Chiral drag force

Krishna Rajagopal[a] **and Andrey V. Sadofyev**[a,b]

[a] *Center for Theoretical Physics, Massachusetts Institute of Technology,*
Cambridge, MA 02139, U.S.A.

[b] *ITEP,*
B. Cheremushkinskaya 25, Moscow, 117218 Russia

E-mail: krishna@mit.edu, sadofyev@mit.edu

ABSTRACT: We provide a holographic evaluation of novel contributions to the drag force acting on a heavy quark moving through strongly interacting plasma. The new contributions are chiral in the sense that they act in opposite directions in plasmas containing an excess of left- or right-handed quarks. The new contributions are proportional to the coefficient of the axial anomaly, and in this sense also are chiral. These new contributions to the drag force act either parallel to or antiparallel to an external magnetic field or to the vorticity of the fluid plasma. In all these respects, these contributions to the drag force felt by a heavy quark are analogous to the chiral magnetic effect (CME) on light quarks. However, the new contribution to the drag force is independent of the electric charge of the heavy quark and is the same for heavy quarks and antiquarks, meaning that these novel effects do not in fact contribute to the CME current. We show that although the chiral drag force can be non-vanishing for heavy quarks that are at rest in the local fluid rest frame, it does vanish for heavy quarks that are at rest in a suitably chosen frame. In this frame, the heavy quark at rest sees counterpropagating momentum and charge currents, both proportional to the axial anomaly coefficient, but feels no drag force. This provides strong concrete evidence for the absence of dissipation in chiral transport, something that has been predicted previously via consideration of symmetries. Along the way to our principal results, we provide a general calculation of the corrections to the drag force due to the presence of gradients in the flowing fluid in the presence of a nonzero chemical potential. We close with a consequence of our result that is at least in principle observable in heavy ion collisions, namely an anticorrelation between the direction of the CME current for light quarks in a given event and the direction of the kick given to the momentum of all the heavy quarks and antiquarks in that event.

Contents

1 Introduction and summary

The analysis of how a heavy quark moving through the strongly coupled liquid quark-gluon plasma produced in ultrarelativistic heavy ion collisions loses energy is motivated by heavy ion collision experiments, in which these heavy quarks are used as probes of the plasma and, for those that lose enough energy, as tracers that follow its flow. If one assumes that the interactions between the heavy quark and the plasma are weak, then perturbative methods that were first introduced in ref. [1] can be employed to analyze heavy quark energy loss. However, the discovery that the plasma produced in heavy ion collisions is itself a strongly coupled liquid has raised the question of how to understand the real-time dynamics of heavy quarks in a strongly coupled non-Abelian plasma. Treating all aspects of the dynamics as strongly coupled is of value first as a benchmark and second because it means that rigorous calculations of novel effects become tractable in plasmas with a gravitational dual.

The simplest plasma in which one can calculate the rate of energy loss of a heavy quark is that in strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in the large number of colors (large N_c) limit, whose plasma with temperature T is dual to classical gravity in a 4+1-dimensional spacetime that contains a 3+1-dimensional horizon with Hawking temperature T and that is asymptotically Anti-de Sitter (AdS) spacetime [2, 3]. In the dual gravitational description, the heavy quark is represented by a string moving through the AdS black hole spacetime, trailing behind its endpoint that follows the trajectory of the infinitely heavy quark along the boundary of the AdS [4–7]. The earliest

work on heavy quark dynamics in the equilibrium plasma of strongly coupled $\mathcal{N} = 4$ SYM theory [5–7] yielded determinations of the drag force felt by a heavy quark moving through the static plasma and the diffusion constant that governs the subsequent diffusion of the heavy quark once its initial motion relative to the static fluid has been lost due to drag. This work has been generalized in many directions since then. We will in particular need the modifications of the spacetime metric that describe a flowing, hydrodynamic, plasma in which there are gradients of the fluid properties as a function of space and time [8–10]. The corresponding modifications of the drag force were worked out to leading order in the fluid gradients in ref. [11]. However, in this calculation the possibility of a nonzero density of some fermion species, and a corresponding nonzero chemical potential, was not taken into account.

We begin in section 2 by introducing the dual gravitational description of a strongly coupled plasma with both a chemical potential and fluid gradients, working to leading non-trivial order in both μ/T and the fluid gradients. At the same time, we introduce the dual gravitational description of the axial anomaly, relevant if the chemical potential is either that for left-handed quarks or for right-handed quarks. In section 3 we turn off the anomaly, for example as appropriate if μ is the chemical potential for baryon number, and calculate the corrections to the drag force in powers of μ/T, working to first order in fluid gradients. This yields the (straightforward although laborious) extension of the results of ref. [11] to the case of a plasma with nonzero μ. The results of this section constitute quantitative modifications to the drag force, but they do not introduce qualitatively new effects.

In section 4 we analyze a chiral plasma. That is, we take μ to be the chemical potential for either left-handed or right-handed quarks, and turn on the anomaly. In the dual gravitational theory this means we turn on the Chern-Simons term in the holographic action that we introduced in section 2. This term gives rise to novel chiral contributions (contributions that change sign if the plasma contains right-handed quarks as opposed to left-handed quarks) to the charge and entropy currents and to the stress-energy tensor [9, 12], reproducing the chiral magnetic effect (CME) and chiral vortical effect (CVE) that had already been introduced without the use of holography [12–15]. These anomalous contributions to the hydrodynamic motion of a chiral fluid have been discussed widely [9, 12, 16–43]. These effects originate in topological aspects of the gauge theory [12, 14, 15, 33, 36, 37], as is of course the case for the axial anomaly itself [44].

All the previously analyzed consequences of the axial anomaly in a chiral plasma — the CME and its cousins — concern the motion of light quarks; in fact the quarks are usually assumed to be massless. We show in section 4 that there are anomalous contributions to the drag force on an infinitely heavy quark that finds itself in a chiral plasma. The standard CME and related effects involve the generation of anomalous currents parallel or anti-parallel to an external magnetic field or the angular velocity vector of the fluid [12–15]. The anomalous contributions to the drag force that we compute, order by order in μ/T, have the same feature. This means that in the presence of a nonzero density of heavy quarks that are initially at rest in the local fluid rest frame, the chiral drag force that we compute can yield a new contribution to the CME electric current, even though the heavy quarks themselves do not participate in the CME. Note, however, that if the plasma

features equal and opposite number densities of some heavy quark and its antiquark, or densities of two species of heavy quarks with opposite electric charges, all the heavy quarks and antiquarks feel a chiral drag force acting in the same direction, and no electric current is generated. For example, the fluid produced in a heavy ion collision is seeded with equal numbers of charm and anti-charm quarks, and equal numbers of bottom and anti-bottom quarks. Since in any volume of the plasma in which there is an excess of, say, right-handed light quarks all the heavy quarks and antiquarks feel a chiral drag force in the same direction, the chiral drag force does not result in an electric current. Of course, if there were a nonzero chemical potential for some species of heavy quark, meaning an excess of those heavy quarks relative to their antiquarks, the push on all heavy quarks and antiquarks from the chiral drag force would result in a heavy quark contribution to the electric current. This would be an example of a correction to the CME or CVE currents, as has been found in other contexts [35, 38, 40–43].

It at first seems odd to find a nonvanishing chiral drag force on a heavy quark even when the heavy quark is at rest in the local fluid rest frame. We show in section 5 that (as long as we neglect the gravitational anomaly) the resolution is related to the previously known fact that in the local fluid rest frame there are anomalous contributions to the entropy current, since if we go instead to a frame in which the local entropy current vanishes at the location of the heavy quark we find no chiral drag force on the heavy quark. In this frame, the heavy quark at rest is immersed in a flowing fluid, with nonzero momentum and charge currents that are both proportional to the axial anomaly coefficient, but the heavy quark feels no drag force. If we think of the heavy quark as a defect placed in these propagating anomaly-induced streams, the fact that the heavy quark feels no drag force is a direct consequence of the dissipationless nature of the CME current. The nondissipative character of the chiral magnetic effect has been discussed before [24]. Our calculation of the chiral drag force provides direct evidence for this fundamental attribute of the chiral magnetic effect.

In section 6 we discuss possible phenomenological consequences of the chiral drag force. The basic effect is that the heavy quarks and antiquarks in a heavy ion collision in which there has been a fluctuation resulting in an excess of right-handed (left-handed) light quarks will feel a force that pushes them in a direction perpendicular to the reaction plane that is antiparallel (parallel) to the direction of any magnetic field or fluid angular velocity vector. We shall show that the effects are small. Furthermore, they will average out in an ensemble of events. And, mesons containing heavy quarks are not so numerous in any single heavy ion collision. For all these reasons, it will be difficult to separate the effects of a small force that acts in the same direction on all the heavy quarks and antiquarks in one event from random forces that act differently on different heavy quarks. Perhaps clever correlation observables can be found. One possibility is to utilize the fact that in each event the direction of the kick that all the heavy quarks and antiquarks in that event receive, regardless of their electric charge, is opposite to the direction of the CME electric current in the light quark sector.

2 Holographic fluid

To compute the drag force on a heavy quark moving through the strongly coupled plasma of $\mathcal{N} = 4$ SYM theory with a nonzero chemical potential and with spatial and temporal gradients in the fluid velocity, temperature and chemical potential we need to start with the perturbations to the dual gravitational theory that correspond to hydrodynamic flow [9, 10]. We shall present the corresponding 4+1-dimensional bulk metric in this section, delaying the introduction of the heavy quark that we are interested in to section 3. The dual gravitational theory is described by the 4+1-dimensional Einstein-Maxwell action

$$S = -\frac{1}{16\pi G_5} \int d^5 x \sqrt{-G} \left(R + 12 - \frac{1}{4}F^2 \right) - \frac{\kappa}{48\pi G_5} \int d^5 x \epsilon^{MNOPQ} A_M F_{NO} F_{PQ} \quad (2.1)$$

whose equations of motion are

$$R_{MN} + 4G_{MN} + \frac{1}{2}F_M{}^K F_{KN} + \frac{1}{12}G_{MN}F^2 = 0$$

$$\partial_N \left(\sqrt{-G}F^{NM} \right) + \kappa \epsilon^{MNOPQ} F_{NO} F_{PQ} = 0 \, . \quad (2.2)$$

Here, G_5 is the 4+1-dimensional Newton constant which according to the holographic dictionary, see e.g. ref. [9], is related to the number of colors in the boundary gauge theory by $G_5 = \frac{\pi}{2N_c^2}$. Furthermore, A_M and F_{MN} are a 4+1-dimensional vector potential and the corresponding field strength and κ is the 4+1-dimensional Chern-Simons coupling, which is dual to the axial anomaly coefficient in the boundary gauge theory. In the case of $\mathcal{N} = 4$ SYM theory at strong coupling,

$$\kappa = -\frac{1}{4\sqrt{3}} \, . \quad (2.3)$$

(See e.g. refs. [12, 45, 46]). We should mention that we have chosen units in which the AdS radius R_{AdS} has been set to 1, meaning that our holographic dictionary coincides with that in ref. [9]. The equations of motion (2.2) have a static black hole solution that corresponds, in the boundary theory, to a plasma with some nonzero temperature T and some nonzero density of right-handed fermions with chemical potential μ. Note that κ is the bulk Chern-Simons coupling constant and it gives the boundary theory the axial anomaly. If we want a model for a theory like QCD in which there are both left- and right-handed quarks, we need to introduce two bulk vector potentials, one with $\kappa = -\frac{1}{4\sqrt{3}}$ corresponding to μ_R and the other with $\kappa = +\frac{1}{4\sqrt{3}}$ corresponding to μ_L. If we then wish to consider only the case $\mu_L = \mu_R$, we could define $\mu_V \equiv (\mu_R + \mu_L)/2$, set the two bulk vector potentials equal to each other, and ignore κ. Of course, κ becomes relevant in any circumstance in which $\mu_A \equiv (\mu_R - \mu_L)/2 \neq 0$.

In Eddington-Finkelstein coordinates, the metric and bulk gauge field that describe a static black hole of mass M and charge Q take the form

$$ds^2 = -r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu - 2u_\mu dx^\mu dr$$

$$A_\mu = -\frac{\sqrt{3}Q}{r^2} u_\mu \quad (2.4)$$

with

$$f(r) \equiv 1 - \frac{M}{r^4} + \frac{Q^2}{r^6} \,, \tag{2.5}$$

where u_μ is a constant vector satisfying $u_\mu u^\mu = -1$, where $P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu$, and where we are working in axial gauge $A_r = 0$. The energy density, charge density, entropy density, temperature and chemical potential of the boundary theory strongly coupled plasma can be related to the M and Q of the dual black hole as follows [45, 46]. μ and T are given by

$$\mu = \frac{\sqrt{3}Q R_{\text{AdS}}}{r_+^2} \,, \qquad T = \frac{r_+}{2\pi R_{\text{AdS}}^2} \left(2 - \left(\frac{r_-}{r_+} \right)^2 - \left(\frac{r_-}{r_+} \right)^4 \right) \,, \tag{2.6}$$

where r_+ and r_- are the larger and smaller real solutions of the equation $f(r) = 0$, and where we have temporarily restored the factors of R_{AdS}. These relations can be rewritten as

$$\frac{r_+}{R_{\text{AdS}}^2} \equiv \frac{\pi T}{2} \left(1 + \sqrt{1 + \frac{2\mu^2}{3\pi^2 T^2}} \right) \quad \text{and} \quad r_-^2 \equiv \frac{1}{2} r_+^2 \left(-1 + \sqrt{9 - \frac{16}{\left(1 + \sqrt{1 + \frac{2\mu^2}{3\pi^2 T^2}} \right)}} \right) . \tag{2.7}$$

The energy density, charge density and entropy density are given in terms of μ and T by [9]

$$\varepsilon = \frac{3N_c^2}{8\pi^2} \frac{r_+^3}{R_{\text{AdS}}^6} \left(3\frac{r_+}{R_{\text{AdS}}^2} - 2\pi T \right) \,, \qquad \rho = \frac{\mu N_c^2}{4\pi^2} \frac{r_+^2}{R_{\text{AdS}}^4} \,, \qquad s = \frac{N_c^2}{2\pi} \frac{r_+^3}{R_{\text{AdS}}^6} \,, \tag{2.8}$$

where N_c is the rank of the gauge group and we are working throughout in the large-N_c limit. The boundary theory plasma is conformal, meaning that it has $T^\mu_\mu = 0$, and so has an equation of state $\varepsilon = 3P$, and has bulk viscosity $\zeta = 0$. Its shear viscosity can be calculated via gauge/gravity duality and is given by $\frac{\eta}{s} = \frac{1}{4\pi}$ [47, 48].

Analogously to the way that hydrodynamics is usually derived, the way to find a bulk metric that is the dual gravitational description of a flowing strongly coupled plasma is to look for a solution to the bulk Einstein-Maxwell equations in which T, μ and u^μ are all slowly varying functions of space and time, and to organize the calculation via a gradient expansion. The metric is expanded in powers of boundary gradients

$$G_{\mu\nu} = G_{\mu\nu}^{(0)} + G_{\mu\nu}^{(1)} + \mathcal{O}(\partial^2) \,, \tag{2.9}$$

where $G_{\mu\nu}^{(0)}$ is defined in (2.4) and where $G_{\mu\nu}^{(1)}$ contains all possible gradient structures of first order, with unknown coefficient functions. The solution has been obtained up to first order in gradients and is given by [9]

$$ds^2 = -r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu - 2u_\mu dx^\mu dr + r^2 F(r) \sigma_{\mu\nu} dx^\mu dx^\nu$$
$$+ r^2 j_\sigma \left(P_\mu^\sigma u_\nu + P_\nu^\sigma u_\mu \right) dx^\mu dx^\nu + \frac{2}{3} r(\partial \cdot u) u_\mu u_\nu dx^\mu dx^\nu \tag{2.10}$$

where

$$\sigma^{\mu\nu} \equiv P^{\mu\alpha} P^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right) - \frac{2}{3} P^{\mu\nu} \partial \cdot u \tag{2.11}$$

and

$$j_\sigma \equiv -\frac{1}{r}(u \cdot \partial)u_\sigma + \frac{2\sqrt{3}Q^3\kappa}{Mr^6}\ell_\sigma + J(r)\partial_\sigma\frac{\mu}{T}\,, \tag{2.12}$$

where $\ell_\mu \equiv \epsilon_{\mu\nu\alpha\beta}u^\nu\partial^\alpha u^\beta$ is the four-vector containing the vorticity of the fluid, and where the functions $F(r)$ and $J(r)$ are, up to leading nontrivial order in μ, given by

$$\begin{aligned}
F(r) \equiv \frac{1}{4\pi T}&\left[2\arctan\left(\frac{\pi T}{r}\right) - \log\left(\frac{r^4}{(r+\pi T)^2(r^2+\pi^2 T^2)}\right)\right]\\
&+\frac{\mu^2}{24\pi^2 T^2}\left[-\frac{3}{T} + \frac{4\pi T}{r^2} + \frac{2}{r+\pi T} + \frac{4(r+\pi T)}{r^2+\pi^2 T^2}\right.\\
&\left.+\frac{6}{\pi T}\left[\arctan\left(\frac{r}{\pi T}\right) + \log\left(\frac{r^2}{r^2+\pi^2 T^2}\right)\right]\right] + \mathcal{O}\left(\mu^4\right)
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
J(r) \equiv \frac{\mu}{24\pi^3 r^4 T^2}&\left[2\pi^2 T^2 r(3r - 2\pi T)\right. \tag{2.14}\\
&\left.-3(r^4 - \pi^4 T^4)\left[2\arctan\left(\frac{\pi T}{r}\right) - \log\left(\frac{(r+\pi T)^2}{r^2+\pi^2 T^2}\right)\right]\right] + \mathcal{O}\left(\mu^3\right)\,.
\end{aligned}$$

Note that since $Q \propto \mu$ the anomalous term in j_σ that is proportional to $\kappa\ell_\sigma$ is of order μ^3, while the leading μ-dependence in the metric (2.10) coming from $F(r)$ and $J(r)$ is of order μ^2. We will see the effects of the terms of order μ^2 in section 3 and shall introduce the effects of order μ^3 that originate from the axial anomaly in section 4. In section 4 we shall also introduce effects that originate from the axial anomaly that are proportional to an external magnetic field, rather than to the fluid vorticity. Doing so will require adding an additional term in j_σ, a term that is of order μ^2, and that is proportional to κ and the magnetic field. Note that, as in ref. [9], we are using the Landau definition of the fluid velocity u^μ, such that the Lorentz frame in which $u^\mu = (1, \vec{0})$ corresponds to the frame in which the fluid momentum T^{0i} vanishes. In this local fluid rest frame, there can still be nonzero charge currents or entropy currents, as we shall see later.

To the same order in μ/T and gradients as above, the bulk gauge field now takes the form

$$A_\mu = -\frac{\sqrt{3}Q}{r^2}u_\mu + \frac{6\kappa Q^2}{Mr^2}\ell_\sigma + a(r)\partial_\sigma\frac{\mu}{T} \tag{2.15}$$

where we are still in $A_r = 0$ gauge and where $a(r)$ is a known function that can be found in ref. [9]. However, the bulk gauge field does not affect the string that we will introduce in the next section that constitutes the dual description of the heavy quark, and for this reason we do not require A_μ for our considerations.

In most contexts relevant to heavy ion collisions, μ/T is small. If μ represents an excess of μ_R over μ_L or vice versa, these only arise due to fluctuations or anomalous effects and are certainly small compared to T. If, as in section 3, we treat a nonchiral plasma and think of μ as representing the chemical potential for quark number (which in turn is one-third that for baryon number), μ/T can be as large as ~ 1 in the lowest energy heavy

ion collisions possible at RHIC but in higher energy collisions it is substantially smaller than 1. We will see in section 6 that the effects of interest are suppressed by powers of $\mu_A/(\pi T)$ or $\mu_V/(\pi T)$; even if the quark number chemical potential μ_V can get as large as $\sim T$ it is always small compared to πT. We will therefore work perturbatively in powers of μ/T.

3 Drag force

In this section we shall only consider $\mu_L = \mu_R$ and shall therefore set $\kappa = 0$ and think of μ as representing the quark number chemical potential $(\mu_L + \mu_R)/2$.

The drag force has been calculated in a static plasma with $\mu = 0$ and no gradients in refs. [5–7]. The basic picture of heavy quark dynamics that emerges, with all but the initially most energetic heavy quarks being rapidly slowed by drag and then becoming tracers diffusing within the (moving) fluid, is qualitatively consistent with early experimental investigations [49]. For a review, see ref. [50]. Subsequently, the holographic calculational techniques were generalized to any static plasmas whose gravitational dual has a 4+1-dimensional metric that depends only on the holographic (i.e. 'radial') coordinate in ref. [51] and heavy quark energy loss and diffusion has by now been investigated in the equilibrium plasmas of many gauge theories with gravitational duals [52–66]. In particular, the drag force on a heavy quark in a static plasma with $\mu \neq 0$ was calculated in refs. [51, 52]. We reproduce these results in section 3.1. More recently, the drag force on a heavy quark moving through the far-from-equilibrium matter present just after the collision of two sheets of energy density [67] has been calculated and compared to that in static strongly coupled plasma in equilibrium [68]. Motivated initially by the need to understand these results, in ref. [11] the drag force was computed in a fluid with $\mu = 0$ in which there are spatial and temporal gradients in the fluid temperature and flow velocity, to leading order in these fluid gradients. In section 3.2, we shall reproduce these results and shall compute the leading order effects of fluid gradients in a strongly coupled $\mathcal{N} = 4$ SYM plasma with $\mu \neq 0$. (As an aside, we note that the effects of fluid gradients on photon emission have also been investigated [69].)

3.1 Fluid at rest

According to refs. [5, 6], the dual picture of an (infinitely) heavy quark or antiquark moving through the strongly coupled plasma of $\mathcal{N} = 4$ SYM theory corresponds to a string whose endpoint moves on the boundary of AdS along the trajectory of the heavy quark or antiquark of interest. The string trails "downward" into the bulk spacetime, toward the black hole horizon. It also trails behind the moving quark or antiquark. The drag force is defined as the force needed to move the heavy quark with constant velocity v through the plasma, so we take the endpoint of the string to move along the AdS boundary with constant velocity v. The drag force is determined in the dual gravitational description by the shape of the trailing string. Because the string trailing behind a heavy quark or a heavy antiquark moving with the same velocity through the same fluid has the same shape, they feel the same force. This conclusion applies throughout: the introduction of gradients

in the fluid, magnetic fields, and anomalous effects later in this section and in section 4 does not change the fact that heavy quarks and antiquarks feel the same force. We shall therefore henceforth refer only to heavy quarks.

Without loss of generality one can consider the heavy quark velocity $\vec{v}$ to be directed along the x-direction and because we are considering a static plasma it makes sense to work in the rest frame of the medium, generalizing this only later. In this setup the probe string is described by the Nambu-Goto action

$$S_{NG} = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{-g(\tau, \sigma)}, \tag{3.1}$$

where λ is the 't Hooft coupling, which we are assuming is large, and with $g(\tau, \sigma)$ the determinant of the induced worldsheet metric. The worldsheet metric is defined as $g_{ab} \equiv G_{AB} \partial_a x^A \partial_b x^B$ with $x^A(\tau, \sigma)$ the string profile. Here, upper case Latin indices correspond to 4+1-dimensional spacetime indices and lower case indices correspond to the 1+1-dimensional worldsheet metric indices. We also have freedom in the choice of the parametrization of the worldsheet and it is convenient to take

$$t(\tau, \sigma) = \tau, \qquad r(\tau, \sigma) = \sigma.$$

For a quark whose trajectory in the boundary theory is $x = vt$, the ansatz for the shape of the string in the bulk that corresponds to the word "trailing" is

$$x(\tau, \sigma) = v\tau + \xi(\sigma), \tag{3.2}$$

with $y(\tau, \sigma) = z(\tau, \sigma) = 0$.

The string profile $\xi(\sigma)$ is found by extremizing the Nambu-Goto action and solving the resulting Euler-Lagrange equation

$$\partial_\tau \left(\frac{\delta \mathcal{L}}{\delta \partial_\tau \vec{x}} \right) + \partial_\sigma \left(\frac{\delta \mathcal{L}}{\delta \partial_\sigma \vec{x}} \right) = \left(\frac{\delta \mathcal{L}}{\delta \vec{x}} \right). \tag{3.3}$$

We shall solve this for the string profile order-by-order in powers of μ/T. The $\mu = 0$ (or equivalently zero quark number) solution was obtained in refs. [5, 6]. We can write the expansion about the $\mu = 0$ solution in Eddington-Finkelstein coordinates as

$$\vec{x}^{0,2} = \vec{v}\tau + \vec{v}\frac{\pi - 2\arctan\left(\frac{\sigma}{\pi T}\right)}{2\pi T} + \vec{\xi}^{0,2}(\sigma) + \mathcal{O}\left(\mu^4\right), \tag{3.4}$$

where the $0, 2$ superscript refers to zeroth order in gradients and second order in μ/T and where the direction of $\vec{\xi}^{0,2}$ is the same as the direction of $\vec{v}$, which we are taking to be the x direction. From the Euler-Lagrange equation (3.3), we find that the equation of motion for $\xi_x^{0,2}(\sigma)$ takes the form

$$\partial_\sigma(\sigma^4 - \pi^4 T^4 \gamma^2)\partial_\sigma \xi_x^{0,2}(\sigma) = \text{r.h.s.} \tag{3.5}$$

where the left-hand side that we have given explicitly has the same form also at higher orders in μ/T while the right-hand side, an expression involving no derivatives with respect

to σ, is different at each order. Since the string endpoint is constrained to move along the constant velocity trajectory $x = vt$, we require $\xi_x^{0,2}(\infty) = 0$. Upon solving the equation of motion, we find that the string profile to order μ^2/T^2 and to zeroth order in gradients takes the form

$$
\vec{x} = \vec{v}\tau + \frac{\vec{v}(\pi - 2\arctan\left(\frac{\sigma}{\pi T}\right))}{2\pi T}
$$
$$
+ \frac{\vec{v}}{12\pi^3 T}\left(\frac{\mu}{T}\right)^2\left[\pi\left(-3 - \frac{1}{\gamma^{5/2}} + \frac{2T(1+\gamma^2)}{\gamma^2\sigma} + \frac{4T\sigma}{\pi^2 T^2 + \sigma^2}\right)\right.
$$
$$
\left. + 6\arctan\left(\frac{\sigma}{\pi T}\right) + \frac{2}{\gamma^2\sqrt{\gamma}}\arctan\left(\frac{\sigma}{\pi T\sqrt{\gamma}}\right)\right] \quad (3.6)
$$

where $\gamma = 1/\sqrt{1-v^2}$. In the zero chemical potential limit, (3.6) is the solution obtained in refs. [5, 6]. In solving the equations of motion to obtain (3.6), we fixed the integration constants by requiring that the worldsheet must be regular at the worldsheet horizon, to zeroth order in μ located at $\sigma = \pi T\gamma^{\frac{1}{2}}$, and by requiring that the string endpoint follows the desired trajectory, as already mentioned.

With the string profile in hand, we can now compute the drag force. The drag force is defined as (see e.g. refs. [5, 6, 11])

$$
f^\mu(\tau) \equiv -\lim_{\sigma\to\infty}\eta^{\mu\nu}\Pi_\mu^\sigma(\tau,\sigma) \quad (3.7)
$$

where the string momentum is

$$
\Pi_\mu^\sigma \equiv -\frac{\sqrt{\lambda}}{2\pi}G_{\mu N}\frac{1}{\sqrt{-g}}\left[g_{\tau\sigma}\partial_\tau X^N - g_{\tau\tau}\partial_\sigma X^N\right] \quad (3.8)
$$

and we are interested in the instantaneous drag force at $\tau = 0$. (Note that f^μ is not a Lorentz 4-vector but one could introduce the proper force γf^μ which has appropriate transformation properties.) Upon substituting the string profile (3.6) into the definition of the string momentum we have

$$
\lim_{\sigma\to\infty}\Pi_\mu^\sigma(0,\sigma) = -\frac{T^2\sqrt{\lambda}}{12\pi}\left[6\pi^2\gamma - (1-3\gamma)\left(\frac{\mu}{T}\right)^2\right]\left(1 - \frac{1}{\gamma^2},\vec{v}\right) \quad (3.9)
$$

from which we find that the drag force is given by

$$
f_{(0)}^\mu = \frac{T^2\sqrt{\lambda}}{12\pi}\frac{1}{\gamma^2}(\gamma w^\mu - \delta_0^\mu)\left(6\pi^2\gamma - (1-3\gamma)\left(\frac{\mu}{T}\right)^2\right) \quad (3.10)
$$

where $w^\mu = \gamma(1,\vec{v})$ is the four velocity of the quark. This result was first obtained in ref. [51]. Note that this force, conventionally called the drag force, is defined as the force that an external agent must exert on the heavy quark in order to keep it moving at constant velocity $\vec{v}$. We see from the result (3.10) that $\vec{f}$ points in the same direction as $\vec{v}$. The force that the fluid itself exerts on the heavy quark is $-\vec{f}$, in the $-\vec{v}$ direction.

We have obtained the drag force in the fluid rest frame, where $u^\mu = (1,\vec{0})$. We can now Lorentz-transform γf^μ to a general frame, and upon doing so we find

$$
f_{(0)}^\mu = -\frac{\sqrt{\lambda}}{2\pi}\frac{\pi^2 T^2}{\gamma}(sw^\mu + u^\mu)\left(1 + \frac{(1+3s)}{6s}\left(\frac{\mu}{\pi T}\right)^2\right) \quad (3.11)
$$

where $s \equiv u \cdot w$ is the only Lorentz scalar that can arise at zeroth order in gradients. It also can be shown by direct calculation, starting from the definitions of the drag force and the string momentum, that $w_\mu f^\mu(\tau) = 0$. Note that in the nonrelativistic limit $\gamma \to 1$ we have $s = -1$ and the μ-dependence of the drag force is $\sim \left(1 + \frac{1}{3}(\frac{\mu}{\pi T})^2\right)$ whereas in the ultra-relativistic limit $\gamma \to \infty$ we have $s = -\infty$ and the μ-dependence of the drag force is $\sim \left(1 + \frac{1}{2}(\frac{\mu}{\pi T})^2\right)$. This means that in the presence of a nonzero chemical potential, the force required to move the heavy quark through the strongly coupled plasma is strictly speaking no longer a drag force, since the magnitude of the force is no longer proportional to the momentum of the heavy quark. This is also the case in the presence of nonzero fluid gradients, as shown in ref. [11]. By convention, we will nevertheless continue to refer to the force needed to move the heavy quark through the fluid as a drag force. As a side remark here, but a side remark that we will need in eq. (5.14) in section 5, note from (2.7) that $r_+ \sim \pi T \left(1 + \frac{1}{6}(\frac{\mu}{\pi T})^2\right)$, indicating that in the nonrelativistic limit the drag force is proportional to r_+^2.

3.2 Corrections to the drag force due to fluid gradients

The drag force was calculated to first order in fluid gradients at zero μ in ref. [11]. We can follow the same logic. We must first obtain the string profile to first order in fluid gradients. Since the string profile at zeroth order in gradients that we obtained in (3.6) above contains τ only at linear order one can make the following ansatz, which satisfies the equation of motion:

$$\vec{x}^{1,n} = \vec{x}^{0,n} + \tau h^{1,n}(\sigma) + \vec{\xi}^{1,n}(\sigma). \tag{3.12}$$

Here, the expression for the string profile to zeroth order in fluid gradients in a general frame can easily be obtained from our previous results and is given by

$$\vec{x}^{0,0}(\tau, \sigma) = \vec{v}\tau - \frac{1}{\pi T} \left(u^0\vec{v} - \vec{u}\right) \left(\arctan\left(\frac{\sigma}{\pi T}\right) - \frac{\pi}{2}\right). \tag{3.13}$$

$\vec{x}^{0,1}$ vanishes, and $\vec{x}^{0,2}$ can also easily be obtained from our previous results. It can then be shown that choosing $h^{1,n}(\sigma) = D_t\vec{x}^{0,n}|_{\tau=0}$, with the medium derivative defined as $D_t \equiv \partial_t + v^i\partial_i$, cancels the τ-dependence in the equation of motion. Next, we expand about the zeroth-order-in-gradients solution, see the τ-dependent terms cancel against contributions introduced by $h^{1,n}(\sigma)$, and find that $\vec{\xi}^{1,n}$ satisfies an equation of the form

$$\partial_\sigma(\sigma^4 - \pi^4 T^4 \gamma^2)\partial_\sigma \xi_i^{1,n}(\sigma) = \text{r.h.s.} \tag{3.14}$$

where the right-hand side depends only on σ, not on derivatives of σ, and where the right-hand side is the same for $i = y$ and $i = z$ but is different for $i = x$. We shall not provide the expressions for the right-hand side here as they are lengthy. In solving the equations of motion (3.14), the boundary conditions are the same as at zeroth order and again we require regularity at the worldsheet horizon. The full solution for $\vec{\xi}^{1,0}(\sigma)$ can be found in ref. [11]. We have obtained $\vec{\xi}^{1,2}(\sigma)$, but again these expressions are very lengthy and we will not give them here, focussing instead on presenting results for the drag force.

To first order in gradients and zeroth order in μ, we reproduce the gradient corrections to the drag force on the heavy quark that were first obtained in ref. [11] and that take the form

$$f_{(1,0)}^{\mu} = -\frac{\sqrt{\lambda}}{2\pi}\frac{\pi T}{\gamma}\Big[c_1(s)\Big(u^{\mu}(w\cdot\partial)s - s\partial^{\mu}s - s(su^{\alpha}+w^{\alpha})\partial_{\alpha}U^{\mu}\Big)$$
$$+c_2(s)U^{\mu}(\partial\cdot u) - \sqrt{-s}(u\cdot\partial)U^{\mu}\Big] \qquad (3.15)$$

where $s \equiv u^{\alpha}w_{\alpha}$ as before and where we have defined

$$c_1(s) \equiv \pi/2 - \arctan\left(\sqrt{-s}\right) - \pi T F^0(s)$$
$$c_2(s) \equiv \frac{1}{3}(\sqrt{-s} + (1+s^2)c_1(s))$$

and introduced the projector $U_{\mu} \equiv u_{\mu} + sw_{\mu}$. Here, the function $F^0(s)$ is obtained by setting $\mu = 0$ in the function F defined in (2.13).

Turning now to the corrections that are leading nonzero order in both gradients and μ/T, following the logic set out above we expand the equations of motion for the string profile in a double expansion in gradients and μ/T, solve them, and obtain the force from the string profile as we did at zeroth order. We find no contribution that is first order in μ/T. To second order in μ/T and first order in gradients, the correction to the drag force is given by

$$f_{(1,2)}^{\mu} = -\frac{\sqrt{\lambda}}{48\gamma\pi^2}\frac{\mu^2}{T} \times \Bigg[-c_3(s)\Big(u^{\mu}(w\partial)s - s\partial^{\mu}s\Big) - U^{\mu}c_4(s)(\partial\cdot u)$$

$$-\frac{2c_5(s)}{s}\left((w\cdot\partial)\log\frac{\mu}{T}\right)u^{\mu} + U^{\mu}\left(\frac{16-4\pi}{(-s)^{\frac{3}{2}}} + 2\frac{c_5(s)}{s}\right)(w\cdot\partial)\log\frac{\mu}{T}$$

$$+U^{\mu}\left(\frac{4c_1(s)}{s} + \frac{8-2\pi}{(-s)^{\frac{5}{2}}}\right)(su^{\alpha}+w^{\alpha})\partial_{\alpha}s + \frac{10-3\pi}{(-s)^{\frac{5}{2}}}U^{\mu}(w\cdot\partial)s$$

$$+\left(s^2c_3(s) + \frac{6s^3 + (3\pi-10)s^2 + 4 - \pi}{(-s)^{\frac{5}{2}}}\right)(u\cdot\partial)U^{\mu}$$

$$+\left(sc_3(s) - \frac{4(\pi-4)}{(-s)^{\frac{3}{2}}}\right)(w\cdot\partial)U^{\mu} + 2c_5(s)\partial^{\mu}\log\frac{\mu}{T}\Bigg] \qquad (3.16)$$

where we have defined

$$c_3(s) \equiv \frac{1}{(-s)^{\frac{5}{2}}}\left[4 - \pi + 6(-s)^{\frac{3}{2}} - 6s^2 + 4(-s)^{\frac{3}{2}}c_1(s) + 3(-s)^{\frac{5}{2}}\log\left(1 - \frac{2\sqrt{-s}}{-1+s}\right)\right]$$

$$c_4(s) \equiv \frac{1}{3(-s)^{\frac{3}{2}}}\left[2\Big(3\sqrt{-s} + s\big(2 + 3s(-1+\sqrt{-s}+s)\big)\Big)\right.$$
$$\left. +\sqrt{-s}\left(8c_1(s) - 3s(1+s^2)\log\left(1 - \frac{2\sqrt{-s}}{-1+s}\right)\right)\right] \qquad (3.17)$$

$$c_5(s) \equiv \frac{1}{\sqrt{-s}}\left[4 - 3s(\sqrt{-s}+2s) + 6\sqrt{-s}(s^2-1)\left(c_1(s) + \log\left(1 + \frac{1}{\sqrt{-s}}\right)\right)\right]$$

To a large extent, all of these contributions to the drag force can be considered corrections to the $\mu = 0$ results (3.15) from ref. [11]; they introduce quantitative changes to the gradient corrections to the drag force, but do not change the story in qualitative ways beyond introducing $\partial_\mu(\mu/T)$ terms that describe the contributions to the drag force on the heavy quark due to gradients in the chemical potential, and the corresponding charge currents.

4 Anomalous contributions

In the previous section, we have computed the complete correction to the drag force to first order in fluid gradients up to order μ^2 in the absence of κ, which is to say in the absence of any chiral anomaly, as for example for the case where $\mu_L = \mu_R$ and the μ in the previous section represents $(\mu_L + \mu_R)/2$. In this setting, the next order contributions to the drag force would either be of order μ^4 or second order in gradients, and as far as we can see would not introduce any qualitatively new effects.

We now allow μ_R and μ_L to differ and hence we turn on κ, the Chern-Simons coupling in the bulk gravitational theory that describes the axial anomaly in the boundary gauge theory. Upon doing so, we shall find contributions to the drag force on a heavy quark that are qualitatively new, arising in two ways. First, we find new effects that arise at order μ^3 because the contribution to the metric (2.10) proportional to κ is proportional to μ^3. These effects are proportional to $\ell_\mu = \epsilon_{\mu\nu\alpha\beta} u^\mu \partial^\alpha u^\beta$ and hence are first order in gradients. And, they only arise when μ represents a chiral chemical potential like μ_L or μ_R, meaning that the effects we shall consider in this section should be smaller in magnitude than those in the previous section. Second, we find new effects that arise at order μ^2 and are proportional to κ that arise only in the presence of a magnetic field.

We shall also describe the contributions to the drag force that are introduced when we include the gravitational anomaly in the boundary theory.

As is by now well studied [9, 12, 19, 20, 35], introducing the Chern-Simons coupling in the bulk corresponds to introducing anomalous contributions to the hydrodynamic equations for a chiral plasma (a plasma with $\mu_L \neq \mu_R$) in the boundary theory, contributions that arise because of the axial anomaly in the gauge theory [44]. The corresponding anomalous transport phenomena have been studied at weak coupling [13, 15, 23] as well as at strong coupling [9, 12, 19, 20, 23, 30, 35, 70]. These anomalous effects can be summarized by noting that in a chiral fluid in the presence of an external magnetic field $\vec{B}$ or of nonzero fluid angular momentum $\vec{\Omega}$ the axial anomaly causes both vector and axial currents to flow [12, 13, 20, 23]:

$$\vec{J}_V(x) = \frac{\mu_A}{2\pi^2}\vec{B} + \frac{\mu_V \mu_A}{\pi^2}\vec{\Omega}$$
$$\vec{J}_A(x) = \frac{\mu_V}{2\pi^2}\vec{B} + \left(\frac{\mu_V^2 + \mu_A^2}{2\pi^2} + \frac{T^2}{6}\right)\vec{\Omega}, \tag{4.1}$$

where $\mu_V \equiv (\mu_R + \mu_L)/2$ and $\mu_A \equiv (\mu_R - \mu_L)/2$ are the vector and axial chemical potentials. The first and second terms in the vector current have been named the chiral magnetic effect (CME) and the chiral vortical effect (CVE), respectively. The expressions (4.1) are

valid for a U(1) gauge theory; in section 5 we will generalize them as appropriate for $\mathcal{N} = 4$ SYM theory. We note also that in (4.1) we have left out terms that are higher order in chemical potentials (order μ^2 in the CME and order μ^3 in the CVE) that we will need, and introduce, later in (5.4). The contribution to $\vec{J}_A$ that is proportional to T^2 arises due to the gravitational anomaly (see e.g. refs. [23, 71, 72]).

We shall introduce the four-vector $B_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$. Note that in the limit in which the fluid velocity is nonrelativistic, B_μ and ℓ_μ take the form $B_\mu = (0, \vec{B})$ and $\ell_\mu = (0, 2\vec{\Omega})$ in the local fluid rest frame, with $\vec{B}$ and $\vec{\Omega}$ being the (externally applied) magnetic field and the local angular velocity of the fluid.

4.1 Chiral vortical drag force

In this section, we shall compute the anomalous contributions to the drag force required to pull a heavy quark through a chiral plasma, in which currents like (4.1) are flowing. We begin by setting the external magnetic field $\vec{B} = 0$, considering only the effects of vorticity in the chiral fluid. We derive and solve equations of the form (3.14) and find that at order μ^3 the string profile contains no term proportional to τ and confirm that a contribution at order μ^3 can only come from the presence of the Chern-Simons term in the bulk metric. Direct calculation then yields a contribution to the string profile given by

$$\vec{\xi}^{\,\ell}_{1,3}(\sigma) = -\frac{2\kappa T}{3\pi^2\gamma}\left(\frac{\mu}{T}\right)^3 \frac{\pi^4 T^4 \gamma^2 + \pi^2 T^2 \gamma^2(1+\gamma)\sigma^2 + \sigma^4}{\sigma^4(\pi^2 T^2 \gamma + \sigma^2)(\pi^2 T^2 + \sigma^2)}\, \vec{\ell} \tag{4.2}$$

and a vorticity-induced contribution to the drag force in a generic frame given by

$$(f^\ell)^\mu_{1,3} = \frac{\kappa\sqrt{\lambda}}{3\gamma\pi^3}\frac{\mu^3}{T^2}\frac{\ell^\mu + (\ell \cdot w)w^\mu}{s}\,. \tag{4.3}$$

This chiral contribution is directly proportional to the anomalous coefficient κ, as expected. In the calculation above, μ represents μ_R. We now repeat the calculation for μ_L, flipping the sign of κ, and find that when μ_L and μ_R are both nonzero the result is obtained by replacing μ^3 in (4.3) by $\mu_R^3 - \mu_L^3 = 6\mu_A\mu_V^2 + 2\mu_A^3$.

4.2 Chiral magnetic drag force

Next, we are interested in the anomalous contribution to the drag force due to the presence of a magnetic field. (We will not attempt the analysis for the more generic case of background electric and magnetic fields.) First, we need to modify the gravitational metric in a way that corresponds to introducing an external magnetic field in the boundary gauge theory, see e.g. ref. [39]. This is done by adding a term to the static metric (2.4) that corresponds to adding a new term to j_σ in (2.12) given by

$$j_\sigma^B \equiv \frac{\kappa}{\pi^2 T^2}\left(\frac{\mu}{\pi T}\right)^2 C_B(r)B_\sigma, \tag{4.4}$$

where

$$C_B(r) \equiv \frac{2\pi^2 T^2}{r^2} - \frac{\pi^4 T^4}{r^4} + 2\left(1 - \frac{\pi^4 T^4}{r^4}\right)\log\left(\frac{r^2}{r^2 + \pi^2 T^2}\right)\,. \tag{4.5}$$

The full consideration of hydrodynamics with external electric and magnetic fields can be found in ref. [39]. We shall follow the same procedure as above. And, as we are interested only in the effects due to the presence of the magnetic field we can work to zeroth order in fluid gradients. Since there is no contribution in it that is linear in μ/T the string equation of motion (3.14) simplifies as in the vortical case above and we find a correction to the string profile given by

$$\vec{\xi}_{1,2}^B(\sigma) = -\frac{\kappa}{\pi^4 T^2}\left(\frac{\mu}{T}\right)^2 \frac{\sigma^2(\pi^2 T^2\gamma^2 + \sigma^2)C_B(\sigma) - \pi^2 T^2\gamma^2(\pi^2 T^2 + \sigma^2)C_B(\pi T\sqrt{\gamma})}{(\pi^4 T^4\gamma^2 - \sigma^4)(\pi^2 T^2 + \sigma^2)}\vec{B},$$
(4.6)

where we are considering the magnetic field itself to be first order in gradients. This results in a magnetic-field-induced correction to the drag force given by

$$(f^B)_{1,2}^\mu = -\frac{\kappa\sqrt{\lambda}}{2\pi^3\gamma}\left(\frac{\mu}{T}\right)^2 s^2 \, C_B(\pi T\sqrt{-s})\left(B_\mu + (B\cdot w)w_\mu\right).$$
(4.7)

We have excluded the term corresponding to the Lorentz force on the quark, as this does not depend on the medium or on the chiral anomaly and is not of interest to us here.

We then repeat the calculation for μ_L, flipping the sign of κ, and hence replace μ^2 in (4.7) by $\mu_R^2 - \mu_L^2 = 4\mu_A\mu_V$. Note also that we have assumed that the magnetic field is a perturbation; it could be interesting to perform a similar calculation in the case of a strong magnetic field.

We defer speculating about possible phenomenological consequences of the vortical and magnetic contributions to the drag force that we have found in (4.3) and (4.7) to section 6. We note, however, that both effects are small both because they are proportional to μ_A, which arises only due to topological fluctuations in the plasma, and because they are proportional to μ_V/T (chiral magnetic drag force) or μ_V^2/T^2 (chiral vortical drag force). This suppression is less severe in lower energy heavy ion collisions in which the quark number chemical potential is higher, but of course the lower the collision energy the rarer it is to produce heavy quarks.

4.3 Contributions to the drag force arising from the gravitational anomaly

Before we continue, we introduce a further generalization. The T^2 term in the anomalous axial current in (4.1) raises the question of whether there is an analogue of this term in the drag force. Such a term could be substantially greater in magnitude in any phenomenological consideration, since T^2 is greater than μ_V^2 in heavy ion collisions, and is much greater than μ_A^2. The T^2 contribution to J_A in (4.1) is connected to the presence of a gravitational Chern-Simons term in the bulk metric (see e.g. ref. [23]), namely a contribution to the bulk action that we have neglected up to this point that takes the form

$$\delta S = -\frac{\kappa_g}{16\pi G_5}\int d^5 x\sqrt{-G}\,\epsilon^{MNPQR}A_M R_{BNP}^A R_{AQR}^B.$$
(4.8)

Here, κ_g fixes the coefficient in front of the gravitational axial anomaly in the boundary theory. If, following ref. [26], we add a single chiral fermion transforming under $\mathrm{U}(1)_L$ to

the holographic theory, we then have

$$\kappa_g = \frac{\kappa}{24} \,. \tag{4.9}$$

We will use this expression when we make estimates. The value of the ratio (4.9) for the axial quark number current in two-flavor QCD is $\kappa_g/\kappa = 3/20$ while in the three-flavor case it is $\kappa_g/\kappa = 3/16$. Introducing $\kappa_g \neq 0$ adds two new terms to the expression for j_σ in (2.12) that appears in the bulk metric (2.10), one in the direction of the fluid angular momentum and one in the direction of the external magnetic field. Working to zeroth order in gradients and in each case keeping the lowest two nonzero terms in the expansion in μ/T we find

$$
\begin{aligned}
j^B_{g,\sigma} &= \kappa_g B_\sigma \Bigg(\frac{8\pi^4 T^4}{r^6} - \frac{4\mu^2 \left(r^2 - \pi^2 T^2\right)^2}{\pi^2 T^2 r^8} \Bigg(\left(r^2 + \frac{3\pi^2 T^2}{2} \right) \\
&\qquad + \frac{r^4 \left(r^2 + \pi^2 T^2\right)}{\pi^2 T^2 \left(r^2 - \pi^2 T^2\right)} \log \left(\frac{r^2}{r^2 + \pi^2 T^2} \right) \Bigg) + \mathcal{O}(\mu^4) \Bigg) \\
j^\ell_{g,\sigma} &= \kappa_g \ell_\sigma \frac{4\mu \left(2\pi^2 r^6 T^2 - \pi^4 r^4 T^4 + 2r^4 \left(r^4 - \pi^4 T^4\right) \log \left(\frac{r^2}{r^2 + \pi^2 T^2} \right) + 3\pi^8 T^8 \right)}{\pi^2 r^8 T^2} \\
&\quad - \kappa_g \ell_\sigma \frac{32\sqrt{2}\mu^3 \pi^2 T^2}{27 r^6} \Bigg(\frac{3 \left(6r^6 - 3\pi^4 r^2 T^4 - \pi^6 T^6\right)}{\pi^6 T^6} \log \left(\frac{r^2}{r^2 + \pi^2 T^2} \right) \\
&\quad + \frac{1}{r^4 \pi^4 T^4} \left(18r^8 - 9\pi^2 r^6 T^2 + \pi^4 r^4 T^4 - 18\pi^6 r^2 T^6 + 8\pi^8 T^8 \right) \Bigg) + \mathcal{O}(\mu^3) \,.
\end{aligned}
\tag{4.10}
$$

Note that κ_g flips sign for left-handed quarks, which means that in a theory in which there are both left- and right-handed quarks the term in $j^B_{g,\sigma}$ that is of order $\kappa_g T^4$ will cancel, and we drop it henceforth. This means that the $j^B_{g,\sigma}$ is only a quantitative correction to the j^B_σ in (4.4) that, given (4.9), is small in magnitude. In contrast, because $j^\ell_{g,\sigma}$ includes a term that is linear in μ it can be substantially greater than the j^ℓ_σ term in (2.12), as anticipated.

Completing the calculation, for the gravitational axial anomaly contribution to the drag force we find two terms, one in the $\vec{B}$ direction and one in the $\vec{\ell}$ direction, given by

$$
\begin{aligned}
(f^B_g)^\mu_{1,2} &= -\frac{2\kappa_g}{\kappa} (f^B)^\mu_{1,2} + \frac{\sqrt{\lambda}\kappa_g \mu^2}{3\pi^3 s^3 T^2 \gamma} \Big(5sB^\mu + (B \cdot w)(-2u^\mu + 3sw^\mu) \Big) \\
(f^\ell_g)^\mu_{1,1} &= -\frac{\sqrt{\lambda}}{2\pi} \frac{4\kappa_g \mu s^2}{\gamma} \left(C_B \left(\pi T \sqrt{-s}\right) + \frac{3}{s^4} \right) \left(\ell^\mu + (\ell \cdot w)w^\mu \right),
\end{aligned}
\tag{4.11}
$$

where we have only kept the lowest order terms in μ that are nonzero in each case, and in particular have dropped the $\mathcal{O}(\mu^3)$ term in f^ℓ_g that could be obtained from the $\mathcal{O}(\mu^3)$ term in $j^\ell_{g,\sigma}$.

The new contribution to the chiral magnetic drag force does not introduce any qualitative change. The new contribution to the chiral vortical drag force, however, is linear in μ/T whereas what we had found previously in (4.3) was of order $(\mu/T)^3$, meaning that the new contribution (4.11) is the leading one. At present no observational evidence of

the T^2 term in (4.1) has been reported (except on the lattice, see refs. [42, 73, 74])). So, if it is ever possible to see the chiral vortical drag force, in which (4.11) is the leading contribution, this could be a way to probe the physics of the gravitational axial anomaly in the gauge theory.

5   The dissipationless character of the chiral magnetic and vortical effects

It was pointed out some time ago [24] using a symmetry argument that the charge currents induced by the chiral magnetic and chiral vortical effects must be dissipationless. The argument in its essence is that these chiral effects are time-reversal invariant, and so cannot result in the production of entropy. In this section we show that our results for the chiral drag force on a heavy quark provide an example of an explicit calculation that confirms the dissipationless character of the CME and CVE. We do so by finding a particular setting in which we can place a heavy quark at rest within momentum and charge currents induced by either the CME or the CVE, much like placing a defect in a current-carrying wire or placing a rock in a flowing stream, and see that in the setting that we construct the CME-or CVE-induced current flows past the heavy quark defect without exerting any drag force upon it.

Throughout this section we shall largely neglect effects arising from the gravitational anomaly, in effect setting $\kappa_g = 0$. In order to achieve our goals in this section, we need not include spatial or temporal gradients in the fluid other than vorticity. Gradients can also be added, again at the expense of adding complication, and again without affecting the conclusion.

We shall make our argument in three steps. First, we begin by noting that a heavy quark that is at rest in the local fluid rest frame feels a chiral drag force in a chiral fluid, in the presence of either a magnetic field $\vec{B}$ or a fluid vorticity $\vec{\Omega}$. The local fluid rest frame is the frame in which the fluid momentum T^{0i} vanishes at the location of the heavy quark. If the heavy quark is at rest in this frame, then in this frame $u^\alpha = w^\alpha = (1, \vec{0})$, $\gamma = 1$ and $s = -1$ and the leading contributions to the chiral drag force due to $\vec{B}$ and $\vec{\Omega}$ that we have calculated in (4.7) and (4.3) simplify to

$$\vec{f}_{\text{rest}}^{B,\ell} = -\frac{\kappa\sqrt{\lambda}}{2\pi^3}\frac{\mu^2}{T^2}\vec{B} - \frac{2\kappa\sqrt{\lambda}}{3\pi^3}\frac{\mu^3}{T^2}\vec{\Omega}\,, \tag{5.1}$$

where we have used the fact that $C_B(\pi T) = 1$. Recall that μ here represents μ_R. When μ_L and μ_R are both nonzero, μ^2 is replaced by $4\mu_A\mu_V$ and μ^3 is replaced by $6\mu_A\mu_V^2 + 2\mu_A^3$. It is striking that a heavy quark at rest in a stationary chiral fluid with $\mu_A > 0$ ($\mu_A < 0$) feels a force exerted on it by the chiral fluid in a direction antiparallel to (parallel to) the direction of a magnetic field or fluid vorticity. (Recall that $\kappa = -\frac{1}{4\sqrt{3}}$ is negative for $\mu = \mu_R$, and recall that $\vec{f}$ is the force that some external agent must exert to keep the heavy quark at rest while $-\vec{f}$ is the force that the heavy quark feels from the fluid.)

Second, we ask what we expect will happen if we release the heavy quark at rest and allow it to move under the action of the force $-\vec{f}$, with $\vec{f}$ as in (5.1). The heavy quark will start to move through the fluid with some initially increasing velocity $\vec{v}$. As long as

$\vec{v}$ is small in magnitude we can neglect the resulting change in the chiral drag force (5.1), but because the heavy quark is now moving it will feel an ordinary drag force (3.10) (as calculated in refs. [5, 6]) in addition, which for small v is given by

$$\vec{f}_{\text{drag}} = \frac{\sqrt{\lambda}}{2\pi}\pi^2 T^2 \vec{v} \ . \tag{5.2}$$

This means that the heavy quark will accelerate until it reaches a terminal velocity

$$\vec{v}_{\text{terminal}} = \kappa \frac{\mu^2 \vec{B}}{(\pi T)^4} + \frac{4\kappa}{3}\frac{\mu^3 \vec{\Omega}}{(\pi T)^4} \tag{5.3}$$

at which it experiences no further net force. We therefore reach the remarkable conclusion that in the local fluid rest frame a heavy quark at rest feels a force (5.1) while a heavy quark moving through the fluid with velocity $\vec{v}_{\text{terminal}}$ feels no force.[1]

Third, we boost by a velocity $\vec{v}_{\text{terminal}}$ to a new frame in which the heavy quark that was moving with velocity $\vec{v}_{\text{terminal}}$ in the original frame is now at rest. In this new frame, the fluid at the location of the heavy quark is flowing with velocity $-\vec{v}_{\text{terminal}}$, the heavy quark is at rest, and the heavy quark feels no force. The fluid momentum current is proportional to κ and is entirely due to the CME and CVE, and the heavy quark placed in the flowing fluid feels no force. This is an explicit demonstration of the dissipationless character of the chiral magnetic and vortical effects.

In the remainder of this section, we will generalize this conclusion in two steps. First, we will show that the frame in which a heavy quark at rest in the flowing fluid feels no force is, in fact, the frame in which the local entropy current vanishes. This seems reasonable since if a heavy quark at rest feels no force even though the fluid is moving past it that means that the heavy quark is in equilibrium with the moving fluid. We will then use holography to demonstrate that this conclusion generalizes.

The full expressions for the charge current J^μ and the entropy current including the chiral magnetic and chiral vortical effects are [12, 13, 20, 23]

$$J_\mu = n u_\mu + C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\varepsilon + P}\right)B_\mu + \frac{1}{2}C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\varepsilon + P}\right)\ell_\mu \,, \tag{5.4}$$

$$s_\mu = s u_\mu - \frac{1}{2}\frac{C\mu^2 s}{\varepsilon + P}B_\mu - \frac{1}{3}\frac{C\mu^3 s}{\varepsilon + P}\ell_\mu \,, \tag{5.5}$$

where n is the number density of right- or left-handed quarks and s is the ordinary entropy density. In these expressions, C is the coefficient of the chiral anomaly, $\partial^\mu J_\mu = C\vec{E}\cdot\vec{B}$, and is given by $C = 1/(4\pi^2)$ in a U(1) gauge theory as in (4.1) while here, in $\mathcal{N} = 4$ SYM theory, we have

$$C \equiv -\frac{N_c^2 \kappa}{\pi^2} = \frac{N_c^2}{4\pi^2\sqrt{3}} \ . \tag{5.6}$$

---

[1]Note that if there were a chemical potential for heavy quarks, meaning an excess density of heavy quarks compared to the density of heavy antiquarks, then once all the heavy quarks and antiquarks are moving at v_{terminal} there would be a resulting heavy quark current in the local fluid rest frame.

We see that when the fluid is at rest there are nonvanishing charge and entropy currents. However, if we boost to a new frame in which the new (primed) fluid velocity is related to the original u^μ by

$$u'_\mu = u_\mu - \frac{1}{2}\frac{C\mu^2}{\varepsilon + P}B_\mu - \frac{1}{3}\frac{C\mu^3}{\varepsilon + P}\ell_\mu \tag{5.7}$$

then in this frame the charge and entropy currents are given by

$$J'_\mu = nu'_\mu + C\mu B_\mu + \frac{1}{2}C\mu^2\ell_\mu\,, \tag{5.8}$$

$$s'_\mu = su'_\mu\,, \tag{5.9}$$

meaning that we have boosted to a frame in which $u'_\mu = (1,\vec{0})$ means no entropy current.[2] Note, of course, that when $u'_\mu = (1,\vec{0})$ there is a nonzero momentum current, $T^{0i} \neq 0$; the fluid momentum vanishes when $u_\mu = (1,\vec{0})$. In the nonrelativistic limit, the boost velocity needed to accomplish (5.7), namely to boost from the local fluid rest frame to the local entropy rest frame, is

$$\vec{v}_{\rm boost} = -\frac{1}{2}\frac{C}{\varepsilon + P}\left(\mu^2\vec{B} + \frac{4}{3}\mu^3\vec{\Omega}\right) \tag{5.10}$$

where we have used the nonrelativistic approximation $\vec{l} \simeq 2\vec{\Omega}$. Upon using (5.6), noting that $\varepsilon + P = N_c^2\pi^2T^4/2$, and comparing to (5.3) we see that

$$v_{\rm boost} = v_{\rm terminal}\,. \tag{5.11}$$

We conclude that the frame in which a heavy quark at rest in the flowing fluid feels no force is in fact the local entropy rest frame.

The generality of this conclusion will become apparent after it is recast holographically, in terms of the dual gravitational description of the fluid. In particular, upon so doing we will see that although in the derivation above we only used the contributions to the chiral magnetic and chiral vortical drag force and to the ordinary drag force that arise at the lowest nontrivial order in μ/T in each case, the conclusion that we have reached in fact holds to any order in μ/T.

The authors of refs. [28, 32, 76, 77] have shown that in the holographic description of a flowing strongly coupled plasma with $\kappa_g = 0$ with the 4+1-dimensional metric (2.10) the local entropy rest frame is the frame in which the metric function j_σ in (2.10) vanishes at $r = r_h$, where r_h is the location of the outer horizon, namely the largest solution of $f(r) = 0$, with $f(r)$ also being a metric function in (2.10). For the case of a static fluid with $\mu = 0$, r_h is given simply by $r_h = \pi T$. In a static fluid with $\mu \neq 0$, we have instead $r_h = r_+$ where r_+ is given in terms of μ and T in (2.7). We must therefore compute the force on a heavy quark *at rest* in the frame in which $j(r_h) = 0$. This computation turns out to be sufficiently tractable that we can push it far enough without expanding in μ/T. We shall include a nonzero magnetic field and vorticity. As throughout this section, we neglect the gravitational anomaly and fluid gradients. In this case, j_σ takes the form

$$j_\sigma = W^B(r)B_\sigma + W^\ell(r)\ell_\sigma \tag{5.12}$$

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[2]In the presence of the gravitational anomaly, i.e. if $\kappa_g \neq 0$, the entropy current (5.5) includes additional terms and, consequently, does not vanish in the "no drag frame" [75].

where the effects of the chiral anomaly enter through the functions $W^B(r)$ and $W^\ell(r)$ whose form we obtained explicitly to leading nontrivial order in μ/T in (4.4) and (2.12), respectively. The force on the heavy quark at rest can be calculated following the procedures developed in previous sections, and one finds

$$\vec{f} = -\frac{\sqrt{\lambda}}{2\pi} \left(r_h^2 W^B(r_h) \vec{B} + r_h^2 W^\ell(r_h) \vec{\ell} \right). \tag{5.13}$$

It can be also verified that if one expands this answer for the anomalous drag force in powers of μ/T, it does indeed coincide with (4.7) and (4.3). If we balance this force against the drag force on a slowly moving quark, which in this general context is given by

$$\vec{f} = \frac{\sqrt{\lambda}}{2\pi} r_h^2 \vec{v}, \tag{5.14}$$

we see that the terminal velocity of the heavy quark is given by

$$\vec{v}_{\text{terminal}} = \left(W^B(r_h) \vec{B} + W^\ell(r_h) \vec{\ell} \right). \tag{5.15}$$

Comparing this result with (5.12), we see that in the local entropy rest frame in which $j_\sigma(r_h) = 0$, the terminal velocity vanishes: $v_{\text{terminal}} = 0$. This means that in the local entropy rest frame, a heavy quark at rest feels no force. Note that in obtaining this result via this holographic calculation we did not need to expand $W^B(r)$ or $W^\ell(r)$ in powers of μ/T. This means that the result (5.15) is valid to all orders in μ/T. Of course, in (4.4) and (2.12) we have explicit expressions for $W^B(r_h)$ and $W^\ell(r_H)$ only to lowest nontrivial order in μ/T; at higher orders, $W^B(r_h)$ and $W^\ell(r_H)$ receive corrections but the form of the expression (5.15) for the terminal velocity of the heavy quark remains unchanged.

Note also that although we have not given the resulting more complicated expressions here, if one adds the effects of fluid gradients into the expressions for j_σ, the entropy current, and the drag force, while keeping κ_g set to zero, the conclusion that a heavy quark at rest in the local entropy rest frame feels no force remains unchanged.

Let us close this section by restating this general result in its simplest form. In a strongly coupled fluid in which $\mu_R > \mu_L$, in the presence of a magnetic field $\vec{B}$ and for simplicity in the absence of any fluid gradients including vorticity, we have analyzed the chiral drag force on heavy quarks with two different velocities:

- If the first heavy quark is at rest in the local fluid rest frame, meaning that it sees around it a fluid with no momentum flow, this heavy quark feels the fluid around it exerting a chiral drag force on it pushing it toward the direction of $-\vec{B}$. It also sees around it a charge current in the direction of $\vec{B}$ and an entropy current in the opposite, $-\vec{B}$, direction. (All signs are reversed if instead the fluid has $\mu_L > \mu_R$.)

- If a second heavy quark is released and allowed to accelerate under the influence of the chiral drag force, it does so until it reaches a terminal velocity in the $-\vec{B}$ direction. If we now boost to the rest frame of *this* heavy quark, we find a heavy quark that feels no force. This heavy quark sees a charge current in the $\vec{B}$ direction that is larger in magnitude than that seen by the first heavy quark. This heavy quark also sees the

fluid around it flowing, with fluid momentum in the $\vec{B}$ direction. (Again, all signs are reversed if instead the fluid has $\mu_L > \mu_R$.)

The second heavy quark provides an example of a "defect" at rest in a flowing chiral fluid, with both fluid momentum and charge flowing past it, and yet feels no force. Thus, it provides an explicit example of the dissipationless character of the chiral magnetic and chiral vortical effects.

This becomes particularly important now that the chiral magnetic effect has been seen [78] in a condensed matter system, namely the Dirac semi-metal ZrTe$_5$, since we can assume that the zirconium pentatelluride crystals used in the experiment contain some defects.

6 Outlook and potential phenomenological consequences

In section 3.2, we obtained the contributions to the drag force on a heavy quark that are second order in μ/T and first order in fluid gradients in a fluid in which $\mu_L = \mu_R$, meaning that there are no contributions to the drag force coming from the chiral anomaly. These results generalize the $\mu = 0$ results of ref. [11] to nonzero μ. In ref. [11], the analytic results for the contributions to the drag force that are first order in gradients were used to obtain an understanding of a variety of curious features of the drag force on a heavy quark caught between two colliding sheets of energy density, features that were discovered numerically in the holographic calculation of ref. [68]. We look forward to seeing our results from section 3.2 used similarly, in conjunction with some future calculation of the collision of two sheets of energy and charge density.

We have calculated the anomalous contributions to the drag force on a heavy quark in a chiral plasma to lowest nontrivial order in μ/T in section 4. These contributions to the drag force change sign in a chiral plasma with $\mu_R > \mu_L$ relative to that with $\mu_L > \mu_R$. This parity-odd symmetry makes the chiral drag force novel, qualitatively different from all contributions to the drag force on heavy quarks that have been calculated previously. Its unique features — namely that it is a force that pushes all heavy quarks and antiquarks in the same direction, either parallel to or antiparallel to the magnetic field vector $\vec{B}$ or the fluid vorticity vector $\vec{\Omega}$, with the direction of the force being opposite in a chiral plasma with $\mu_R > \mu_L$ relative to that with $\mu_L > \mu_R$ — make it detectable at least in principle via suitable observables defined for this purpose. In practice the effect is likely to be small in quantitative terms. We shall provide a rough estimate of the magnitude of the chiral drag force, before describing a possible observable.

The full leading order expressions for the chiral drag force are given in section 4 for a quark moving through a chiral plasma with any velocity. Here we recapitulate our result for the anomalous force on a heavy quark at rest in the local fluid rest frame

$$\frac{2\pi}{\sqrt{\lambda}}\vec{f} = -\frac{(3\kappa - 16\kappa_g)\mu^2}{3\pi^2 T^2}\vec{B} - 32\kappa_g\mu\vec{\Omega} - \frac{4\mu^3(\kappa - 16\kappa_g)}{3\pi^2 T^2}\vec{\Omega}\,, \qquad (6.1)$$

where we have restored the effects due to the gravitational anomaly, proportional to κ_g, that we dropped in (5.1). We can compare this result for the chiral drag force on a heavy

quark with $v = 0$ to the result for a heavy quark in the $v \to 1$ limit, where we find

$$\frac{2\pi}{\sqrt{\lambda}}\vec{f} = -\frac{4(\kappa - 2\kappa_g)\mu^2}{3\pi^2 T^2}(\vec{B} \cdot \vec{v})\vec{v} - \frac{32\kappa_g \mu}{3}(\vec{\Omega} \cdot \vec{v})\vec{v} - \frac{4\mu^3(3\kappa - 40\kappa_g)}{9\pi^2 T^2}(\vec{\Omega} \cdot \vec{v})\vec{v} . \qquad (6.2)$$

There are certainly differences between (6.2) and (6.1), for example the fact that in the ultrarelativistic limit the force is in the $\vec{v}$ direction rather than in the $\vec{B}$ or $\vec{\Omega}$ direction, although its magnitude is greatest when $\vec{v}$ is parallel or antiparallel to $\vec{B}$ or $\vec{\Omega}$. The principal message coming from (6.2), however, is that the anomalous drag force on a heavy quark with $v \to 1$ is comparable in magnitude to that on a heavy quark at rest. All heavy quarks and antiquarks feel a chiral drag force whose magnitude is only weakly dependent on their velocity. Given that, it seems clear that the fractional effects of the chiral drag force are largest when $v \sim 0$ rather than when $v \to 1$. Although in principle there are anomalous forces on acting on the ultrarelativistic b-quarks that become b-jets, it seems a better bet to focus on b-quarks that are almost at rest relative to the fluid. We shall see that even in this case the effects are small.

Let us consider a heavy quark with mass m that is initially at rest in the local fluid rest frame in a chiral plasma in the presence of a magnetic field $\vec{B}$. This quark feels a chiral drag force

$$\vec{f}^B = -\kappa\sqrt{\lambda}\left(1 - \frac{16\kappa_g}{3\kappa}\right)\frac{\mu^2}{2\pi^3 T^2}\vec{B} . \qquad (6.3)$$

As we described in section 5, this heavy quark starts to move due to this force and as it moves it begins to feel the standard drag force (5.2) also, meaning that the heavy quark accelerates until it is moving with a terminal velocity

$$\vec{v}_{\text{terminal}} = \kappa\left(1 - \frac{16\kappa_g}{3\kappa}\right)\frac{\mu^2}{(\pi T)^4}\vec{B}. \qquad (6.4)$$

How long does it take for the heavy quark to accelerate from rest to a velocity that is close to v_{terminal}? This timescale is parametrically of order $m|\vec{v}_{\text{terminal}}|/|\vec{f}^B| \sim 2\pi m/(\sqrt{\lambda}(\pi T)^2)$, meaning that for charm (bottom) quarks it is of order $1\,\text{fm}/c$ (a few fm/c). For our rough purposes we can therefore estimate that the chiral drag force gives all heavy quarks a momentum that is around $m\,\vec{v}_{\text{terminal}}$. Bottom quarks produced at rest in the fluid therefore pick up a contribution to their momenta arising from the chiral anomaly that is of order

$$\begin{aligned}
\vec{p}_{\text{terminal}} &\equiv m_b\,\vec{v}_{\text{terminal}} \\
&= m_b\,\kappa\left(1 - \frac{16\kappa_g}{3\kappa}\right)\frac{\mu^2\vec{B}}{(\pi T)^4} \\
&= -0.449\,m_b\frac{\mu_V\mu_A\vec{B}}{(\pi T)^4} \\
&\simeq -3\,\text{MeV}\,\frac{m_b}{4.2\,\text{GeV}}\frac{\mu_V}{0.1\,\text{GeV}}\frac{\mu_A}{0.1\,\text{GeV}}\frac{\vec{B}}{(0.1\,\text{GeV})^2}\left(\frac{0.5\,\text{GeV}}{\pi T}\right)^4 , \quad (6.5)
\end{aligned}$$

where we have used (2.3) and (4.9), and replaced μ^2 by $4\mu_A\mu_V$. For concreteness, we have used the $\mathcal{N} = 4$ SYM values in evaluating the purely numerical factor $\kappa(1 - \frac{16\kappa_g}{3\kappa})$; in QCD

all components of this coefficient — namely κ, κ_g/κ and the $16/3$ — will take on different values. There will certainly also be other places where our $\mathcal{N} = 4$ SYM calculation differs from QCD by factors of order unity. Note, finally, that we have throughout used units in which a factor of e has been absorbed into $\vec{B}$ and $\vec{E}$. Restoring this means replacing $\vec{B}$ by $e\vec{B}$ in (6.5).

Reading the final expression (6.5) from left to right, we see that:

- The effect is small.

- The effect can be enhanced by lowering the collision energy, as doing so increases μ_V, but one should not lower the collision energy too far since doing so makes heavy quark production rarer and also reduces the initial temperatures reached in the collision. Generating the chiral effects we are discussing requires the presence of quark-gluon plasma. Recent measurements [79] indicate that the observables that are sensitive to the chiral magnetic effect that were seen previously in high energy heavy ion collisions [80–83] persist robustly down to collision energies corresponding to baryon chemical potentials $\mu_B > 0.3\,\text{GeV}$, meaning $\mu_V > 0.1\,\text{GeV}$.

- It is difficult to estimate the magnitude of the fluctuations in μ_A in heavy ion collisions. The authors of ref. [84] have recently estimated that μ_A can locally be as large as $0.1\,\text{GeV}$ in heavy ion collisions at top RHIC energies. Estimating this more reliably will require the development of a relativistic viscous chiral magnetohydrodynamic code for heavy ion collisions.

- In heavy ion collisions with a nonzero impact parameter, a magnetic field is created initially by the charged spectators. The conductivity of the quark-gluon plasma slows the decay of the magnetic field, delaying the decay of the magnetic field in the plasma to long after the spectators are gone. Reliable estimates here also await a future magnetohydrodynamic analysis of heavy ion collisions, but perhaps $|\vec{B}|$ can remain as large as $(0.1\ \text{GeV})^2$ for a few fm/c [85–88], although this is more likely to be an overestimate than an underestimate.

- Taking the average temperature seen by the heavy quark as $\pi T = 0.5\,\text{GeV}$ is on the low side for a benchmark value with which to make an estimate, although maybe not unreasonable for heavy ion collisions with $\mu_V > 0.1\,\text{GeV}$.

Our estimates suggest that the chiral magnetic drag force on bottom (charm) quarks and antiquarks pushes them all to the extent that it gives them all a common momentum that might be as large as $\sim 3\,\text{MeV}$ ($\sim 1\,\text{MeV}$), although it should be clear that this estimate is first of all at present very crude and second of all is more likely to be an overestimate than an underestimate.[3] Furthermore, in events in which there are some regions of the plasma

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[3]From (6.1) we see that there is a chance that the chiral vortical drag force could be larger than the chiral magnetic drag force since although $\kappa_g \ll \kappa$ the chiral vortical drag force is larger by some purely numerical factors and because it is not suppressed by $\mu_V/(\pi T)$. However, at present the magnitude of $\vec{\Omega}$ in the droplets of fluid produced in heavy ion collisions is poorly constrained and we have also only just seen the first measurements of observables that can receive a contribution from the chiral vortical effect [89]. We will therefore not attempt an estimate of the magnitude of the chiral vortical drag force.

with $\mu_R > \mu_L$ and other regions with $\mu_R < \mu_L$, some heavy quarks will be pushed in the direction antiparallel to $\vec{B}$ while others will be pushed parallel to $\vec{B}$, reducing the net effect in the event as a whole.

Although the effects of the chiral drag force are small, we do not wish to underestimate the ingenuity of our experimentalist colleagues. Perhaps they can devise sufficiently sensitive correlation observables to see the small effects of the chiral drag force on heavy quarks. For example, it is worth constructing event-by-event observables that can see whether the B and D mesons in an event have picked up a net momentum perpendicular to the reaction plane in the "downward" ("upward") direction in those events in which the chiral magnetic effect has resulted in an electric current "upwards" ("downwards") with positive light hadrons pushed "upwards" ("downwards") and negative light hadrons pushed "downwards" ("upwards"). A correlation observable like this uses the CME current to define the direction in which the heavy quarks should be pushed by the chiral drag force, and then checks whether a net push on all the heavy quarks and antiquarks in the event in this direction is seen. A nice feature of such a correlation observable is that the observable effects of both the CME current and the chiral drag force should each be equally suppressed by the partial cancellation between regions of the plasma with $\mu_R > \mu_L$ and regions with $\mu_R < \mu_L$.

Given the estimates that we have made, it is certainly our impression that the chiral drag force is of interest principally from a theoretical perspective, rather than as a phenomenon that experimentalists should expect to observe in heavy ion collisions. Even if they are only observable in principle, though, it is remarkable to see effects due to the chiral anomaly giving all the heavy quarks and antiquarks in a chiral plasma a kick. In section 5 we have also used our calculation of the chiral drag force to provide explicit evidence for the dissipationless character of the chiral magnetic and chiral vortical effects, showing that the currents that they describe can flow around defects without any dissipation.

Acknowledgments

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