alpha} \frac{C_1}{S_1 p_1} \quad (20)$$

$$\text{and } \frac{1}{1 + \left(\frac{S_2}{S_1}\right)^{\frac{\alpha}{\alpha-1}}} \leq \frac{1}{\alpha} \frac{C_2}{S_2 p_2}. \quad (21)$$

Therefore, equation (20) and equation (21) together are necessary and sufficient conditions for type CC equilibrium to exist. \square

In any 2-player game, out of 9 possible types, 6 types of equilibria, namely AC, CA, BB, BC, CB and CC may exist depending on operating cost, fare and seating capacity values. Furthermore, all the necessary and sufficient conditions for the existence and uniqueness of each type of equilibrium can be expressed in terms of only 5 dimensionless parameters namely, $r_1 = \frac{C_1}{p_1 S_1}$, $r_2 = \frac{C_2}{p_2 S_2}$, $k = \frac{C_1 p_2}{C_2 p_1}$, $l = \frac{S_1}{S_2}$ and α , out of which l can be expressed as a function of the rest as $l = k \frac{r_2}{r_1}$. So there are only 4 independent parameters, which completely describe a 2-player frequency game. The total passenger demand M plays no part in any of the conditions.

5.2 Realistic Parameter Ranges

Up to 6 different pure strategy Nash equilibria may exist for a 2-player game depending on game parameters. Apart from α , the flight operating costs, seating capacities and fares are the only determinants of these parameters. In order to identify realistic ranges of these parameters, we looked at all the domestic segments in the United States with exactly 2 carriers providing non-stop service. There are 157 such segments that satisfied this criteria. Many of these markets cannot be classified as pure duopoly situations because passenger demand on many of these origin-destination pairs is served not only by the nonstop itineraries, but also by connecting itineraries offered by several carriers, often including the two carriers providing the nonstop service. Moreover, one or both endpoints for many of these non-stop segments are important hubs of one or both of these nonstop carriers, which means that connecting passengers traveling on this segment also play an important role in the profitability of this segment. Therefore, modeling these nonstop markets as pure duopoly cases can be a gross approximation. Our aim is not to capture all these effects into our frequency competition model but rather to identify realistic relative values of flight operating costs, seating capacities and fares. Despite these complications, these 157 segments are the real-world situations that come closest to the simplified frequency competition model that we have considered. Therefore, data from these markets were used to narrow down our modeling focus. Figure 5 shows the histograms of k , $\frac{S_1}{S_2}$ and $\frac{C}{pS}$. All k values were found to lie in the range 0.4 to 2.5, all $\frac{S_1}{S_2}$ were in the range 0.5 to 2 and all $\frac{C}{pS}$ values were found to lie in the range 0.18 to 0.8. We will restrict our further analysis to these ranges of values only. In particular, for later analysis, we will need only one of these assumptions, which is given by,

Assumption 3. $0.4 \leq k \leq 2.5$

For $\alpha = 1.5$, the conditions for type BB equilibrium were satisfied in 144 out of these 157 markets, i.e. almost 92% of the times. Conditions for type AC (or CA) equilibrium were satisfied in 71 markets, of which 8 were such that the conditions for both type AC and type CA equilibrium were satisfied together. Conditions for type BC (or CB) equilibrium were satisfied in only 1 out of 157 markets and conditions for type CC equilibrium were never satisfied. In all the markets, the conditions for the existence of at least one pure strategy Nash equilibrium were satisfied. Out of 157 markets, almost 55% (86 markets) were such that type BB was the unique pure strategy Nash equilibrium.

We have already proved that AA, AB and BA type equilibria do not exist. Further, as discussed above, AC, CA, BC, CB and CC type equilibria are suspect in terms of portrayal of reality. Therefore, type BB equilibrium appears to be the most reasonable type of equilibrium. Indeed, the data analysis suggested that the existence conditions for type BB equilibrium were

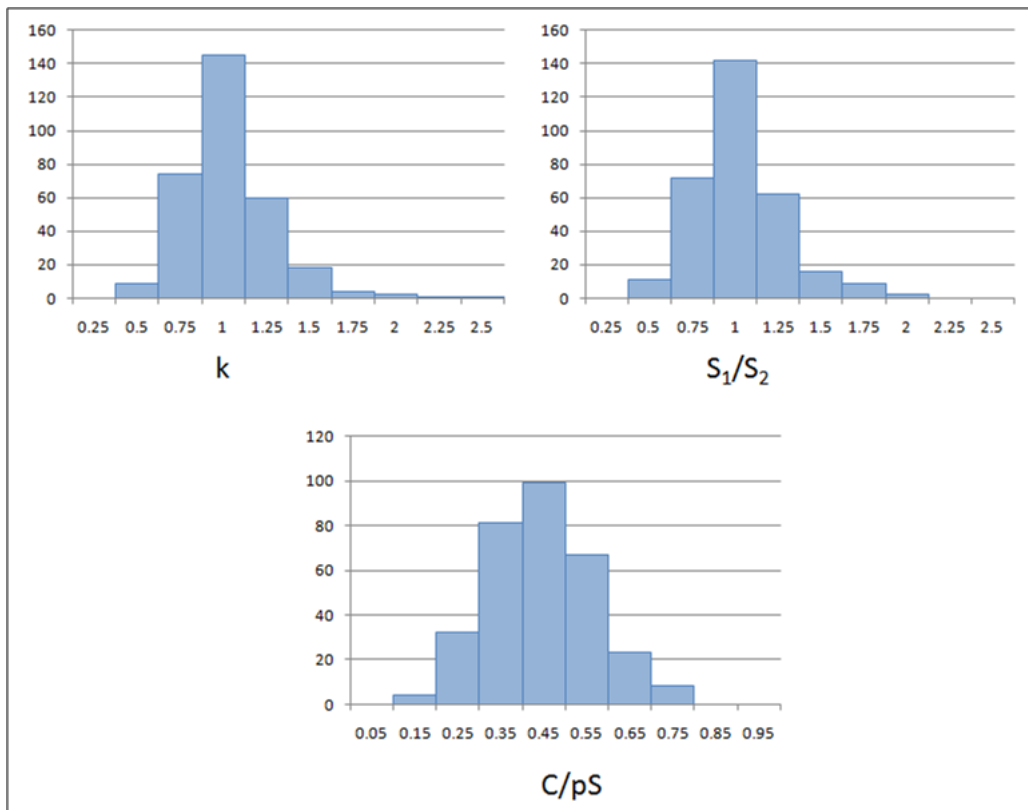


Figure 5: Histograms of Parameter Values

satisfied in most of the markets. So for the purpose of analyzing learning dynamics we will only consider the type BB equilibrium.

Now, we propose two alternative dynamics for the non-equilibrium situations.

5.3 Myopic Best Response Dynamic

Consider an adjustment process where the two players take turns to adjust their own frequency decision so that each time it is the best response to the frequency chosen by the competitor in the previous period. If x^i and y^i is the frequency decision by each carrier in period i , then x^i is the best response to y^{i-1} and y^{i-1} is the best response to x^{i-2} etc. We will prove the convergence of this dynamics for two representative values of α namely $\alpha = 1$ and $\alpha = 1.5$. We chose these two values because they correspond to two disparate beliefs about the market share-frequency share relationship. There is nothing specific about these two values that makes the algorithm converge. In fact given any value in between, we would probably be able to come up with a proof of convergence. But due to space constraints we will restrict our attention to these two specific values of α .

Let us define $\chi = x^\alpha$ and $\gamma = y^\alpha$. We will often use the $\chi - \gamma$ coordinate system in this section. Without any loss of generality, we assume that $k = \frac{C_1 p_2}{C_2 p_1} \leq 1$. We will denote the best response functions as $x_{BR}(y)$ and $y_{BR}(x)$ in the $x - y$ coordinate system and as $\chi_{BR}(\gamma)$ and $\gamma_{BR}(\chi)$ in the $\chi - \gamma$ coordinate system. Consider a two-dimensional interval I given by $x_{lb} \leq x \leq x_{ub}$, $y_{lb} \leq y \leq y_{ub}$ where,

$$\begin{aligned} y_{ub} &= \frac{\alpha M p_2}{4 C_2} \\ x_{ub} &= x_{BR}(y_{ub}) \\ y_{lb} &= y_{BR}(x_{ub}) \\ x_{lb} &= x_{BR}(y_{lb}) \end{aligned}$$

Figure 6 provides a pictorial depiction of interval I .

Proposition 7. *As long as the competitor frequency for each carrier remains in region B , regardless of the starting point: (a) the myopic best response algorithm will reach a point in interval I in a finite number of iterations, (b) once inside interval I , it will never leave the interval.*

Proof. Let us denote the frequency decisions of the two carriers after the i^{th} iteration by x^i and y^i respectively. At the beginning of the algorithm the frequency values are arbitrarily chosen

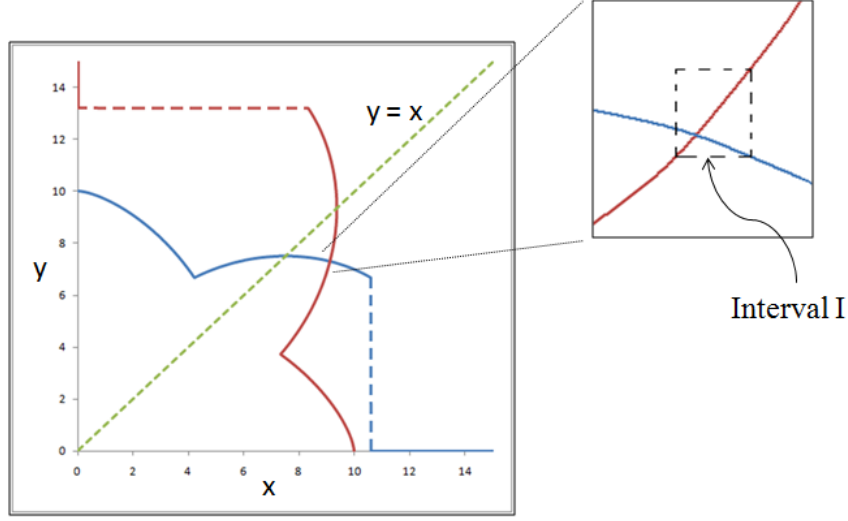


Figure 6: Best Response Curves in 2-player Game

to be x^0 and y^0 . If $i \geq 0$ is odd, then $x^i = x_{BR}(y^{i-1})$ and $y^i = y^{i-1}$. If $i \geq 0$ is even, then $y^i = y_{BR}(x^{i-1})$ and $x^i = x^{i-1}$.

Therefore for all $i \geq 2$, x_i is a best response to some y and y_i is a best response to some x . Best response curve $x_{BR}(y)$ in region B has a unique maximum at $y = \frac{\alpha M p_1}{4C_1}$ with $x_{BR}\left(\frac{\alpha M p_1}{4C_1}\right) = \frac{\alpha M p_1}{4C_1}$. By symmetry, the best response curve $y_{BR}(x)$ in region B has a unique maximum at $x = \frac{\alpha M p_2}{4C_2}$ with $y_{BR}\left(\frac{\alpha M p_2}{4C_2}\right) = \frac{\alpha M p_2}{4C_2}$. $k \leq 1$ implies that $\frac{\alpha M p_2}{4C_2} \leq \frac{\alpha M p_1}{4C_1}$. Therefore, $y^i \leq \frac{\alpha M p_2}{4C_2} = y_{ub}$ for all $i \geq 2$. $\frac{\partial x_{BR}}{\partial y} > 0$ for $y < \frac{\alpha M p_1}{4C_1}$. Therefore, for all $i \geq 3$, $x^i = x_{BR}(y^{i-1}) \leq x_{BR}(y_{ub}) = x_{ub}$. So for all $i \geq 3$, $y^i \leq y_{ub}$ and $x^i \leq x_{ub}$.

Let us now prove that the type BB equilibrium point (x_{eq}, y_{eq}) is contained inside interval I . y_{eq} is a best response to x_{eq} . Therefore, $y_{eq} \leq \frac{\alpha M p_2}{4C_2} = y_{ub}$. For $k \leq 1$, $x_{eq} = \frac{\alpha M p_2}{4C_2} \frac{4k^{\alpha-1}}{(1+k^\alpha)^2} \geq \frac{\alpha M p_2}{4C_2}$ and $y_{eq} = \frac{\alpha M p_1}{4C_1} \frac{4k^{\alpha+1}}{(1+k^\alpha)^2} \leq \frac{\alpha M p_1}{4C_1}$.

For all $y_{eq} \leq y \leq y_{ub}$, $\frac{\partial x_{BR}}{\partial y} \geq 0 \Rightarrow x_{eq} = x_{BR}(y_{eq}) \leq x_{BR}(y_{ub}) = x_{ub}$.

For all $x_{eq} \leq x \leq x_{ub}$, $\frac{\partial y_{BR}}{\partial x} \leq 0 \Rightarrow y_{eq} = y_{BR}(x_{eq}) \geq y_{BR}(x_{ub}) = y_{lb}$.

For all $y_{lb} \leq y \leq y_{eq}$, $\frac{\partial x_{BR}}{\partial y} \geq 0 \Rightarrow x_{eq} = x_{BR}(y_{eq}) \geq x_{BR}(y_{lb}) = x_{lb}$.

Thus, we have proved that $x_{lb} \leq x_{eq} \leq x_{ub}$, $y_{lb} \leq y_{eq} \leq y_{ub}$, that is, the type BB equilibrium is contained inside interval I .

Because of existence of a unique type BB equilibrium, the best response curves intersect each other at exactly one point denoted by (x_{eq}, y_{eq}) . Further, for all $x < x_{eq}$ and for all $y < y_{eq}$,

the y_{BR} curve is above the x_{BR} curve and x_{BR} curve is to the right of y_{BR} curve. Also, for all $y < y_{eq}$, $x_{BR}(y) < x_{eq}$. Therefore, for all $x_i < x_{eq}$, if i is odd then $x_{i+1} = x_i$, $y_i < y_{i+1} \leq y_{ub}$ and if i is even then $x_i < x_{i+1} < x_{eq}$, $y_{i+1} = y_i$. So in each iteration, either x_i or y_i keeps strictly increasing until $y_i \geq y_{eq}$. In the very next iteration, $x_{i+1} = x_{BR}(y_i) \geq x_{eq}$ and $y_{i+1} = y_i \geq y_{eq}$. Thus, $x_{lb} \leq x_{eq} \leq x_{i+1} \leq x_{ub}$ and $y_{lb} \leq y_{eq} \leq y_{i+1} \leq y_{ub}$. We have proved part (a) of the proposition.

We have already proved that at the end of any iteration $i \geq 2$, $x_i \leq x_{ub}$ and $y_i \leq y_{ub}$. So for all i such that $x_{lb} \leq x_i \leq x_{ub}$ and $y_{lb} \leq y_i \leq y_{ub}$, all that remains to be proved is that $x_{lb} \leq x_{i+1}$ and $y_{lb} \leq y_{i+1}$. We first consider the case where i is even. $y_{i+1} = y_i$. As proved earlier, for all y such that $y_{lb} \leq y \leq y_{ub}$, $\frac{\partial x_{BR}}{\partial y} \geq 0$. Therefore, $y_{lb} \leq y_i \leq y_{ub} \Rightarrow x_{lb} = x_{BR}(y_{lb}) \leq x_{BR}(y_i) = x_{i+1} \leq x_{BR}(y_{ub}) = x_{ub}$. Therefore, $x_{lb} \leq x_{i+1} \leq x_{ub}$ and $y_{lb} \leq y_{i+1} \leq y_{ub}$. Now consider the case where i is odd. $x_{i+1} = x_i$. For all x_i such that $x_{eq} \leq x_i \leq x_{ub}$, $\frac{\partial y_{BR}}{\partial x} \leq 0$. Therefore, $y_{lb} = y_{BR}(x_{ub}) \leq y_{BR}(x_i) = y_{i+1}$. On the other hand, for all $x_i < x_{eq}$, $y_i < \frac{\alpha M p_1}{4 C_1}$, $y_{i+1} = y_{BR}(x_i) > y_i \geq y_{lb}$. Therefore, if $x_{lb} \leq x_i \leq x_{ub}$, then $y_{lb} \leq y_{i+1}$. Thus we have proved that $x_{lb} \leq x_{i+1} \leq x_{ub}$ and $y_{lb} \leq y_{i+1} \leq y_{ub}$, if i is odd. Therefore, for any i such that (x_i, y_i) is in interval I , (x_{i+1}, y_{i+1}) is also in interval I . We have proved part (b) of the proposition. \square

Now we will prove that the absolute value of the slope of each of the best response curves inside interval I is less than 1 in the $\chi - \gamma$ coordinates. We will prove this for two representative values of α namely, $\alpha = 1.5$ and $\alpha = 1$.

Proposition 8. *For $\alpha = 1.5$, the absolute value of the slope of each of the best response curves inside interval I is less than 1 in the $\chi - \gamma$ coordinates.*

Proof. We will first prove that at $x = x_{ub}$, $|\frac{\partial \gamma_{BR}(\chi)}{\partial \chi}| < 1$.

$$\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} = -\alpha \frac{\gamma}{\chi} \frac{1 - \frac{\gamma}{\chi}}{(\alpha + 1) \frac{\gamma}{\chi} - (\alpha - 1)}$$

The denominator of the right hand side (RHS) is always positive, due to the second order conditions. At $x = x_{ub}$, $x \geq y_{BR}(x)$, and hence, $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} \leq 0$. For $\alpha = 1.5$, solving for the point where $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} = -1$, leads to a unique solution given by (x_{-1}, y_{-1}) , where,

$$y_{-1} = \frac{9}{32} \frac{M p_2}{C_2} \text{ and } x_{-1} = 3^{\frac{2}{3}} \frac{9}{32} \frac{M p_2}{C_2}.$$

Because $x_{ub} = x_{BR} \left(\frac{\alpha Mp_2}{4C_2} \right)$, we get,

$$\frac{4}{k} = \left(\frac{4C_2 x_{ub}}{1.5Mp_2} \right)^{2.5} + 2 \left(\frac{4C_2 x_{ub}}{1.5Mp_2} \right) + \left(\frac{4C_2 x_{ub}}{1.5Mp_2} \right)^{-0.5}.$$

Define $f(x) = \left(\frac{4C_2 x}{1.5Mp_2} \right)^{2.5} + 2 \left(\frac{4C_2 x}{1.5Mp_2} \right) + \left(\frac{4C_2 x}{1.5Mp_2} \right)^{-0.5}$. $f(x)$ is a strictly increasing function of x for $x \geq \frac{1.5Mp_2}{4C_2}$. $f(x_{ub}) = \frac{4}{k}$ and $f(x_{-1}) \approx 6.96$. $f(x_{ub}) < f(x_{-1})$ if and only if $k \geq 0.575$, which is always satisfied because one of the necessary conditions for the existence of type BB equilibrium requires that $k \geq (\alpha - 1)^{\frac{1}{\alpha}} = 0.5^{\frac{2}{3}} > 0.575$. Therefore, $x_{ub} < x_{-1}$. Thus, we have proved that at $x = x_{ub}$, $-1 < \frac{\partial \gamma_{BR}(x)}{\partial x} < 0$.

Also for $x \geq \frac{\alpha Mp_2}{4C_2}$, $\frac{\partial y_{BR}}{\partial x} \leq 0$, therefore $y_{-1} = y_{BR}(x_{-1}) < y_{BR}(x_{ub}) = y_{lb}$. Next, we will obtain the coordinates of the point (which turns out to be unique) such that $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} = 1$ and prove that the y-coordinate at this point is less than y_{lb} . The condition,

$$\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} = 1.5 \frac{\chi}{\gamma} \frac{\frac{\chi}{\gamma} - 1}{(1.5 + 1) \frac{\chi}{\gamma} - (1.5 - 1)} = 1,$$

can be simplified to obtain,

$$x \approx 0.2029 \frac{1.5Mp_1}{C_1} \text{ and } y \approx 0.1091 \frac{1.5Mp_1}{C_1}.$$

Because $k \geq (\alpha - 1)^{\frac{1}{\alpha}} = 0.5^{\frac{2}{3}} > 0.589$, we get $y_{lb} > y_{-1} > 0.1091 \frac{1.5Mp_1}{C_1}$. So the y-coordinate of the point at which $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} = 1$ is less than y_{lb} . Because $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} \geq 0$ throughout interval I , $0 \leq \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} < 1$ for the $\chi_{BR}(\gamma)$ curve at $y = y_{lb}$.

Now, let us obtain the coordinates of the point (which turns out to be unique) such that $\frac{\partial \gamma_{BR}(x)}{\partial x} = 1$ and prove that the x-coordinate of this point is less than x_{lb} . Solving for $\frac{\partial \gamma_{BR}(x)}{\partial x} = 1$ we get,

$$y \approx 0.2029 \frac{1.5Mp_2}{C_2} \text{ and } x \approx 0.1091 \frac{1.5Mp_2}{C_2}.$$

In order to prove that $0.1091 \frac{1.5Mp_2}{C_2} < x_{lb} = x_{BR}(y_{lb})$, it is sufficient to prove that the y-coordinate of the point on the lower part of $x_{BR}(y)$ curve at which $x = 0.1091 \frac{1.5Mp_2}{C_2}$ is less than $y_{-1} = \frac{9}{32} \frac{Mp_2}{C_2}$. This is easy to prove because for $y < \frac{\alpha Mp_1}{4C_1}$, the $x_{BR}(y)$ curve lies below $y = x$

line. Therefore, the y -coordinate corresponding to $x = 0.1091 \frac{1.5Mp_2}{C_2}$ is less than $0.1091 \frac{1.5Mp_2}{C_2}$ which is less than $\frac{9}{32} \frac{Mp_2}{C_2}$. Therefore, at $x = x_{lb}$, $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} < 1$.

So far we have proved that $-1 < \frac{\partial \gamma_{BR}(\chi)}{\partial \chi} \leq 0$ at $x = x_{ub}$ and $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} < 1$ at $x = x_{lb}$. Therefore, $-1 < \frac{\partial \gamma_{BR}(\chi)}{\partial \chi} < 1$ for all x such that $x_{lb} \leq x \leq x_{ub}$. Also we have proved that $0 \leq \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} < 1$ at $y = y_{lb}$ and $0 \leq \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma}$ at $y = y_{ub}$. Therefore, $-1 < \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} < 1$ for all y such that $y_{lb} \leq y \leq y_{ub}$.

Therefore for $\alpha = 1.5$, the absolute value of the slopes of each of the best response curves inside interval I is less than 1 in the $\chi - \gamma$ coordinates. \square

Proposition 9. *For $\alpha = 1$, the absolute value of the slope of each of the best response curves inside interval I is less than 1 in the $\chi - \gamma$ coordinates.*

Proof. For $\alpha = 1$, the $\chi - \gamma$ coordinate system is the same as the $x - y$ coordinate system. We will first prove that at $x = x_{ub}$, $|\frac{\partial \gamma_{BR}(\chi)}{\partial \chi}| < 1$.

For $\alpha = 1$,

$$\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} = -\frac{1}{2} \left(1 - \frac{\gamma}{\chi} \right) > -\frac{1}{2}.$$

We know that at $x = x_{ub}$, $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} \leq 0$. Therefore, $x = x_{ub}$, $|\frac{\partial \gamma_{BR}(\chi)}{\partial \chi}| < 1$.

Next, we will obtain the coordinates of the point (which turns out to be unique) such that $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} = 1$ and prove that the y -coordinate at this point is less than y_{lb} . Solving for $\frac{\partial \chi(\gamma_{BR})}{\partial \gamma} = \frac{\chi - 1}{2} = 1$, we get,

$$x = \frac{3Mp_1}{16C_1} \text{ and } y = \frac{Mp_1}{16C_1}.$$

For $x \geq \frac{Mp_2}{4C_2}$, we have $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} \leq 0$ and for $y \leq \frac{Mp_1}{4C_1}$, we have $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} \geq 0$. $y_{ub} = \frac{Mp_2}{4C_2} \leq \frac{Mp_1}{4C_1}$. So $x_{ub} = x_{BR}(y_{ub}) \leq x_{BR}\left(\frac{Mp_1}{4C_1}\right) = \frac{Mp_1}{4C_1}$. So we get $y_{lb} = y_{BR}(x_{ub}) \geq y_{BR}\left(\frac{Mp_1}{4C_1}\right)$. As per the first order conditions,

$$\frac{\frac{Mp_1}{4C_1}}{\left(y_{BR}\left(\frac{Mp_1}{4C_1}\right) + \frac{Mp_1}{4C_1}\right)^2} = \frac{C_2}{Mp_2} \iff y_{BR}\left(\frac{Mp_1}{4C_1}\right) = \frac{Mp_1}{4C_1} (2\sqrt{k} - 1).$$

$y_{lb} \geq y_{BR}\left(\frac{Mp_1}{4C_1}\right) = \frac{Mp_1}{4C_1} (2\sqrt{k} - 1) > \frac{Mp_1}{16C_1}$ because $k \geq 0.4$. Therefore, the y coordinate of the point where $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} = 1$ is less than y_{lb} .

Now, let us obtain the coordinates of the point (which turns out to be unique) such that $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} = 1$ and prove that the x-coordinate of this point is less than x_{lb} . Solving for $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} = 1$, we get,

$$x = \frac{Mp_2}{16C_2} \text{ and } y = \frac{3Mp_2}{16C_2}.$$

Because $\frac{Mp_1}{4C_1} \geq y_{lb} > \frac{Mp_1}{16C_1}$, and $\frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} > 0$ for $y < \frac{Mp_1}{4C_1}$, we get $x_{lb} = x_{BR}(y_{lb}) > x_{BR}\left(\frac{Mp_1}{16C_1}\right) = \frac{3Mp_1}{16C_1} > \frac{Mp_2}{16C_2}$. The last inequality holds because $k \leq 1$. Therefore, the x coordinate at the point where $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} = 1$ is less than x_{lb} .

Thus we have proved that $-1 < \frac{\partial \gamma_{BR}(\chi)}{\partial \chi} \leq 0$ at $x = x_{ub}$ and $\frac{\partial \gamma_{BR}(\chi)}{\partial \chi} < 1$ at $x = x_{lb}$. Therefore, $-1 < \frac{\partial \gamma_{BR}(\chi)}{\partial \chi} < 1$ for all x such that $x_{lb} \leq x \leq x_{ub}$. Also we have proved that $0 \leq \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} < 1$ at $y = y_{lb}$ and $0 \leq \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma}$ at $y = y_{ub}$. Therefore, $-1 < \frac{\partial \chi_{BR}(\gamma)}{\partial \gamma} < 1$ for all y such that $y_{lb} \leq y \leq y_{ub}$.

Therefore, for $\alpha = 1$, the absolute value of the slopes of each of the best response curves inside interval I is less than 1 in the $\chi - \gamma$ coordinates. \square

In order to prove the next proposition, we assume that the absolute value of slope of each of the best response curves is less than 1 in interval I .

Proposition 10. *If the absolute value of slope of each of the best response curves is less than 1 in interval I , then as long as the competitor frequency for each carrier remains in region B , regardless of the starting point, the myopic best response algorithm converges to the unique type BB equilibrium.*

Proof. We have assumed that the absolute value of slope of each of the best response curves is less than 1 in interval I . Also we have proved that as long as the competitor frequency for each carrier remains in region B , regardless of the starting point the myopic best response algorithm will reach a point in interval I in a finite number of iterations and once inside interval I , it will never leave the interval.

Let (χ_{eq}, γ_{eq}) be the type BB equilibrium point in the $\chi - \gamma$ coordinate system. We define a sequence $L(i)$ as follows:

$$L(i) = \begin{cases} |\chi_i - \chi_{eq}| & \text{if } i \text{ is odd} \\ |\gamma_i - \gamma_{eq}| & \text{if } i \text{ is even.} \end{cases}$$

Let us consider any iteration i after the algorithm has reached inside the interval I . We will prove that once inside interval I , $L(i)$ is strictly decreasing.

Let us first consider the case where i is odd. $L(i) = |\chi_i - \chi_{eq}|$. In the $(i+1)^{th}$ iteration, χ value remains unchanged. Only the γ value changes from γ_i to γ_{i+1} .

$$\begin{aligned} L(i+1) &= |\gamma_{i+1} - \gamma_{eq}| = |\gamma_{BR}(\chi_i) - \gamma_{BR}(\chi_{eq})| = \left| \int_{\chi_{eq}}^{\chi_i} \left(\frac{\partial \gamma(\chi)}{\partial \chi} \right) d\chi \right| \\ &\leq \left| \int_{\chi_{eq}}^{\chi_i} \left| \frac{\partial \gamma(\chi)}{\partial \chi} \right| d\chi \right| < \left| \int_{\chi_{eq}}^{\chi_i} 1 d\chi \right| = |\chi_i - \chi_{eq}| = L(i) \end{aligned}$$

We have proved that once inside interval I , $L(i)$ is strictly decreasing for odd values of i . By symmetry, the same is true for even values of i . Moreover, $L(i) = 0$ if and only if $x = x_{eq}$ and $y = y_{eq}$. Therefore, $L(i)$ is a decreasing sequence which is bounded below. So it converges to the unique type BB equilibrium point. \square

Proposition 11. *Regardless of the starting point, the myopic best response algorithm converges to the unique type BB equilibrium as long as the following conditions are satisfied.*

$$\begin{aligned} \frac{\alpha M p_1}{4C_1} &< x_{th} \\ \frac{\alpha M p_2}{4C_2} &< y_{th} \\ x_{cr} &< x_{BR}(y_{th}) \\ y_{cr} &< y_{BR}(x_{th}) \\ x_{cr} &< x_{BR}(y_{cr}) \\ y_{cr} &< y_{BR}(x_{cr}) \end{aligned}$$

Proof. First we develop sufficient conditions under which the competitor frequency for each carriers remains in region B for all iterations $i \geq 2$, regardless of the starting point.

As proved earlier, the shape of the best response curve $y_{BR}(x)$ is such that at $x = 0$, $y = \frac{M}{S_2}$. It is initially strictly decreasing followed by a point of nondifferentiability (at x_{cr}) beyond which it is strictly increasing until a local maximum is reached at $x = \frac{\alpha M p_2}{4C_2}$. Beyond the local maximum, it is strictly decreasing again to a point of discontinuity (at x_{th}), beyond which it takes a constant value 0. For $x \leq x_{th}$, the only candidates for global minima of the best response

curve $y_{BR}(x)$ are x_{cr} and x_{th} . The only candidates for global maxima are $x = 0$ and $x = \frac{\alpha Mp_2}{4C_2}$. If the y-coordinate at each of these four important points lies in the range $y_{cr} < y < y_{th}$, then $y_{cr} < y_{BR}(x) < y_{th}$ for all $x < x_{th}$. Similarly, if $x_{BR}(y)$ at $y = 0, y = y_{cr}, y = \frac{\alpha Mp_1}{4C_1}$, and $y = y_{th}$ are all in the range $x_{cr} < x < x_{th}$, then $x_{cr} < x_{BR}(y) < x_{th}$ for all $y < y_{th}$. So for any starting point x_0 such that $x_0 < x_{th}$, the algorithm will remain in the region B of both carriers for all subsequent iterations. The only remaining case is when $x \geq x_{th}$ or $y \geq y_{th}$. This does not pose any problem because for all $x \geq x_{th}, x_{BR}(y_{BR}(x)) = x_{BR}(0)$ and $x_{cr} < x_{BR}(0) = \frac{M}{S_2} < x_{th}$. So if the aforementioned conditions are satisfied, then regardless of the starting point, the algorithm will remain in the region B of both carriers for all iterations i such that $i \geq 2$.

For all the aforementioned conditions to be satisfied, it is sufficient to ensure that the upper bound conditions on the points of local maxima are satisfied and the lower bound conditions on the points of local minima are satisfied. Let us first look at the upper bounds on the points of local maxima. There are 4 such conditions per carrier, namely $\frac{M}{S_1} < x_{th}, \frac{M}{S_2} < y_{th}, \frac{\alpha Mp_1}{4C_1} < x_{th}$ and $\frac{\alpha Mp_2}{4C_2} < y_{th}$. $\frac{M}{S_1} < x_{th}$ simplifies to,

$$\frac{C_2}{S_2 p_2} < \frac{S_1}{S_2} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$$

which is the exact negation of the condition for existence of type CA equilibrium. Because we have assumed that the only unique PSNE in this game is a type BB equilibrium, a type CA equilibrium cannot exist. Hence this condition is automatically satisfied. By symmetry, due to the non-existence of a type AC equilibrium, the condition $\frac{M}{S_2} < y_{th}$ is automatically satisfied.

The remaining six conditions are as follows:

$$\begin{aligned} \frac{\alpha Mp_1}{4C_1} &< x_{th} \\ \frac{\alpha Mp_2}{4C_2} &< y_{th} \\ x_{cr} &< x_{BR}(y_{th}) \\ y_{cr} &< y_{BR}(x_{th}) \\ x_{cr} &< x_{BR}(y_{cr}) \\ y_{cr} &< y_{BR}(x_{cr}) \end{aligned}$$

If each of these conditions is satisfied then the myopic best response algorithm converges to the unique type BB equilibrium, regardless of the starting point. \square

Out of the 157 records of 2-player cases analyzed for the domestic US segments, 86 segments were such that the only PSNE was a type BB equilibrium. In each and every one of these 86

cases, all the 6 conditions mentioned above were satisfied. Therefore, for each of these 86 cases, the myopic best response dynamic converges to the unique type BB equilibrium point regardless of the starting point. Thus the data analysis suggests that these conditions are very mild.

5.4 Alternative Dynamic

This dynamic is applicable only in the part of region B where the utility function is strictly concave i.e. we will consider the region where $\frac{\alpha-1}{\alpha+1} + \epsilon \leq \left(\frac{x}{y}\right)^\alpha \leq \frac{\alpha+1}{\alpha-1} - \epsilon$, where ϵ is any sufficiently small positive number. This requirement is not very restrictive. This condition is always satisfied at the type BB equilibrium, due to the second order conditions. Moreover, the $\frac{x}{y}$ values satisfying this condition cover a large region surrounding the type BB equilibrium. For example, for $\alpha = 1$, this condition is always satisfied for all values of $\frac{x}{y}$, while for $\alpha = 1.5$, the condition translates approximately to $0.342 \leq \frac{x}{y} \leq 2.924$, which is a large range. In order to provide a complete specification of the player utilities, we will define the player i utility outside this region by means of a quadratic function of a single variable x_i . The coefficients are such that $u_i(x_i)$ and its first and second order derivatives with respect to x_i are continuous.

Multiplying the utility function by a positive real number is an order preserving transformation, which does not affect the properties of the game. We will multiply the utility of player i by $\frac{1}{p_i}$. So $u_i = \frac{\Pi_i}{p_i}$. This dynamic was proposed by Rosen [19]. Under this dynamic, each player changes his strategy such that his own utility would increase if all the other players held to their current strategies. The rate of change of each player's strategy with time is equal to the gradient of his utility with respect to his own strategy, subject to constraints. For the frequency competition game, where each player's strategy space is 1-dimensional, the rate of change of each player's strategy simply equals the derivative of the player's utility with respect to the frequency decision, subject to the upper and lower bound on allowable frequency values. Therefore, the rate of adjustment of each player's strategy is given by,

$$\frac{dx_i}{dt} = \frac{du_i(x)}{dx_i} + b_{min} - b_{max}.$$

The only purpose of the summation term is to ensure that the frequency values stay within the allowable range, $x_{min} \leq x \leq x_{max}$. b_{min} will be equal to 0 for all $x > x_{min}$ and will take an appropriate positive value at $x = x_{min}$ to ensure that the lower bound is respected. Similarly, b_{max} will be equal to 0 for all $x < x_{max}$ and will take an appropriate positive value at $x = x_{max}$ to ensure that the upper bound is respected. As long as the competitor frequencies remain in region B for each carrier, the utilities are given by:

$$u_1(x, y) = M \frac{x^\alpha}{x^\alpha + y^\alpha} - \frac{C_1}{p_1} x \text{ and } u_2(x, y) = M \frac{y^\alpha}{x^\alpha + y^\alpha} - \frac{C_2}{p_2} y.$$

The vector of utility functions $u(x, y)$ is given by: $u(x, y) = [u_1(x, y), u_2(x, y)]$. The vector of first order derivatives of each player's utility with respect to his own frequency is given by:

$$\nabla u(x, y) = \left[\frac{\partial u_1(x, y)}{\partial x}, \frac{\partial u_2(x, y)}{\partial y} \right]. \text{ The Jacobian of } \nabla u \text{ is given by: } U(x, y) = \begin{pmatrix} \frac{\partial^2 u_1(x, y)}{\partial x^2} & \frac{\partial^2 u_1(x, y)}{\partial x \partial y} \\ \frac{\partial^2 u_2(x, y)}{\partial y \partial x} & \frac{\partial^2 u_2(x, y)}{\partial y^2} \end{pmatrix}$$

The first order derivatives are given by,

$$\begin{aligned} \frac{\partial u_1(x, y)}{\partial x} &= \frac{M\alpha x^{\alpha-1} y^\alpha}{(x^\alpha + y^\alpha)^2} - \frac{C_1}{p_1} \\ \text{and } \frac{\partial u_2(x, y)}{\partial y} &= \frac{M\alpha y^{\alpha-1} x^\alpha}{(x^\alpha + y^\alpha)^2} - \frac{C_2}{p_2}, \end{aligned}$$

and the second order derivatives are given by,

$$\begin{aligned} [U(x, y)]_{11} &= \frac{\partial^2 u_1(x, y)}{\partial x^2} = \frac{M\alpha x^{\alpha-2} y^\alpha}{(x^\alpha + y^\alpha)^3} ((\alpha - 1) y^\alpha - (\alpha + 1) x^\alpha) < 0 \\ [U(x, y)]_{22} &= \frac{\partial^2 u_2(x, y)}{\partial y^2} = \frac{M\alpha y^{\alpha-2} x^\alpha}{(x^\alpha + y^\alpha)^3} ((\alpha - 1) x^\alpha - (\alpha + 1) y^\alpha) < 0 \\ [U(x, y)]_{12} &= \frac{\partial^2 u_1(x, y)}{\partial x \partial y} = \frac{M\alpha^2 x^{\alpha-1} y^{\alpha-1}}{(x^\alpha + y^\alpha)^3} (x^\alpha - y^\alpha) \\ [U(x, y)]_{21} &= \frac{\partial^2 u_2(x, y)}{\partial y \partial x} = \frac{M\alpha^2 x^{\alpha-1} y^{\alpha-1}}{(x^\alpha + y^\alpha)^3} (y^\alpha - x^\alpha) \\ \Rightarrow [(U(x, y) + U^T(x, y))]_{11} &= 2[U(x, y)]_{11} \\ \text{and } [(U(x, y) + U^T(x, y))]_{22} &= 2[U(x, y)]_{22} \\ \text{and } [(U(x, y) + U^T(x, y))]_{12} &= [(U(x, y) + U^T(x, y))]_{12} = 0. \end{aligned}$$

Therefore, $(U(x, y) + U^T(x, y))$ is a diagonal matrix with both diagonal elements strictly negative. Therefore, $(U(x, y) + U^T(x, y))$ is negative definite. This is sufficient to prove that the payoff functions are diagonally strictly concave [19]. Therefore, under the alternative dynamic mentioned above, the frequencies of the competing carriers will converge to the unique type BB equilibrium frequencies.

6 N-Player Symmetric Game

Now we will extend the analysis to the N-player symmetric case, where $N \geq 2$. By symmetry, we mean that the operating cost C_i , the seating capacity S_i and the fare p_i is the same for all carriers. For the analysis presented in this section, it is sufficient to have $\frac{C_i}{p_i}$ constant for all carriers. However, for computing the price of anarchy in the next section we need the remaining assumptions. We will simplify the notation and denote the operating cost for each carrier as C , seating capacity as S and fare as p . Under symmetry, the necessary and sufficient conditions for the existence of a type BB equilibrium for a 2-player game reduce to a single condition, $\frac{\alpha PS}{C} > 2$. We will assume that this condition holds throughout the following analysis.

Assumption 4. $\frac{\alpha PS}{C} > 2$

Proposition 12. *In an N-player symmetric game, a symmetric equilibrium with excess seating capacity exists at $x_i = \frac{\alpha Mp}{C} \frac{N-1}{N^2}$ for all i if and only if $N \leq \frac{\alpha}{\alpha-1}$ and if it exists, then it is the unique symmetric equilibrium.*

Proof. The utility of each carrier i is given by $u_i(x_i, y_i) = M \frac{x_i^\alpha}{x_i^\alpha + y_i^\alpha} - \frac{C}{p} x_i$, where $y_i = \left(\sum_{j=1, j \neq i}^N x_j^\alpha \right)^{\frac{1}{\alpha}}$ is the effective competitor frequency for player i . From the FOCs, we get $x_i = \frac{\alpha Mp}{C} \frac{x_i^\alpha y_i^\alpha}{(x_i^\alpha + y_i^\alpha)^2}$.

In the symmetric game, $\frac{C_i}{p_i}$ is the same for every player i . In general, this symmetric game may have both symmetric and unsymmetric equilibria. In a symmetric equilibrium, $x_1 = x_2 = \dots = x_N$. Assume excess seating capacity for each carrier. Substituting in the FOCs we get $y_i = (N-1)^{\frac{1}{\alpha}} x_i$. Therefore, $x_i = \frac{\alpha Mp}{C} \frac{N-1}{N^2}$ for all i is the unique solution. Therefore, we have proved that if an equilibrium exists at this point, then it must be the unique symmetric equilibrium of this game.

In order to prove that this point is an equilibrium point, we need to prove that the SOC is satisfied, the profit at this point is non-negative and seating capacity is at least as much as the demand for each carrier.

The SOC is satisfied if and only if,

$$\frac{\partial^2 U_i}{\partial x_i^2} = \frac{M \alpha x_i^{\alpha-2} y_i^\alpha}{(x_i^\alpha + y_i^\alpha)^3} ((\alpha-1) y_i^\alpha - (\alpha+1) x_i^\alpha) \leq 0 \iff N \leq \frac{2\alpha}{\alpha-1}.$$

The condition on non-negativity of profit is satisfied if and only if,

$$\frac{\alpha Mp}{C} \frac{N-1}{N^2} * C \leq \frac{Mp}{N} \iff N \leq \frac{\alpha}{\alpha-1}.$$

The condition of excess seating capacity is satisfied if and only if,

$$\frac{\alpha Mp}{C} \frac{N-1}{N^2} * S > \frac{M}{N} \iff N > \frac{\alpha \frac{pS}{C}}{\alpha \frac{pS}{C} - 1},$$

which is always true for $\alpha \frac{pS}{C} > 2$.

Thus the symmetric equilibrium exists if and only if $N \leq \frac{\alpha}{\alpha-1}$. \square

Proposition 13. *In a symmetric N-player game, there exists no unsymmetric equilibrium where all players have a non-zero frequency and excess seating capacity.*

Proof. Let us assume the contrary. For a symmetric N-player game, let there exist an unsymmetric equilibrium such that all players have a non-zero frequency and excess seating capacity. Let us define $\beta = \sum_{j=1}^N x_j^\alpha$ and $\omega_i = \frac{x_i^\alpha}{\sum_{j=1}^N x_j^\alpha}$. So $x_i = (\omega_i \beta)^{\frac{1}{\alpha}}$. Substituting in the FOC, we get,

$$\begin{aligned} (\omega_i \beta)^{\frac{1}{\alpha}} &= \frac{\alpha Mp}{C} \omega_i (1 - \omega_i) \\ \Rightarrow \frac{C}{\alpha Mp} \beta^{\frac{1}{\alpha}} &= \omega_i^{\frac{\alpha-1}{\alpha}} - \omega_i^{\frac{2\alpha-1}{\alpha}} \end{aligned}$$

Let us define a function $h(\omega_i) = \omega_i^{\frac{\alpha-1}{\alpha}} - \omega_i^{\frac{2\alpha-1}{\alpha}}$. The value of $h(\omega_i)$ is the same across all the players at equilibrium. For all $\omega_i > 0$, $h(\omega_i)$ is a strictly concave function. So it can take the same value at at most two different values of ω_i . So all ω_i can take at most two different values. Let $\omega_i = v_1$ for $m (\leq N)$ players, and $\omega_i = v_2$ for the remaining $N - m$ players. Let $v_1 > v_2$, without loss of generality. $h(\omega_i)$ is maximized at $\omega_i = \frac{\alpha-1}{2\alpha-1}$. So $v_2 < \frac{\alpha-1}{2\alpha-1} < v_1$.

At equilibrium, each player's profit must be non-negative. Therefore, the profit for each player i such that $\omega_i = v_2$ is given by $Mp\omega_i - Cx_i$. But $x_i = \frac{\alpha Mp}{C} \omega_i (1 - \omega_i)$. So the condition on non-negativity of profit simplifies to, $v_2 \geq \frac{\alpha-1}{\alpha}$. Therefore, $\frac{\alpha-1}{2\alpha-1} > v_2 \geq \frac{\alpha-1}{\alpha}$, which can be true only if $\alpha < 1$. This leads to a contradiction. So we have proved that for a symmetric N-player game, there exists no unsymmetric equilibrium such that all players have a non-zero frequency and excess seating capacity. \square

Proposition 14. *In a symmetric N-player game, there exists some n_{min} such that for any integer n with $\max(2, n_{min}) \leq n \leq \min(N-1, \frac{\alpha}{\alpha-1})$ there exist exactly $\binom{N}{n}$ unsymmetric equilibria such that exactly n players have non-zero frequency and all players with nonzero frequency have excess seating capacity. There exists at least one such integer for $N \geq \frac{\alpha}{\alpha-1}$. The frequency of each player with non-zero frequency equals $\frac{\alpha Mp}{C} \frac{n-1}{n^2}$.*

Proof. Let us denote this game as G . Consider any equilibrium having exactly n players with non-zero frequency. Let us rearrange the player indices such that players $i = 1$ to $i = n$ have non-zero frequencies. Let us consider a new game which involves only the first n players. We will denote this new game as G' . An equilibrium of G where only the first n players have a non-zero frequency is also an equilibrium for the game G' where all players have non-zero frequency. As we have already proved, the equilibrium frequencies of each of the first n players must be equal to $\frac{\alpha Mp}{C} \frac{n-1}{n^2}$. This ensures that any of the first n players will not benefit from unilateral deviations from this equilibrium profile. In order to ensure that none of the remaining $N - n$ players has an incentive to deviate, we must ensure that the effective competitor frequency for any player j such that $j > n$ must be at least equal to y_{th} . This condition is satisfied if and only if,

$$\begin{aligned} n^{\frac{1}{\alpha}} \frac{n-1}{n^2} \frac{\alpha Mp}{C} &\geq (\alpha-1)^{\left(\frac{\alpha-1}{\alpha}\right)} \frac{Mp}{\alpha C} \\ \iff n^{\frac{1-\alpha}{\alpha}} - n^{\frac{1-2\alpha}{\alpha}} &\geq \frac{(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{\alpha^2}. \end{aligned} \quad (22)$$

LHS is an increasing function of n for $n \leq \frac{\alpha}{\alpha-1}$. Also the RHS is a decreasing function of α (this can be verified by differentiating the log of RHS with respect to α). Also it can be easily verified that at $n = \frac{\alpha}{\alpha-1}$, the inequality holds for every α . Therefore, for any given α value, there exists some $n_{min} \geq 0$ such that for all n such that $\frac{\alpha}{\alpha-1} \geq n \geq n_{min}$, this inequality is satisfied. As proved earlier, the condition for existence of an equilibrium with all players having non-zero frequency in game G' is $n \leq \frac{\alpha}{\alpha-1}$.

So all the conditions for an equilibrium of game G are satisfied if $\max(2, n_{min}) \leq n \leq \min(N-1, \frac{\alpha}{\alpha-1})$. Therefore, any equilibrium of game G' where all players have non-zero frequency is also an equilibrium of game G where all the remaining players have zero frequency and vice versa. The players in game G' can be chosen in $\binom{N}{n}$ ways. Therefore, we have proved that in a symmetric N-player game, for any integer n such that $\max(2, n_{min}) \leq n \leq \min(N-1, \frac{\alpha}{\alpha-1})$, there exist exactly $\binom{N}{n}$ unsymmetric equilibria such that exactly n players have non-zero frequency. To show that there exists at least one such integer n , consider 2 cases. If $\alpha > 1.5$, then it is easy to verify that the inequality (22) is always satisfied for $n = 2$. If $\alpha \leq 1.5$, then we see that (22) is satisfied by $n = \frac{1}{\alpha-1} = \frac{\alpha}{\alpha-1} - 1$. In either case, $\frac{\alpha}{\alpha-1} > 2$ is always satisfied. So there always exist some such n . The frequency of each player with non-zero frequency equals $\frac{\alpha Mp}{C} \frac{n-1}{n^2}$. \square

From here onwards, we will denote each such equilibrium as an n-symmetric equilibrium of an N-player game.

Proposition 15. *Among all equilibria with exactly n players ($n \leq N$) having nonzero frequency, the total frequency is maximum for the symmetric equilibrium.*

Proof. As proved earlier, any possible unsymmetric equilibria with exactly n players having nonzero frequency must involve at least one player with no excess seating capacity. Let player i be such a player with nonzero frequency and no excess seating capacity at equilibrium. So the effective competitor frequency y must be at most equal to y_{cr} and $x_i \geq x_{cr} = \frac{M}{S} \left(1 - \frac{C}{\alpha p S}\right) > \frac{M}{2S}$. Therefore, each such player must carry at least $\frac{M}{2}$ passengers. Therefore, at equilibrium there can be at most one such player. So each of the remaining $n - 1$ players has excess capacity. Using the same argument as the one used in proving proposition 13, we can prove that each player with non-zero frequency and excess capacity will have equal frequency at equilibrium.

Let us denote the equilibrium frequency of the sole player with no excess capacity by x_1 and that of each of the remaining players as x_2 . We will denote the equilibrium market share of the player with no excess capacity as l . Therefore, the total frequency under the unsymmetric equilibrium equals,

$$\begin{aligned} (n-1)x_2 + x_1 &= \frac{\alpha M p}{C} \frac{(n-1)x_2^\alpha}{(n-1)x_2^\alpha + x_1^\alpha} \left(1 - \frac{x_2^\alpha}{(n-1)x_2^\alpha + x_1^\alpha}\right) + \frac{M}{S} \frac{x_1^\alpha}{(n-1)x_2^\alpha + x_1^\alpha} \\ &= \frac{\alpha M p}{C} (1-l) \left(1 - \frac{1-l}{n-1}\right) + \frac{M}{S} l \end{aligned}$$

Let us assume that there exists an unsymmetric equilibrium where the total frequency is greater than that under the corresponding n -symmetric equilibrium, which equals $\frac{\alpha M p n-1}{C}$. This condition translates into,

$$\frac{\alpha M p}{C} (1-l) \left(1 - \frac{1-l}{n-1}\right) + \frac{M}{S} l > \frac{\alpha M p n-1}{C},$$

which further simplifies to, $nl(5-n-2l) > 2$. But we know that $n \in \mathbb{I}^+$, $n \geq 2$ and $l \geq \frac{1}{2}$. So $5-n-2l > 0$ only if $n < 5-2l \leq 4$. So $n = 2$ or $n = 3$. For $n = 2$, the conditions for existence of type BC equilibrium in the 2-player case require $\frac{\alpha p S}{C} \leq 2$, which contradicts our assumption. For $n = 3$, we need some l such that $3l^2 - 3l + 1 < 0$, which is true if and only if $3(l-0.5)^2 + 0.25 < 0$, which is also impossible. Thus our assumption leads to a contradiction. So we have proved that among all equilibria with exactly n players ($n \leq N$) having nonzero frequency, the total frequency is maximum for the symmetric equilibrium. \square

Proposition 16. *There exists no equilibrium with exactly n players with non-zero frequency such that $n > \frac{\alpha}{\alpha-1}$.*

Proof. We already proved that if $n > \frac{\alpha}{\alpha-1}$, there exists no equilibrium with all n players having excess capacity. We have also proved that the number of players without excess capacity can be at most one. So consider some equilibrium with one player with no excess capacity. Let the market share of that player be l and let the equilibrium frequency of each of the remaining players be x_2 . Because $n > \frac{\alpha}{\alpha-1}$, therefore $\alpha > \frac{n}{n-1}$.

For non-negative profit at equilibrium we require, $\frac{Mp}{C} \frac{1-l}{n-1} \geq x_2$. From the FOC, we get $x_2 = \frac{\alpha Mp}{C} \frac{1-l}{n-1} \left(1 - \frac{1-l}{n-1}\right)$. Combining the two we get,

$$\begin{aligned} 1 &\geq \alpha \left(1 - \frac{1-l}{n-1}\right) \\ \Rightarrow \frac{n}{n-1} &< \alpha \leq \frac{n-1}{n+l-2} \leq \frac{n-1}{n-1.5} \\ \Rightarrow n &< 2, \end{aligned}$$

which is impossible.

Therefore, we have proved that there exists no equilibrium with exactly n players with non-zero frequency such that $n > \frac{\alpha}{\alpha-1}$. \square

In this section, we proved that for an N -player symmetric game, if $N \leq \frac{\alpha}{\alpha-1}$, then there exists a fully symmetric equilibrium where the equilibrium frequency of each carrier at equilibrium is $\frac{\alpha Mp}{C} \frac{N-1}{N^2}$ and there exists no unsymmetric equilibrium with all N players having a non-zero frequency. On the other hand, if $N > \frac{\alpha}{\alpha-1}$, then there exists no equilibrium with all players having non-zero frequency. In either case, there exist exactly $\binom{N}{n}$ n -symmetric equilibria for each integer $n < N$ such that $\max(2, n_{\min}) \leq n \leq \min(N-1, \frac{\alpha}{\alpha-1})$ for some $n_{\min} \geq 0$. Additionally, there may be unsymmetric equilibria such that each unsymmetric equilibrium has exactly one player with 100% load factor, $n-1$ more players with non-zero frequency and excess seating capacity and $N-n$ players with zero frequency. We also proved that there always exists at least one equilibrium for an N -player symmetric game. The aforementioned types of equilibria are exhaustive, that is there exist no other types of equilibria. As before, we realize that all the equilibria except those where all players have a nonzero frequency and excess capacity are suspect in terms of their portrayal of reality. So the fully symmetric equilibrium appears to be the most realistic one. In addition, the fully symmetric equilibrium is also the worst case equilibrium in the sense that it is the equilibrium which has the maximum total frequency, as will be apparent in the next section.

We proved that for some $n' < N$, if there exists no symmetric equilibrium for all $n \geq n'$, then there exists no unsymmetric equilibrium for all $n \geq n'$ either. We also proved that for any given n , the total frequency at each unsymmetric equilibrium having n non-zero frequency

players is at most equal to the total frequency at the corresponding n-symmetric equilibrium. These results will help us obtain the price of anarchy in the next section.

7 Price of Anarchy

In any equilibrium, the total revenue earned by all carriers remains equal to Mp . The total flight operating cost to all carriers is given by $\sum_{i=0}^n Cx_i = C \sum_{i=0}^n x_i$. On the other hand, if there were a central controller trying to minimize the total operating cost, the minimum number of flights for carrying all the passengers would be equal to $\frac{M}{S}$ and the total operating cost would be $\frac{MC}{S}$. Similar to the notion introduced by Koutsoupias and Papadimitriou [15], let us define the price of anarchy as the ratio of total operating cost at Nash equilibrium to the total operating cost under the optimal frequency. The denominator is a constant and the numerator is proportional to the total number of flights.

A large proportion of airport delays are caused by congestion. Congestion related delay at an airport is an increasing (often nonlinearly) function of the total number of flights. Therefore, the greater the total number of flights, more is the delay. Total profit earned by all the airlines in a market is also a decreasing function of the total frequency. Also, because the total number of passengers remains constant, the average load factor in a market is inversely proportional to the total frequency. Lower load factors mean more wastage of seating capacity. Thus total frequency is a good measure of airline profitability, total operating cost, airport congestion and load factors. Higher total frequency across all carriers in a market means lower profitability, more cost, more congestion and lower average load factor, assuming constant aircraft size. Greater the price of anarchy, more is the inefficiency introduced by the competitive behavior of players at equilibrium.

Proposition 17. *In a symmetric N-player game, the price of anarchy is given by $\frac{\alpha p S}{C} \frac{n-1}{n}$, where n is the largest integer not exceeding $\min(N, \frac{\alpha}{\alpha-1})$.*

Proof. As proved earlier, a symmetric N-player game has $\sum_{n=\max(2, n_{\min})}^{\min(N, \frac{\alpha}{\alpha-1})} \binom{N}{n}$ equilibria (for some $n_{\min} \geq 0$). such that each equilibrium has a set of exactly n players each with frequency $\frac{\alpha Mp}{C} \frac{n-1}{n^2}$ and excess capacity, whereas remaining $N - n$ players have zero frequency. Also, for any $n < \min(N, \frac{\alpha}{\alpha-1})$, there may exist equilibria with exactly n players having non-zero frequency and one of them having no excess capacity at equilibrium. However, the frequency under any equilibrium with exactly n players having non-zero frequency is at most equal to the corresponding n-symmetric equilibrium. In any equilibrium having n players with non-zero frequency, the total flight operating cost is given by $\frac{\alpha Mp}{C} \frac{n-1}{n^2}$, which is an increasing function of n . The total cost under minimum cost scheduling would be $\frac{MS}{C}$. Therefore, the ratio of

total cost under equilibrium to total cost under minimum cost scheduling is $\frac{\alpha p S}{C} \frac{n-1}{n}$, which is an increasing function of n . Also, no equilibrium exists for $n > \frac{\alpha}{\alpha-1}$. Therefore, the price of anarchy is given by $\frac{\alpha p S}{C} \frac{n-1}{n}$, where n is the greatest integer less than equal to $\min(N, \frac{\alpha}{\alpha-1})$. \square

This expression has several important implications. Greater the α value, more is the price of anarchy. This means that as the market share-frequency share relationship becomes more and more curved, and goes away from the straight line, greater is the price of anarchy. So the S-curve phenomenon has a direct impact on airline profitability and airport congestion. Also, more the airfare compared to the operating cost per seat (i.e. more is the value of $\frac{pS}{C}$), greater is the price of anarchy. In other words, for short-haul, high-fare markets the price of anarchy is greater. Finally, more the number of competitors, greater is the price of anarchy (up to a threshold value beyond which it remains constant).

The equilibrium results from this simple model help substantiate some of the claims mentioned earlier. The price of anarchy increases because of the S-shaped (rather than linear) market share-frequency share relationship. Therefore, similar to the suggestions by Button and Drexler [10] and O'connor [17], the S-curve relationship tends to encourage airlines to provide excess capacity and schedule greater numbers of flights. Total profitability of all the carriers in a market under the worst case equilibrium provides a lower bound on airline profitability under competition. This lower bound is an increasing function of the price of anarchy, which in turn increases with number of competitors. Therefore, similar to Kahn's [14] argument, this raises the question of whether the objectives of a financially strong and highly competitive airline industry are inherently conflicting. In addition, these results also establish the link between airport congestion and airline competition. Airport congestion under the worst-case equilibrium is directly proportional to the price of anarchy. So greater the number of competitors and more the curvature of the market share-frequency share relationship, greater is the airport congestion and delays.

8 Summary

In this paper, we modeled airline frequency competition based on the S-curve relationship which has been well documented in airline literature. Regardless of the exact value of α parameter, it is usually agreed that market share is an increasing (linear or S-shaped) function of frequency share. Our model is general enough to accommodate somewhat differing beliefs about the market share-frequency share relationship. We characterize the best response curves for each player in a multi-player game. Due to complicated shape of best response curves, we proved that there exist anywhere between 0 to 6 different equilibria depending on the exact parameter values. All the existence and uniqueness conditions can be completely described by 3 dimensionless

parameters (in addition to α) of the game. Only one out of the 6 possible equilibria seemed reasonable in terms of portrayal of reality. This equilibrium corresponds to both players having nonzero frequency and less than 100% load factors. In order to narrow down the modeling effort, realistic parameter ranges were identified based on real world data that come closest to the simplified models analyzed in this paper. We proposed 2 different myopic learning algorithms for the 2-player game and proved that under mild conditions, either of them converges to Nash equilibrium. For the N-player (for any integer $N \geq 2$) game with identical players, we characterized the entire set of possible equilibria and proved that at least one equilibrium always exists for any such game. The worst case equilibrium was identified. The price of anarchy was found to be an increasing function of number of competing airlines, ratio of fare to operating cost per seat and the curvature of S-curve relationship.

This paper presented two central results. First, there are simple myopic learning rules under which less than perfectly rational players would converge to an equilibrium. This substantiates the predictive power of the Nash equilibrium concept. Second, the S-curve relationship between market share and frequency share has direct and negative implications to airline profitability and airport congestion, as speculated in multiple previous studies.

References

- [1] Nicole Adler. Competition in a deregulated air transportation market. *European Journal of Operational Research*, 129(2):337–345, March 2001.
- [2] V. Aguirregabiria and C. Y. Ho. A dynamic oligopoly game of the US airline industry: Estimation and policy experiments. *Manuscript. University of Toronto*, 2009.
- [3] R. Baseler. Airline fleet revenue management: Design and implementation. *Handbook of Airline Economics*, 2002.
- [4] Peter Belobaba. The airline planning process. In Peter Belobaba, Amedeo Odoni, and Cynthia Barnhart, editors, *The Global Airline Industry*, pages 153–181. Wiley, June 2009.
- [5] Peter Belobaba. Overview of airline economics, markets and demand. In Peter Belobaba, Amedeo Odoni, and Cynthia Barnhart, editors, *The Global Airline Industry*, pages 47–71. Wiley, June 2009.
- [6] Philippe Bonnefoy. *Scalability of the air transportation system and development of multi-airport systems : a worldwide perspective*. Thesis, Massachusetts Institute of Technology, Cambridge, 2008.

- [7] J. A Brander and A. Zhang. Dynamic oligopoly behaviour in the airline industry. *International Journal of Industrial Organization*, 11(3):407435, 1993.
- [8] J. K Brueckner. Schedule competition revisited. *Unpublished paper, Department of Economics, UC Irvine*, 2008.
- [9] Jan Brueckner and Ricardo Flores-Fillol. Airline schedule competition. *Review of Industrial Organization*, 30(3):161–177, May 2007.
- [10] Kenneth Button and Jonathan Drexler. Recovering costs by increasing market share: An empirical critique of the S-Curve. *Journal of Transport Economics and Policy*, 39(3):391–404, September 2005.
- [11] G. Dobson and P. J Lederer. Airline scheduling and routing in a hub-and-spoke system. *Transportation Science*, 27(3):281297, 1993.
- [12] Mark Hansen. Airline competition in a hub-dominated environment: An application of noncooperative game theory. *Transportation Research Part B: Methodological*, 24(1):27–43, 1990.
- [13] S. Hong and P. T Harker. Air traffic network equilibrium: Toward frequency, price and slot priority analysis. *Transportation Research Part B: Methodological*, 26(4):307323, 1992.
- [14] A. E Kahn. Change, challenge, and competition: A review of the airline commission report. *Regulation*, 3:110, 1993.
- [15] E. Koutsoupas and C. Papadimitriou. Worst-case equilibria. *Computer Science Review*, 3(2):6569, 2009.
- [16] V. D Norman and S. P Strandenes. Deregulation of scandinavian airlines: A case study of the Oslo-Stockholm route. *Empirical Studies of Strategic Trade Policy; Chicago*, 1994.
- [17] William E. O'Connor. *An introduction to airline economics*. Greenwood Publishing Group, 2001.
- [18] Eric Pels, Peter Nijkamp, and Piet Rietveld. A note on the optimality of airline networks. *Economics Letters*, 69(3):429–434, December 2000.
- [19] J. B. Rosen. Existence and uniqueness of equilibrium points for concave N-Person games. *Econometrica*, 33(3):520–534, July 1965.

- [20] C. E. Schumer and C. B. Maloney. Your flight has been delayed again: flight delays cost passengers, airlines, and the US economy billions. *The US Senate Joint Economic Committee*, 2008.
- [21] R. W Simpson. *A market share model for U.S. domestic airline competitive markets*. Massachusetts Institute of Technology, Flight Transportation Laboratory, Cambridge, Mass, 1970.
- [22] N. K Taneja. *The commercial airline industry*. DC Heath, 1976.
- [23] Nawal K Taneja. *Airline competition analysis*. M.I.T. Flight Transportation Laboratory, Cambridge, 1968.
- [24] Wenbin Wei and Mark Hansen. Impact of aircraft size and seat availability on airlines' demand and market share in duopoly markets. *Transportation Research Part E: Logistics and Transportation Review*, 41(4):315–327, July 2005.
- [25] Wenbin Wei and Mark Hansen. Airlines' competition in aircraft size and service frequency in duopoly markets. *Transportation Research Part E: Logistics and Transportation Review*, 43(4):409–424, July 2007.