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BITS AND BUCKS: MODELING COMPLEX SYSTEMS BY INFORMATION FLOW

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Modeling complex systems by information flow

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Abstract: This paper presents a general method for modeling and characterizing complex systems in terms of flows of information together with flows of conserved or quasi-conserved quantities such as energy or money. Using mathematical techniques borrowed from statistical mechanics and from physics of computation, a framework is constructed that allows general systems to be modeled in terms of how information, energy, money, etc. flow between subsystems. Physical, chemical, biological, engineering, and commercial systems can all be analyzed within this framework.

By their very nature, complex systems resist analysis. Intricate, consisting of many parts, complex systems often behave in unpredictable and unforeseen ways. For the engineers who design and build complex engineered systems, this inscrutable character of complex systems is a considerable challenge: how does one engineer a system that is sufficiently complex to meet an equally complex set of operational requirements, while maintaining its stability and robustness? A wide variety of techniques have been developed over the years to cope with the problem of engineering complexity, but there is no universal method for solving the problems posed by complex engineered systems other than hard work, attention to detail, and lots of computation and simulation.

This paper presents a formal theory of the behavior of complex systems. This theory is universal in that it can in principle be used to describe and characterize the behavior of any complex physical system. Of course, by its very generality, it is far from providing solutions to the problems of engineering complex systems. Rather, the theory presented here is a 'hopeful' one: the goal is to present a mathematically well-defined framework that applies in a straightforward fashion to the analysis of well-known complex systems such as statistical mechanical systems in physics or game-theoretic models of financial markets, in the hope that some of the results derived may eventually throw light on some of the mysterious and inscrutable behavior of complex systems in general. No claims are made for solving the 'problem of complexity' in general.

The basic idea explored here is the trade off between conserved or approximately conserved quantities such as energy, money, commodoties, etc., and statistically defined quantitites such as information. First, a framework is developed to describe the causal structure and probabilistic dynamics of complex physical systems. The framework models complex systems as *information networks* in which information and energy/money flow between subsystems over time. The framework is constructed so as to allow the easy definition of quantities and flows of information using conventional Shannon information theory. The information-related part of the framework is closely related to the theory of Bayesian networks. But unlike Bayesian networks, the framework also allows the quantification of flows of conserved quantities such as energy and money between the parts of the complex system. Finally, the trade offs between information and money/energy are investigated in detail, allowing the definition of quasi-thermodynamic quantities analogous to temperature and measured in dollars or joules per bit.

Take, for example, trading over the internet. Each flow of information (measured in bits per second) is associated with a flow of energy (measured in watts). The energy per bit – effectively, a form of temperature – is a crucial quantity in characterizing the communications performance of the network in the presence of noise and loss. But each bit can also be associated with a monetary value (bucks), as when the title to some commodity is transferred electronically to a buyer and an electronic draft to pay for the commodity is transferred to the seller. The bucks per bit – again, a form of temperature – is a crucial quantity in deciding whether to buy or sell. Clearly, some bits are worth more than others!

This paper shows that in complex systems that can be accurately described by such a modeling framework, different structures for interconnects and protocols for exchange can lead to qualitative and quantitative differences in behavior. In some cases, such as thermodynamic systems, stable behavioral equilibria exist and exhibit gaussian fluctuations. In other cases, such as phase transitions and systems of economic exchange, quasi-stable or unstable equilibria exist and exhibit power-law fluctuations. Finally, some types of flows yield no equilibrium at all. The framework makes quantitative predictions for the efficacy, flexibility, stability, and robustness of complex systems characterized by flows of information together with energy, money, etc.

The paper proceeds as follows. First, the bits/bucks framework is defined mathematically: formal definitions are given, theorems are stated, and the analogy to statistical mechanics is defined in a relatively precise fashion. The formal treatment is followed by a series of examples, in which applications for the framework are suggested. The reader who is not interested in the formal details may wish to skip directly to the examples, with the usual caveat that fully understanding the results and their implications requires delving into the mathematics.

1. Information networks

To quantify and relate flows of information and energy/money, it is useful to construct a ‘space-time’ representation of the behavior of a complex system. In this space-time picture the behavior of a complex system over time is modeled in the form of a causal structure represented by a directed graph in which each vertex represents the state of a subsystem at a given point in time (figures 1 and 2). When the vertices of the graph are associated with quantities of information and energy/money, and the edges are associated with flows of these quantities, the graph will be called an information network. The vertices in the graph represent the states of the various subsystems of the complex system at different points in time. The information network then represents a ‘space-time’ picture of the history of a complex system, in analogy in Einstein’s theory of general relativity, where the behavior of complex gravitational systems is modeled in terms of a directed graph representing the way in which particle trajectories and interactions are embedded in space and time.

The space-time picture of a complex system in terms of information networks should be contrasted with the usual block diagram picture of a complex system. A block diagram models a complex system as a directed graph in which vertices represent subsystems, and directed edges represent input/output relationships. The space-time picture of the same complex system ‘unpacks’ the block diagram over time: a single subsystem is represented by different vertices, each one representing the state of that subsystem at a different point in time (figure 3). At each point in time, a cross-section of the space-time graph for the complex system looks like the original block diagram, with the directed arrows of the block diagram going from one time step to the next. Clearly, the block-diagram picture of a complex system is more compact. The information network picture of a complex system is more flexible: any system that can be described by a block diagram can be described by an information network, but the converse is not true. In particular, unlike block diagrams, information networks readily handle complex systems where the dynamics of subsystems, the input/output relationships between them, and the very decomposition into subsystems changes over time (figure 4).

2. Mathematical framework

Let us now make this information network framework mathematically precise. We will use methods of probability and information theory as applied to coupled dynamical systems as developed in (1-2). Model the behavior of a complex system over time as a directed graph, $G = (V_i, E_{ij})$. Each vertex V_i of the graph corresponds to a subsystem at a particular point in time t_i. Each directed edge E_{ij} from the i-th vertex to the j-th vertex represents a path along which information and energy/money can flow (figure 1). To preserve our ordinary notions of causality, flows are directed from past to future. The graphs are acyclic (no time travel). Let v_i label the state of the i'th subsystem at time t_i. Let e_{ij} label the state of the edge E_{ij}. Let $x(v_i)$ be the quantity of the conserved quantity x (energy/money) in the i'th subsystem at time t_i. (There can be many conserved quantities, x_ℓ, but for the moment just deal with one.) Similarly, let $x(e_{ij})$ be the amount of the conserved quantity associated with the edge E_{ij}. The v_i, e_{ij}, and x can be either continuous or discrete. Conservation of the overall quantity x over time implies that

$$\sum_i x(e_{ij}) = x(v_j) = \sum_k x(e_{jk}), \tag{1}$$

where the first sum is over inputs i to the vertex j, and the last sum is over outputs k from the vertex j. (In the interests of compactness, we will always use a discrete notation: if the variables are continuous, the sums should be replaced by integrals.) That is, conservation of x leads to a constraint over the possible states of the inputs and outputs to a vertex.

Flows are inherently dynamical. To specify the dynamics of the model in the most general way possible that respects the causal structure, consider a Markovian dynamics in which the probabilities that a system is in a particular state at a particular time are functions of the states of its inputs. That is, $p(v_j|e_{i_1j}...e_{i_mj})$ is the probability that the j'th subsystem is in state v_j given that its m inputs are in states $e_{i_1j}...e_{i_mj}$. Note that the conservation restrictions imply that

$$p(v_j|e_{i_1j}...e_{i_mj}) \propto \delta(\sum_i x(e_{ij}) - x(v_j)). \tag{2}$$

Equation (2) shows how the probabilities for a node are to be set as a function of its inputs. Similarly, let

$$p(e_{jk_1}...e_{jk_n}|v_j) \propto \delta(x(v_j) - \sum_k x(e_{jk})) \tag{3}$$

be the probability that the n outputs of the j'th vertex are in the the states $e_{jk_1}...e_{jk_n}$ given that the vertex is in the state v_j. Equations (2-3) give a Markovian probabilistic dynamics that respects the conservation constraint (1). Given a set of probabilities for the states of the initial vertices of the directed graph, equations (2-3) completely determine the joint probability distribution for the states of all vertices at all times in a way that is familiar from the theory of Bayesian networks (3). Indeed, an information network can be thought of as a Bayesian network in which vertices and edges are associated with conserved quantitites such as energy and money. A convenient way to visualize the construction of this joint probability distribution is to imagine events unfolding in time: first, the edges leading from the initial vertices are assigned probabilities based on the probabilities of the initial vertices together with the conditional probabilities of equation (3). Then the vertices to which these edges lead are assigned probabilities for their states according to the joint probabilities of the edges leading into them together with the conditional probabilities of equation (2). As time progresses, each vertex and edge is eventually assigned its probabilities in their proper order within the directed, acyclic graph. The result is a joint probability distribution $p(v_1 v_2 \ldots v_M e_{j_1 k_1} \ldots e_{j_N k_N})$ for the M vertices and N edges of the graph. This joint probability distribution contains all dynamic information about the complex system as modeled in this probabilistic fashion. It contains all correlation functions, information about flows of information and energy/money, probabilities for future events, etc. This completes the formal dynamic description.

The simplest example (after the trivial case of a graph with one vertex) is that in which V_1 represents a system at time t_1 and V_2 represents a system at a later time t_2 (figure 2a). Here the flow along E_{12} is from a single system at one point in time to the same system at a later point in time. Similarly, a chain of vertices $V_1 E_{12} V_2 E_{23} ... E_{(n-1)n} V_n$ can represent the same system at n successive points in time (figure 2b). A graph containing cycles that represents the flows between subsystems averaged over time can always be expanded into the correponsing acyclic graph representing flows over time (figure 3).

This framework is potentially very powerful: it can represent the probabilistic dynamics of essentially any physical system (discretized in space and time). For example, the action of a digital computer can readily be mapped onto this framework, where the vertices represent logic gates, the edges represent wires, and the graph as a whole represents the wiring diagram for the computation. So this framework can clearly represent any process that can be simulated on a digital computer. Indeed, if the nodes represent quantum logic gates, and conditional probabilities are replaced by conditional probability amplitudes, the framework can be used to represent any computable quantum system, including a quantum computer. Since quantum computers can simulate all known quantum systems to an arbitrary degree of accuracy, this framework is capable of representing an arbitrary physical dynamics.

The power of the framework renders it necessarily abstract. The usefulness of the framework for any given system will depend on the ease with which the system's operation can be mapped onto the framework, and on the complexity of computing the dynamics of the resulting model. Below, several examples of this framework will be presented. First, however, investigate how the framework allows one to measure the tradeoffs between information and energy/money.

The probabilistic directed graph allows one to assign a joint probability distribution $p(v_1, \ldots, v_n, e_{12}, \ldots e_{(n-1)n})$ to the states of all vertices and edges. As a result, each vertex and edge has a corresponding information, $I(V_i) = -\sum_{v_i} p(v_i) \log_2 p(v_i)$, $I(E_{jk}) = -\sum_{e_{jk}} p(e_{jk}) \log_2 p(e_{jk})$. Conditional and mutual informations can be defined similarly. The amount of information flowing onto edge E_{jk} from vertex V_j is the mutual information $I(V_j : E_{jk}) = I(V_j) + I(E_{jk}) - I(V_j, E_{jk})$. Similarly, the amount of information flowing onto vertex V_k from edge E_{jk} is the mutual information $I(E_{jk} : V_k)$. Note that a vertex can receive the same information from more than one edge. Accordingly, the total amount of information $I(V_k : E_{j_1 k} \ldots E_{j_m k})$ flowing into vertex V_k from its inputs is less than or equal to the sum of the amounts flowing in from the input edges individually.

The average amount of conserved quantity x each vertex and edge is similarly well defined, as is the amount flowing into a vertex from an edge and *vice versa*. The average amount of conserved quantity x on vertex V_j is $\langle x_j \rangle = \sum_{v_j} p(v_j) x(v_j)$, and the amount on edge E_{jk} is $\langle x_{jk} \rangle = \sum_{e_{jk}} p(e_{jk}) x(e_{jk})$. The x in a vertex is the sum of the x flowing in. It is also equal to the sum of the x flowing out. In other words, x is conserved: for any closed surface surrounding any subset of vertices, the amount of x flowing in is equal to the amount of x flowing out.

Since the amounts of information and energy/money on each vertex and on each edge are well-defined, one can define effective temperatures $T_j = \langle x_j \rangle / I(V_j)$, $T_{jk} = \langle x_{jk} \rangle / I(E_{jk})$ for each vertex V_j and edge E_{jk}. In keeping with the notation of statistical mechanics, define the inverse temperatures $\beta_j = 1/T_j, \beta_{jk} = 1/T_{jk}$. T_j is the average number of bucks or joules per bit on the j'th vertex, and T_{jk} is the number of bucks or joules per bit flowing from j to k. Note that while the usual quantity of temperature in statistical mechanics is only well-defined for quasi-equilibrium situations, here the temperature is well-defined everywhere even in situations that are far from equilibrium. Effective temperatures will prove useful in determining the average direction and quantity of flows of information between subsystems.

3. Relation to statistical mechanics

To make the connection with statistical mechanics, fluid mechanics, etc., it is helpful to distinguish between several different types of dynamics. First, deterministic dynamics make up a useful subset of probabilistic dynamics. In deterministic dynamics, all conditional probabilities are either 1 or 0. Second, a useful subset of deterministic dynamics includes one-to-one dynamics: in one-to-one dynamics the conditional probability of an output given an input is 1 for exactly one input, and 0 for the remainder of the inputs. That is, each state of a vertex has probability 1 for one and only one of the joint states of its input edges. And each joint state for the set of edges emanating from a vertex has probability 1 for exactly one state of the vertex. In the case of continuous variables, one-to-one dynamics should be volume-preserving on the underlying state space, in analog with Liouvillian dynamics in classical mechanics. A third type of useful dynamics consists of probabilistic mixtures of one-to-one dynamics: these are analogs of the familiar double-stochastic dynamics for Markovian systems.

In fact, only one-to-one dynamics are required to describe all probabilistic systems. Any probabilistic dynamics can be embedded in a one-to-one dynamics by adjoining a suitable 'environment' and by inducing a one-to-one dynamics for system and environment (2). The uncertain state of the environment then supplies the fluctuations that drive the stochastic dynamics, and the environment can absorb information to give a deterministic, many-to-one dynamics. This situation is familiar from classical statistical mechanics, in which the underlying dynamics are one-to-one and volume-preserving in phase space. This fact will be used extensively below.

Now examine the behavior of information and energy/money under these different types of dynamics. Define $V < V'$ if there is a directed path going from the vertex V to the vertex V'. The *past* of V' is the set $P(V') = \{V : V < V'\}$. Similarly, the *future* of V is the set $F(V) = \{V' : V' > V\}$. A *Cauchy surface* for the directed, acyclic graph is a set of vertices C such that no vertex in the set lies in the past or future of another vertex in the set (in space-time terms, the vertices are 'spacelike') and such that all paths from the past of the vertices in the set to the future of vertices in the set pass through one vertex in the set (figure 5). A Cauchy surface for the graph is just the analog for a Cauchy surface in space time. Define $C' > C$ if all the vertices in C' lie in the future of all the vertices in C.

Let $I(C)$ be the joint information for all vertices on the Cauchy surface, and $X(C)$ be the average quantity of energy/money on the surface. Then the following results can easily be shown. First, $X(C') = X(C)$ for all Cauchy surfaces. Second, if the dynamics are deterministic, $C' > C \rightarrow I(C') \leq I(C)$, with equality if and only if the dynamics are one-to-one. Third, if the dynamics are double-stochastic (a probilistic mix of one-to-one dynamics), then $C' > C \rightarrow I(C') \geq I(C)$. This third result corresponds to the increase of entropy in statistical mechanical systems (figure 6).

The previous results hold only for Cauchy surfaces as a whole — they are global results. But similar results hold locally. In particular, consider a set of vertices Z, and let $P(Z)$, $(F(Z)$), be the set consisting of the union of the pasts (futures) of all vertices in Z. It is straightforward to see that $X(C \cap P(Z)) \geq X(Z)$ for any Cauchy surface in the past of Z, and $X(C \cap F(Z)) \geq X(Z)$ for any Cauchy surface in the future of Z. Similarly, for deterministic dynamics, $I(C \cap P(Z)) \geq I(Z)$ for any Cauchy surface in the past of Z. And for double-stochastic dynamics, $I(C \cap F(Z)) \geq I(Z)$ for any Cauchy surface in the future of Z. These relationships merely reflect the conservation of X, the non-increasing nature of I in deterministic settings, and the non-decreasing nature of I in double-stochastic settings.

From the results of the previous paragraphs, it immediately follows that under deterministic dynamics, the average information per unit energy/money (bits per buck) decreases or remains constant, while under double-stochastic dynamics, the bits per buck increases or remains constant. Under one-to-one dynamics, the average number of bits per buck remains constant, though it may go up and down in different parts of the system.

Even though the total information in a system remains constant under one-to-one dynamics, the sum of the amount of information in different parts of the system is not in general conserved: $I(V_1) + I(V_2) \neq I(V_1 V_2)$. In fact, $I(V_1) + I(V_2) - I(V_1 V_2) = I(V_1 : V_2) \geq 0$ is just the mutual information between the two vertices V_1 and V_2. In the language of statistical mechanics, information is a nonextensive quantity. Entropy, by contrast, is normally taken to be an extensive quantity due to the typically small amounts of mutual information in statistical mechanical systems. However, as the example of Maxwell's demon shows, the non-extensive character of information must be recognized if one is to preserve the second law of thermodynamics for correlated systems (1,4,5).

Examples

Example 1: statistical mechanics

Now turn to examples. The first example is that of statistical mechanics. Here the underlying dynamics is one-to-one, in analog with the Hamiltonian dynamics of classical mechanical systems. Figure 7 applies the information network framework to a heat engine undergoing a Carnot cycle.

The heat engine consists of two parts, a 'working fluid' such as gas in a cylinder, and an energy storage device such as a flywheel. There are two reservoirs, a hot reservoir at temperature T_H and a cold reservoir at temperature T_L. Initially, the working fluid is in contact with the high-temperature reservoir. In the first step, energy $E = k_B T_H S$ and entropy S flow from the high-temperature reservoir to the working fluid, while the working fluid does work on the energy storage device, for example by expanding to move a piston that does work on the energy storage device (k_B is Boltzmann's constant). Entropy-laden energy is heat, and entropy is a form of information — information that we do not possess. The information inherent in entropy S is $I = S/k_B \ln 2$. For the purposes of heat engines, entropy can be thought of as 'junk' information.

In the second step, the working fluid is removed from contact with the high-temperature reservoir, and more work is extracted from it (e.g., by further expansion, now adiabatic), cooling it to temperature T_L. The working fluid is now put in contact with the low-temperature reservoir and compressed using energy stored in the energy storage device. During the compression process, the fluid transfers entropy S (information $I = S/k_B \ln 2$) and energy $E = k_B T_L S$ (heat) to the low-temperature reservoir. Finally, the fluid is removed from contact with the low-temperature reservoir and adiabatically compressed using energy from the energy storage device to restore it to the original temperature T_H.

The Carnot cycle is now complete: heat $k_B T_H S$ is extracted from the high-temperature reservoir, energy $k_B(T_H - T_L)S$ is converted to work, waste heat $k_B T_L S$ is deposited in the low-temperature reservoir, and entropy S is pumped from the high-temperature reservoir to the low-temperature reservoir.

The information network picture of the Carnot cycle clearly reveals the flows of energy and information/entropy in the course of the engine's operation. In the idealized Carnot cycle, entropy is conserved. In any real Carnot cycle entropy will in fact increase: typically this happens because some of the underlying dynamic processes are double stochastic. The information network framework can readily be applied to arbitrary statistical mechanical systems. It is particularly useful for dealing with tradeoffs between information and entropy, as in the case of Maxwell's demon. In such cases, information networks provides the proper accounting of the trade off between information and entropy required to preserve the second law of thermodynamics. If the reader desires excercise in the use of information networks to characterize statistical mechanical systems, he or she may construct an information network to characterize flows of information, energy, and entropy in the quantum-mechanical Maxwell's demon of reference (5), and verify that the second law of thermodynamics is indeed preserved in this case.

It is not surprising that the information network framework reproduces conventional statistical mechanics, as it was developed specifically to construct a fully-information theoretic picture of nonequilibrium statistical mechanics. (The application of information networks to nonequilibrium statistical mechanics will be explored elsewhere.) Information networks are also useful in describing trade offs between information, controllability and observability in feedback control (2). Now we apply information networks to game theory and economic models — to the trade off between bits and bucks.

Most games and economic systems can be modeled as interactions between agents (individuals, corporations, etc.) that are the subsystems of the overall complex economic system. Agents exchange information and money, as well as additional commodities. First look at the case of games where only information and money change hands. Set up an information network representing the different players and their interactions with eachother and with the overall game. Then analyze the flows of information and rewards (money) over the course of the game.

Example 2: a simple game

As an example, consider the following simple game. Two players bet on the outcome of a coin. Before each coin flip, each player bets part of their money on heads and the remainder on tails. If heads comes up, the total money is redistributed between the players in proportion to the amount that they bet. The coin may be fair or weighted. An information net describing this game is shown in figure 8. This game has been studied in detail (6). If the coin is weighted so that the probability of heads is p and the probability of tails is $1-p$, then the long-range optimum strategy is to bet a fraction p of one's money on heads and the remainder on tails. Interestingly, the long-term optimal strategy is not the one that maximizes one's average return: the maximum average return strategy is to bet all one's money on the most likely outcome. But unless $p = 1$ or 0, betting all one's money on one outcome will leave one broke very rapidly in all cases except the one in which one has luckily guessed right every time. The strategy of betting a fraction p of one's money on heads maximizes one's profit on the sequences of coin flips with measure one, in which heads occur with frequency p and tails occur with frequency $1-p$.

The coin-flip game has a simple information-theoretic description in the long term (6). If player one bets a fraction q_1 of her money on heads at each flip and player two bets a fraction q_2 of his on heads, then after n flips the expected amounts of money m_1, m_2 of the two players obeys the following formula:

$$\langle \log m_1/m_2 \rangle = n\big(p \log q_1/q_2 + (1-p)\log(1-q_1)/(1-q_2)\big) = -n(\Delta(q_1 : p) - \Delta(q_2 : p)), \quad (1)$$

where $\Delta(q : p) = -p \log q/p - (1-p)\log(1-q)/(1-p)$ is the Kullbach-Liebler distance between the probability distributions with p for heads and q for heads. The Kullbach-Liebler distance is an entropy-like quantity that is minimized for $q = p$. In other words: the player who wins in the long run is the one whose bets match most closely the actual probabilities of the weighted coin. The rate at which she profits over her less knowledgeable opponents is governed directly by an information-like quantity.

The coin-flip game is the basis for the theory of axiomatic gambling (6) and can be made very general: the probabilities for outcomes of coin flips can depend on the outcomes of previous flips, or the coin can have a predetermined but difficult to analyze sequence of outcomes. No matter how complicated the sequence becomes, the relative degree of profit of the individual players is still governed by the information theoretic formulae of equation (1). This result is confirmed by the information network picture. Here the probabilistic dynamics of the information network can be solved exactly as a Markov process. A decomposition of the probabilities for the different outcomes in terms of the eigenvectors and eigenvalues of this Markov process confirm (as they must) the results of the previous paragraph. The coin flip game is an example of a game where the interaction of the players with an underlying probabilistic process (coin flipping) results in an allocation of money that is directly dependent on information theoretic quantities.

Because of the relative simplicity and closed form solution of the coin-flipping game, the analysis in terms of information networks adds little to the existing treatment. But it is useful to try out the framework on existing systems for which closed form solutions exist, simply to verify that the framework functions properly, as it does here. The usefulness of information networks arises when the game is more complicated, and one wishes to extract general features of the behavior of the players. The situation is very much like that of non-equilibrium statistical mechanics, in which one wishes to extract general features such as energy flows and entropy production from the complicated behavior of the individual degrees of freedom of the system.

A natural domain where information networks may be able to contribute is the case of complicated, multi-player games that offer the opportunity for exchanges of information and cooperation between players. In such a game, the average profit of a group of players is typically increased by cooperation and exchange of information (it cannot be decreased by cooperation for the simple reason that one possible group strategy is not to cooperate or exchange information!). Take for example the case of two players: if they have a jointly more profitable strategy, then one can assign a value to the bits of information exchanged in the course of collaboration. This value, measured in bucks per bit, is once again a temperature-like quantity. If the game has a Nash equilibrium, in which neither player has an incentive to break their cooperative agreement, the information network picture of the game possesses a stable dynamics: the Nash equilibrium corresponds to an eigenvector of the Markovian dynamics of the information net with non-degenerate eigenvalue one. But if the game has no Nash equilibrium, as the case for the well-known game Prisoner's Dilemma, then the information net picture will not in general possess a stable dynamics for strategies that have no memory of past behavior. But such games can possess stable time-dependent strategies, as in the 'Tit for Tat' strategy in Prisoner's Dilemma. Information networks provide a natural framework for examining such strategies, as they allow for arbitrarily complicated time-dependent probilistic strategies. For example, in the context of information networks, it may be possible to prove the stability of Tit for Tat as a strategy for time-dependent Prisoner's Dilemma.

To prove the existence of a time-dependent Nash equilibrium for a game such as Prisoner's Dilemma in the context of information networks requires the investigation of the Markovian dynamics corresponding to the game for n time steps to look at the properties of eigenvalues and eigenvectors of this dynamics. In a Markovian system, the dominant eigenvalue has value 1. For the system to be stable under perturbations to the equilibrium behavior (for example, a change in strategy by one of the players), the equilibrium probability distribution for the system must correspond to the dominant eigenvector and the dominant eigenvalue must be nondegenerate.

To summarize, information nets provide a natural way to map games to probabilistic dynamical systems. Accordingly, one can investigate the possibility of stable strategies in games by analyzing the stability of the fixed points of the information net dynamics.

Example 3: a complex game

Now turn to a more entertaining game, for which closed form optimal solutions are not known to exist. Consider n interacting agents. The k'th agent begins with money $\$_k$ and with a quantity x_k of the commodity x. Both the amounts of money and the amounts of the commodity are conserved over all agents. Agents can exchange money, information, and quantities of x between eachother. For example, an agent can offer to pay a certain amount of money in a return for a certain amount of the commodity. Or an agent can sell information to another agent in the form of an option to purchase a certain amount of the commodity at a certain price at some point in the future. The goal of the agents is to assemble the maximum amount of money and the maximum amount of the commodity possible, though different agents may have different goals for the proportions of money and commodity that they seek. Call this game the market game (figure 9).

For the purposes of the market game, it is useful to think of information in terms not only of Shannon information, but also in terms of actual messages that can be sent from one agent to another. The information content of a message can be measured in terms of algorithmic information content (7) — the length of the shortest program in a suitable computer language that specifies the content of the message. Algorithmic information content is a useful surrogate for Shannon information in cases where the probabilities for different messages are not easily obtainable.

For the market game, the flexibility of information networks is allows one to capture fluid relationships between different agents, the addition and departure of agents from the game, and varying strategies for different agents at different times. The market game is a complex game in which the optimum strategy is not known. Note that the game is complicated by the possibility of credit: one agent may loan money or commodity to another player in return for a promise to pay back more at a later date. Accordingly, an agent can end up with a negative balance sheet, with more obligations than assets. Many common strategies for profiting from actual markets can be mapped with little change to the market game.

If the agents in the market game all adopt simple strategies then market equilibria may exist, as in the case where each agent desires a ratio of money to commodity equal to the total amount of money divided by the total amount of commodity. (This ratio is the 'natural price' of the commodity.) Each agent will buy at the natural price if they have too little of the commodity and sell at the natural price if they have too much. Here the price remains constant and equilibrium is attained after a period of buying and selling. The information network picture of this process is straightforward: agents interact in randomly selected pairs and exchange a few bits of information ('buy an amount up to x' or 'sell an amount up to y'). If one wishes to buy and the other wishes to sell, then they exchange the maximum desired amount. The process continues until no more transactions can take place. The Markovian dynamics of this information net is simple and stable, and converges to a distribution in which each agent has the same ratio of money to commodity.

Similarly, if each agent has a different desired ratio of money to commodity and is willing to buy or sell at a price that sends their actual ratio of money to commodity closer to their desired ratio, then a stable market equilibrium will also be established. The information network now possesses a more complicated Markovian dynamics, though the dominant eigenvalue is still nondegenerate, indicating the stability of the dynamics. But if agents have more complicated goals and strategies, then simulations of artificial stock markets suggest that the market game will in general exhibit complex behavior, including booms, busts, and bubbles.

How can we analyze the market game using information networks? First, construct an information network representing the individual agents and their interactions with eachother. It may be useful, for instance, to imagine a special agent such as a market maker who collects bids and and offers and who matches up buyers and sellers. The actual dynamics of a particular game will depend sensitively on the set of strategies allowed. For the simple cases above in which each agent buys at a price below what that agent considers to be the natural price, and sells above that price, then the dynamics are simple and stable. But if agents adopt more complicated strategies of betting on trends of prices, then the artificial stock market results show that the system exhibits complex behavior. If the agents are allowed to exchange information in the form of contracts (loans, options, etc.), then the behavior is more complex still. The hope of introducing an information network analysis is to see if one can characterize a variety of these complex behaviors in terms of flows of information, money, and commodities.

For example, one might attempt to reproduce the spectrum of price fluctuations in artificial stock markets by analyzing the stability of Markovian dynamics of the market game's information network. There is evidence that real and artificial stock markets exhibit a scale-free (power-law) spectrum of fluctuations. Such spectra arise in statistical mechanics at critical points such as phase transitions. An information network can exhibit such behavior if the dominant eigenvalue of the Markovian dynamics is non-degenerate, so that perturbations of the system can cause it to move among the various behaviors corresponding to the different eigenvectors of the dominant eigenvalue. We are investigating the stability of information networks for the market game under a variety of strategies for the individual agents.

Like statistical mechanics, the market game admits natural notions of temperature. When two agents get together, they communicate to determine if they are willing to exchange commodity, information, and money. For example, if agent one is willing to pay an amount of money up to $\$_1(x)$ for an amount of commodity x, and agent two is willing to sell an amount x at a price $\$_2(x)$ then the two agents have a deal for exchanging the amount x such that $\$_1(x) = \$_2(x)$. The price, $\$_i/x$, is a temperature-like quantity, and two agents who assign different prices to a quantity will in general offer and counter offer until they attain a common equilibrium price where the two 'temperatures' $\$_1/x$ and $\$_2/x$ are the same (as in the normal micro-economic picture of markets). The market game evolves by pairs of agents coming to equilibrium with eachother (figure 10). Nontrivial dynamics arise when the strategies of agents involve predicting the future dynamics of the price as a function of past and current prices.

Similarly, agents can attach a price to information, so that agent 1 can be willing to exchange money for information that is possessed by agent 2. This information could be a promise to repay a higher amount of money at a later time (a loan), a signature on a check, or insider information about a transaction to occur. The primary difference between the exchange of a commodity for money and the exchange of information for money is that agent 2 can sell the information to agent 1 while retaining a copy of the information. This feature leads to quite different dynamics for the sale of information than for a commodity (although some pieces of information, like dollar bills, can have a relatively constant value). We are also investigating the price of information in the market game.

Discussion

This paper presented a basic formalism for information networks. Information networks are a formalism for analyzing systems that evolve by exchanges of information and conserved quantities such as energy and money. Information networks are a very general formalism that can be applied to virtually any physical system. A number of examples were suggested in which information networks might be able to elucidate a variety of apparently complex behaviors. Whether or not they will prove effective in capturing complex behavior remains to be seen, but information networks potentially allow one to extend techniques that have been successful in statistical mechanics and information processing to problems of complexity in general.

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Figures

(1) An information network. Vertices represent states of subystems at different points in time. Edges represent flows of information and conserved quantities such as energy and money between subsystems.

(2) Information networks for single (non-complex) systems. (2a) represents a single system at two points in time. (2b) represents a system at multiple points in time.

(3) A block diagram (bottom) and the corresponding information network. The information network 'unpacks' the block diagram at successive instants in time The feedback loop in the block diagram becomes a closed loop in the information network.

(4) An information network for a complex system whose subsystems together with the relationships between them change over time. Initially, the system is composed of two interacting subsystems, A and B. Then A and B cease to interact, A ceases to exist, and a new subsystem C appears and sporadically interacts with B.

(5) A 'Cauchy surface' for an information network divides the past from the future and contains all information about the entire system at one point in time. A Cauchy surface can be thought of as a 'spacelike slice' of a complex system.

(6) The intersection of a Cauchy surface (dashes) with the set of points in the future of V_1 and V_2 (dots). The amount of energy/money on the intersection is greater than or equal to the amount of energy/money on V_1 and V_2. If the system has a one-to-one dynamics, then the points on the intersection contain all information about V_1 and V_2.

(7) An information network picture of a Carnot cycle. The working fluid absorbs heat $T_H S$ from a reservoir at high temperature T_H, doing work on an energy storage device in the process. Then the fluid is removed from contact with the high-temperature reservoir and expanded further, doing more work and having its temperature reduced to T_L. Now the energy storage device does work on the working fluid, compressing it to drive heat $T_L S$ to the low-temperature reservoir. Finally the working fluid is removed from contact with the reservoir and compressed further until it reaches temperature T_H. The Carnot cycle is complete, and energy $(T_H - T_L)S$ has been stored in the energy storage device, while pumping entropy S from the high-temperature reservoir to the low-temperature reservoir.

(8) An information network for the coin-tossing game. Players A and B each bet part of their money on heads and the remainder on tails. In the network this flow of money and information is represented by the edges between A and B and the coin tosser. The

coin is tossed, and the money distributed to A and B in proportion to the amount that they bet on the winning outcome. This information net shows each of the flows of money and information, including the information about past successes and failures that A and B retain.

(9) The market game. In the market game, agents can exchange information, money, and a commodity x. This piece of an information network shows two transactions in the market game. In the first, agents B and C exchange information (e.g., prices), and on the basis of that information decline to make a transaction. In the second, agents A and B exchange information, and on the basis of that information A sells B a quantity of commodity x for an amount of money \$.

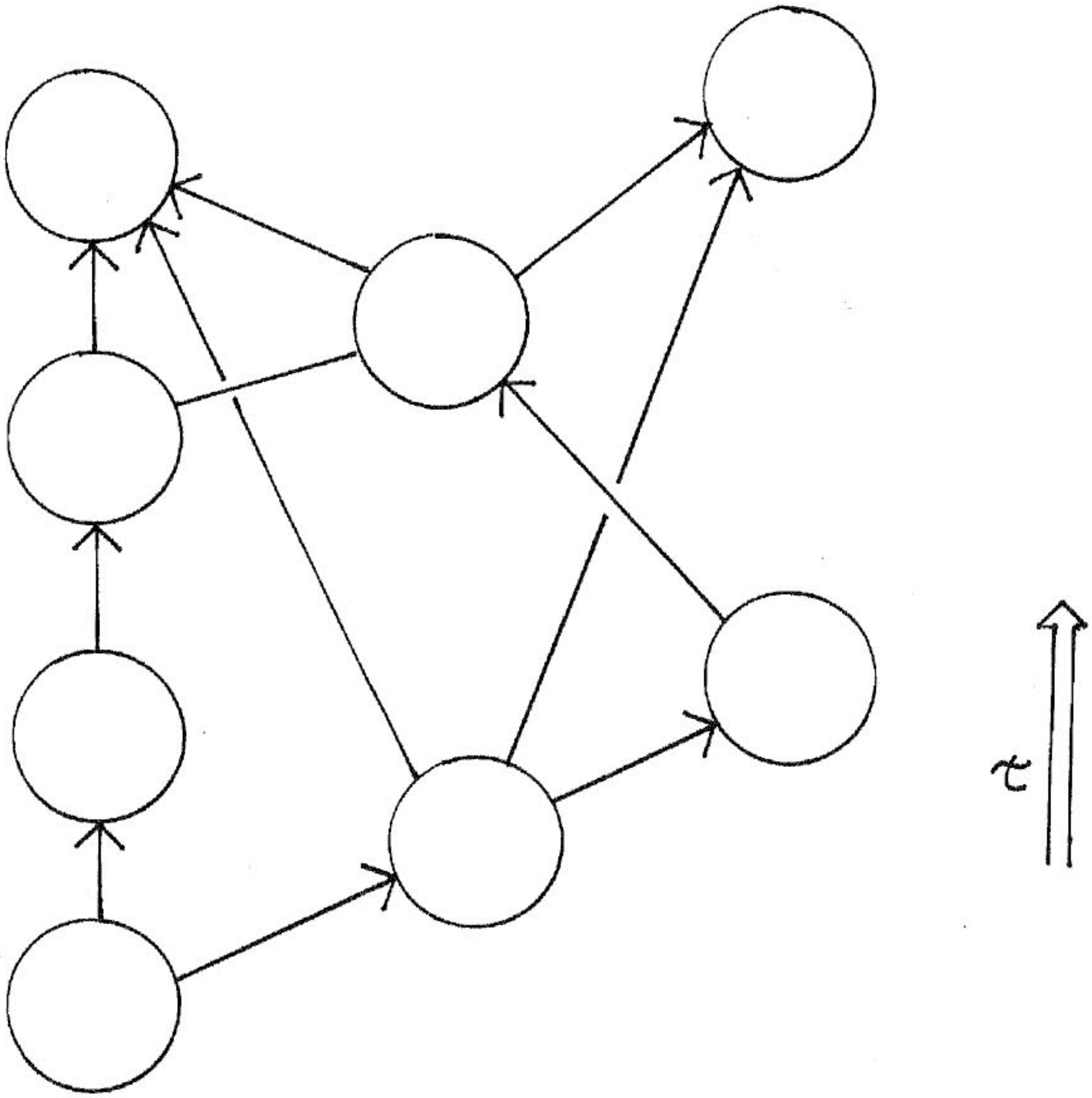


figure 1

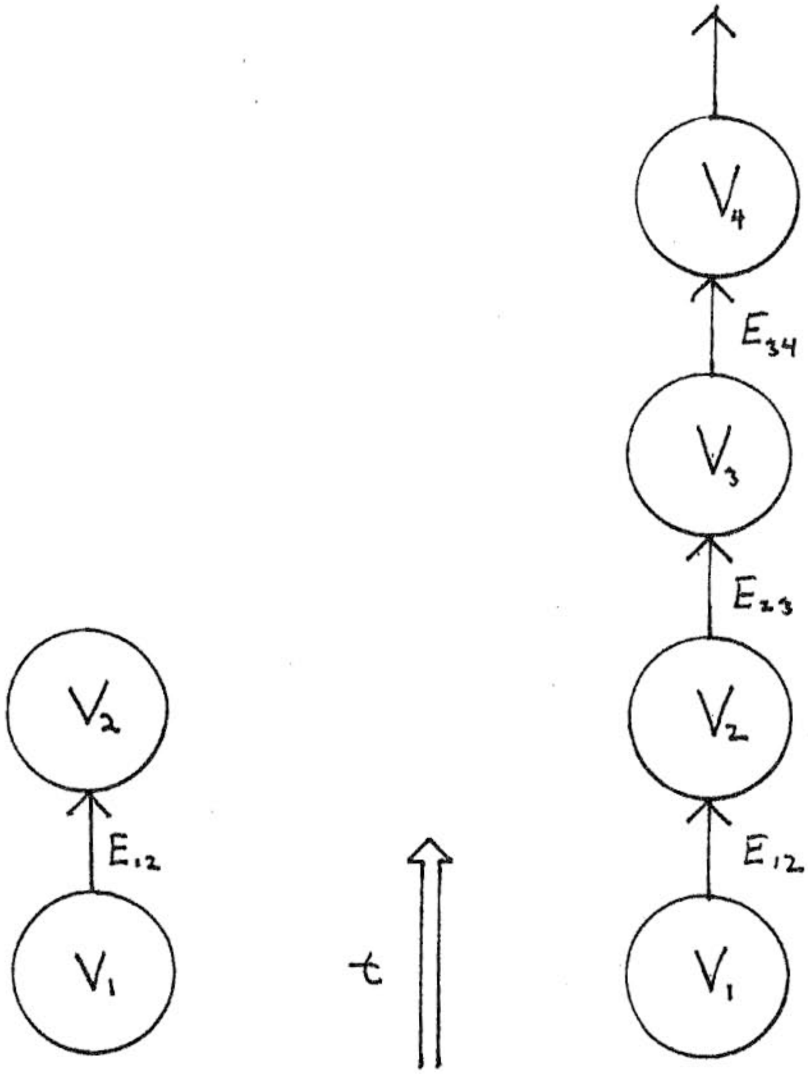
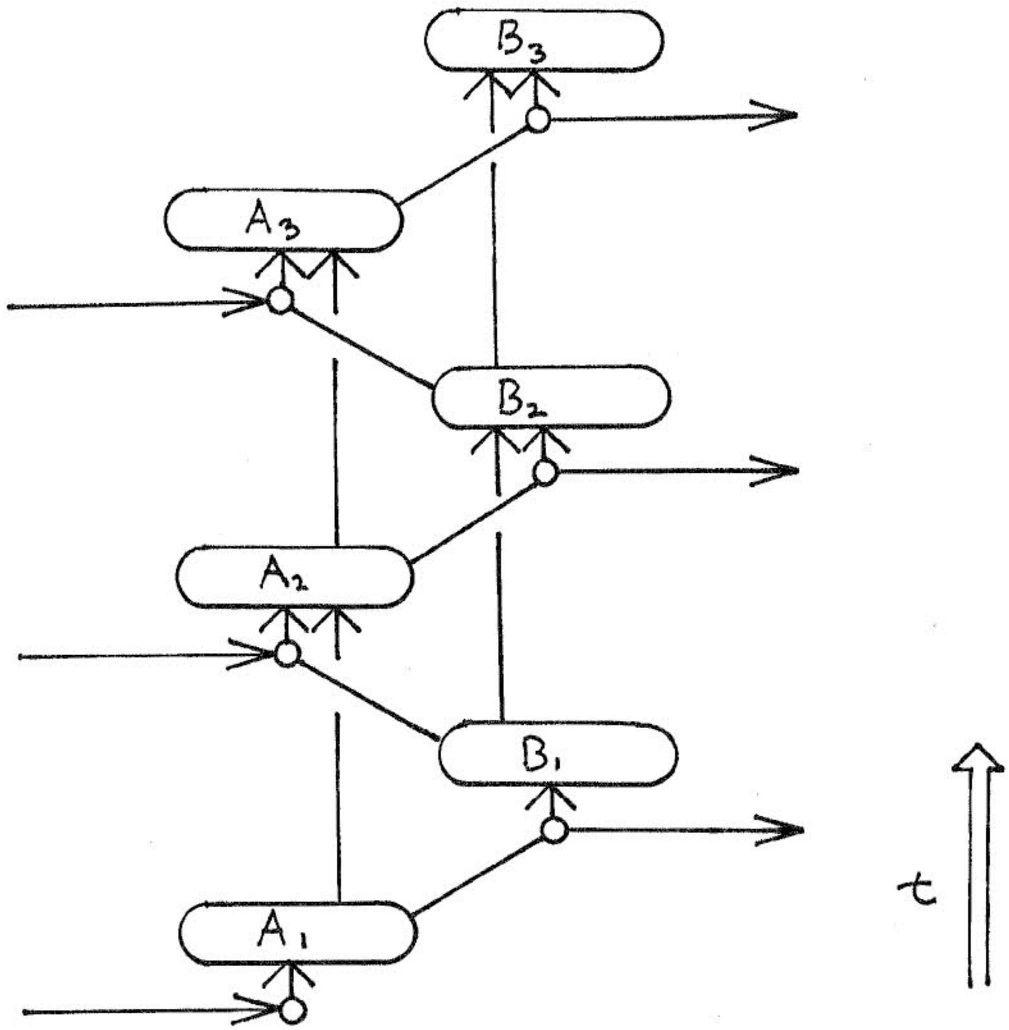
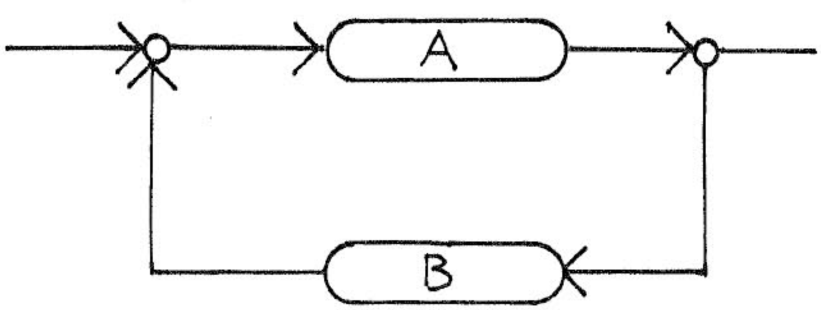


figure 2a, b





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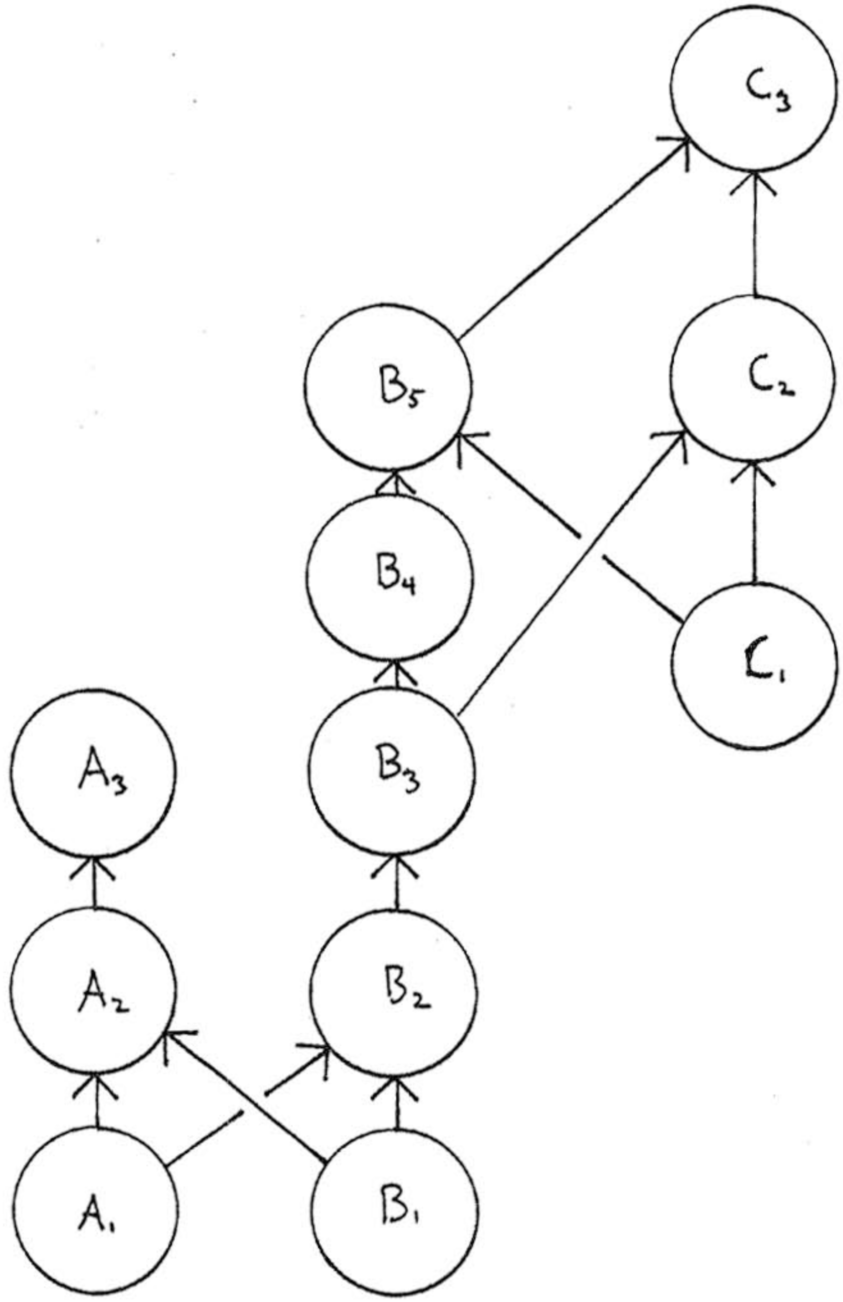

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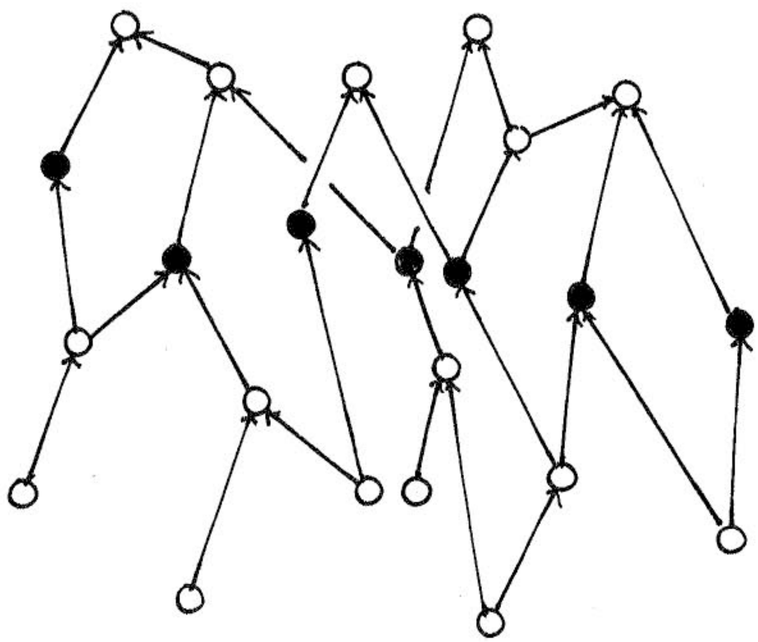

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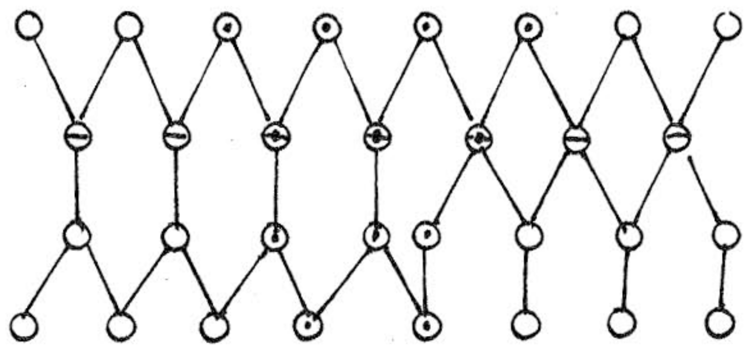
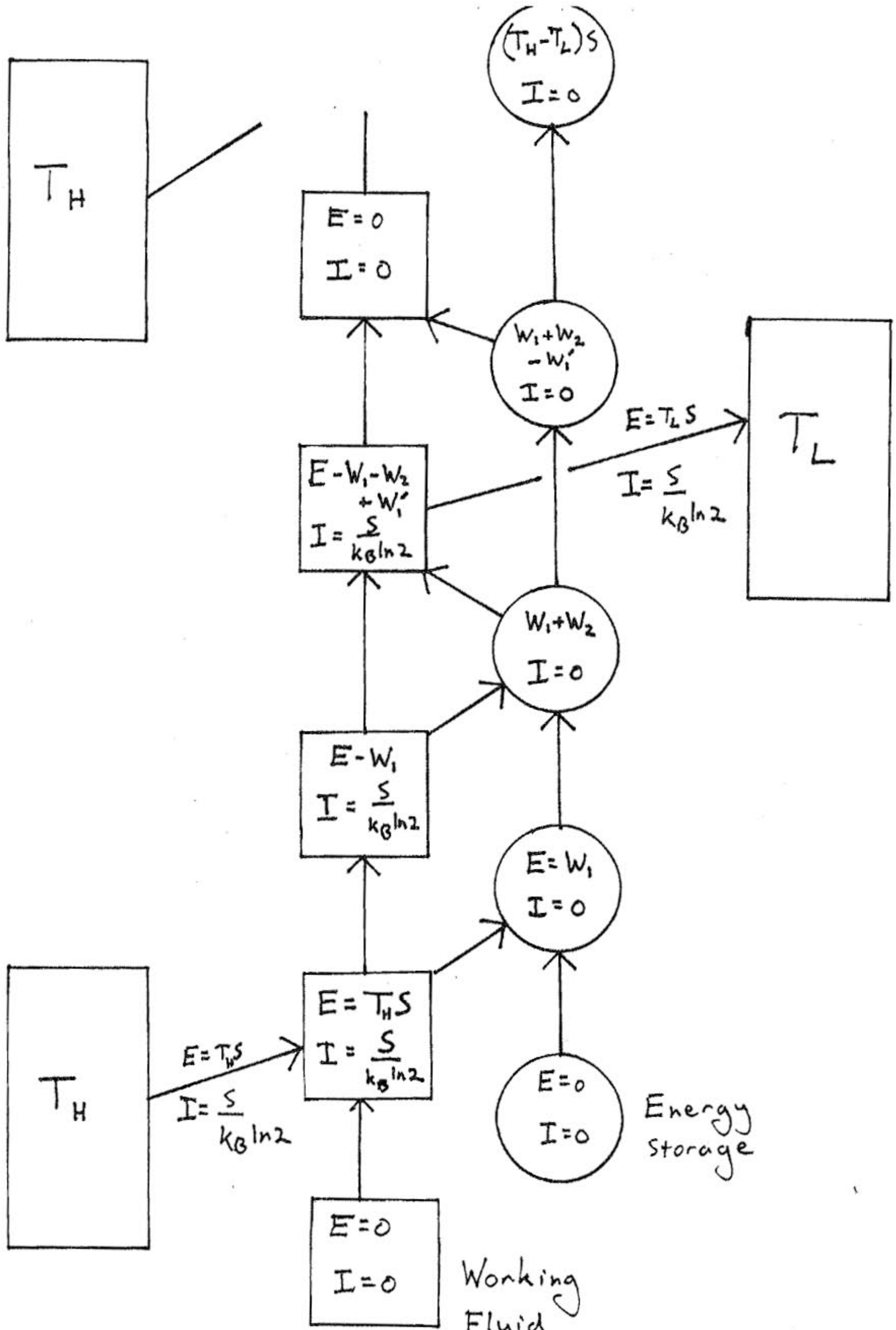


figure 6



$(T_H - T_L)S$
$I=0$
T_H
$E=0$
$I=0$
$W_1+W_2-W_1'$
$I=0$
$E=T_L S$
$I=\frac{S}{k_B \ln 2}$
T_L
$E-W_1-W_2+W_1'$
$I=\frac{S}{k_B \ln 2}$
W_1+W_2
$I=0$
$E-W_1$
$I=\frac{S}{k_B \ln 2}$
$E=W_1$
$I=0$
$E=T_H S$
$I=\frac{S}{k_B \ln 2}$
$E=T_H S$
$I=\frac{S}{k_B \ln 2}$
T_H
$E=0$
$I=0$
Energy Storage
$E=0$
$I=0$
Working Fluid

figure 7

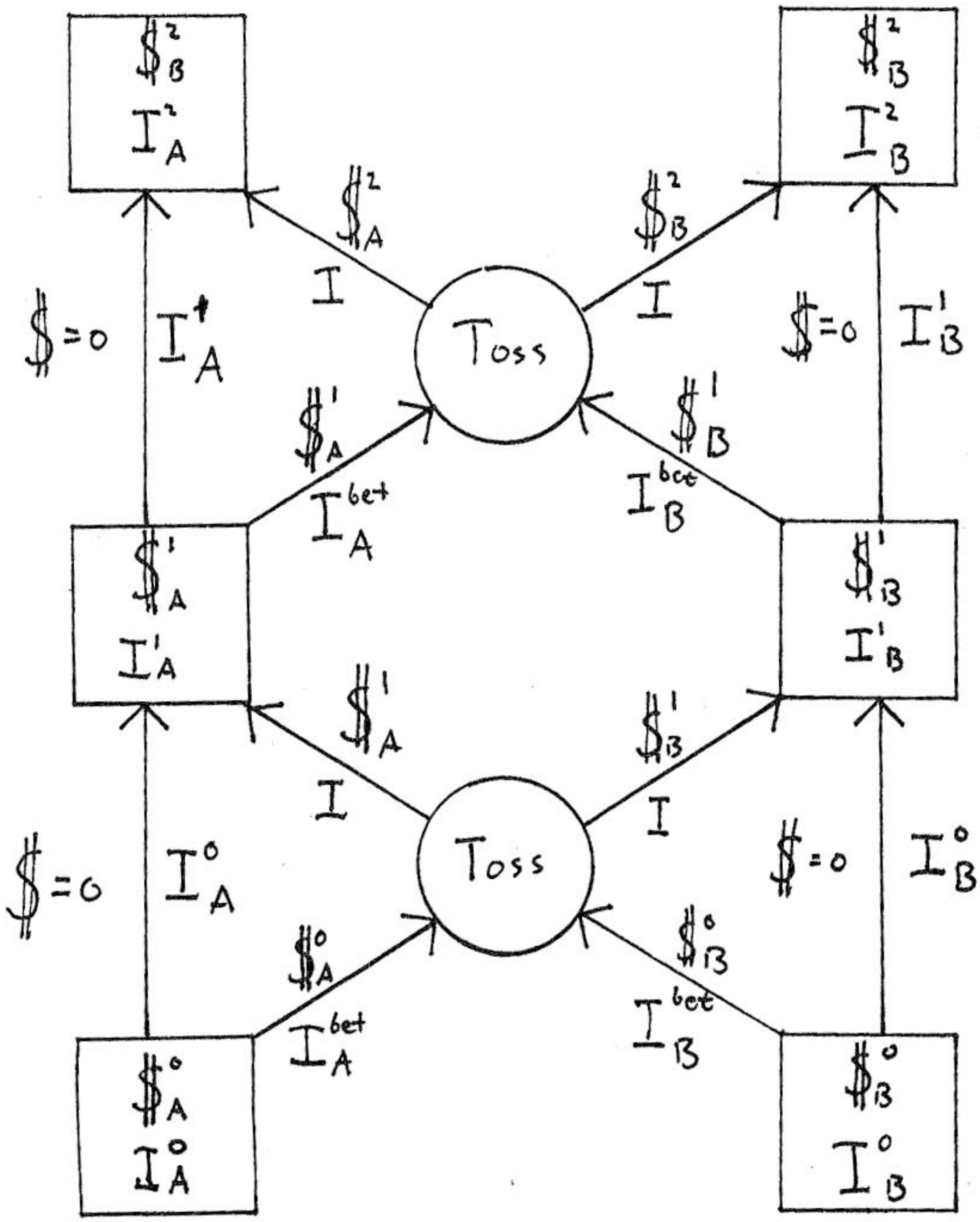


figure 8

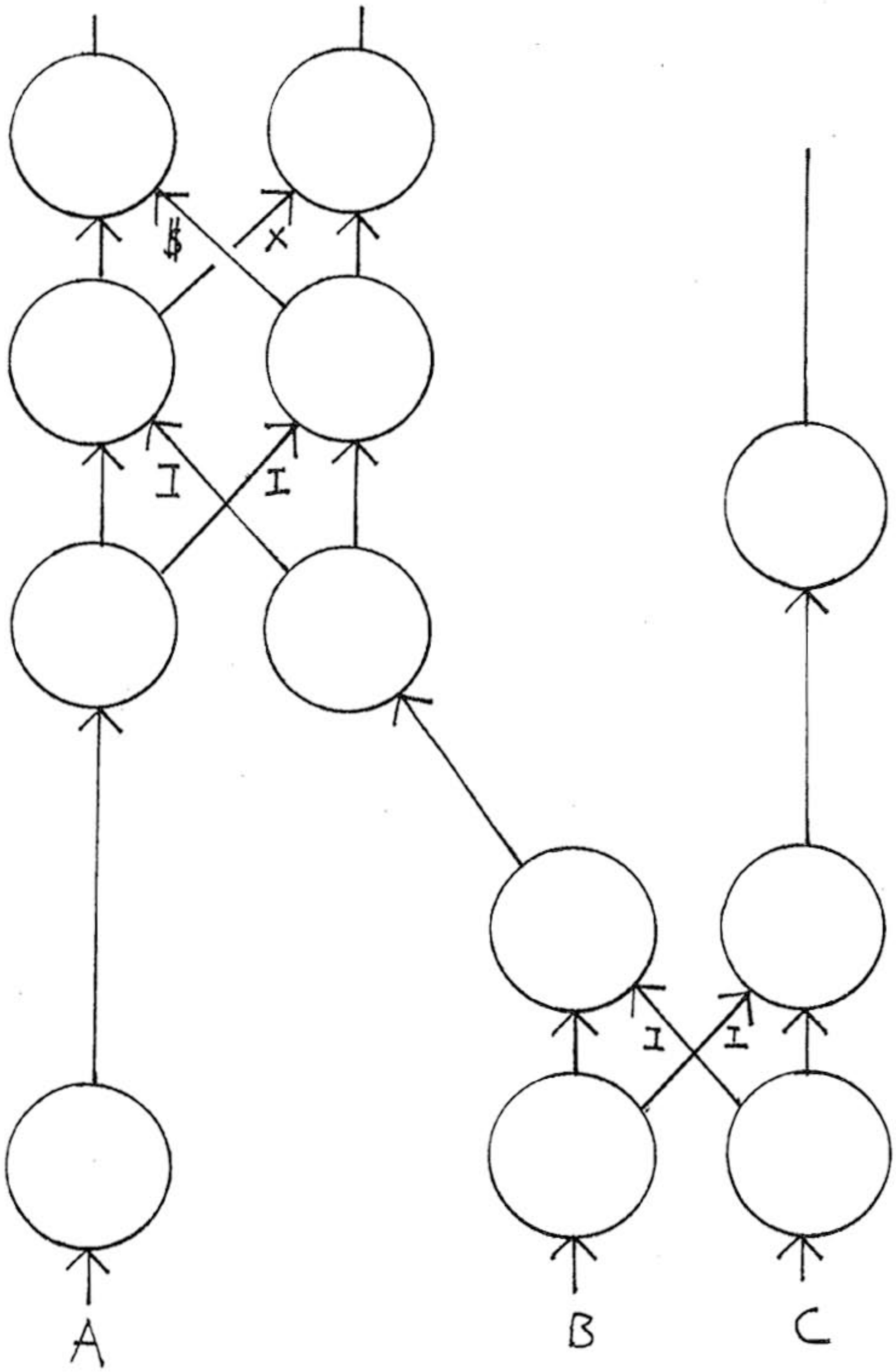


figure 9