Multikink topological terms and charge-binding domain-wall condensation induced symmetry-protected topological states: Beyond Chern-Simons/BF field theories

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Quantum disordering a discrete-symmetry-breaking state by condensing domain walls can lead to a trivial symmetric insulator state. In this work, we show that if we bind a one-dimensional representation of the symmetry (such as a charge) to the intersection point of several domain walls, condensing such modified domain walls can lead to a nontrivial symmetry-protected topological (SPT) state. This result is obtained by showing that the modified domain-wall condensed state has a nontrivial SPT invariant, the symmetry-twist-dependent partition function. We propose two different kinds of field theories that can describe the above-mentioned SPT states. The first one is a Ginzburg-Landau–type nonlinear sigma model theory, but with an additional multikink domain-wall topological term. Such theory has an anomalous $U^k(1)$ symmetry but an anomaly-free Z_N^k symmetry. The second one is a gauge theory, which is beyond Abelian Chern-Simons/BF gauge theories. We argue that the two field theories are equivalent at low energies. After coupling to the symmetry twists, both theories produce the desired SPT invariant.

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I. INTRODUCTION

A. SPT states and their effective field theories

Recently, it has been realized that many-body ground states can be divided into two classes [1]: long-range entangled (LRE) states and short-range entangled (SRE) states. The LRE states can belong to many different phases that correspond to topologically ordered phases [2,3]. When there is a global symmetry (described by a group G), even SRE states can belong to many different phases, and these phases are called symmetry-protected topological (SPT) states [4–9]. A large class of bosonic SPT states whose boundary has a pure "gauge anomaly" [10–12] can be systematically classified via group cohomology classes $H^{d+1}(G,\mathbb{R}/\mathbb{Z})$ [13–15]. All these SPT states can be realized by exactly soluble lattice nonlinear σ model with the symmetry group G as the target space plus a 2π quantized topological θ term. They can also be realized by exactly soluble lattice Hamiltonians that contain many body interactions. In addition, bosonic SPT states whose boundary has a "gauge gravitational mixed anomaly" can all be realized by lattice nonlinear σ model with $SO_\infty \times G$ as the target space and with a 2π quantized topological θ term [16]. The potentially possible SPT invariants of the first and the second classes of SPT states can also be studied directly via cobordism theory [17–20], but the cobordism theory does not give rise to a realization of the SPT states.

Many of the SPT states protected by discrete group symmetry can also be realized by condensing domain walls in symmetry-breaking states, if we decorate the domain walls with lower-dimensional SPT states and/or invertible topologically ordered states [16,21–23]. In this work, we will realize some additional SPT states by condensing domain walls, such that the intersection point of several domain walls carries the quantum number of the unbroken symmetries. More general SPT states protected by discrete group symmetry can be obtained by decorating the intersection lines (or surfaces) of several domain walls with one-dimensional (1D) [or two-dimensional (2D)] SPT states (as indicated by the Kunneth formula for the group cohomology [16,21]).

In addition to the above systematic constructions of all the bosonic SPT phases, people have also developed many field theory realizations for some special simple SPT states (under the name of bosonic topological insulator (BTI) [23–32]) which lead to some simple physical pictures and mechanisms for bosonic SPT states. Due to the incompressibility of topological phases, it is sufficient to only consider quantum fluctuations of collective modes at low energies and long wavelengths, e.g., density and current fluctuations. Such an approach is the so-called "hydrodynamical approach" or effective quantum field theory for topological phases. The field theory realizations of SPT states belong to this approach.

Historically, the "hydrodynamical approach" turns out to be extremely powerful to understand the underlying physics of topological phases. For example, the fractional quantum Hall effect (FQHE) can be understood by the Ginzburg-Landau Chern-Simons theory [33] or more systematically by pure Chern-Simons theory [34–40]. Those bulk dynamic effective theories that capture the low energies and long-wavelength physics are also very useful to study phase transitions among different topological phases, e.g., phase transitions between FQHE at different filling fractions. Thus, the bulk dynamical Chern-Simons action approach to FQHE phases can be viewed as the Ginzburg-Landau action approach to symmetry-breaking phases. Therefore, it is very natural to ask what is the "hydrodynamical approach" to SPT states.

Very recently, Chern Simons/BF theories have been proposed [23,41–46] as bulk dynamical effective actions to describe 2D/3D bosonic SPT states protected by Abelian symmetry group (the so-called Abelian SPT states). Nevertheless, it has been pointed out [43] that the Abelian Chern Simons/BF theory approach is incomplete. Therefore, a much

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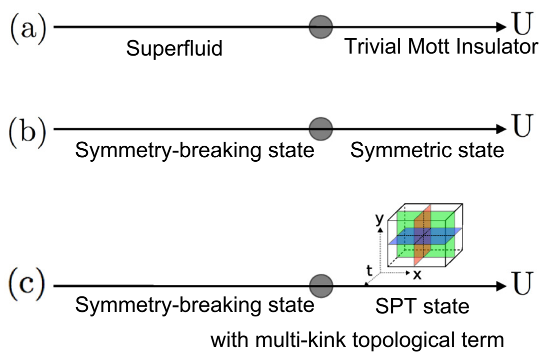

FIG. 1. (a) Disordering a $U(1)$ symmetry-breaking superfluid with an action by condensing the vortices, e.g., tuning some coupling constant U to increase the charge repulsion [47–49]. (b) Disordering a discrete-symmetry-breaking state by condensing the domain walls. The gray region qualitatively indicates the phase transition region, such as a critical point or a different phase. (c) In this work, we generalize the previous process by condensing domain walls with multikink topological terms. We obtain nontrivial SPT states with SPT invariants listed in Table I.

more general theoretical framework for bulk dynamical actions of SPT states is very desired. In this paper, we will focus on the mechanisms and bulk dynamical effective actions for bosonic SPT states with finite Abelian group symmetry within group cohomology classification. We propose a class of new topological actions to characterize bosonic Abelian SPT states in arbitrary dimensions that are beyond Abelian Chern-Simons/BF theory. We will show that such a class of generalized topological actions serves as a complete description for bosonic Abelian SPT states in 1D and 2D. In 3D, there are still some Abelian SPT states beyond the proposed bulk dynamical effective action; however, we believe that the basic principle and method developed in this paper are still applicable. We will leave these studies for future work. It is also worthwhile to mention that in a parallel work [32], a bulk dynamical effective action for Abelian SPT states beyond group cohomology classifications is also proposed. In principle, the "hydrodynamical approach" can also be generalized into interacting fermionic systems.

B. Summary of results

1. A mechanism of SPT states

Let us start by summarizing the mechanism that generates SPT states at intuitive level. It is well known that if we disorder a discrete-symmetry-breaking state by condensing domain walls, we can obtain a symmetry-restored state. Our approach is basically analogous to this line of thinking, except that we generalize the approach by including additional multikink topological terms to the domain walls (see Fig. 1).

There are two ways to view the multikink topological terms: the space picture and the space-time picture. In the space picture, we create the symmetry-breaking domain walls and trap some charges (not fractionalized) of the remained unbroken symmetry at the intersecting points, then we proliferate and condense the domain walls to restore the broken symmetry. On the other hand, in the space-time picture, we have an intersecting profile that contributes a nontrivial phase to the path integral [see Fig. 1(b)], and we then disorder the symmetry-breaking state with such nontrivial multikink topological term. As we will show explicitly and quantitatively using field theories, both processes lead to a nontrivial SPT state.

Using the above domain-wall condensation picture, we also obtain two kinds of field theory realization of the corresponding $Z_{N_1} \times Z_{N_2} \times Z_{N_3} \times \dots$ SPT states (see Table I). The first one is a $U^k(1)$ nonlinear σ model with a multikink topological term. The second one is a dynamical gauge theory that is beyond Abelian Chern-Simons/BF theory. Throughout the paper, we will implement the Euclidean space-time approach with the Euclidean time $t_E = it$ as the Minkowski time Wick-rotated by an imaginary i. We define the derivative ∂_0 as ∂_{t_E}. We choose the Euclidean space-time for the future convenience of the lattice regularization.

In the first column of Table I, we list the $U^k(1)$ nonlinear σ models with the multikink topological terms of the form $\frac{i}{d} C_{IJK\dots} \varepsilon^{\mu\nu\lambda\dots} \partial_\mu\theta^I \partial_\nu\theta^J \partial_\lambda\theta^K \dots$ with $C_{IJK\dots}$ a fully antisymmetric tensor and d the space-time dimension. In the second column of Table I, we list the gauge theory realization of the same $Z_{N_1} \times Z_{N_2} \times Z_{N_3} \times \dots$ SPT states. Our local field theories in the first and the second columns can produce the desired SPT invariants dictated by group cohomology [20]

TABLE I. First column: the $U^k(1)$ nonlinear σ model (NLσM) realization of the $Z_{N_1} \times Z_{N_2} \times Z_{N_3} \times \dots$ SPT states in the $\chi < \chi_c$ disordered limit. The additional multikink topological term [bikink for (1+1)D, trikink for (2+1)D, quadkink for (3+1)D, etc.] are listed. The phase fluctuating term $\partial_\mu\theta^I \equiv \partial_\mu\theta^I_{\mathrm{s}} + b^I_\mu$ contains a smooth piece $\partial_\mu\theta^I_{\mathrm{s}}$ and a singular piece b^I_μ. Here, $C_{IJ\dots}$ is a totally antisymmetric tensor, with $C_{12} = \frac{1}{(2\pi)} \frac{N_1 N_2\ p_{\mathrm{II}}}{N_{12}}$, $C_{123} = \frac{1}{(2\pi)^2 2!} \frac{N_1 N_2 N_3\ p_{\mathrm{III}}}{N_{123}}$, $C_{1234} = \frac{1}{(2\pi)^3 3!} \frac{N_1 N_2 N_3 N_4\ p_{\mathrm{IV}}}{N_{1234}}$, etc., with $N_{12\dots} \equiv \gcd(N_1, N_2, \dots)$. Second column: the dynamical gauge theory realization of the $Z_{N_1} \times Z_{N_2} \times Z_{N_3} \times \dots$ SPT states. The important global constraints on the fields are not specified, moreover, we need to well define the SPT path integral more than just the SPT Lagrangian; we will discuss this issue of path integral in depth in Sec. VII. Third column: the SPT invariants after integrating out the matter fields. Here, the nondynamical flat A^I field describes the Z_{N_I}-symmetry twist, which satisfies $\oint A^I_\mu dx^\mu = 0 \bmod 2\pi/N_I$. The main result of our work is that the field theories in the first and the second columns are equivalent at low energies at the $\chi < \chi_c$ disordered limit. We can derive their SPT invariants by integrating out the matter field. The SPT invariant is of the form $\int d^d x \frac{i C_{I_1 I_2 \dots I_d}}{d} \varepsilon^{\mu\nu\dots\sigma} A^{I_1}_\mu A^{I_2}_\nu \dots A^{I_d}_\sigma$ given in [20].

	Ginzburg-Landau NLσM	Dynamical gauge theory	SPT invariants: Probed field theory
1D	$\frac{\chi}{2}(\partial_\mu\theta^I)^2 + \frac{i}{2} C_{IJ} \varepsilon^{\mu\nu} \partial_\mu\theta^I \partial_\nu\theta^J$	$\frac{i}{2\pi} \varepsilon^{\mu\nu} b^I_\mu \partial_\nu a^I + \frac{-i}{2} C_{IJ} \varepsilon^{\mu\nu} b^I_\mu b^J_\nu$	$\frac{-i}{2} C_{IJ} \varepsilon^{\mu\nu} A^1_\mu A^2_\nu$
2D	$\frac{\chi}{2}(\partial_\mu\theta^I)^2 + \frac{i}{3} C_{IJK} \varepsilon^{\mu\nu\lambda} \partial_\mu\theta^I \partial_\nu\theta^J \partial_\lambda\theta^K$	$\frac{i\varepsilon^{\mu\nu\lambda}}{2\pi} b^I_\mu \partial_\nu a^I_\lambda + \frac{iC_{IJK}}{3} \varepsilon^{\mu\nu\lambda} b^I_\mu b^J_\nu \lambda^K_b$	$\frac{iC_{IJK}}{3} \varepsilon^{\mu\nu\lambda} A^1_\mu A^2_\nu A^3_\lambda$
3D	$\frac{\chi}{2}(\partial_\mu\theta^I)^2 + \frac{i}{4} C_{IJKL} \varepsilon^{\mu\nu\lambda\sigma} \partial_\mu\theta^I \partial_\nu\theta^J \partial_\lambda\theta^K \partial_\sigma\theta^L$	$\frac{i\varepsilon^{\mu\nu\rho\sigma}}{4\pi} b^I_\mu \partial_\nu a^I_{\sigma\rho} + \frac{-iC_{IJKL}}{4} \varepsilon^{\mu\nu\sigma\rho} b^I_\mu b^J_\nu b^K_\sigma b^L_\rho$	$\frac{-iC_{IJKL}}{4} \varepsilon^{\mu\nu\rho\sigma} A^I_\mu A^J_\nu A^K_\rho A^L_\sigma$

(after integrating out the dynamical fields). We list the SPT invariants in the third column of Table I.

2. Field theory with anomalous $U(1)$ symmetry

We stress that although the proposed $U^k(1)$ nonlinear σ model with the multikink topological terms formally has a $U^k(1)$ global symmetry $\theta_I(x^\mu) \to \theta_I(x^\mu) + \Delta f_I$. However, due to the presence of multikink topological terms, the $U^k(1)$ global symmetry is actually anomalous, i.e., cannot be realized by an onsite symmetry [12] in any lattice regularization of the field theories. Or, more precisely, the $U^k(1)$ nonlinear σ models have anomalous $U^k(1)$ symmetry if the multikink topological terms are quantized as $C_{12} = 0 \bmod \frac{1}{(2\pi)}$ in (1+1)D, $C_{123} = 0 \bmod \frac{1}{(2\pi)^2 2!}$ in (2+1)D, and $C_{1234} = 0 \bmod \frac{1}{(2\pi)^3 3!}$ in (3+1)D.

Here, we use a (1+1)D example to explain the above statement (the higher-dimensional cases can be understood in a similar way). Let us consider an ideal experiment by inserting a 2π flux corresponding to the first $U(1)$ symmetry through a closed 1D ring, the bikink topological term $\frac{i}{2} C_{IJ} \varepsilon^{\mu\nu} \partial_\mu \theta^I \partial_\nu \theta^J$ will induce a charge $2\pi C_{12}$ associate with the second $U(1)$ symmetry. Therefore, if the $2\pi C_{IJ}$ is not an integer, the $U^k(1)$ nonlinear σ model does not even have the $U^k(1)$ symmetry at quantum level. When $2\pi C_{IJ} \in \mathbb{Z}$, the $U^2(1)$ symmetry is anomalous since adding the flux of the first $U(1)$ can cause a nonconservation of the second $U(1)$.

The above charge pumping phenomena via flux insertion can happen on a boundary of a (2+1)D system, where an integer charge is created in the bulk and the total $U^k(1)$ charges are conserved.

However, the above charge pumping phenomena cannot happen in a strict $(1+1)$D system with onsite $U^k(1)$ symmetry. This is because the onsite $U^k(1)$ symmetry is gaugeable [i.e. , we can add $U(1)$ flux without breaking the $U^k(1)$ symmetry]. The presence of the charge pumping phenomena implies that, at quantum level, the $U^k(1)$ symmetry is broken by the $U(1)$ flux, which in turn implies that the $U^k(1)$ symmetry is anomalous (or non-onsite). Or, in other words, in a strict (1+1)D system with $U^2(1)$ onsite symmetry, C_{12} must vanish.

On the other hand, if the $2\pi C_{12} = 0 \bmod \frac{N_1 N_2}{N_{12}}$, the $Z_{N_1} \times Z_{N_2}$ subgroup of the $U^2(1)$ corresponds to an anomaly-free symmetry (i.e., an onsite symmetry). This is because the 2π flux of $U(1)$ induces a charge $2\pi C_{12} = \frac{N_1 N_2}{N_{12}} \times$ integer, which is essentially trivial since Z_{N_2} charge is only conserved mod N_2. Therefore, the $Z_{N_1} \times Z_{N_2}$ subgroup of the $U^2(1)$ is not anomalous. The $U^2(1)$ nonlinear σ model describes a system with $Z_{N_1} \times Z_{N_2}$ onsite symmetry, if $2\pi C_{12} = 0 \bmod \frac{N_1 N_2}{N_{12}}$.

Similarly, the $U^k(1)$ nonlinear σ model has a $Z_{N_1} \times Z_{N_2} \times Z_{N_3} \times \ldots$ onsite symmetry only when proper quantized values are assigned for $C_{IJK\ldots}$. For example, in $(1+1)$D, $(2+1)$D, and $(3+1)$D, we require that $C_{12} = \frac{1}{(2\pi)} \frac{N_1 N_2\, p_{\mathrm{II}}}{N_{12}}$, $C_{123} = \frac{1}{(2\pi)^2 2!} \frac{N_1 N_2 N_3\, p_{\mathrm{III}}}{N_{123}}$, and $C_{1234} = \frac{1}{(2\pi)^3 3!} \frac{N_1 N_2 N_3 N_4\, p_{\mathrm{IV}}}{N_{1234}}$, where $p_{\mathrm{I}}, p_{\mathrm{II}}, p_{\mathrm{III}} \in \mathbb{Z}$.

C. Organization of the paper

The rest of the paper is organized as follows: In Sec. II, we briefly review how to use SPT invariants to define SPT states. In Sec. III, we propose a bulk dynamical effective action to describe (1+1)D bosonic Abelian SPT states and use it to derive the corresponding SPT invariants. In Sec. IV, we briefly review the Chern-Simons action approach for (2+1)D bosonic Abelian SPT states and discuss its limitation. In Sec. V, we compute the SPT invariants for (2+1)D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state and propose a bulk dynamical effective action to describe such (2+1)D SPT states. In Sec. VI, we generalize our results to (3+1)D bosonic Abelian SPT states and propose a bulk dynamical action beyond BF theory. In Sec. VII, we verified that the partition function with the proposed SPT action has the ground state degeneracy (GSD) $= 1$. In Sec. VIII, edge theories for Abelian SPT states beyond Chern-Simons/BF actions are discussed via a standard dimension reduction scheme. Finally, there are conclusion remarks and discussions for future directions.

In Appendix A, we review the derivation of disordering the superfluid state to the Mott insulator (see the pioneering work [47–49] and Refs. [50,51]. In Appendix B, we provide an explicit calculation of an effective bulk action of SPT state. In Appendix C, we provide some words of caution by comparing our effective action of SPT state to topological gauge theories with non-semisimple Lie algebra. In Appendix D, we compute the edge mode GSD by counting the degenerate zero modes.

II. A REVIEW OF SPT STATES DEFINED BY SPT INVARIANTS

It has been shown that SPT states (within group cohomology or beyond group cohomology classifications) can be probed or even defined through the so-called SPT invariants [21,52] that may completely characterize different SPT states. In this section, we will review and discuss such a point of view.

A. Universal wave-function overlap: A complete SPT invariant for SPT orders

We start from reviewing the results of the SPT invariants in [52], using (2+1)D systems as examples. It was conjectured that the degenerate ground states $|\Psi_\alpha\rangle$, $\alpha = 1,2,\ldots,$ of a (2+1)D topological phase on a torus have the following properties [53]:

$$S_{\alpha\beta}\; e^{-f_S L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{S} | \Psi_\beta \rangle,$$
$$T_{\alpha\beta}\; e^{-f_T L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{T} | \Psi_\beta \rangle, \qquad (1)$$

where $\hat{S}$ is the 90° rotation operation $(x,y) \to (-y,x)$ and $\hat{T}$ is the Dehn twist rotation operation $(x,y) \to (x+y,y)$. It was conjectured that while the complex numbers f_S and f_T are not universal, the complex matrices $S_{\alpha\beta}$ and $T_{\alpha\beta}$ are universal. $S_{\alpha\beta}$ and $T_{\alpha\beta}$ can change only via phase transitions. Thus, we can use them to characterize different topological orders. In fact, we believe that $S_{\alpha\beta}$ and $T_{\alpha\beta}$ completely define (2+1)D topological ordered phases with gappable edges.

Can we use the similar idea to completely define (2+1)D SPT order? The wave-function overlap for SPT state also has the following universal structure:

$$S\; e^{-f_S L^2 + o(L^{-1})} = \langle \Psi_0 | \hat{S} | \Psi_0 \rangle,$$
$$T\; e^{-f_T L^2 + o(L^{-1})} = \langle \Psi_0 | \hat{T} | \Psi_0 \rangle, \qquad (2)$$

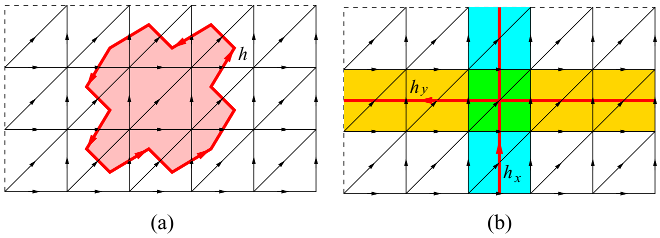

FIG. 2. (a) Symmetry twist along the boundary ∂R is generated by the symmetry transformation that acts only within R. (b) The symmetry twist h_x, h_y on torus gives rise to the twisted ground state $|\Psi_{(h_x,h_y)}\rangle$.

where the 1×1 unitary matrices S and T are universal. In fact, $S = T = 1$, due to the trivial bulk excitations in SPT state. Thus, S and T are trivial and could not be used to distinguish different SPT states.

To obtain a nontrivial wave-function overlap, we introduce symmetry twist: a symmetric transformation generated by $h \in G$ within the region R. The Hamiltonian is not invariant under such a local symmetry transformation [see Fig. 2(a)]:

$$H = \sum H_{ijk} \to H_h = \sum_{\text{in } R,\bar{R}} H_{ijk} + \sum_{\text{on } \partial R} H^h_{ijk}, \tag{3}$$

where H_{ijk} acts on sites i,j,k and H^h_{ijk} is on the boundary of R, ∂R, if the sites i,j,k are not all on one side of ∂R. We call $\sum_{\text{on } \partial R} H^h_{ijk}$ the h-symmetry twist.

Note that H and H_h have the same energy spectrum. So, the symmetry twist costs no energy. Let $|\Psi_{(h_x,h_y)}\rangle$ be the ground state of H_{h_x,h_y} on a torus with symmetry twists h_x,h_y in x and y directions. $|\Psi_{(h_x,h_y)}\rangle$ simulates the degenerate ground states for topologically ordered phases. We can use $|\Psi_{(h_x,h_y)}\rangle$ to construct S,T matrices that characterize the SPT order (see Figs. 3 and 4):

$\hat{S}$ move: $\langle \Psi_{(h_y^{-1},h_x)}|\hat{S}|\Psi_{(h_x,h_y)}\rangle = S_{h_x,h_y}\, e^{-f_S L^2 + o(L^{-1})}$,

$\hat{T}$ move: $\langle \Psi_{(h_x,h_y h_x)}|\hat{T}|\Psi_{(h_x,h_y)}\rangle = T_{h_x,h_y}\, e^{-f_T L^2 + o(L^{-1})}$,

$\hat{U}$ move: $\langle \Psi_{(h_t h_x h_t^{-1},h_t h_y h_t^{-1})}|\hat{U}(h_t)|\Psi_{(h_x,h_y)}\rangle = U_{h_x,h_y}(h_t)$.

Note that in addition to the $\hat{S}$ and $\hat{T}$ moves, the SPT invariants also contain $\hat{U}$ move generated by the global symmetry transformation $h_t \in G$.

The $\hat{S}$, $\hat{T}$, and $\hat{U}$ moves shift $(h_x,h_y) \to (h'_x,h'_y)$:

$$\begin{aligned} \hat{S} &: (h_x,h_y) \to (h'_x,h'_y) = \left(h_y^{-1},h_x\right); \\ \hat{T} &: (h_x,h_y) \to (h'_x,h'_y) = (h_x,h_y h_x); \\ \hat{U}(h_t) &: (h_x,h_y) \to (h'_x,h'_y) = \left(h_t h_x h_t^{-1},h_t h_y h_t^{-1}\right). \end{aligned} \tag{4}$$

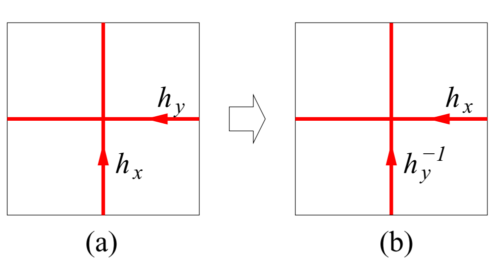

FIG. 3. $\hat{S}$ move is 90° rotation.

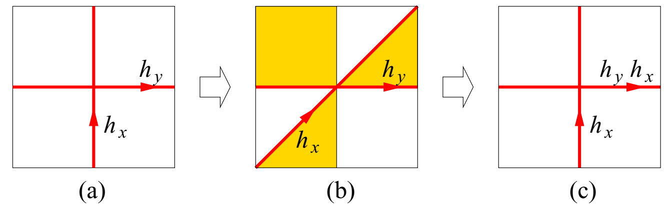

FIG. 4. $\hat{T}$ move is the Dehn twist followed by a symmetry transformation h_x in the shaded area.

When $(h'_x,h'_y) \neq (h_x,h_y)$, the complex phases S_{h_x,h_y}, T_{h_x,h_y}, $U_{h_x,h_y}(h_t)$ are not well defined since they depend on the choices of the phases of $|\Psi_{(h_x,h_y)}\rangle$ and $|\Psi_{(h'_x,h'_y)}\rangle$. However, the product of S_{h_x,h_y}, T_{h_x,h_y}, $U_{h_x,h_y}(h_t)$ around a closed orbit $(h_x,h_y) \to (h'_x,h'_y) \to \cdots \to (h_x,h_y)$ is universal (see Fig. 5). We believe that those products for various closed orbits completely characterize the (2+1)D SPT states.

For example, N $\hat{T}$ moves always form a closed orbit for Abelian $\mathbb{Z}_N = \{h = 0,\ldots,N-1\}$ group. For (2+1)D Z_N SPT state labeled by $k \in H^3[Z_N,U(1)] = \mathbb{Z}_N$, it has one SPT invariant:

$$T_{h_x h_y^{N-1},h_y} \cdots T_{h_x h_y^2,h_y} T_{h_x h_y,h_y} T_{h_x,h_y} = e^{2\pi i (h_x-1)^2 k/N}, \quad h_x,h_y \in \mathbb{Z}_N. \tag{5}$$

Such an SPT invariant completely characterizes the (2+1)D Z_N SPT state.

B. Universal wave-function overlap in (1+1)D

In (1+1)D, the SPT invariants are very simple. We only have the $\hat{U}$ move: $\langle \Psi_{(h_t h_x h_t^{-1})}|\hat{U}(h_t)|\Psi_{(h_x)}\rangle = U_{h_x}(h_t)$, which generates the shift $h_x \to h_t h_x h_t^{-1}$. Similar to the (2+1)D cases, the product of $U_{h_x}(h_t)$ around a closed orbit is well defined and universal (see Fig. 6). In particular, for Abelian symmetry group, $U_{h_x}(h_t)$ itself is universal.

III. A (1+1)D $Z_{N_1} \times Z_{N_2}$ SPT STATE AND ITS BIKINK BULK DYNAMICAL ACTION

A. A simple example

Now, let us apply the results obtained in the last section to a (1+1)D $Z_{N_1} \times Z_{N_2}$ bosonic SPT state, which is classified by

$$H^2[Z_{N_1} \times Z_{N_2},U(1)] = \mathbb{Z}_{N_{12}} = \{0,1,\ldots,N_{12}-1\}, \tag{6}$$

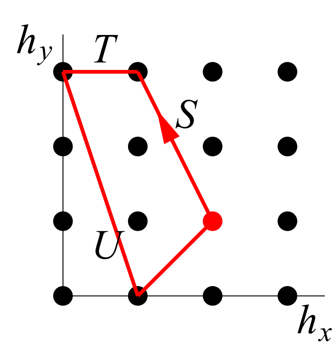

FIG. 5. A closed orbit in the (h_x,h_y) space.

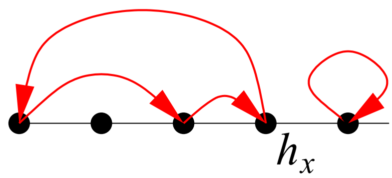

FIG. 6. Two closed orbits in h_x space.

where $N_{12} = \gcd(N_1,N_2)$. We consider an SPT state labeled by $k \in \mathbb{Z}_{N_{12}}$.

The group elements of $Z_{N_1} \times Z_{N_2}$ are labeled by $h = (h^1,h^2)$, $h^1 \in \mathbb{Z}_{N_1}$, $h^2 \in \mathbb{Z}_{N_2}$. The universal wave-function overlap [the SPT invariant $U_{h_x}(h_t)$] is

$$\langle \Psi_{(h_x^1,h_x^2)} | \hat{U}(h_t^1,h_t^2) | \Psi_{(h_x^1,h_x^2)} \rangle = U_{h_x^1,h_x^2}(h_t^1,h_t^2) = e^{ik\frac{2\pi}{N_{12}}(h_x^1 h_t^2 - h_x^2 h_t^1)}, \quad (7)$$

which can also be viewed as the fixed-point partition function on space-time $T^2 = S^1 \times S^1$ with symmetry twists in x,t directions (see Fig. 7):

$$Z_{\text{fixed point}} = U_{h_x^1,h_x^2}(h_t^1,h_t^2) = e^{ik\frac{2\pi}{N_{12}}(h_x^1 h_t^2 - h_x^2 h_t^1)}. \quad (8)$$

Both wave-function overlap and partition function pictures imply the following physical meaning for the above SPT invariant: a symmetry twist of Z_{N_1} carries Z_{N_2} charge k:

$$\langle \Psi_{(h_x^1,h_x^2)=(1,0)} | \hat{U}(h_t^1 = 0, h_t^2 = 1) | \Psi_{(h_x^1,h_x^2)=(1,0)} \rangle = e^{ik\frac{2\pi}{N_{12}}}. \quad (9)$$

Let us discuss a concrete example for the above (1+1)D SPT invariant. We consider a spin-1 chain with the spin-rotation symmetry $Z_2 \times Z_2 = D_2 = 180^\circ$ in S^x, S^z. The Hamiltonian on a ring is given by (untwisted case)

$$H_{D_2} = \sum_{i=1}^{L-1} (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) + J_x S_L^x S_1^x + J_y S_L^y S_1^y + J_z S_L^z S_1^z, \quad (10)$$

where $J_x = J_y = J_z > 0$. The ground state carries a trivial quantum number $e^{i\pi \sum S_i^z}$ with $e^{i\pi \sum S_i^z} = 1$.

If we add a twist by $e^{i\pi \sum S_i^x}$, the Hamiltonian becomes

$$H_{D_2}^{\text{twist}} = \sum_{i=1}^{L-1} (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) + J_x S_L^x S_1^x - J_y S_L^y S_1^y - J_z S_L^z S_1^z. \quad (11)$$

The twisted ground state carries a nontrivial quantum number $e^{i\pi \sum S_i^z}$ with $e^{i\pi \sum S_i^z} = -1$. Such a dependence of the ground-state quantum number $e^{i\pi \sum S_i^z}$ on the $e^{i\pi \sum S_i^x}$ twist is the (1+1)D SPT invariant discussed above.

The above SPT invariant also suggests a mechanism for the (1+1)D $Z_{N_1} \times Z_{N_2}$ SPT state. We notice that the SPT invariant

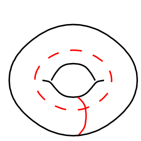

FIG. 7. Space-time $S^1 \times S^1$ with two symmetry twists in x,t directions.

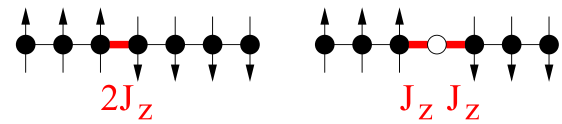

FIG. 8. Two kinds of domain walls with the same energy, but different Z_2^z charges, 0 (mod 2) and 1 (mod 2), respectively, on a lattice. Equation (14)'s H_1^{hop} is a hopping operator for the first kind of domain wall. Equation (15)'s H_2^{hop} is a hopping operator for the second kind of domain wall.

implies a symmetry twist of Z_{N_1} that carries a "charge" of Z_{N_2}. Since the symmetry twist of Z_{N_1} is the domain wall of Z_{N_1} in a Z_{N_1} symmetry-breaking state, we may (1) start with a Z_{N_1} symmetry-breaking state, (2) bind k Z_{N_2} charge to the domain wall of Z_{N_1}, and (3) restore the Z_{N_1} symmetry by proliferating the domain walls. In this way, we obtain a (1+1)D $Z_{N_1} \times Z_{N_2}$ SPT state labeled by $k \in \mathcal{H}^2[Z_{N_1} \times Z_{N_2}, U(1)]$.

For example, let us consider a 1D $Z_2^x \times Z_2^z$ spin-1 chain with symmetry

$$Z_2^x : (|\uparrow_x\rangle, |0_x\rangle, |\downarrow_x\rangle) \to (|\uparrow_x\rangle, -|0_x\rangle, |\downarrow_x\rangle),$$
$$Z_2^z : (|\uparrow_z\rangle, |0_z\rangle, |\downarrow_z\rangle) \to (|\uparrow_z\rangle, -|0_z\rangle, |\downarrow_z\rangle). \quad (12)$$

The following Hamiltonian has the $Z_2^x \times Z_2^z$ symmetry

$$H_{Z_2 \times Z_2}^0 = \sum_i -J_z S_i^z S_{i+1}^z, \quad (13)$$

but its ground state breaks the Z_2^x symmetry. Such a symmetry-breaking state has two kinds of domain walls which happen to have the same energy, but different Z_2^z charges. The two kinds of domain walls, shown in Fig. 8, have different hopping operators:

$$H_1^{\text{hop}} = \sum_i -\frac{K}{2}[(S_i^+)^2 + (S_i^-)^2] = \sum_i -K[(S_i^x)^2 - (S_i^y)^2], \quad (14)$$

$$H_2^{\text{hop}} = -\sum_i \frac{J_{xy}}{2}(S_i^+ S_{i+1}^+ + S_i^- S_{i+1}^-) = \sum_i J_{xy}(-S_i^x S_{i+1}^x + S_i^y S_{i+1}^y). \quad (15)$$

Here, we used the fact that $S_i^+ \equiv S_i^x + iS_i^y$ and $S_i^- \equiv S_i^x - iS_i^y$. It is straightforward to see the $(S_i^+)^2$ operator hops the first kind of domain wall of Fig. 8 in one direction, while the $(S_i^-)^2$ operator hops the first kind of domain wall of Fig. 8 in the opposite direction. On the other hand, the $S_i^+ S_{i+1}^+$ operator hops the second kind of domain wall of Fig. 8 in one direction, while the $S_i^- S_{i+1}^-$ operator hops the second kind of domain wall of Fig. 8 in the opposite direction.

Adding a strong enough hopping operator can make a domain wall subject to a negative energy cost, which restores the Z_2^x symmetry by proliferating the domain walls. We find that $H_{Z_2 \times Z_2}^0 + H_1^{\text{hop}}$ leads to a trivial SPT state, while $H_{Z_2 \times Z_2}^0 + H_2^{\text{hop}}$ leads to a nontrivial $Z_2 \times Z_2$ SPT state. Via a unitary transformation, the Hamiltonian $H_{Z_2 \times Z_2}^0 + H_2^{\text{hop}}$ is equivalent to the Hamiltonian of Eq. (10) discussed above, as the Haldane phase of a spin-1 antiferromagnetic Heisenberg chain.

B. Bikink topological term NLσM and dynamic gauge theory

The underlying physics of the above (1+1)D $Z_{N_1} \times Z_{N_2}$ SPT state can also be captured by the following Higgs action with a *bikink topological term*:

$$\begin{aligned}\mathcal{L}_{\text{bikink}} &= \frac{\chi}{2}(\partial_\mu \theta^I)^2 + \frac{i}{2} C^{IJ} \varepsilon^{\mu\nu} \partial_\mu \theta^I \partial_\nu \theta^J \\ &\simeq \frac{\chi}{2}\big(\partial_\mu \theta^I_{\text{s}} + b^I_\mu\big)^2 \\ &\quad + \frac{i}{2} C^{IJ} \varepsilon^{\mu\nu}\big(\partial_\mu \theta^I_{\text{s}} + b^I_\mu\big)\big(\partial_\nu \theta^J_{\text{s}} + b^J_\nu\big) + \mathcal{L}^b_{\text{Maxwell}}, \quad (16)\end{aligned}$$

where $I = 1,2$ and the structure constant C_{IJ} is totally antisymmetric with $C_{IJ} = -C_{JI}$. We assume Einstein summations for repeated indices throughout the whole paper. The quantum phase fluctuation can be captured by a real scalar compact field $\theta^I \equiv \theta^I_{\text{s}} + \theta^I_{\text{v}}$ with a smooth piece and a singular piece θ^I_{s} and θ^I_{v}. To achieve the disordered insulator state, we can condense the vortex, namely, strongly disorder the superfluid coherent phase. We will write $\partial_\mu \theta^I_{\text{s}} + \partial_\mu \theta^I_{\text{v}} \equiv \partial_\mu \theta^I_{\text{s}} + b^I_\mu$. The $\partial_\mu \theta^I_{\text{s}}$ captures the smooth piece $\partial_\mu \theta^I_{\text{s}}$, and the additional b^I_μ captures the singular piece $\partial_\mu \theta^I_{\text{v}}$. We note that the real scalar fields θ^I_{s} can be viewed as the phase fluctuations of Z_{N_I} symmetry in a $Z_{N_1} \times Z_{N_2}$ symmetry-breaking phase while vector fields b^I_μ (with $\mathcal{L}^b_{\text{Maxwell}}$ the corresponding Maxwell term) describe the proliferations of domain walls, which restore the $Z_{N_1} \times Z_{N_2}$ symmetry. Such a Higgs action with a bikink topological term will enforce a Z_{N_1} domain wall that carries a "charge" of Z_{N_2}, and vice versa. It is clear that the bikink topological term is just a boundary term in the absence of gauge fields b^I_μ. In the following, we will show that such a bulk action with the bikink topological term indeed describes the $Z_{N_1} \times Z_{N_2}$ SPT physics in (1+1)D.

After dropping the total derivative term, we can rewrite the above action as

$$\begin{aligned}\mathcal{L}_{\text{bikink}} &= \frac{\chi}{2}\big(\partial_\mu \theta^I_{\text{s}} + b^I_\mu\big)^2 \\ &\quad + \frac{i}{2} C^{IJ} \varepsilon^{\mu\nu}\big(-2\theta^I_{\text{s}} \partial_\mu b^J_\nu + b^I_\mu b^J_\nu\big) + \mathcal{L}^b_{\text{Maxwell}}. \quad (17)\end{aligned}$$

Next, we introduce the Hubbard-Stratonovich fields j^μ_I to decouple the quadratic term as

$$\begin{aligned}\mathcal{L}_{\text{bikink}} &= \frac{1}{2\chi}\big(j^\mu_I\big)^2 - i\theta^I_{\text{s}} \partial_\mu j^\mu_I + i b^I_\mu j^\mu_I \\ &\quad + \frac{i}{2} C^{IJ} \varepsilon^{\mu\nu}\big(-2\theta^I_{\text{s}} \partial_\mu b^J_\nu + b^I_\mu b^J_\nu\big) + \mathcal{L}^b_{\text{Maxwell}}.\end{aligned}$$

Integrating out the smooth fluctuations θ^I_{s} leads to the following constraint:

$$\partial_\mu\big(j^\mu_I + C^{IJ} \varepsilon^{\mu\nu} b^J_\nu\big) = 0. \quad (18)$$

The above constraints can be solved by

$$j^\mu_I = \frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_\nu a^I - C^{IJ} \varepsilon^{\mu\nu} b^J_\nu, \quad (19)$$

where a^I do not need to be globally defined. To disorder the $U(1)$ phase, we take $\chi \ll \chi_c$, we can drop out the $\frac{1}{2\chi}(j^\mu_I)^2$ term as well as the Maxwell term of gauge fields b^I_μ thanks to their renormalization group (RG) irrelevancy [51]. We end up with an effective topological action

$$\mathcal{L}_{\text{top}} = \frac{i}{2\pi} \varepsilon^{\mu\nu} b^I_\mu \partial_\nu a^I + \frac{-i}{2} C^{IJ} \varepsilon^{\mu\nu} b^I_\mu b^J_\nu. \quad (20)$$

The gauge transformation of b^I_μ in the above action will induce a shift on the scalar fields a^I:

$$a^I \to a^I + 2\pi C^{IJ} g^J; \quad b^I_\mu \to b^I_\mu + \partial_\mu g^I. \quad (21)$$

The above functions do not necessarily need to be *globally defined*. In fact, the compactness condition of a^I and b^I_μ implies the closed loop or the closed surface integral has the constraints

$$\oint da/(2\pi) \in \mathbb{Z}, \quad \oiint db_I/(2\pi) \in \mathbb{Z}. \quad (22)$$

In Sec. VII, we will derive the same constraints in the path-integral level, from the constraints of $U(1)$ charge and the vortex number on a closed surface.

Now, let us compute the quantization condition for the coefficients C_{IJ}. We note that the average of $\theta^I = \theta^I_{\text{s}} + \theta^I_{\text{v}}$ is quantized as $2\pi/N_I \times$ integer. In the disordered phase which restores the $Z_{N_1} \times Z_{N_2}$ symmetry, θ^I's have many fluctuating kinks. Let us consider a configuration where θ^1 has a kink $\Delta\theta^1 = 2\pi k_1/N_1$ on the t axis and θ^2 has a kink $\Delta\theta^2 = 2\pi k_2/N_2$ on the x axis. For such a configuration, the action from the bikink topological term is given by

$$\begin{aligned}S &= \int dx\, dt\, \frac{i}{2} C_{IJ} \varepsilon^{\mu\nu} \partial_\mu \theta^I \partial_\nu \theta^J \\ &= 8\pi^2 i C_{12} \frac{k_1 k_2}{N_1 N_2}. \quad (23)\end{aligned}$$

This means that the θ^1 kink carries a Z_{N_2} charge $2\pi C_{12} \frac{k_1}{N_1}$ mod N_2. Since $k_1 = 0 \sim k_1 = N_1$, C_{12} must be quantized:

$$2\pi C_{12} = 0 \bmod N_2, \;\; 2\pi C_{12} = 0 \bmod N_1. \quad (24)$$

Thus,

$$C_{12} = \frac{p_{\text{II}}}{2\pi} \frac{N_1 N_2}{N_{12}}, \quad p_{\text{II}} = 0, \ldots, N_{12} - 1 \quad (25)$$

where $N_{12} = \gcd(N_1, N_2)$. Also we note that C_{12} has only N_{12} distinct quantized values, corresponding to N_{12} distinct charge assignments.

The above argument for the quantization condition of C_{12} due to global $Z_{N_1} \times Z_{N_2}$ symmetry can also be derived in a rigorous way by adding a coupling term to external background gauge field A^I:

$$\mathcal{L}_{\text{coupling}} = \frac{i}{2\pi} \varepsilon^{\mu\nu} A^I_\mu \partial_\nu a^I. \quad (26)$$

As the physical meanings of A^1 and A^2 are Z_{N_1} and Z_{N_2} symmetry twists, A^I must be a flat connection with $dA^I = 0$ and $\oint A^I = 2\pi n_I/N_I$. On the other hand, since $\int dx\, dt \mathcal{L}_{\text{coupling}}$ must be invariant under gauge transformation (21), C_{12} can not take arbitrary value. A short calculation suggests the same quantization condition (24).

In Sec. VII, we will define a rigorous SPT internal gauge theory path integral, and we confirm that the GSD of our theory is unique on a closed manifold, GSD $= 1$, in agreement with SPT state. We will also derive the *SPT invariant* in Ref. [20] by coupling the internal gauge theory to semiclassical probed field A. In this way, it becomes manifested that C_{12} can only take N_{12} distinguishable value derived in Eq. (24). In the following, we generalize the above results to higher dimensions.

IV. A REVIEW OF CHERN-SIMONS ACTION APPROACH TO (2+1)D ABELIAN SPT STATES

In this section, we will start with a brief review on the Chern-Simons action approach for (2+1)D Abelian SPT states. Then, we explain the physical meaning of the Chern-Simons action approach and discuss its limitations.

It is well known that a vortex condensation can turn a boson superfluid into a trivial bosonic insulator. A bosonic $U(1)$ SPT state is also a bosonic insulator, but a nontrivial one. It turns out that a condensation of vortex-charge bound state can turn a boson superfluid into a nontrivial $U(1)$ SPT state.

To show this, let us consider a boson superfluid for one species of bosons, which can be described by an XY model:

$$\mathcal{L}_{\text{XY}} = \tfrac{1}{2}(\partial_\mu \theta)^2. \tag{27}$$

If the vortex of the boson condenses, θ in the XY model is no longer a smooth function of space-time. We can introduce the singular part by replacing $\partial_\mu\theta$ by $\partial_\mu\theta_{\text{s}} + b_\mu$, where the field strength of gauge field b_μ corresponds to the vortex current density $\tilde{J}^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$.

The charge of gauge field b_μ is the number of vortices minus the number of antivortices and is quantized. In the vortex condensed phase, the phase fluctuation of the vortex condensate can be described by another XY model, which is dual to the Maxwell term of the gauge field b_μ. Now, the boson superfluid is described by the following Lagrangian:

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}\left[(\partial_\mu\theta_{\text{s}} + b_\mu)^2 + \frac{1}{4\pi^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right], \tag{28}$$

where $\tilde{F}_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ and we have normalized with $v = 1$, $\chi = 1$.

We can introduce a Hubbard-Stratonovich field j_μ to decouple the quadratic term as

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}(j^\mu)^2 - i\theta_{\text{s}}\partial_\mu j^\mu + i b_\mu j^\mu + \frac{1}{8\pi^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}.$$

Integrating out the θ_{s} field results in a constraint $\partial_\mu j^\mu = 0$. From this constraint, we can write $j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$. The charge of a_μ is equal to the boson number and is quantized. With these results, the path integral becomes

$$\mathcal{L}_{\text{BF}} = \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu a_\lambda + \frac{1}{8\pi^2}[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}], \tag{29}$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Note that the boson current $j^b_\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$, while the vortex current $j^v_\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$.

The above can be generalized to the case with k species of bosons with $U^k(1)$ symmetry. The bosonic insulator induced by the vortex condensation is described by the following Chern-Simons action:

$$\mathcal{L}_{\text{CS}} = \frac{i}{4\pi}\varepsilon^{\mu\nu\lambda}K^0_{IJ}a^I_\mu\partial_\nu a^J_\lambda, \quad I = 1,2,\ldots,2k \tag{30}$$

with

$$K^0_{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{I}_{k\times k}, \tag{31}$$

where $a^{2k}_\mu \sim a_\mu$ and $a^{2k-1}_\mu \sim b_\mu$. Since $|\det[K]| = 1$, the above Chern-Simons action has a unique ground state 1 on any closed manifold. The chiral central charge for the edge states is given by the signature of K which is zero. So, the bosonic insulator has a trivial topological order.

However, the bosonic insulator may have a nontrivial $U^k(1)$ SPT order. To see this, we turn on the external $U^k(1)$ gauge field A^α_μ to reveal the $U^k(1)$ symmetry of the theory:

$$\mathcal{L}_{\text{coupling}} = \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda}q^I_\alpha A^\alpha_\mu\partial_\nu a^I_\lambda, \quad \alpha = 1,2,\ldots,k. \tag{32}$$

Here, $\boldsymbol{q}_\alpha$ are integer-value charge vectors. q^{2l-1}_α is the A^α charge carried by the lth species of bosons, and $q^{2\beta}_\alpha$ is the A^α charge carried by the vortex of the lth species of bosons. We see that charge vectors $\boldsymbol{q}_\alpha$ reveal the information on what kinds of vortex-charge bound states are condensing to produce the bosonic insulator. Different vortex-charge bound states (i.e., different charge vectors) will lead to different $U^k(1)$ SPT orders.

The full theory is given by $\mathcal{L} = \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{coupling}}$. After integrating out internal gauge fields a^I_μ (the matter fields), we obtain an effective theory for the external fields A^α:

$$\mathcal{L}_{\text{eff}} = -\frac{i}{4\pi}\varepsilon^{\mu\nu\lambda}A^\alpha_\mu q^I_\alpha K^0{}_{IJ}q^J_\beta\partial_\nu A^\beta_\lambda. \tag{33}$$

By considering equivalent class of response K matrix $\widetilde{K}_{\alpha\beta} \equiv q^I_\alpha K^0{}_{IJ}q^J_\beta$, we can "classify" (2+1)D $U^k(1)$ SPT states described by the Chern-Simons theory (30). We can also break the $U^k(1)$ symmetry down to $Z_{N_1} \times \ldots \times Z_{N_k}$ symmetry and obtain a "classification" of $Z_{N_1} \times \cdots \times Z_{N_k}$ SPT states in (2+1)D. Since Z_{N_α} group can always be embedded into $U^k(1)$ group, it is not a surprise that the $Z_{N_1} \times \cdots \times Z_{N_k}$ SPT state can be described by the same Chern-Simons action.

However, since $H^3[Z_{N_1} \times \cdots \times Z_{N_k}, U(1)] = \oplus_i \mathbb{Z}_{N_i} \oplus_{i<j} \mathbb{Z}_{N_{ij}} \oplus_{i<j<k} \mathbb{Z}_{N_{ijk}}$ $[N_{ijk} = \gcd(N_i,N_j,N_k)]$, the above classification turns out to be incomplete and it can only describe a subclass of Abelian SPT states labeled by $\oplus_i \mathbb{Z}_{N_i} \oplus_{i<j} \mathbb{Z}_{N_{ij}}$, namely, the type I and type II SPT phases. In the following, we will develop an effective field theory description for type-III SPT order in (2+1)D, which is labeled by $\oplus_{i<j<k}\mathbb{Z}_{N_{ijk}}$.

V. A (2+1)D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT STATE AND ITS TRIKINK BULK DYNAMICAL ACTION

A. A (2+1)D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state

Without loss of generality, it is sufficient to discuss a (2+1)D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ bosonic SPT state, which is classified by

$$H^3[Z_{N_1}\times Z_{N_2}\times Z_{N_3},U(1)] = \mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\oplus\mathbb{Z}_{N_3}\oplus\mathbb{Z}_{N_{12}}\oplus\mathbb{Z}_{N_{23}}\oplus\mathbb{Z}_{N_{13}}\oplus\mathbb{Z}_{N_{123}}. \quad (34)$$

We consider a type-III SPT state labeled by $k\in\mathbb{Z}_{N_{123}}$.

The group elements of $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ are labeled by $h=(h^1,h^2,h^3)$, $h^1\in\mathbb{Z}_{N_1}$, $h^2\in\mathbb{Z}_{N_2}$, $h^3\in\mathbb{Z}_{N_3}$. The SPT invariant $U_{h_x,h_y}(h_t)$ for the above SPT state is the fixed-point partition function on space-time $T^3=(S^1)^3$ with symmetry twists in x,y,t directions:

$$Z_{\text{fixed point}}=U_{h_x,h_y}(h_t)=e^{ik\frac{2\pi}{N_{123}}\epsilon_{abc}h_x^a h_y^b h_t^c}. \quad (35)$$

The physical meaning of the SPT invariant is the following: Consider the ground state of the Hamiltonian with symmetry twists in Z_{N_1} and Z_{N_2}, the intersection of the symmetry twist in Z_{N_1} and the symmetry twist in Z_{N_2} carries Z_{N_3} charge k.

The above SPT invariant also allows us to calculate the dimension reduction of the (2+1)D SPT state to a (1+1)D SPT state: We view the space-time as $T^3=T^2_{x,t}\times S^1_y$, and put Z_{N_3} symmetry twist $(h_y^1,h_y^2,h_y^3)=(0,0,1)$ in the small circle S^1_y. The (2+1)D partition function reduces to a (1+1)D partition function

$$Z_{\text{fixed point}}=e^{ik2\pi(h_x^1h_t^2-h_x^2h_t^1)}, \quad (36)$$

which is the SPT invariant of a (1+1)D SPT state. We find that the resulting (1+1)D SPT state is the one labeled by $k\in H^2[Z_{N_1}\times Z_{N_2},U(1)]=\mathbb{Z}_{N_{12}}$. The boundary of such a (1+1)D SPT state carries degenerated states that form a projective representation of $Z_{N_1}\times Z_{N_2}$. This leads to an experimental probe of the $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ SPT state: a Z_{N_3} "vortex" (end of Z_{N_3} symmetry twist) carries degenerated states that form a projective representation of $Z_{N_1}\times Z_{N_2}$.

The result of the above dimension reduction can also be viewed as each Z_{N_3} twist (which is a 1D curve in 2D space) carries a (1+1)D $Z_{N_1}\times Z_{N_2}$ SPT state labeled by $k\in H^2[Z_{N_1}\times Z_{N_2},U(1)]$. This picture leads to another mechanism for the (2+1)D $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ SPT state: (1) start with a Z_{N_3} symmetry-breaking state, (2) bind a (1+1)D $Z_{N_1}\times Z_{N_2}$ SPT state to the domain wall of Z_{N_3}, and (3) restore the Z_{N_3} symmetry by proliferating the domain walls. In this way, we obtain a (2+1)D $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ SPT state labeled by $k\in\mathbb{Z}_{N_{123}}$.

The (2+1)D SPT invariant (35) on space-time $T^3=(S^1)^3$ can also be expressed as a topological term of probe fields A_I:

$$Z^{\text{twist}}_{\text{fixed point}}(T^3)=e^{ip_{\text{III}}\frac{N_1N_2N_3}{(2\pi)^2N_{123}}\int A_1\wedge A_2\wedge A_3},\quad dA_I=0, \quad (37)$$

with an integer p_{III}. Again, since A_I describes symmetry twists on the boundary, it must be flat connection with $dA_I=0$. $\int A_1\wedge A_2\wedge A_3$ is also gauge invariant if $dA_I=0$. The field theory representation of the SPT invariants [Eq. (37)] should be valid for any space-time topologies. In the following, we will show how to derive such a topological response from a bulk dynamical effective action.

B. Trikink topological term NLσM

To describe the so-called type-III $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ SPT orders in $(2+1)$D, we consider the following effective action for three species of bosons with vortex condensation. The action contains a new *trikink topological term*, the C_{IJK} term [the following is a generalization of Eq. (28)]:

$$\begin{aligned}\mathcal{L}_{\text{trikink}}&=\frac{1}{2}(\partial_\mu\theta^I)^2+\frac{i}{3}C_{IJK}\varepsilon^{\mu\nu\lambda}\partial_\mu\theta^I\partial_\nu\theta^J\partial_\lambda\theta^K\\&\simeq\frac{1}{2}\left(\partial_\mu\theta^I_{\text{s}}+b^I_\mu\right)^2+\mathcal{L}^b_{\text{Maxwell}}\\&\quad+\frac{i}{3}C_{IJK}\varepsilon^{\mu\nu\lambda}\left(\partial_\mu\theta^I_{\text{s}}+b^I_\mu\right)\left(\partial_\nu\theta^J_{\text{s}}+b^J_\nu\right)\left(\partial_\lambda\theta^K_{\text{s}}+b^K_\lambda\right),\end{aligned} \quad (38)$$

where $I=1,2,3$ and the structure constant C_{IJK} is totally antisymmetric with $C_{IJK}=-C_{JIK}=-C_{IKJ}$. It is clear that the trikink topological term is just a boundary term in the absence of gauge fields b^I_μ.

To understand the physical meaning of the trikink topological term, we first note that the type-III SPT orders in (2+1)D only exist for a finite group $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$. So, we need to break the $U(1)^3$ symmetry down to $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ symmetry. The average of θ^I is quantized as $2\pi/N_I\times$ integer. In the disordered phase which restores the $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ symmetry, θ^I's have many fluctuating kinks along space-time surfaces. Let us consider a configuration in the space-time where θ^1 has a kink $\Delta\theta^1=2\pi k_1/N_1$ on the y-t plane, θ^2 has a kink $\Delta\theta^2=2\pi k_2/N_2$ on the t-x plane, and θ^3 has a kink $\Delta\theta^3=2\pi k_3/N_3$ on the x-y plane. For such a configuration ($b^I_\mu=0$), the action from the trikink topological term is given by

$$\begin{aligned}S&=\int dxdydt\ \frac{i}{3}C_{IJK}\varepsilon^{\mu\nu\lambda}\partial_\mu\theta^I\partial_\nu\theta^J\partial_\lambda\theta^K\\&=16\pi^3iC_{123}\frac{k_1k_2k_3}{N_1N_2N_3}.\end{aligned} \quad (39)$$

This means that the intersection of the kinks in θ^1 and θ^2 carries a Z_{N_3} charge $8\pi^2C_{123}\frac{k_1k_2}{N_1N_2}$ mod N_3. Since $k_1=0\sim k_1=N_1$, C_{123} must be quantized:

$$8\pi^2C_{123}\frac{k_2}{N_2}=0 \bmod N_3,\ \ 8\pi^2C_{123}\frac{k_1}{N_1}=0 \bmod N_3. \quad (40)$$

Thus,

$$C_{123}=\frac{p_{\text{III}}}{(2\pi)^22!}\frac{N_1N_2N_3}{N_{123}},\quad p_{\text{III}}=0,\ldots,N_{123}-1 \quad (41)$$

where $N_{123}=\gcd(N_1,N_2,N_3)$. Also, we note that C_{123} has only N_{123} distinct quantized values, corresponding to N_{123} distinct charge assignments.

Now, the physical meaning of the trikink topological term is clear: It is well known that the fluctuations of the kinks will turn a $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ symmetry-breaking state into a $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ symmetric state with a trivial SPT order. However, if we bound a Z_{N_3} charge to the intersection of the kinks in θ^1 and θ^2, etc., the resulting $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ symmetric state will have a nontrivial SPT order, as we will show below. In this way, we can produce N_{123} distinct type-III $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ SPT orders, consistent with the group cohomology result.

By integrating out the smooth fluctuations θ^I and introducing auxiliary gauge fields a^I_λ and λ^I_μ, we can derive the following bulk dynamical action:

$$\mathcal{L}_{\text{trikink}} = \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda}\lambda^I_\mu \partial_\nu a^I_\lambda + \frac{i}{3}C_{IJK}\varepsilon^{\mu\nu\lambda} \times \left[\lambda^I_\mu \lambda^J_\nu \lambda^K_\lambda + \left(b^I_\mu - \lambda^I_\mu\right)\left(b^J_\nu - \lambda^J_\nu\right)\left(b^K_\lambda - \lambda^K_\lambda\right)\right] + \frac{1}{2}\left(b^I_\mu - \lambda^I_\mu\right)^2 + \mathcal{L}^b_{\text{Maxwell}}. \quad (42)$$

The derivation from Eq. (38) to (42) is preserved in Appendix B with details. Interestingly, the field strength of gauge field a^I_μ is formally akin to a non-Abelian gauge field and its infinitesimal gauge transformation should be modified as

$$a^I_\mu \to a^I_\mu + \partial_\mu f^I - 4\pi C_{IJK}\left(g^J \lambda^K_\mu + \tfrac{1}{2} g^J \partial_\mu g^K\right);$$
$$b^I_\mu \to b^I_\mu + \partial_\mu g^I; \quad \lambda^I_\mu \to \lambda^I_\mu + \partial_\mu g^I. \quad (43)$$

C. Saddle-point approximation and internal gauge theory

If we assume the field b^I_μ has a weak fluctuation, we can apply the saddle-point approximation for b^I_μ. The saddle-point equation reads as

$$C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b^J_\nu - \lambda^J_\nu\right)\left(b^K_\lambda - \lambda^K_\lambda\right) + \left(b^I_\mu - \lambda^I_\mu\right) + \text{ higher-order terms} = 0, \quad (44)$$

clearly $b^I_\mu = \lambda^I_\mu$ is a stable saddle point. Since the λ field is a Lagrangian multiplier and b is a more-restricted $U(1)$ field, we should replace λ by b. At the level of this approximation, we can simplify the bulk effective action by

$$\mathcal{L}_{\text{eff}} = \frac{i\varepsilon^{\mu\nu\lambda}}{2\pi} b^I_\mu \partial_\nu a^I_\lambda + \frac{iC_{IJK}}{3}\varepsilon^{\mu\nu\lambda} b^I_\mu b^J_\nu b^K_\lambda, \quad (45)$$

and with the gauge redundancy given by

$$b^I_\mu \to b^I_\mu + \partial_\mu g^I,$$
$$a^I_\mu \to a^I_\mu + \partial_\mu f^I - 4\pi C_{IJK}\left(g^J b^K_\mu + \tfrac{1}{2} g^J \partial_\mu g^K\right). \quad (46)$$

We also have the global constraints

$$\oiint da/(2\pi) \in \mathbb{Z}, \quad \oiint db_I/(2\pi) \in \mathbb{Z}. \quad (47)$$

Similar to the (1+1)D case, there is a rigorous way to compute the quantization of coefficients C_{123} protected by global $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ symmetry. Let us add a coupling term to the external gauge field A^I:

$$\mathcal{L}_{\text{coupling}} = iA^I_\mu j^\mu_I = \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda} A^I_\mu \partial_\nu a^I_\lambda. \quad (48)$$

Again, A^I are Z_{N_I} symmetry twists, thus A^I must be a flat connection with $dA^I = 0$ and $\oint A^I = 2\pi n_I/N_I$. Similar to the (1+1)D case, since $\int dxdydt\mathcal{L}_{\text{coupling}}$ must be invariant under gauge transformation (46), C_{123} can not take arbitrary value, and a short calculation gives rise to exactly the same condition (40).

It turns out that the above gauge transformation corresponds to a *non-semisimple Lie algebra* of symmetry. We will discuss a generic class of such Lie algebra, called *the symmetric self-dual Lie algebra* in Appendix C. In Sec. VII, we will define a rigorous SPT internal gauge theory path integral, and we confirm that the GSD of our theory is unique on a closed manifold, GSD $= 1$, just like the SPT state. We will also derive the *SPT invariant* by coupling the internal gauge theory to semiclassical probed field A claimed in Ref. [20], which suggests that Eq. (41) indeed gives rise to N_{123} distinguishable SPT phases.

VI. A (3+1)D GENERALIZATION

The above trikink topological term can be generalized into higher dimensions as well, such as a quadkink topological action in (3+1)D:

$$\mathcal{L}_{\text{q-kink}} = \frac{1}{2}(\partial_\mu \theta^I)^2 + \frac{i}{4}C_{IJKL}\varepsilon^{\mu\nu\lambda\sigma}\partial_\mu\theta^I\partial_\nu\theta^J\partial_\lambda\theta^K\partial_\sigma\theta^L$$
$$\simeq \frac{1}{2}\left(\partial_\mu\theta^I_s + b^I_\mu\right)^2 + \frac{i}{4}C_{IJKL}\varepsilon^{\mu\nu\lambda\sigma}\left(\partial_\mu\theta^I_s + b^I_\mu\right) \times \left(\partial_\nu\theta^J_s + b^J_\nu\right)\left(\partial_\lambda\theta^K_s + b^K_\lambda\right)\left(\partial_\sigma\theta^L_s + b^L_\sigma\right). \quad (49)$$

The quantization condition on C_{IJKL} can be worked out in a similar way, and finally we obtain $C_{1234} = \frac{p_{\text{IV}}}{(2\pi)^3 3!}\frac{N_1N_2N_3N_4}{N_{1234}}$, where p_{IV} is an integer on $p_{\text{IV}} = 0,\ldots,N_{1234}-1$.

For example, in (3+1)D, we can use the following quartic-kink term to describe the so-called type-IV SPT state. Parallel to our previous derivation in Sec. V B, we can derive the SPT bulk dynamical action

$$\mathcal{L}_{\text{q-kink}} = \frac{i}{4\pi}\varepsilon^{\mu\nu\lambda\rho}\lambda^I_\mu\partial_\nu a^I_{\lambda\rho} - \frac{i}{4}C_{IJKL}\varepsilon^{\mu\nu\lambda\rho} \times \left[\lambda^I_\mu\lambda^J_\nu\lambda^K_\lambda\lambda^L_\rho - \left(b^I_\mu - \lambda^I_\mu\right)\left(b^J_\nu - \lambda^J_\nu\right) \times \left(b^K_\lambda - \lambda^K_\lambda\right)\left(b^L_\rho - \lambda^L_\rho\right)\right] + \frac{1}{2}\left(b^I_\mu - \lambda^I_\mu\right)^2 + \mathcal{L}^b_{\text{Maxwell}} \quad (50)$$

and its gauge transformation

$$a^I_{\mu\nu} \to a^I_{\mu\nu} + \partial_\mu f^I_\nu - \partial_\nu f^I_\mu + 12\pi C_{IJKL} g^J \lambda^K_\mu \lambda^L_\nu + \ldots;$$
$$b^I_\mu \to b^I_\mu + \partial_\mu g^I + \ldots; \quad \lambda^I_\mu \to \lambda^I_\mu + \partial_\mu g^I + \ldots. \quad (51)$$

Here, $\mathcal{L}^b_{\text{Maxwell}}$ terms contain nontopological Maxwell term. If we further apply the saddle-point approximation, we obtain

$$\mathcal{L}_{\text{eff}} = \frac{i\varepsilon^{\mu\nu\rho\sigma}}{4\pi} b^I_\mu \partial_\nu a^I_{\sigma\rho} - \frac{iC_{IJKL}}{4}\varepsilon^{\mu\nu\sigma\rho} b^I_\mu b^J_\nu b^K_\sigma b^L_\rho. \quad (52)$$

The corresponding infinitesimal gauge transformation (we only keep the leading-order term here and use ... to represent higher-order terms) of arbitrary functions f and g reads as

$$a^I_{\mu\nu} \to a^I_{\mu\nu} + \partial_\mu f^I_\nu - \partial_\nu f^I_\mu + 12\pi C_{IJKL} g^J b^K_\mu b^L_\nu + \ldots,$$
$$b^I_\mu \to b^I_\mu + \partial_\mu g^I + \ldots. \quad (53)$$

Here, g^I and b^I are globally defined, but f^I is not globally defined. The analogous global constraint can be derived:

$$\oiiint da/(2\pi) \in \mathbb{Z}, \quad \oiint db_I/(2\pi) \in \mathbb{Z}. \quad (54)$$

VII. PARTITION FUNCTION, GSD, AND SPT INVARIANTS COMPUTED FROM THE SPT INTERNAL GAUGE THEORY

Here, we will analytically show the path-integral definition of internal gauge field theory: Eq. (20) for (1+1)D, Eq. (45)

for (2+1)D, Eq. (52) for (3+1)D. In particular, we will show three key issues:

(i) Define the partition function $\mathbf{Z}$ using field theory path integral.

(ii) Derive the SPT invariants of semiclassical flat probed field theory in Ref. [20] by coupling the SPT internal gauge theory to probed fields A.

(iii) The internal gauge theory on any compact closed spatial manifold has a unique ground state, namely, GSD = 1. This means that the absolute value of the phase space volume ratio between the case with topological term and the case without topological term: $|\frac{\mathbf{Z}}{\mathbf{Z}(p=0)}| = 1$.

This procedure also applies to the SPT internal gauge field theory in any other dimensions. We know that the SPT state has no intrinsic topological order and the SPT's GSD = 1 on any compact closed spatial manifold. Therefore, this GSD computation serves as the consistency check that the internal field theory shows a gapped phase with nontrivial symmetry transformation: the internal gauge field theory realizes SPT state.

We emphasize that knowing the field theory action is *not enough* to fully understand the SPT field theory. We stress that defining the partition function $\mathbf{Z}$ using field theory path integral is *necessary* to fully understand the SPT field theory. In the following, we especially remark the global constraints of fields in order to define the SPT path integral. The partition function in terms of the path-integral form with a total space-time dimension d is

$$\mathbf{Z} = \int [Db][Da] \exp\left[\int \left(\frac{i}{2\pi} b^I \wedge da^I + \frac{i(-1)^{d-1} C_{IJK\dots}}{d} b^I \wedge b^J \wedge b^K \wedge \dots \right) \right], \tag{55}$$

here $I,J,K,\dots \in \{1,2,3,\dots,d\}$. Here, b is 1-form, a is $(d-2)$-form, and $f = da$ is $(d-1)$-form. In the presence of symmetry-twist semiclassical background 1-form gauge field A, we can write the partition function $\mathbf{Z}$ as

$$\begin{aligned}\mathbf{Z} &= \int [Db][Da] \exp\left[\int \left(\frac{i}{2\pi} (b^I - A^I) \wedge da^I + \frac{i(-1)^{d-1} C_{IJK\dots}}{d} b^I \wedge b^J \wedge b^K \wedge \dots \right) \right] \\ &= \int [Db][Df] \exp\left[\int \left(\frac{i}{2\pi} (b^I - A^I) \wedge f^I + \frac{i(-1)^{d-1} C_{IJK\dots}}{d} b^I \wedge b^J \wedge b^K \wedge \dots \right) \right],\end{aligned} \tag{56}$$

with the field strength of charges $f \equiv da$. Importantly, we view b^I and a^I all *dynamical internal gauge fields*, so they are involved in the path-integral measure.

Now, let us define this path integral properly. Let us impose the constraints for this field function in the path integral, based on the dual equivalent theory using the nonlinear σ model. We recall that the a is related to the *current density* j specified by the $U(1)$ or Z_N charge, where we have the total number of charges quantized:

$$\oiint *j = \oiint da/(2\pi) = \oiint f/(2\pi) \in \mathbb{Z}. \tag{57}$$

The current density $*j$ is a $(d-1)$-form, thus $\oiint$ of da represents the surface integral of a $(d-1)$-closed manifold, such as a 1-surface for (1+1)D space-time, 2-surface for (2+1)D space-time.

Now, we integrate over the field variable f for the partition function (56), which procedure analogous to the discrete Fourier summation yields a constraint

$$\sum_{n\in\mathbb{Z}} e^{i\varphi n} = \delta(\varphi \bmod 2\pi). \tag{58}$$

For $\int [Df] e^{\int \frac{i}{2\pi}(b^I - A^I)\wedge f^I}$ with $\oiint f/(2\pi) \in \mathbb{Z}$ or $\mathbb{Z}_N$, we obtain an analogous constraint on a 1D loop:

$$\oint (b^I - A^I) = 0 \bmod 2\pi \tag{59}$$

$$\Rightarrow \oint b^I = \oint A^I \bmod 2\pi = \frac{2\pi n_I}{N_I} \bmod 2\pi. \tag{60}$$

The first line constraint is true for both $U(1)$ charge and Z_N charge. The second line constraint (60) with $n_I \in \mathbb{Z}$ is an additional constraint if $\oiint f/(2\pi) \in \mathbb{Z}_{N_I}$ for our case of discrete Z_N charge for SPT state with Z_N symmetry. We can still view b field as a $U(1)$ connection but with a constraint from the Z_N symmetry-twist probed field A. This means that the internal gauge field b is subject to the global constraint from the semiclassical symmetry-twist probed field A. After integrating out the f, the partition function (56) subject to the global constraint (60) of the symmetry-twist fields A becomes

$$\begin{aligned}\mathbf{Z} &= \int [Db] \exp\left(\int \frac{i(-1)^{d-1} C_{I_1 I_2 \dots I_d}}{d} b^{I_1} \wedge b^{I_2} \wedge \dots \wedge b^{I_d} \right) \\ &= \exp\left[\int \left(\frac{i(-1)^{d-1} C_{I_1 I_2 \dots I_d}}{d} A^{I_1} \wedge A^{I_2} \wedge \dots \wedge A^{I_d} \right) \right].\end{aligned} \tag{61}$$

Thus, so far by using SPT internal gauge theory path integral, we have recovered the SPT invariant of Ref. [20] claimed in the item (ii). Next, without losing generality, let us take (2 + 1)D SPT as an example, with an explicit $C_{IJK} = \frac{1}{(2\pi)^2 2!} \frac{N_1 N_2 N_3\, p_{\text{III}}}{N_{123}}$. Let us do the explicit partition function calculation on the two topologies, a sphere and a torus, respectively, by comparing the nontrivial class $\mathbf{Z}$ to the trivial class $\mathbf{Z}(p_{\text{III}} = 0)$. For each calculation below we will fix a particular set of n_I for the global constraint (60).

The first topology. On a spatial sphere S^2 with a time loop S^1, there is only a noncontractible loop along the time direction. So, there is only a nonzero n_I for the global constraints in Eq. (60), and other n_J must be zeros. We have

$$\frac{\mathbf{Z}}{\mathbf{Z}(p_{\text{III}}=0)}=\frac{\exp\left(i\frac{2\pi p_{\text{III}}}{N_{123}}0\,0\,n_I\right)}{1}=1. \tag{62}$$

The second topology. On a space-time T^3 torus, without losing generality, let us assume A^1, A^2, A^3 along x, y, t directions have nontrivial global constraints with some generic n_1, n_2, and n_3. For example, analogous to Sec. V B's setup, we can assume $dx^\mu=dx$, $dx^\nu=dy$, $dx^\rho=dt$:

$$\frac{\mathbf{Z}}{\mathbf{Z}(p_{\text{III}}=0)}=\frac{\exp\left(ip_{\text{III}}\frac{2\pi\,n_1n_2n_3}{N_{123}}\right)}{1}=e^{ip_{\text{III}}\frac{2\pi\,n_1n_2n_3}{N_{123}}}. \tag{63}$$

Since for both on a sphere and on a torus, the absolute value of the above, $|\frac{\mathbf{Z}}{\mathbf{Z}(p_{\text{III}}=0)}|$, measures the GSD ratio between the nontrivial phases and the trivial insulator. Since the trivial insulator has GSD $=1$ here, all other phases have GSD $=1$, so the $p_{\text{III}}\neq 0$ phase is a generic SPT state.

We thus confirm that the path integral (55) with dynamical variables describes nontrivial type-III SPT states in (2+1)D. The same procedure can be generalized to other dimensions, such as Eq. (20) as SPT states in (1+1)D and Eq. (52) as SPT states in (3+1)D. The GSD for these theories defined by the partition function is 1. The procedure works in more general closed topology; we thus show the claim in the item (iii).

One further extension of our work is to study the *duality* [10] between SPT (which is nontopologically ordered) and dynamical topological gauge theory (which is topologically ordered). More precisely, we can start from the SPT internal gauge theory path integral of Eq. (56) and then *dynamically gauge* the theory to a dynamical topological gauge theory equivalent to the Dijkgraaf-Witten theory [54]. In Sec. IX, we will outline such a procedure using field theory path integral, and we will propose the *continuous dynamical topological gauge theory* dual to the *Dijkgraaf-Witten theory with a discrete gauge group*.

VIII. EDGE THEORY

The bulk effective field theory can also describe interesting edge physics. For the (1+1)D case, by integrating out the Lagrange multiplier fields a^I in Eq. (20), the corresponding edge theory takes a very simple form

$$\mathcal{L}^0_{\text{edge}}=\frac{i}{2}C_{IJ}\varphi^I\partial_0\varphi^J, \tag{64}$$

with scalar fields φ^I define the gauge transformation $\varphi^I\to\varphi^I-g^I$ to cancel the gauge transformation of $b^I\to b^I+dg^I$, which is nothing but a quantized topological term for a quantum mechanical system with degenerate ground states. Such a Berry phase implies the following quantization condition:

$$[\varphi^1,\varphi^2]=\frac{i}{C_{12}}=\frac{2\pi i N_{12}}{p_{\text{II}}N_1N_2}=\frac{2\pi i}{p_{\text{II}}N^{12}}. \tag{65}$$

Here, N^{12} is defined as the least common multiplier (lcm) where $N^{12}\equiv\text{lcm}(N_1,N_2)=N_1N_2/N_{12}$. Due to the compactification and the quantization constraint, shown in Appendix D 3, the symmetry generators are $S_{\varphi^1}=e^{iN_1\varphi^1\frac{p_{\text{II}}}{N_{12}}}$ and $S_{\varphi^2}=e^{iN_2\varphi^2\frac{p_{\text{II}}}{N_{12}}}$. It is straightforward to check that $S_{\varphi^I}(\int dt\mathcal{L}^0_{\text{edge}})S^{-1}_{\varphi^I}=(\int dt\mathcal{L}^0_{\text{edge}})+2\pi$ integer, so the partition function $\mathbf{Z}=\int D\varphi^1D\varphi^2e^{-\int dt\mathcal{L}^0_{\text{edge}}}$ is invariant under the symmetry transformation S_{φ^I}. We find that the symmetry is realized in a projective representation manner on the 0D edge because the symmetry generators do not commute:

$$S_{\varphi^1}S_{\varphi^2}=e^{-\frac{2\pi i p_{\text{II}}}{N_{12}}}S_{\varphi^2}S_{\varphi^1}. \tag{66}$$

Here, p_{II} is defined as a p_{II} (mod N_{12}) variable. If $\gcd(p_{\text{II}},N_{12})=1$, it is the $Z_{N_{12}}$ Heisenberg algebra and requires an N_{12}-dimensional representation for the symmetry generators S_{φ^1} and S_{φ^2}. This implies the (0+1)D edge mode of the ground state has a N_{12}-fold degeneracy, consistent with the edge mode physics analysis via the dimensional reduction approach in Ref. [46]. In general, even if $\gcd(p_{\text{II}},N_{12})\neq 1$, we have a generic $\frac{N_{12}}{\gcd(p_{\text{II}},N_{12})}$-dimensional representation for the symmetry generators, thus, the zero-mode degeneracy is

$$\text{GSD}=\frac{N_{12}}{\gcd(p_{\text{II}},N_{12})}. \tag{67}$$

For the (2+1)D bulk system with its (1+1)D edge theory, we have an analogous derivation as follows. Integrating out a_μ leads to the constraint

$$\varepsilon^{\mu\nu\lambda}\partial_\mu\lambda^I_\nu=0. \tag{68}$$

The constraint can be solved by requiring

$$\lambda^I_\nu=\partial_\nu\varphi^I. \tag{69}$$

We see that $\mathcal{L}_{\text{eff}}$ is nothing but a total derivative

$$\mathcal{L}_{\text{eff}}=\frac{i}{3}C_{IJK}\varepsilon^{\mu\nu\lambda}\partial_\mu\varphi^I\partial_\nu\varphi^J\partial_\lambda\varphi^K, \tag{70}$$

which actually describes a (1+1)D edge with effective action

$$\mathcal{L}^1_{\text{edge}}=\frac{i}{3}C_{IJK}\varepsilon^{\mu\nu}\varphi^I\partial_\mu\varphi^J\partial_\nu\varphi^K. \tag{71}$$

The higher-dimensional generalization is also straightforward, e.g., the type-IV SPT in (3+1)D can have a (2+1)D edge theory described by

$$\mathcal{L}^2_{\text{edge}}=\frac{i}{4}C_{IJKL}\varepsilon^{\mu\nu\rho}\varphi^I\partial_\mu\varphi^J\partial_\nu\varphi^K\partial_\rho\varphi^L. \tag{72}$$

The gapless nature of these boundary terms can be proved via dimension reduction to the (1+1)D case we discussed at the beginning of this section. Finally, we note that if we view φ^I as scaling dimension zero fields, $\mathcal{L}^1_{\text{edge}}$ and $\mathcal{L}^2_{\text{edge}}$ can be regarded as a fractionalized version of $O(3)$ and $O(4)$ topological theta terms. For future work, it would be of great interest to understand the underlying conformal field theory described by these fractionalized theta terms.

IX. TOPOLOGICAL FIELD THEORY FOR DIJKGRAAF-WITTEN LATTICE MODEL

In Sec. VII, we had established the SPT field theory by defining the SPT path integral. It is known that there exists a *duality* [10] between SPT (which is nontopologically ordered) and dynamical topological gauge theory (which is topologically ordered). More precisely, we can start from the SPT internal gauge theory path integral of Eq. (56) and then *dynamically gauge* the theory by coupling the SPT matter field to external probed fields A, and make the A dynamical gauge fields. This procedure of gauging SPT with a finite symmetry group in principle yields a dynamical topological gauge theory equivalent to the Dijkgraaf-Witten theory [54]. Here, we describe such a procedure using field theory path integral, and we propose some *continuous dynamical topological gauge theory* dual to the *Dijkgraaf-Witten theory with a discrete gauge group*.

Naively, one approach is starting from the path integral (56), if we promote the semiclassical probed field A to a dynamical field by including the path-integral measure $[DA]$, we obtain

$$\mathbf{Z} = \int [Db][Da][DA] \exp\left[\int \left(\frac{i}{2\pi}(b^I - A^I) \wedge da^I + \frac{iC_{IJK\dots}}{N} b^I \wedge b^J \wedge b^K \wedge \dots\right)\right]. \tag{73}$$

One can see that if A is still subject to some global constraint

$$\oint A^I \bmod 2\pi = \frac{2\pi n_I}{N_I} \bmod 2\pi, \tag{74}$$

but now $n_I \in \mathbb{Z}_{N_I}$ needs *not* to be fixed. The dynamical gauge theory of A would sum over all possible n_I. If we compute the GSD of this field theory on a space-time manifold, then we essentially reproduce the same calculation using the group cohomology cocycle while summing over all possible group elements $n_I \in \mathbb{Z}_{N_I}$. Equation (73) can produce the same physical observables such as GSD of Dijkgraaf-Witten theory. This suggests that Eq. (73) can be an equivalent description of Dijkgraaf-Witten theory.

Another approach to obtain the dynamical gauge theory is through the minimal coupling of the internal gauge field a to the external gauge field A, and then integrating out all the internal gauge fields a and b. We describe it below.

(2+1)D. Now, we are ready to discuss the bulk response theory. The external probe gauge field A_μ^I will couple to the internal charge current in a standard way:

$$\mathcal{L}_{\text{coupling}} = i A_\mu^I j_I^\mu = \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda} A_\mu^I \partial_\nu a_\lambda^I. \tag{75}$$

However, since A_μ^I is in the Higgs phase with Z_{N_I} charge condensation, we need to introduce a BF term [43] for response gauge field A_μ^I as well:

$$\frac{iN_I}{2\pi}\varepsilon^{\mu\nu\lambda} B_\mu^I \partial_\mu A_\nu^I. \tag{76}$$

Actually, such a term is crucial for maintaining the gauge invariance for the total action. (It is easy to check that $\mathcal{L}_{\text{coupling}}$ is not gauge invariant under the gauge transformation of a_μ^I and we need to shift B_μ^I to restore the gauge invariance.) Finally, by integrating out the internal gauge field a_μ^I and b_μ^I, we end up with an effective action $\frac{iN_I}{2\pi}\varepsilon^{\mu\nu\lambda} B_\mu^I \partial_\mu A_\nu^I + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} A_\mu^I A_\nu^J A_\lambda^K$:

$$\mathcal{L}_{\text{response}} = \frac{iN_I}{2\pi}\varepsilon^{\mu\nu\lambda} B_\mu^I \partial_\mu A_\nu^I + \frac{ip_{\text{III}} N_1 N_2 N_3}{(2\pi)^2 N_{123}}\varepsilon^{\mu\nu\lambda} A_\mu^1 A_\nu^2 A_\lambda^3. \tag{77}$$

If we view A_μ^I as background gauge fields describing the symmetry twists on the boundary, the above action is equivalent to the SPT invariants (37). However, if we view both A_μ^I and B_μ^I as dynamical gauge fields, the above action potentially describes non-Abelian Berry phases, although the original global symmetry is Abelian and all the gauge fields are Abelian in its own sectors. The whole Lie algebra becomes a non-Abelian feature due to the central extension (C4). It will be interesting to verify whether the fully dynamical topological gauge theory is equivalent to the Dijkgraaf-Witten gauge theory [54]. Our word of caution is that the non-semisimple Lie algebra detailed in Appendix C suggests a more conservative side of this claim. It is also likely that the method beyond the saddle-point approximation is required to capture the global constraints and missing pieces that we may omit in Eqs. (42) and (45).

(3+1)D. Similarly, we can discuss the bulk response theory. The external probe gauge field A_μ^I will couple to the internal charge current in a standard way:

$$\mathcal{L}_{\text{coupling}} = i A_\mu^I j_I^\mu = \frac{i}{4\pi}\varepsilon^{\mu\nu\rho\sigma} A_\mu^I \partial_\nu a_{\rho\sigma}^I. \tag{78}$$

Similar to the (2+1)D case, we also need to introduce a BF term to describe the Z_{N_I} external gauge field in (3+1)D:

$$\frac{iN_I}{4\pi}\varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^I \partial_\rho A_\sigma^I. \tag{79}$$

By integrating out the internal gauge fields $a_{\mu\nu}^I$ and λ_μ^I, we end up with an effective action

$$\mathcal{L}_{\text{response}} = \frac{iN_I}{4\pi}\varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^I \partial_\rho A_\sigma^I - \frac{i}{4} C_{IJKL}\varepsilon^{\mu\nu\rho\sigma} A_\mu^I A_\nu^J A_\rho^K A_\sigma^L + \dots. \tag{80}$$

We warn the reader that there is a potential danger to view Eq. (80) as the dynamical topological gauge theory, as one needs to further confirm the physical properties such as topological GSD and braiding statistics must match with the (3+1)D Dijkgraaf-Witten topological gauge theory [54] computed in Ref. [55]. We will leave the study of topological gauge theories for future work. The minimum claim of our approach is that viewing the B field as a Lagrangian multiplier constrains the flatness of A with $dA = 0$, we essentially derive the SPT invariant in terms of the semiclassical probed field A agreed with [20]. This confirms our multikink topological term and vortex condensation mechanism do generate nontrivial SPT states.

X. CONCLUSIONS AND DISCUSSIONS

In conclusion, we have discussed the multikink topological term and vortex condensation mechanism for bosonic Abelian SPT states that cannot be described by Abelian Chern-Simons/BF actions. We have pointed out that nontrivial SPT states can be viewed as certain Higgs phases via defects proliferating in various nontrivial ways. Thus, the formalism and concepts developed in this paper can provide further insights for understanding the universal mechanism for bosonic SPT states, especially for those protected by non-Abelian symmetry.

Moreover, the general concept of "hydrodynamical approach" is applicable for fermion systems as well, if the spin manifold is taken into account. Just like we can use the spin Chern-Simons theory to describe certain special Abelian fermionic SPT states [43], the bulk effective actions beyond Chern-Simons/BF theory proposed here should also have their corresponding "spin" version that can describe new classes of fermion Abelian SPT states.

The field theory based on the saddle-point approximation (detailed in Appendix C) *may or may not* fully capture the topological properties of the gapped SPT state. However, in Sec. VII, we show that at least for the level-1 *trivial* class of our theory, it has GSD $= 1$ on a compact closed manifold just like the SPT state. Moreover, so far as the SPT invariant is concerned, we confirm that the bulk SPT response theory induced by the multikink topological term does reproduce the desired SPT invariant. Even though our theory exhibits the so-called symmetric self-dual non-semisimple Lie algebra [56], however, due to the *extra set of global constraints* [Eqs. (57) and (60)], our theory is not equivalent to the usual gauge theory with non-semisimple Lie algebra studied in the high-energy literature (see Appendix C). We believe our theory is *unitary* and has *finite ground-state degeneracy on a closed manifold.*

Another important research direction is to study the phase transition between superfluids and SPT states, analogous to the usual case where we have superfluid and insulator phase transition. We will leave these further developments for future work.

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APPENDIX A: DISORDER A SUPERFLUID STATE INTO A MOTT INSULATOR OR AN SPT STATE

To guide the readers understanding our formalism, here we briefly review this approach using field theory (see the pioneering work [47–49] and Refs. [50,51] for a field theory approach). We plan to study SPT states for a discrete Abelian symmetry group. First, we will embed our discrete Abelian symmetry group into the symmetry group of several $U(1)$ symmetries. Instead of starting with a discrete-symmetry-breaking state, we will start with a symmetry-breaking state that breaks several $U(1)$ symmetries. When we restore the $U(1)$ symmetries, we also restore our real discrete symmetry.

The superfluid state [the $U(1)$ symmetry-breaking state] in any d-space-time dimension is described by a bosonic $U(1)$ quantum phase kinetic term, whose partition function $\mathbf{Z}$ is

$$\mathbf{Z} = \int [D\theta] \exp\left(-\int d^d x\, \frac{\chi}{2}(\partial_\mu\theta_{\rm s} + \partial_\mu\theta_{\rm v})^2\right) \quad \text{(A1)}$$

with a smooth piece $\theta_{\rm s}$ and a singular piece $\theta_{\rm v}$ for the bosonic phase, and the superfluid compressibility χ. We stress that the $\theta_{\rm v}$ is essential to capture the vortex core. We can introduce an auxiliary field j^μ and implement the Hubbard-Stratonovich technique [50]

$$\mathbf{Z} = \int [D\theta][Dj^\mu] \times \exp\left(-\int d^d x\, \frac{1}{2\chi}(j_I^\mu)^2 - i j^\mu(\partial_\mu\theta_{\rm s} + \partial_\mu\theta_{\rm v})\right). \quad \text{(A2)}$$

By integrating out the smooth part $\int[D\theta_{\rm s}]$, we obtain a constraint $\delta(\partial_\mu j^\mu)$ in the measure of the path integral. We can define a generic form

$$j^\mu = \frac{1}{2\pi(d-2)!}\epsilon^{\mu\mu_2\ldots\mu_d}\partial_{\mu_2}a_{\mu_3\ldots\mu_d},$$

with an antisymmetric a and the total space-time dimension d to satisfy this constraint. More conveniently, in the differential form notation, the constraint is $d(*j) = 0$ and the resolution is $j = \frac{1}{2\pi}(*da)$ with $*$ the Hodge star, with an a gauge field in real values. To disorder the superfluid, we have to make the θ angle strongly fluctuate, namely, we should take the $\chi < \chi_c$ or $\chi \to 0$ limit [51] to achieve large $(\partial_\mu\theta)^2$. We will, however, drop the Maxwell term due to its irrelevancy in the renormalization group (RG) sense. The partition function becomes $\mathbf{Z} = \int[D\theta_{\rm v}][Da]\exp[i\int\frac{1}{2\pi}a\wedge(d^2\theta_{\rm v})]$. Hereafter, we compensate the dropped $\pm$ sign by redefining the fields. Even though naively $d^2 = 0$, due to the singularity core of $\theta_{\rm v}$, the $(d^2\theta_{\rm v})$ can be nonzero. Thus, $(d^2\theta_{\rm v})$ describes the vortex core density and the vortex current, which we shall denote $(d^2\theta_{\rm v})/(2\pi) = *j_{\rm vortex}$. In addition, the action has a symmetry of $a \to a + d\xi$ or, more explicitly, $a_{\mu_3\ldots\mu_d} \to a_{\mu_3\ldots\mu_d} + \partial_{[\mu_3}\xi_{\mu_4\ldots\mu_d]}$. By Noether theorem, this symmetry leads to the conservation of the vortex current: the continuity equation $d * j_{\rm vortex} = 0$, this implies that

$$*j_{\rm vortex} \equiv (d^2\theta_{\rm v})/(2\pi) = db/(2\pi)$$

for some gauge field b. We can thus define the singular part of bosonic phase $d\theta_{\rm v} = b$ as a 1-form gauge field to describe the vortex core, so

$$d\theta_{\rm s} + d\theta_{\rm v} = d\theta_{\rm s} + b. \quad \text{(A3)}$$

The partition function in the disordered state away from the superfluid now becomes that of an insulator state $\mathbf{Z} = \int[Db][Da]\exp(\frac{i}{2\pi}\int b\wedge da)$ with a topological BF action. More explicitly, the path-integral formalism shows

$$\mathbf{Z} = \int[Db][Da]\exp\left(i\int\frac{d^d x}{2\pi(d-2)!}\epsilon^{\mu\mu_2\ldots\mu_d}b_\mu\partial_{\mu_2}a_{\mu_3\ldots\mu_d}\right). \quad \text{(A4)}$$

The Hamiltonian of Eq. (A4) is zero, which describes an insulator with an energy gap separating the ground state from excitations. It has no intrinsic topological order in the sense that it has a unique ground-state degeneracy (GSD, see Ref. [57], this action is a level-1 BF theory with GSD $= 1$). This is known as the mechanism of disordering the charge, while condensing the vortices generates a trivial insulator: a Mott insulator without SPT order.

APPENDIX B: DERIVATION OF THE DYNAMICAL EFFECTIVE BULK ACTION OF SPT

In the following, we list some details for deriving the internal field theory of SPT in Sec. V B, specifically for type-III (2+1)D SPT with $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ symmetry. We note that up to a total derivative, the trikink action (38) can be simplified as

$$\mathcal{L}_{\text{trikink}} = \frac{1}{2}\left(\partial_\mu \theta_{\text{s}}^I + b_\mu^I\right)^2 + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda}\left[-3\theta_{\text{s}}^I\partial_\mu\left(b_\nu^J b_\lambda^K\right) - 3\theta_{\text{s}}^I\partial_\mu\left(\partial_\nu\theta_{\text{s}}^J b_\lambda^K\right) + b_\mu^I b_\nu^J b_\lambda^K\right] + \mathcal{L}^b_{\text{Maxwell}}. \tag{B1}$$

Again, we can introduce Hubbard-Stratonovich fields j_I^μ to decouple the quadratic term as

$$\mathcal{L}_{\text{trikink}} = \frac{1}{2}\left(j_I^\mu\right)^2 - i\theta_{\text{s}}^I\partial_\mu j_I^\mu + i b_\mu^I j_I^\mu + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda}\left[-3\theta_{\text{s}}^I\partial_\mu\left(b_\nu^J b_\lambda^K\right) - 3\theta_{\text{s}}^I\partial_\mu\left(\partial_\nu\theta_{\text{s}}^J b_\lambda^K\right) + b_\mu^I b_\nu^J b_\lambda^K\right] + \mathcal{L}^b_{\text{Maxwell}}. \tag{B2}$$

We further introduce Lagrangian multiplier fields ξ_I^μ and λ_μ^I to decouple the $-C_{IJK}\varepsilon^{\mu\nu\lambda}\theta^I\partial_\mu(\partial_\nu\theta^J b_\lambda^K)$ term. We have

$$\begin{aligned}
\mathcal{L}_{\text{trikink}} &= \frac{1}{2}\left(j_I^\mu\right)^2 - i\theta_{\text{s}}^I\partial_\mu j_I^\mu + i b_\mu^I j_I^\mu + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda}\left[-3\theta_{\text{s}}^I\partial_\mu\left(b_\nu^J b_\lambda^K\right) + b_\mu^I b_\nu^J b_\lambda^K\right] - i\theta_{\text{s}}^I\partial_\mu\xi_I^\mu \\
&\quad + i\lambda_\mu^I\left(\xi_I^\mu - C_{IJK}\varepsilon^{\mu\nu\lambda}\partial_\nu\theta_{\text{s}}^J b_\lambda^K\right) + \mathcal{L}^b_{\text{Maxwell}} \\
&= \frac{1}{2}\left(j_I^\mu\right)^2 - i\theta_{\text{s}}^I\partial_\mu\left(j_I^\mu + \xi_I^\mu + C_{IJK}\varepsilon^{\mu\nu\lambda} b_\nu^J b_\lambda^K\right) + i b_\mu^I j_I^\mu + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K \\
&\quad + i\lambda_\mu^I\left(\xi_I^\mu - C_{IJK}\varepsilon^{\mu\nu\lambda}\partial_\nu\theta_{\text{s}}^J b_\lambda^K\right) + \mathcal{L}^b_{\text{Maxwell}} \\
&= \frac{1}{2}\left(j_I^\mu\right)^2 - i\theta_{\text{s}}^I\partial_\mu\left[j_I^\mu + \xi_I^\mu + C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K\right)\right] + i b_\mu^I j_I^\mu + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K + i\lambda_\mu^I\xi_I^\mu + \mathcal{L}^b_{\text{Maxwell}}.
\end{aligned} \tag{B3}$$

Integrating out the θ_{s}^I fields results in a constraint $\partial_\mu[j_I^\mu + \xi_I^\mu + C_{IJK}\varepsilon^{\mu\nu\lambda}(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K)] = 0$. From this constraint, we can write the conserved $j_I^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^I - \xi_I^\mu - C_{IJK}\varepsilon^{\mu\nu\lambda}(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K)$. Finally, we obtain

$$\begin{aligned}
\mathcal{L}_{\text{trikink}} &= \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda} b_\mu^I\partial_\nu a_\lambda^I - \frac{2i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K + i C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I\lambda_\nu^J b_\lambda^K + \frac{1}{2}\left[\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^I - C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K\right)\right]^2 \\
&\quad + \frac{1}{2}\left(\xi_I^\mu\right)^2 + \left[i\left(\lambda_\mu^I - b_\mu^I\right) - \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^I + C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K\right)\right]\xi_I^\mu + \mathcal{L}^b_{\text{Maxwell}}.
\end{aligned} \tag{B4}$$

Integrating out the ξ_I^μ fields, we end up with

$$\begin{aligned}
\mathcal{L}_{\text{trikink}} &= \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda} b_\mu^I\partial_\nu a_\lambda^I - \frac{2i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K + i C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I\lambda_\nu^J b_\lambda^K + \frac{1}{2}\left[\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^I - C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K\right)\right]^2 \\
&\quad - \frac{1}{2}\left[i\left(\lambda_\mu^I - b_\mu^I\right) - \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^I + C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K\right)\right]^2 + \mathcal{L}^b_{\text{Maxwell}} \\
&= \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda} b_\mu^I\partial_\nu a_\lambda^I - \frac{2i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K + i C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I\lambda_\nu^J b_\lambda^K + \frac{1}{2}\left(\lambda_\mu^I - b_\mu^I\right)^2 \\
&\quad + i\left(\lambda_\mu^I - b_\mu^I\right)\left[\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^I - C_{IJK}\varepsilon^{\mu\nu\lambda}\left(b_\nu^J b_\lambda^K - \lambda_\nu^J b_\lambda^K\right)\right] + \mathcal{L}^b_{\text{Maxwell}} \\
&= \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda}\lambda_\mu^I\partial_\nu a_\lambda^I + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K - i C_{IJK}\varepsilon^{\mu\nu\lambda} b_\mu^I\lambda_\nu^J b_\lambda^K + i C_{IJK}\varepsilon^{\mu\nu\lambda}\lambda_\mu^I\lambda_\nu^J b_\lambda^K + \frac{1}{2}\left(\lambda_\mu^I - b_\mu^I\right)^2 + \mathcal{L}^b_{\text{Maxwell}} \\
&= \frac{i}{2\pi}\varepsilon^{\mu\nu\lambda}\lambda_\mu^I\partial_\nu a_\lambda^I + \frac{i}{3} C_{IJK}\varepsilon^{\mu\nu\lambda}\left(\lambda_\mu^I\lambda_\nu^J\lambda_\lambda^K + \left(b_\mu^I - \lambda_\mu^I\right)\left(b_\nu^J - \lambda_\nu^J\right)\left(b_\lambda^K - \lambda_\lambda^K\right)\right) + \frac{1}{2}\left(b_\mu^I - \lambda_\mu^I\right)^2 + \mathcal{L}^b_{\text{Maxwell}}.
\end{aligned} \tag{B5}$$

APPENDIX C: COMMENTS ON NON-SEMISIMPLE LIE ALGEBRA AND TOPOLOGICAL FIELD THEORY

In Sec. V C, we learn that the saddle-point approximation leads us to an intrinsic field theory and a bulk dynamical theory with non-semisimple Lie algebra. If we write the gauge connection in terms of its gauge field components and its generators,

$$\tilde{a}_\mu^\alpha T^\alpha \equiv b_\mu^I X_I + a_\mu^I H_I^*, \tag{C1}$$

$$\left(\tilde{a}_\mu^1 T^1, \tilde{a}_\mu^2 T^2, \tilde{a}_\mu^3 T^3\right) = \left(b_\mu^1 X_1, b_\mu^2 X_2, b_\mu^3 X_3\right), \tag{C2}$$

$$\left(\tilde{a}_\mu^4 T^4, \tilde{a}_\mu^5 T^5, \tilde{a}_\mu^6 T^6\right) = \left(a_\mu^1 H_1^*, a_\mu^2 H_2^*, a_\mu^3 H_3^*\right). \tag{C3}$$

Here, $\alpha = 1, \ldots, 6$ and $I = 1, \ldots, 3$.

The corresponding generators H^I and X_I satisfy

$$[H_I^*, H_J^*] = [H_I^*, X_J] = 0; \quad [X_I, X_J] = C_{IJK} H_K^*, \tag{C4}$$

where C_{IJK} serves as the structure constant now. The full Lie algebra consists of an Abelian Lie algebra $\mathcal{X}(X)$ with a central extension by another Abelian Lie algebra $\mathcal{H}^*(H^*)$. Here, $\mathcal{X}(X)$ contains the set of generators X_I, and $\mathcal{H}^*(H^*)$ contains the set of generators H_I^*.

For the specific case of level-1 Chern-Simons theory in Sec. VII, we are able to show the GSD $= 1$. However, for the general level-k case, the structure of the phase space volume is changed (see for example, Appendix D). This appendix is meant to provide some word of caution to prevent us from making a stronger claim that the Chern-Simons theory with this non-semisimple Lie algebra is exactly the dynamical Dijkgraaf-Witten field theory we look for, unless we carefully specify the global constraints analogous to Sec. VII.

The particular type of the non-semisimple Lie algebra we derived in Eq. (C4) is in the class of *symmetric self-dual Lie algebra* [56]. Even if the Killing form κ_{ab} degenerates, we can replace the κ_{ab} by an invariant nondegenerate symmetric bilinear form $\mathcal{K}^G_{a\alpha'}$ if it satisfies the criteria below.

For a Lie algebra given by $[T_a,T_b] = f_{ab}{}^c T_c$, the structure constant f_{ab} satisfies the Jacobi identity $f_{bc}{}^d f_{ad}{}^e + f_{ca}{}^d f_{bd}{}^e + f_{ab}{}^d f_{cd}{}^e = 0$. The Killing form as a bilinear matrix in the adjoint representation can be determined from the structure constant

$$\kappa_{ab} = \kappa(T_a,T_b) = -\mathrm{Tr}(T_a,T_b) = -\sum_{\alpha,\beta} f_{a\alpha}{}^{\beta} f_{b\beta}{}^{\alpha}. \quad \text{(C5)}$$

The Killing form is called degenerate, if there exists a nonzero generator T' such that $\kappa(T',T) = 0$ for any T.

In the Euclidean space-time, we have a Chern-Simons theory

$$L = \frac{i}{4\pi}\epsilon^{\mu\nu\rho}\mathcal{K}^G_{a\alpha'}\bigg(\mathcal{A}^a_\mu(x)\partial_\nu\mathcal{A}^{\alpha'}_\rho(x) + \frac{1}{3} f_{bc}{}^{a}\mathcal{A}^{\alpha'}_\mu(x)\mathcal{A}^b_\nu(x)\mathcal{A}^c_\rho(x)\bigg). \quad \text{(C6)}$$

Even if the Killing form is degenerate, as long as this $(\mathcal{K}^G)_{IJ}$ can be found, the $(\mathcal{K}^G)_{IJ}$ can replace the degenerate Killing form to make sense of the Chern-Simons theory [Eq. (C6)] with the symmetric self-dual Lie algebra.

The $(\mathcal{K}^G)_{IJ}$ is a symmetric nondegenerate invariant bilinear form, constrained by

$$f_{a\ell}{}^i(\mathcal{K}^G)_{bi} + f_{ab}{}^i(\mathcal{K}^G)_{\ell i} = 0. \quad \text{(C7)}$$

The finite and infinitesimal gauge transformations are

$$\mathcal{A}_\mu \to \mathcal{A}^U_\mu = U^{-1}(\mathcal{A}_\mu + \partial_\mu)U = e^{-\alpha^a T_a}(\mathcal{A}_\mu + \partial_\mu)e^{\alpha^a T_a},$$
$$\mathcal{A}^a_\mu(x) \to \big[\mathcal{A}^a_\mu(x) + f_{bc}{}^a\mathcal{A}^b_\mu(x)\alpha^c(x) + \partial_\mu\alpha^a(x)\big]. \quad \text{(C8)}$$

The Lie algebra we find out in Sec. V C is a subalgebra of the most generic symmetric self-dual Lie algebra [56]

$$[X_a,X_b] = i f^{(X)c}_{ab} X + i f^{(H^*)\alpha}_{ab} H^*_\alpha, \quad \text{(C9)}$$

$$[H_a,H_b] = i f^{(H)c}_{ab} H_c, \quad \text{(C10)}$$

$$[X_a,H_b] = i f^{(xH)c}_{ab} X_c, \quad \text{(C11)}$$

$$[H_a,H^*_b] = -i f^{(H)b}_{ac} H^*_c, \quad \text{(C12)}$$

$$[X_a,H^*_\alpha] = [H^*_\alpha,H^*_\beta] = 0. \quad \text{(C13)}$$

Notice that the subalgebra spanned by $\mathcal{X}(X)$ and $\mathcal{H}^*(H^*)$ is the Abelian extension of $\mathcal{X}(X)$ by $\mathcal{H}^*(H^*)$. The full algebra is the semidirect product of $\mathcal{H}(H)$ by this Abelian extension. The particular non-semisimple symmetric self-dual Lie algebra in Eq. (C4) is nilpotent, non-Abelian, nonreductive, and solvable. The corresponding Lie group is noncompact.

Our theory in Sec. VII is a special case such that the GSD is still 1 which can describe the gapped SPT. Due to the noncompact Lie group, however, it is likely the generic gauge theory of symmetric self-dual Lie algebra can capture an infinite degenerate gapless phase instead of a phase with finite topological degenerate ground states. The concern of (non)unitarity has been investigated, for example, in Ref. [58].

We believe that the generic difference between our SPT path integral and the usual non-semisimple Lie algebra gauge theory is the set of global constraints: Eqs. (57) and (60). For our SPT path integral, the global constraints lead to the finite ground-state degeneracy; for the usual non-semisimple Lie algebra gauge theory, the ground-state degeneracy can be infinite. It is possible a more generic theory can describe a state close to the potential gapless phase transition between superfluids, symmetry-breaking states, and SPT/topologically ordered states. We will leave the further investigation open for future work.

APPENDIX D: COUNTING THE DEGENERATE ZERO MODES

1. GSD for a gapped system with a (0+1)D topological term

We first review a simple ground-state degeneracy (GSD) calculation by counting the zero mode for a (0+1)D system. Namely, we will count the volume of the phase space volume

$$\text{GSD} = \text{the volume of the phase space}, \quad \text{(D1)}$$

up to some normalization factor.

The first system we consider is described by a Berry phase term $\mathcal{L}^0 = \dot{X}P$. On one hand, in the path-integral formalism, we have a partition function

$$\mathbf{Z} = \int [DX][DP]\exp\bigg[i\,k\int \dot{X}P\bigg], \quad \text{(D2)}$$

$\dot{X} = \partial_0 X$ is the time derivative X.

On the other hand, in the quantum operator formalism, we have the commutator $[X,\frac{\partial\mathcal{L}^0}{\partial\dot{X}}] = i$:

$$[X,P] = i\frac{1}{k}. \quad \text{(D3)}$$

X and P are some matrix operators acting on the (0+1)D space. Here, we will consider a compact phase space, so that the phase space volume is finite. In particular, without losing generality, the identification we assume is $X \sim X + 2\pi$ and $P \sim P + 1$. Since the Hamiltonian is essentially $H = \dot{X}P - \mathcal{L}^0 = 0$, the system seems to be trivial without kinetic terms or potential terms. However, there can be degenerated ground states. All ground states Ψ satisfy $H\Psi = 0$. But, these Ψ may not be all independent. To count the GSD thus to count the independent degree of freedom, we can construct a generic ground state Ψ in terms of the function of X if we choose X

as the basis:

$$\Psi(X) = \sum_{n\in\mathbb{Z}} c_n e^{inX}. \tag{D4}$$

The form is obtained by satisfying the constraint $\Psi(X) = \Psi(X + 2\pi)$ as $X \sim X + 2\pi$. The 2π shift in the exponent will not affect the form of the $\Psi(X)$ function. On the other hand, by doing the Fourier transformation, we can transform the X basis to the P basis via $\tilde{\Psi}(P) = \int e^{ikPX}\Psi(X)\, dP$. Up to some normalization factor, this yields

$$\tilde{\Psi}(P) = \sum_{n\in\mathbb{Z}} c_n \delta(kP + n). \tag{D5}$$

Meanwhile, the form satisfies the constraint $\tilde{\Psi}(P) = \tilde{\Psi}(P + 1)$ as $P \sim P + 1$. This implies that $c_n\delta(kP + n) = c_{n-k}\delta[kP + k + (n - k)]$. This means that

$$c_n = c_{n-k} \tag{D6}$$

with $k \in \mathbb{Z}$. The volume of the phase space is $|k|$. We have $|k|$ independent degenerate ground states determined by k independent coefficients, thus GSD $= |k|$. The strategy for this example is basically the same as the approach in Ref. [57].

2. Compactification and quantization

For the later convenience, we now set up a relation between the constraint of compactification and quantization using an angular rotational system as an example, with the angle Θ and the angular momentum L. First, Θ is compactified and identified via

$$\Theta \sim \Theta + 2\pi.$$

The compactness of Θ leads to the quantization or the discretization of its dual variable L, in order to have $e^{i\Theta L}$ stay invariant as $\Theta \to \Theta + 2\pi$. That means, the quantization is

$$\Delta L = 1.$$

On the other hand, if we consider the angle Θ is also discretized as rotor angle with

$$\Delta\Theta = \frac{2\pi}{N},$$

then this quantization must come from the compactification of L, with

$$L \sim L + N.$$

In short, due to the constraint of compactification and quantization, we have a set of relations:

$$\Theta \sim \Theta + 2\pi \Leftrightarrow \Delta L = 1, \tag{D7}$$

$$L \sim L + N \Leftrightarrow \Delta\Theta = \frac{2\pi}{N}. \tag{D8}$$

The volume of the phase space is N. It can be counted in Θ space as well as in L space as $(2\pi/\Delta\Theta) = (N/\Delta L) = N$.

3. GSD for a gapped system at the (0+1)D edge of (1+1)D SPTs

After the previous simple first part of calculation, in the second part, we consider the (0+1)D edge of (1+1)D SPT. The system we consider is described by a Berry phase term in the partition function for the path-integral formalism:

$$\mathbf{Z} = \int [D\varphi^1][D\varphi^2] \exp\left[\frac{i}{2}\int C_{IJ}\varphi^I \partial_0 \varphi^J\right], \tag{D9}$$

with $C_{12} = \frac{p_{\text{II}} N_1 N_2}{2\pi N_{12}}$.

On the other hand, for the canonical quantization with quantum operators, the commutation relation satisfies

$$[\varphi^1,\varphi^2] = \frac{i}{C_{12}} = \frac{2\pi i N_{12}}{p_{\text{II}} N_1 N_2}. \tag{D10}$$

To well define the denominator for the trivial class $p_{\text{II}} = 0$, the trivial class's p_{II} is identified as $p_{\text{II}} = N_{12}$. We may define the conjugate variables as $[\varphi^1, P_{\varphi^1}] = [\varphi^1, C_{12}\varphi^2] = i$ and $[\varphi^2, P_{\varphi^2}] = [\varphi^2, -C_{12}\varphi^1] = i$.

The first approach. The compactified size of φ^1 and φ^2 is no larger than 2π:

$$\varphi^1 \sim \varphi^1 + 2\pi, \quad \varphi^2 \sim \varphi^2 + 2\pi. \tag{D11}$$

The quantization and the discreteness of this rotor clock is no smaller than

$$\Delta\varphi^1 = \frac{2\pi}{N_1}, \quad \Delta\varphi^2 = \frac{2\pi}{N_2}. \tag{D12}$$

Due to the conjugation relation, following the logic of Eq. (D7), the compactness in Eq. (D11) of $\varphi^1 \sim \varphi^1 + 2\pi$ leads to $\Delta P_{\varphi^1} = C_{12}\Delta\varphi^2 = 1$. Similarly, the compactness of φ^2 leads to $\Delta P_{\varphi^2} = C_{12}\Delta\varphi^1 = 1$. Namely, the quantization can be

$$\Delta\varphi^1 = \frac{2\pi N_{12}}{p_{\text{II}} N_1 N_2}, \quad \Delta\varphi^2 = \frac{2\pi N_{12}}{p_{\text{II}} N_1 N_2}. \tag{D13}$$

On the other hand, following the logic of Eq. (D8), the quantization Eq. (D12) implies the possible compactness size of P_{φ^1} and P_{φ^2} as $P_{\varphi^1} \sim P_{\varphi^1} + N_1$ and $P_{\varphi^2} \sim P_{\varphi^2} + N_2$, namely,

$$\varphi^1 \sim \varphi^1 + \frac{2\pi N_{12}}{p_{\text{II}} N_1}, \quad \varphi^2 \sim \varphi^2 + \frac{2\pi N_{12}}{p_{\text{II}} N_2}. \tag{D14}$$

To construct the refined phase space, we need to take the largest quantization size in the discretized lattice among Eqs. (D12) and (D13), and the smallest compactification size among Eqs. (D11) and (D14). This means that we will require Eqs. (D12) and (D14):

$$\Delta\varphi^1 = \frac{2\pi}{N_1}, \quad \Delta\varphi^2 = \frac{2\pi}{N_2},$$

$$\varphi^1 \sim \varphi^1 + \frac{2\pi N_{12}}{p_{\text{II}} N_1}, \quad \varphi^2 \sim \varphi^2 + \frac{2\pi N_{12}}{p_{\text{II}} N_2}.$$

Therefore, the phase space volume counting from both φ^1 space and its dual space φ^2 is both $\frac{2\pi N_{12}}{p_{\text{II}} N_1}/\Delta\varphi^1 = \frac{2\pi N_{12}}{p_{\text{II}} N_2}/\Delta\varphi^2 = \frac{N_{12}}{p_{\text{II}}}$. However, $\frac{N_{12}}{p_{\text{II}}}$ may not be integer in general. We will need to multiply a minimal factor on the size of the phase space until it becomes an integer. This means that in general we will multiply it by the minimal phase factor $\frac{p_{\text{II}}}{\gcd(p_{\text{II}}, N_{12})}$ until we have an integer size of phase volume: $\frac{N_{12}}{\gcd(p_{\text{II}},N_{12})} = \frac{N_{12}}{p_{\text{II}}} \frac{p_{\text{II}}}{\gcd(p_{\text{II}},N_{12})}$. The phase space volume counting from both φ^1 space and φ^2 space results in

$$\text{GSD} = \frac{N_{12}}{\gcd(p_{\text{II}},N_{12})}. \tag{D15}$$

It is straightforward to construct the functional $\Psi(\varphi^1)$ and its Fourier transformation $\tilde{\Psi}(\varphi^2)$ with a number of $\frac{N_{12}}{\gcd(p_{\text{II}},N_{12})}$ independent coefficients as in Appendix D 1.

The second approach. We can verify this GSD result from an alternative viewpoint by considering the projective representation of the symmetry group $G = Z_{N_1} \times Z_{N_2}$: We propose the symmetry generators as

$$S_{\varphi^1} = e^{i N_1 \varphi^1 \frac{p_{\text{II}}}{N_{12}}}, \tag{D16}$$

$$S_{\varphi^2} = e^{i N_2 \varphi^2 \frac{p_{\text{II}}}{N_{12}}}, \tag{D17}$$

in order to have the symmetry generators invariant under the shift over a full compactification size $\varphi^1 \to \varphi^1 + \frac{2\pi N_{12}}{p_{\text{II}} N_1}$ and $\varphi^2 \to \varphi^2 + \frac{2\pi N_{12}}{p_{\text{II}} N_2}$. Namely, our choice is guaranteed to satisfy $S_{\varphi^1}(\varphi^1) = S_{\varphi^1}(\varphi^1 + \frac{2\pi N_{12}}{p_{\text{II}} N_1})$ and $S_{\varphi^2}(\varphi^2) = S_{\varphi^2}(\varphi^2 + \frac{2\pi N_{12}}{p_{\text{II}} N_2})$. Our choice also obeys the Z_{N_1} and Z_{N_2} symmetries: $(S_{\varphi^1})^{N_1} = (S_{\varphi^2})^{N_2} = 1$ when we impose the discretization as Eq. (D12).

One can check that $S_{\varphi^I}(\int dt \mathcal{L}^0_{\text{edge}}) S^{-1}_{\varphi^I} = (\int dt \mathcal{L}^0_{\text{edge}}) + 2\pi$ integer, so the partition function $\mathbf{Z} = \int D\varphi^1 D\varphi^2 e^{-\int dt \mathcal{L}^0_{\text{edge}}}$ is invariant under the symmetry transformation S_{φ^I}. To calculate the ground-state degeneracy at the (0+1)D edge, we can study the projective representation of the symmetry group acting on the zero-energy modes, we find

$$S_{\varphi^1} S_{\varphi^2} = e^{-\frac{2\pi i p_{\text{II}}}{N_{12}}} S_{\varphi^2} S_{\varphi^1}. \tag{D18}$$

If $p_{\text{II}} = 0$, the symmetry generators are commutative, so it can be written as a linear representation, and the GSD = 1. In general, the symmetry generators are not commutative, so it shall be written as a higher-dimensional matrix representation. If $\gcd(p_{\text{II}},N_{12}) = 1$, it is the $Z_{N_{12}}$ Heisenberg algebra and it requires an N_{12}-dimensional representation. This implies the (0+1)D edge mode of the ground state has GSD = N_{12}, consistent with the edge mode physics analysis via the dimensional reduction approach in Ref. [46]. If $\gcd(p_{\text{II}},N_{12}) \neq 1$, we can reduce the rank of the representation matrix to a smaller rank, and rewrite

$$S_{\varphi^1} S_{\varphi^2} = e^{-2\pi i \frac{1}{\frac{N_{12}}{\gcd(p_{\text{II}},N_{12})}} \frac{p_{\text{II}}}{\gcd(p_{\text{II}},N_{12})}} S_{\varphi^2} S_{\varphi^1}.$$

In this way, we obtain a relative prime factor $\frac{p_{\text{II}}}{\gcd(p_{\text{II}},N_{12})}$, and the GSD is the reduced rank of the matrix representation of the symmetry generators:

$$\text{GSD} = \frac{N_{12}}{\gcd(p_{\text{II}},N_{12})}.$$

We have shown the degenerate zero modes happening on the 0D edge of 1D bulk SPT. In general, if we create various symmetry-breaking domain wall to gap the gapless boundary mods of the higher-dimensional boundaries, we can study the zero modes trapped at the gapped domain wall via the dimensional reduction approach. As an example, we can look into the (0+1)D kink on a (1+1)D domain-wall edge [Eq. (71)] of (2+1)D bulk SPTs. This result is consistent with Refs. [20,46].

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