Supplementary Materials to "Adaptive Output-Feedback Control for A Class of Multi-Input-Multi-Output Plants with Applications to Very Flexible Aircraft"[1]

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Appendix

A. Proof of Lemma 1

Proof: Define $(\cdot)_{/[\cdot]_0} = \frac{\partial(\cdot)}{\partial[\cdot]}\big|_{[\cdot]_0}$ as partial differential variables. Linearizing (3) around a trim point $[\ddot{\epsilon}_0, \dot{\epsilon}_0, \epsilon_0, \dot{\beta}_0, \beta_0, \lambda_0, u_0]^T$ yields

$$\left(\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & (M_{FF})_{\epsilon_0} & (M_{FB})_{\epsilon_0} \\ 0 & (M_{BF})_{\epsilon_0} & (M_{BB})_{\epsilon_0} \end{bmatrix}}_{\overline{Q}_1(0,0,\epsilon_0,0,\beta_0)} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta M_{FF} & \Delta M_{FB} \\ 0 & \Delta M_{BF} & \Delta M_{BB} \end{bmatrix}}_{\Delta Q_1^*(\ddot{\epsilon}_0,\dot{\epsilon}_0,\dot{\beta}_0)}\right)\begin{bmatrix} \dot{\epsilon} \\ \ddot{\epsilon} \\ \dot{\beta} \end{bmatrix}$$
$$= \left(\underbrace{\begin{bmatrix} 0 & I & 0 \\ -(K_{FF})_{\epsilon_0} + (J_{h\epsilon}^T)_{\epsilon_0} F_{/\epsilon_0}^{load} & -C_e & (J_{h\epsilon}^T)_{\epsilon_0} F_{/\beta_0}^{load} \\ 0 & 0 & -C_{RB} + (J_{hb}^T)_{\epsilon_0} F_{/\beta_0}^{load} \end{bmatrix}}_{\overline{Q}_2(0,0,\epsilon_0,0,\beta_0)}\right.$$
$$\left. + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \Delta K_{FF} & \Delta C_{FF} & \Delta C_{FB} \\ \Delta K_{BF} & \Delta C_{BF} & \Delta C_{BB} \end{bmatrix}}_{\Delta Q_2^*(\ddot{\epsilon}_0,\dot{\epsilon}_0,\dot{\beta}_0)}\right)\underbrace{\begin{bmatrix} \epsilon \\ \dot{\epsilon} \\ \beta \end{bmatrix}}_{x_p} + \underbrace{\begin{bmatrix} 0 \\ B_{F/u_0} \\ B_{B/u_0} \end{bmatrix}}_{Q_3} u_p \tag{33}$$

where, following the definition of terms in (4), the unknown deviation terms are

$$\begin{aligned}
\Delta M_{FF} &= -(J_{h\epsilon}^T)_{\epsilon_0} F_{/\ddot{\epsilon}_0}^{load} \qquad & \Delta M_{FB} &= -(J_{h\epsilon}^T)_{\epsilon_0} F_{/\dot{\beta}_0}^{load} \\
\Delta M_{BF} &= -(J_{hb}^T)_{\epsilon_0} F_{/\ddot{\epsilon}_0}^{load} & \Delta M_{BB} &= -(J_{hb}^T)_{\epsilon_0} F_{/\dot{\beta}_0}^{load} \\
\Delta B_{F/\lambda_0} &= (J_{h\epsilon}^T)_{\epsilon_0} F_{/\lambda_0}^{load} & \Delta B_{B/\lambda_0} &= (J_{hb}^T)_{\epsilon_0} F_{/\lambda_0}^{load}
\end{aligned}$$
$$\begin{aligned}
\Delta K_{FF} &= M_{FF/\epsilon_0}\ddot{\epsilon}_0 + M_{FB/\epsilon_0}\dot{\beta}_0 \\
\Delta K_{BF} &= M_{BF/\epsilon_0}\ddot{\epsilon}_0 + M_{BB/\epsilon_0}\dot{\beta}_0 \\
\Delta C_{FF} &= -(C_{FF})x_0 - C_{FF/\dot{\epsilon}_0}\dot{\epsilon}_0 - C_{FB/\dot{\epsilon}_0}\beta_0 + (J_{h\epsilon}^T)_{\epsilon_0} F_{/\dot{\epsilon}_0}^{load} \\
\Delta C_{BF} &= -(C_{BF})x_0 - C_{BF/\dot{\epsilon}_0}\dot{\epsilon}_0 - C_{BB/\dot{\epsilon}_0}\beta_0 + (J_{hb}^T)_{\epsilon_0} F_{/\dot{\epsilon}_0}^{load} \\
\Delta C_{FB} &= -(C_{FB})x_0 - C_{FF/\beta_0}\dot{\epsilon}_0 - C_{FB/\beta_0}\beta_0 \\
\Delta C_{BB} &= -(C_{BB})x_0 - C_{BF/\beta_0}\dot{\epsilon}_0 - C_{BB/\beta_0}\beta_0
\end{aligned} \tag{34}$$

$B_{F/u_0} = (J_{h\epsilon}^T)_{\epsilon_0} F_{/u_0}^{load}$ and $B_{B/u_0} = (J_{hb}^T)_{\epsilon_0} F_{/u_0}^{load}$. Without loss of generality, we scale each input so that $F_{/u_0}^{load} = I$. In realistic application, only $[\epsilon_0, \beta_0, u_0]^T$ can be measured accurately and therefore variables that depend on them can be gain scheduled. $[\dot{\epsilon}_0, \ddot{\epsilon}_0, \dot{\beta}_0, \lambda_0]^T$ cannot be measured accurately and therefore variables that depends on them are generally unknown. As a result, $\overline{Q}_1$, $\overline{Q}_2$ and Q_3 are known but ΔQ_1^* and ΔQ_2^* are unknown. Examination on (34) using (4) shows that

$$\Delta Q_1^* = \begin{bmatrix} 0 \\ J_{h\epsilon}^T \\ J_{hb}^T \\ 0 \end{bmatrix}_{\epsilon_0} \underbrace{\begin{bmatrix} 0 & F_{/\ddot{\epsilon}_0}^{load} & F_{/\dot{\beta}_0}^{load} & 0 \end{bmatrix}}_{\Theta_{q_1}^{*T}} = Q_3\Theta_{q_1}^{*T} \tag{35}$$

and

$$\Delta Q_2^* = \begin{bmatrix} 0 \\ J_{h\epsilon}^T \\ J_{hb}^T \end{bmatrix}_{\epsilon_0} \underbrace{\begin{bmatrix} M_e\left(\frac{\partial J_{h\epsilon}}{\partial \epsilon}\ddot{\epsilon} + \frac{\partial J_{hb}}{\partial \epsilon}\beta\right) & M_e\left(\dot{J}_{h\epsilon} + \frac{\partial \dot{J}_{h\epsilon}}{\partial \dot{\epsilon}}\dot{\epsilon} + \frac{\partial \dot{J}_{hb}}{\partial \dot{\epsilon}}\beta\right) \\ & M_e\left(\dot{J}_{hb} + \frac{\partial \dot{J}_{h\epsilon}}{\partial \beta}\dot{\epsilon} + \frac{\partial \dot{J}_{hb}}{\partial \beta}\beta\right) \end{bmatrix}_{x_0}}_{\Theta_{q_2}^{*T}}$$
$$= Q_3\Theta_{q_2}^{*T} \tag{36}$$

which is used to rewritten (33) as

$$\left(\overline{Q}_1 + Q_3\Theta_{q_1}^{*T}\right)\dot{x}_p = \left(\overline{Q}_2 + Q_3\Theta_{q_2}^{*T}\right)x_p + Q_3 u_p. \tag{37}$$

Assume that $\overline{Q}_1$, $(\overline{Q}_1 + Q_3\Theta_{q_1}^{*T})$ and $(I + \Theta_{q_2}^{*T}\overline{Q}_1^{-1}Q_3)$ are invertible around the equilibrium. Taking inverse on both sides, and noting

$$(\overline{Q}_1 + Q_3\Theta_{q_1}^{*T})^{-1} = \overline{Q}_1^{-1} - \overline{Q}_1^{-1}Q_3\underbrace{(I + \Theta_{q_1}^{*T}\overline{Q}_1^{-1}Q_3)^{-1}\Theta_{q_1}^{*T}}_{\overline{\Theta}_{q_1}^{*T}}\overline{Q}_1^{-1} \tag{38}$$

yields

$$\begin{aligned}
\dot{x}_p &= \left(\overline{Q}_1^{-1} - \overline{Q}_1^{-1}Q_3\overline{\Theta}_{q_1}^{*T}\overline{Q}_1^{-1}\right)\left(\overline{Q}_2 + Q_3\Theta_{q_2}^{*T}\right)x_p \\
&\quad + \left(\overline{Q}_1^{-1} - \overline{Q}_1^{-1}Q_3\overline{\Theta}_{q_1}^{*T}\overline{Q}_1^{-1}\right)Q_3 u_p \qquad (39) \\
&= \left[\overline{Q}_1^{-1}\overline{Q}_2 + \overline{Q}_1^{-1}Q_3\left(\Theta_{q_2}^{*T} - \overline{\Theta}_{q_1}^{*T}\overline{Q}_1^{-1}\overline{Q}_2 - \overline{\Theta}_{q_1}^{*T}\overline{Q}_1^{-1}Q_3\Theta_{q_2}^{*T}\right)\right]x_p \qquad (40) \\
&\quad + \overline{Q}_1^{-1}Q_3\left(I - \overline{\Theta}_{q_1}^{*T}\overline{Q}_1^{-1}Q_3\right)u_p \\
&= \left[A_p + B_p\underbrace{\left(\Theta_{q_2}^{*T} - \overline{\Theta}_{q_1}^{*T}A_p - \overline{\Theta}_{q_1}^{*T}B_p\Theta_{q_2}^{*T}\right)}_{\Theta_p^{*T}}\right]x_p + B_p\underbrace{\left(I - \overline{\Theta}_{q_1}^{*T}B_p\right)}_{\Lambda_p^*}u_p \qquad (41)
\end{aligned}$$

with $A_p = \overline{Q}_1^{-1}\overline{Q}_2$, $B_p = \overline{Q}_1^{-1}Q_3$. C_p as in $y_p = C_p x_p$ is the selection matrix that picks out measurable states from x_p. ■

B. Proof of Lemma 3

Proof: The proof will be performed in a transformed coordinate. Similar to $\overline{B}_2$, we part $C^T = \begin{bmatrix} C_2^T & C_1^T \end{bmatrix}$. For a square plant model that has nonuniform input relative degree two, there exists an invertible transformation $T_{in} = \begin{bmatrix} (\mathfrak{CB})^{-1}\mathfrak{C} \\ \mathfrak{N} \end{bmatrix}$, $T_{in}^{-1} = \begin{bmatrix} \mathfrak{B} & \mathfrak{M} \end{bmatrix}$, where $\mathfrak{C}^T = \begin{bmatrix} C_2^T & A^T C_2^T & C_1 \end{bmatrix}$, $\mathfrak{B} = \begin{bmatrix} B_2 & AB_2 & B_{s1} \end{bmatrix}$, $\mathfrak{N}$ and $\mathfrak{M}$ are chosen to satisfy $\mathfrak{NB} = 0$, $\mathfrak{CM} = 0$ and $\mathfrak{NM} = I$, that transforms (10) into a new coordinate called "*input normal form*" (See [21, Corollary 2.2.5] for proof). In this proof, matrices in input normal form coordinate will be denoted with the subscript $(\cdot)_{in}$, as in $x_{in} = T_{in}x$, $A_{in} = T_{in}AT_{in}^{-1}$, $\overline{B}_{2,in} = T_{in}\overline{B}_2$ (and therefore $B_{2,in} = T_{in}B_2$ and $B_{s1,in} = T_{in}B_{s1}$), $B_{1,in} = T_{in}B_1$, $B_{z,in} = T_{in}B_z$, $C_{in} = CT_{in}^{-1}$, $\Psi_{1,in}^{*T} = \Psi_1^{*T}T_{in}^{-1}$ and $\Psi_{2,in}^{*T} = \Psi_2^{*T}T_{in}^{-1}$. The input normal form of the plant model (10) is

$$\begin{aligned}
\begin{bmatrix} \dot{\xi}_1^2 \\ \dot{\xi}_2^2 \\ \hline \dot{\xi}_1^1 \\ \hline \dot{\eta} \end{bmatrix} &= \underbrace{\left[\begin{array}{cc|c|c} 0 & R_{2,1}^2 & R_{2,1}^1 & V_2 \\ I & R_{2,2}^2 & R_{2,2}^1 & 0 \\ \hline 0 & R_{1,1}^2 & R_{1,1}^1 & V_1 \\ \hline 0 & U_2 & U_1 & Z \end{array}\right]}_{A_{in}}\underbrace{\begin{bmatrix} \xi_1^2 \\ \xi_2^2 \\ \hline \xi_1^1 \\ \hline \eta \end{bmatrix}}_{x_{in}} + \underbrace{\begin{bmatrix} I_m \\ 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}}_{B_{2,in}}\Lambda^* u \\
&\quad + B_{2,in}\underbrace{\left[\begin{array}{cc|c|c} \psi_{20}^{2*T} & \psi_{21}^{2*T} & \psi_{21}^{1*T} & \psi_{(n-r_s)}^{2*T} \end{array}\right]}_{\Psi_{2,in}^{*T}}x_{in} \\
&\quad + \underbrace{\begin{bmatrix} I_m \\ \frac{1}{c}I_m \\ \hline 0 \\ \hline 0 \end{bmatrix}}_{B_{1,in}}\underbrace{\left[\begin{array}{cc|c|c} 0 & \psi_{11}^{2*T} & \psi_{11}^{1*} & \psi_{(n-r_s)}^{1*T} \end{array}\right]}_{\Psi_{1,in}^{*T}}x_{in} + B_{z,in}z_{cmd} \qquad (42) \\
y &= \underbrace{\left[\begin{array}{cc|c|c} 0 & CAB_2 & CB_{s1} & 0 \end{array}\right]}_{C_{in}}x_{in}.
\end{aligned}$$

Matrix $Z \in \mathbb{R}^{(n-r_s)\times(n-r_s)}$, where $r_s = \sum_i r_i$, is the zero dynamics matrix whose eigenvalues are transmission zeros of the plant model (see [21, Section 2.3]). It is noted that $B_{1,in} = \begin{bmatrix} \times & \times & 0 & 0 \end{bmatrix}^T$ and $\Psi_1^{*T}T_{in}^{-1} = \begin{bmatrix} 0 & \times & \times & \times \end{bmatrix}$ since Assumption 4 holds.

Define $A_{in}^* = A_{in} + B_{1,in}\Psi_{1,in}^{*T} + B_{2,in}\Psi_{2,in}^{*T} = T_{in}A^*T_{in}^{-1}$. Examination of the elements of $\overline{B}_{2,in}^1$ and $\overline{B}_{2,in}^{1*}$, which are

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defined as $\overline{B}_{2,in}^{1*} = T_{in}\overline{B}_2^{1*}$, and $\overline{B}_{2,in}^{1} = T_{in}\overline{B}_2^{1}$, respectively, shows that

$$\overline{B}_{2,in}^{1} = \left[\begin{array}{cc} B_{2,in}^{1} & B_{s1,in} \end{array}\right] = \left[\begin{array}{cc} a_1^0 I_m & 0 \\ a_1^1 I_m & 0 \\ \hline 0 & I_{m_s} \\ \hline 0 & 0 \end{array}\right], \tag{43}$$

and

$$\overline{B}_{2,in}^{1*} = \left[\begin{array}{cc} B_{2,in}^{1*} & B_{s1,in} \end{array}\right] = \left[\begin{array}{cc} a_1^0 I_m + a_1^1 \psi_{20}^{2*T} & 0 \\ a_1^1 I_m & 0 \\ \hline 0 & I_{m_s} \\ \hline 0 & 0 \end{array}\right]. \tag{44}$$

where $B_{2,in}^1 = T_{in}B_{2,in}^1$, $B_{s1,in} = T_{in}B_{s1}$ and $B_{2,in}^{1*} = T_{in}B_2^{1*}$. It is noted that $C_{in}\overline{B}_{2,in}^{1} = C_{in}\overline{B}_{2,in}^{1*} = \left[\begin{array}{cc} a_1^1 CAB_2 & CB_{s1} \end{array}\right]$ has full rank by Assumption 3 and Lemma 2. Examination on elements of $\overline{B}_{2,in}^{1*}$ and $\overline{B}_{2,in}^{1}$ shows that

$$\overline{B}_{2,in}^{1*} \quad = \quad \overline{B}_{2,in}^{1} + B_{2,in} a_1^1 \Psi_{in,m}^{*T}. \tag{45}$$

where

$$\Psi_{in,m}^{*T} = \left[\begin{array}{cc} \psi_{20}^{2*T} & 0_{m\times m_s} \end{array}\right] \in \mathbb{R}^{m\times p} \tag{46}$$

where ψ_{20}^{2*T} is a subset of the elements in $\Psi_{2,in}^{*}$ as shown in (42). It is noted that (44) also holds for $(A_{in}^* - L_{in}C_{in})$ for $\forall L_{in} \in \mathbb{R}^{n\times m}$. Transformation back to the original coordinate proves the Lemma. ■

C. Proof of Lemma 4

Proof: It has been proved that the $\mathcal{Z}\{A^*, B_2^{1*}, C\}$ is exactly the eigenvalues of $\overline{N}_{2,in}^{1*} A_{in}^* M_{in}$ with $M_{in} = \left[\begin{array}{ccc|c} I_m & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-r_s} \end{array}\right]^T$ and $\overline{N}_{2,in}^{1} = (M_{in}^T M_{in})^{-1} M_{in}^T \left[I - \overline{B}_{2,in}^{1}(\overline{C}_{in}\overline{B}_{2,in}^{1})^{-1}\overline{C}_{in}\right]$ (see [18]). Some algebra shows that

$$\overline{N}_{2,in}^{1*} A_{in}^* M_{in} = \left[\begin{array}{cc} -\frac{a_1^0}{a_1^1} & \times \\ 0 & Z \end{array}\right] \tag{47}$$

where Z is the zero dynamics matrix as in 42 whose eigenvalues are $\mathcal{Z}\{A, B_2, C\}$ (see [21, Section 2.3]). ■

D. Proof of Theorem 1

Proof: We propose a Lyapunov function candidate

$$V = e_{mx}^T P^* e_{mx} + Tr\left[\widetilde{\Psi}_\Lambda^T \Gamma_{\psi_\lambda}^{-1} \Lambda^* \widetilde{\Psi}_\Lambda\right] + Tr\left[\widetilde{\Psi}_m^T \Gamma_{\psi_m}^{-1} \widetilde{\Psi}_m\right] \tag{48}$$

where $P^* = P^{*T} > 0$ is the matrix that guarantees the SPR properties of $\{A_{L^*}^*, \overline{B}_2^{1*}, SC\}$, satisfying

$$P^* A_{L^*}^* + A_{L^*}^* P^* = -Q^* < 0 \tag{49}$$

$$P^* \overline{B}_2^{1*} = C^T S^T, \tag{50}$$

for a $Q^* = Q^{*T} > 0$. Partition on both sides of (50) yields

$$P^*\left[B_2^{1*} B_1\right] = C^T \left[\begin{array}{cc} S_2^T & S_1^T \end{array}\right]. \tag{51}$$

By appealing to (27)(28)(29)(49)(51), the derivative of V has the following bound

$$\begin{aligned} \dot{V} &= e_{mx}^T \left[A_{L^*}^{*T} P^* + P^* A_{L^*}^*\right] e_{mx} \\ &\quad - 2e_{mx}^T [P^* B_2^{1*} - C^T S_2^T] \Lambda^* \widetilde{\Psi}_\Lambda^T \overline{\chi} \\ &\quad - 2e_{mx}^T [P^* B_2^{1*} - C^T S_2^T] \widetilde{\Psi}_m^T \overline{e}_{sy} \\ &= -e_{mx}^T Q^* e_{mx} \le 0. \end{aligned} \tag{52}$$

Then $e_{mx}(t)$, $\widetilde{\Psi}_\Lambda(t)$ and $\widetilde{\Psi}_m(t)$ are bounded as $t \to \infty$, which proves i). Applying Barbalat's Lemma (using the fact that $\dot{e}_{mx}(t)$ is bounded) shows that $e_{mx}(t) \to 0$ as $t \to \infty$, which proves ii). From (28) and (16), the fact $e_{mx}(t) \to 0$ implies that $e_y(t) \to 0$, $e_{sy}(t) \to 0$ and $\overline{e}_{sy}(t) \to 0$ as $t \to \infty$, which in turn implies that x_m, as well as $\overline{x}_m$ and $\overline{u}_{bl}$, is bounded. Further, denote

$$e_{pz}(t) = z - z_{cmd}, \; e_{mz}(t) = z_m - z_{cmd}. \tag{53}$$

From (9), it is noted that $\int e_{pz}(t)dt$ is an element of x. From (14), it is noted that $\int (e_{mz}(t) - L_I e_y)dt$ is an element of x_m where L_I are the rows of L corresponding to e_{mz} dynamics. As a result, $e_{mx}(t) \to 0$ as $t \to \infty$, which, together with the definition of e_{mx} as

$$\begin{aligned} e_{mx} = e_x + B_2 \Lambda^* a_1^1 \overline{\Psi}_1^{*T} x_m + B_2 a_1^1 \left[\widetilde{\Psi}_m^T(t)\overline{e}_{sy}\right] \\ - B_2 \Lambda^* a_1^1 \widetilde{\Psi}_\Lambda^T \overline{\chi}, \end{aligned} \tag{54}$$

implies $e_x(t) \to \int (B_2[\cdot])dt$ and therefore

$$\int (e_{pz} - e_{mz} + L_I e_y)dt \to \int (B_{2,I}[\cdot])dt = 0 \tag{55}$$

as $t \to \infty$ (since $B_{2,I}$, the rows of B_2 corresponding to w_z dynamics, is zero). Eq.(55), together with the fact that

$$e_z(t) = z - z_m = e_{pz} - e_{mz} \tag{56}$$

implies

$$\int e_z(t)dt \to -\int (L_I e_y)dt \tag{57}$$

which has a bounded limit as $t \to \infty$ (since $e_y(t) \to 0$). Further, from (9) and (14), $\dot{e}_z(t)$ is bounded as $t \to \infty$. Applying Barbalat's Lemma shows that $e_z(t) \to 0$ as $t \to \infty$, which proves iii). ■