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Double metric, generalized metric, and α'-deformed double field theory

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We relate the unconstrained "double metric" of the "α'-geometry" formulation of double field theory to the constrained generalized metric encoding the spacetime metric and b-field. This is achieved by integrating out auxiliary field components of the double metric in an iterative procedure that induces an infinite number of higher-derivative corrections. As an application, we prove that, to first order in α' and to all orders in fields, the deformed gauge transformations are Green-Schwarz–deformed diffeomorphisms. We also prove that to first order in α' the spacetime action encodes precisely the Green-Schwarz deformation with Chern-Simons forms based on the torsionless gravitational connection. This seems to be in tension with suggestions in the literature that T-duality requires a torsionful connection, but we explain that these assertions are ambiguous since actions that use different connections are related by field redefinitions.

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I. INTRODUCTION

In this paper, we will elaborate on the double field theory constructed in Ref. [1], the defining geometric structures of which are α'-deformed and the action of which, including higher-derivative corrections, is exactly gauge invariant and duality invariant. Concretely, we will report progress relating this theory to conventional theories written in terms of the standard target space fields of string theory such as the spacetime metric, the antisymmetric tensor field, and the dilaton.[1]

The original two-derivative double field theory (DFT) [15–19] can be formulated in terms of the generalized metric $\mathcal{H}$, which takes values in the T-duality group $O(D,D)$ [18], where D denotes the total number of dimensions. More precisely, we can view the metric and b-field as parametrizing the coset space $O(D,D)/O(D)\times O(D)$, which encodes D^2 degrees of freedom. The generalized metric is a constrained symmetric matrix that can be parametrized as

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}b \\ bg^{-1} & g - bg^{-1}b \end{pmatrix}. \tag{1.1}$$

Thus, given a generalized metric, we may read off the spacetime metric g and the b-field. In contrast, the formulation of Ref. [1] is based on a symmetric field $\mathcal{M}$, in the following called the "double metric," that is unconstrained and so cannot be viewed as a generalized metric. Therefore, the question arises of how to relate $\mathcal{M}$ to the standard string fields, the metric g and the b-field.

In Ref. [9], we showed perturbatively, expanding around a constant background, how to relate the double metric $\mathcal{M}$ to the standard perturbative field variables. For a constant background, the field equations of Ref. [1] do in fact imply that $\mathcal{M}$ is a *constant* generalized metric, thus encoding precisely the background metric and b-field. The fluctuations can then be decomposed into the physical metric and b-field fluctuations plus extra fields. These extra fields are, however, auxiliary and can be eliminated by their own algebraic field equations in terms of the physical fluctuations. The resulting action has been determined to cubic order in Ref. [9].

It is desirable to have a systematic procedure to relate the double metric $\mathcal{M}$ to standard fields g and b rather than their fluctuations. In this paper, we will provide such a procedure. In the first part, we will show that the double metric can be written, perturbatively in α' but nonperturbatively in fields, in terms of the generalized metric as

$$\mathcal{M} = \mathcal{H} + F, \tag{1.2}$$

where F starts at order α' and can be systematically determined in terms of $\mathcal{H}$ to any order in α'; see Eq. (2.28). This systematizes and completes tentative results given in Ref. [1]. While the original formulation in terms of $\mathcal{M}$ is *cubic* with a finite number of derivatives (up to six), the procedure of integrating out the auxiliary F leads to an action with an infinite number of higher-derivative corrections. As an application, we compute the gauge transformations $\delta^{(1)}\mathcal{H}$ to first order in α', see Eq. (2.53), thereby determining the $\mathcal{O}(\alpha')$ gauge transformations of g and b, and show that they are equivalent to those required by the Green-Schwarz mechanism. In

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[1]This theory describes a particular T-duality invariant dynamics that is an ingredient of heterotic string theory. References relevant for the DFT description of α' corrections and/or heterotic strings include Refs. [2–14].

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Ref. [8], this was shown perturbatively to cubic order in fields; here, it is shown nonperturbatively in fields. We show that up to and including $\mathcal{O}(\alpha'^2)$ the gauge transformations of $\mathcal{H}$ are independent of the dilaton. We have no reason to suspect that this feature persists to all orders in α'.

From these results and gauge invariance, it follows that the three-form curvature $\hat{H}$ of the b-field contains higher-derivative terms due to the Chern-Simons modification. This curvature enters quadratically as a kinetic term and thus introduces a number of higher-derivative terms in the action. Does the action contain other gauge invariant terms built with $\hat{H}$ and other fields? In the second part of the paper, we partially answer this question by proving that the cubic $\mathcal{O}(\alpha')$ action determined in Ref. [9] is precisely given by the Chern-Simons modification of $\hat{H}$ based on the (torsion-free) Levi-Civitá connection. This result seems to be in tension with suggestions that T-duality requires a torsionfull connection with torsion proportional to $H = \mathrm{d}b$ [11–13]. We use the opportunity to clarify this point by recalling that field redefinitions can be used to transform an $\hat{H}$ based on a torsion-free connection to an $\hat{H}$ based on a torsionfull connection, up to further covariant terms [20]. Therefore, by itself, the statement that T-duality prefers one connection over the other is not meaningful (although it could well be that writing the theory to all orders in α' in terms of conventional fields simplifies for a particular connection). Moreover, our results confirm that the action does not contain the square of the Riemann tensor, as already argued in Ref. [9]. It leaves open, however, the possibility of order α' terms that would not contribute to cubic order as well as the structure of the action to order α'^2 and higher.

A few remarks are in order about how the α' corrections for the theory discussed in this paper relate to the known α' corrections of various string theories. Given that we obtain the Green-Schwarz deformation, it does not correspond to bosonic string theory, but rather it encodes ingredients of heterotic string theory. It does not coincide with it, however, since it does not describe the Riemann-square term, which is known to arise for heterotic strings. It also does not include the gauge connections for $SO(32)$ or $E_8 \times E_8$.[2] In order to encode all corrections appearing in bosonic and/or heterotic string theories, more general deformations of DFT, which are not yet known exactly and nonperturbatively, are needed [9] (see, however, Ref. [14] for recent progress). Moreover, supersymmetry (which requires both the Green-Schwarz deformation and the Riemann-square term) is only compatible with a particular combination of $O(D,D)$ invariants. Other combinations then describe, for instance, bosonic string theory.

Returning to the α'-deformed DFT of Ref. [1], the results of this paper show that it can be related to actions written in terms of conventional fields in a systematic (and hence algorithmic) fashion. It would be increasingly difficult in practice to perform this algorithm as we go to higher orders in α', but one may still wonder if there is a closed form of the theory in terms of conventional fields. In any case, it strikes us as highly significant that using a double-metric one can encode an *infinite* number of α' corrections in a cubic theory with only finitely many derivatives. This seems to provide a radically simpler way of organizing the stringy gravity theories. Even if the theory admits a tractable formulation in terms of g and b, the computation of physical observables may be simpler when working in terms of the fields of the α'-deformed DFT.

II. FROM THE DOUBLE METRIC TO THE GENERALIZED METRIC

A. Constraints and auxiliary fields

We start from the double metric $\mathcal{M}_{MN}$, with $O(D,D)$ indices $M,N = 1,\ldots,2D$, which is symmetric but otherwise unconstrained. Our goal is to decompose it into a 'generalized metric' $\mathcal{H}_{MN}$, which is subject to $O(D,D)$ covariant constraints, and auxiliary fields that can be integrated out algebraically. We use matrix notation for the doubled metric and for the generalized metric, with index structure $\mathcal{M}_{\bullet\bullet}$ and $\mathcal{H}_{\bullet\bullet}$, as well as for the $O(D,D)$ invariant metric, with index structure $\eta^{\bullet\bullet}$. The generalized metric is then subject to the constraints

$$\mathcal{H}\eta\mathcal{H} = \eta^{-1} \Leftrightarrow (\mathcal{H}\eta)^2 = (\eta\mathcal{H})^2 = 1. \tag{2.1}$$

As as a consequence, we can introduce projectors, that we take here to act on objects with indices down. Specifically, acting from the left, they have index structure $P_\bullet{}^\bullet$ and are given by

$$P = \frac{1}{2}(1 - \mathcal{H}\eta), \qquad \bar{P} = \frac{1}{2}(1 + \mathcal{H}\eta). \tag{2.2}$$

Similarly, acting from the right, they have index structure $P^\bullet{}_\bullet$ and are given by $P^T = \frac{1}{2}(1 - \eta\mathcal{H})$ and $\bar{P}^T = \frac{1}{2}(1 + \eta\mathcal{H})$. One can quickly verify that we then have

$$\bar{P}\mathcal{H}P^T = 0. \tag{2.3}$$

In order to be compatible with the constraints (2.1), any variation $\delta\mathcal{H}$ of a generalized metric needs to satisfy

$$\delta\mathcal{H} = \bar{P}\delta\mathcal{H}P^T + P\delta\mathcal{H}\bar{P}^T; \tag{2.4}$$

see e.g. the discussion in Sec. 3.3 in Ref. [6]. This constraint, translated in projector language, becomes

[2]From this fact, it follows that in this theory the Green-Schwarz–deformed field strength $\hat{H}$ satisfies the Bianchi identity $\mathrm{d}\hat{H} = \frac{1}{2}\mathrm{tr}(R \wedge R)$. As a result, $R \wedge R$ is exact, and hence there are certain topological constraints on the spacetimes described by this theory. Such constraints play no role, however, in the map from DFT to conventional variables to be developed here.

$$\delta P = \bar{P}\delta P P + P\delta P\bar{P} = -\delta\bar{P}. \tag{2.5}$$

We now aim to relate the double metric to the generalized metric. To this end, we recall that the $\mathcal{M}$ field variation in the α'-extended double field theory takes the form (Eq. (7.16) in Ref. [1])

$$\delta_{\mathcal{M}} S = -\frac{1}{2}\int e^{\phi}\mathrm{tr}(\delta\mathcal{M}\eta E(\mathcal{M})\eta), \tag{2.6}$$

where $E(\mathcal{M})$, with both indices down, is given by

$$E(\mathcal{M}) \equiv \mathcal{M}\eta\mathcal{M} - \eta^{-1} - 2\mathcal{V}(\mathcal{M}) = 0, \tag{2.7}$$

and setting it equal to zero is the field equation for $\mathcal{M}$. Here, $\mathcal{V}$ contains terms with two and with higher derivatives. The tensor $\mathcal{V}$ is thus of order α' and higher relative to the algebraic terms without derivatives, but we suppress explicit factors of α'. Thus, to zeroth order in α', the field equation implies $\mathcal{M}\eta\mathcal{M} = \eta^{-1}$, from which we conclude with Eq. (2.1) that $\mathcal{M}$ is a generalized metric, $\mathcal{M} = \mathcal{H}$. We next write an ansatz for the double metric $\mathcal{M}$ in terms of a generalized metric $\mathcal{H}$ and a (symmetric) correction F that we take to be of order α' and higher,

$$\mathcal{M} = \mathcal{H} + F. \tag{2.8}$$

Here, $\mathcal{H}$ satisfies the constraints above, while we will constrain F to satisfy

$$\bar{P}FP^T = 0, \qquad PF\bar{P}^T = 0, \tag{2.9}$$

where the second equation follows by transposition of the first. We can motivate this constraint as follows. If F had a contribution with projection $\bar{P}FP^T + PF\bar{P}^T$ (both terms are needed since F is symmetric), by Eq. (2.4) this contribution takes the form of a linearized variation of $\mathcal{H}$, and hence it may be absorbed into a redefinition of $\mathcal{H}$, at least to linearized order. Given the above constraints, we can decompose F into its two independent projections, for which we write

$$F = \bar{F} + \underline{F}, \quad \text{with} \quad \bar{F} = \bar{P}\bar{F}\bar{P}^T, \quad \underline{F} = P\underline{F}P^T. \tag{2.10}$$

Additionally, we see that

$$F = \bar{P}F\bar{P}^T + PFP^T. \tag{2.11}$$

We will now show that, perturbatively in α', the double metric can always be decomposed as in Eq. (2.8). Let us emphasize, however, that there may well be solutions for $\mathcal{M}$ that cannot be related to a generalized metric in this fashion and hence are nonperturbative in α'. We first note that with (2.3) and (2.9) we have[3]

$$\bar{P}\mathcal{M}P^T = 0. \tag{2.12}$$

More explicitly, this equation takes the form

$$(1 + \mathcal{H}\eta)\mathcal{M}(1 - \eta\mathcal{H}) = 0. \tag{2.13}$$

In this form, one may view this equation as an *algebraic* equation that determines the matrix $\mathcal{H}$ in terms of the matrix $\mathcal{M}$. If $\mathcal{H}$ is a solution, so is $-\mathcal{H}$, as follows by transposition of the equation, but this ambiguity is naturally resolved by the physical parametrization of $\mathcal{H}$ in terms of a metric of definite signature. While one can quickly show that for $D = 1$ [corresponding to $O(1,1)$] an arbitrary symmetric two-by-two matrix $\mathcal{M}$ leads to a unique $\mathcal{H}$ (up to sign), a general discussion of the solvability for $\mathcal{H}$ seems quite intricate and will not be done here. This is the issue, alluded to above, that some general $\mathcal{M}$ configurations may not be describable via generalized metrics. It is also clear from Eq. (2.12) that different values of $\mathcal{M}$ may be consistent with the same $\mathcal{H}$. For example, given a field $\mathcal{M}$ that works for some $\mathcal{H}$, replacing

$$\mathcal{M} \to \mathcal{M} + \bar{P}\,\bar{\Lambda}\,\bar{P}^T + P\Lambda P^T, \tag{2.14}$$

with Λ and $\bar{\Lambda}$ symmetric, still leads to a solution for the same $\mathcal{H}$. Thus, Eq. (2.12) does not determine $\mathcal{M}$ in terms of $\mathcal{H}$. As we will see in the following section, this is done with the help of field equations.

It is useful to consider Eq. (2.12) [or (2.13)] more explicitly. We begin by parametrizing the general symmetric double metric as

$$\mathcal{M} = \begin{pmatrix} m_1 & c \\ c^T & m_2 \end{pmatrix}, \qquad m_1^T = m_1, \qquad m_2^T = m_2, \qquad c \text{ arbitrary}. \tag{2.15}$$

Using the standard parametrization (1.1) for the generalized metric $\mathcal{H}(g,b)$ and building the projectors $P, \bar{P}$ from it, a direct computation shows that the condition (2.12) gives rise to four equations, which are all equivalent to

$$\mathcal{E}m_1\mathcal{E} + \mathcal{E}c - c^T\mathcal{E} - m_2 = 0, \quad \text{with} \quad \mathcal{E} \equiv g + b. \tag{2.16}$$

The general solvability of Eq. (2.12) requires that for arbitrary symmetric matrices m_1, m_2 and arbitrary c there is always a matrix $\mathcal{E}$ that solves the above equation. We do not address this general solvability question but establish perturbative solvability.

We have seen that to zeroth order in α' the doubled metric is equal to some generalized metric $\bar{\mathcal{H}}$. We have to show that for an $\mathcal{M} = \bar{\mathcal{H}} + \delta\mathcal{M}$ that deviates from $\bar{\mathcal{H}}$ by a small deformation $\delta\mathcal{M}$ Eq. (2.16) can be solved for $\mathcal{E}$. We will show this perturbatively by writing

[3]This equation was proposed by Ashoke Sen. A number of the results that follow were obtained in collaboration with him.

$$\mathcal{E} = \bar{\mathcal{E}} + \delta\mathcal{E}, \qquad m_1 = \bar{m}_1 + \delta m_1,$$
$$m_2 = \bar{m}_2 + \delta m_2, \qquad c = \bar{c} + \delta c, \tag{2.17}$$

assuming that the background values of $\mathcal{M}$ correspond to a generalized metric $\bar{\mathcal{H}}$ parametrized by $\bar{\mathcal{E}} = \bar{g} + \bar{b}$:

$$\bar{m}_1 = \bar{g}^{-1}, \qquad \bar{m}_2 = \bar{g} - \bar{b}\bar{g}^{-1}\bar{b}, \qquad \bar{c} = -\bar{g}^{-1}\bar{b}. \tag{2.18}$$

By construction, these background values solve Eq. (2.16), as one may verify by a quick computation. To first order, the perturbations should then solve

$$\delta\mathcal{E}\bar{m}_1\bar{\mathcal{E}} + \bar{\mathcal{E}}\delta m_1\bar{\mathcal{E}} + \bar{\mathcal{E}}\bar{m}_1\delta\mathcal{E} + \delta\mathcal{E}\bar{c} + \bar{\mathcal{E}}\delta c$$
$$- \delta c^T\bar{\mathcal{E}} - \bar{c}^T\delta\mathcal{E} - \delta m_2 = 0 \tag{2.19}$$

or, after regrouping the terms,

$$\delta\mathcal{E}(\bar{m}_1\bar{\mathcal{E}} + \bar{c}) + (\bar{\mathcal{E}}\bar{m}_1 - \bar{c}^T)\delta\mathcal{E}$$
$$= \delta m_2 - \bar{\mathcal{E}}\delta m_1\bar{\mathcal{E}} + \delta c^T\bar{\mathcal{E}} - \bar{\mathcal{E}}\delta c. \tag{2.20}$$

With Eq. (2.18), we see that the matrices multiplying $\delta\mathcal{E}$ are the identity, and therefore

$$\delta\mathcal{E} = \frac{1}{2}(\delta m_2 - \bar{\mathcal{E}}\delta m_1\bar{\mathcal{E}} + \delta c^T\bar{\mathcal{E}} - \bar{\mathcal{E}}\delta c), \tag{2.21}$$

showing the perturbative solvability of Eq. (2.16). As we explained before, this confirms that, perturbatively, we can write $\mathcal{M} = \mathcal{H} + F$ with F satisfying Eq. (2.9).

B. Eliminating the auxiliary fields

After having shown that the double metric $\mathcal{M}$, at least perturbatively in α', can always be written in terms of a generalized metric $\mathcal{H}$ and an additional (constrained) field F, we now show that F is an auxiliary field that can be eliminated algebraically by solving its own field equations. Thus, we can systematically eliminate F from the action to get an action for $\mathcal{H}$.

Let us consider the variation of the action S with respect to F. Since $S(\mathcal{M}) = S(\mathcal{H} + F)$, this variation gives the same result as what we would get by varying the original action with respect to $\mathcal{M}$,

$$\delta_F S = -\frac{1}{2}\int e^{\phi}\mathrm{tr}(\delta F\eta E(\mathcal{M})\eta), \tag{2.22}$$

where we used Eq. (2.6). In order to read off the equations of motion, we have to recall that F is a constrained field. The variation δF needs to respect the constraint (2.11), which implies that varying F keeping $\mathcal{H}$ fixed requires

$$\delta F = P\delta F P^T + \bar{P}\delta F\bar{P}^T. \tag{2.23}$$

Using this variation in Eq. (2.22), we find two equations corresponding to the two projections in Eq. (2.23). These are the field equations for $\bar{F}$ and $\underline{F}$, respectively, which are given by

$$\bar{P}^T\eta E(\mathcal{M})\eta\bar{P} = 0,$$
$$P^T\eta E(\mathcal{M})\eta P = 0. \tag{2.24}$$

By moving the η matrices across the projectors and multiplying by η from the left and from the right, these equations are equivalent to a form without explicit η's:

$$\bar{P}E(\mathcal{M})\bar{P}^T = 0,$$
$$PE(\mathcal{M})P^T = 0. \tag{2.25}$$

We now use these equations to solve for $\underline{F}$ and $\bar{F}$ in terms of $\mathcal{H}$. The solutions, that take a recursive form, could be inserted back in the action to find a theory written solely in terms of $\mathcal{H}$ and the dilaton.

For this purpose, we first return to the full equations of motion $E(\mathcal{M})$ given in Eq. (2.7). Substitution of $\mathcal{M} = \mathcal{H} + F$ into this equation yields

$$E(\mathcal{M}) = \mathcal{H}\eta F + F\eta\mathcal{H} - 2\mathcal{V}(\mathcal{M}) + F\eta F = 0, \tag{2.26}$$

where we used $\mathcal{H}\eta\mathcal{H} = \eta^{-1}$. Substituting this into Eq. (2.25) and using $\bar{P}\mathcal{H}\eta = \bar{P}$, $P\mathcal{H}\eta = -P$, which follow immediately from Eq. (2.2), we get

$$2\bar{P}F\bar{P}^T = \bar{P}(2\mathcal{V}(\mathcal{M}) - F\eta F)\bar{P}^T,$$
$$2PFP^T = -P(2\mathcal{V}(\mathcal{M}) - F\eta F)P^T. \tag{2.27}$$

Using the constraint (2.10), we finally obtain

$$\bar{F} = \bar{P}\left(\mathcal{V}(\mathcal{H} + F) - \frac{1}{2}F\eta F\right)\bar{P}^T,$$
$$\underline{F} = -P\left(\mathcal{V}(\mathcal{H} + F) - \frac{1}{2}F\eta F\right)P^T. \tag{2.28}$$

We can now solve these equations iteratively, recalling that F is of order α' relative to $\mathcal{H}$. Thus, on the right-hand side, to the lowest order, we keep only the two-derivative terms in $\mathcal{V}$, denoted by $\mathcal{V}^{(2)}$, and use $\mathcal{H}$ for the argument of $\mathcal{V}$, dropping the term $F\eta F$. This determines the leading term in F in terms of $\mathcal{H}$:

$$\bar{F}^{(1)} = +\bar{P}\mathcal{V}^{(2)}(\mathcal{H})\bar{P},$$
$$\underline{F}^{(1)} = -P\mathcal{V}^{(2)}(\mathcal{H})P. \tag{2.29}$$

We can then substitute this leading-order solution for F into the right-hand side and get the next-order solution for F. After we have determined F to the desired order, we can

substitute it into the action to determine the action in terms of $\mathcal{H}$ to the desired order.

Let us note that the above result (2.29) determines $\mathcal{M}$ in terms of $\mathcal{H}$ to first order in α' in precise agreement with Eq. (7.30) in Ref. [1]. The improvement of the present analysis is to make manifest that the determination of $\mathcal{M}$ in terms of $\mathcal{H}$ corresponds to integrating out auxiliary fields to arbitrary orders in α'.

Since the full equation of motion in terms of $\mathcal{M}$ is given by $E(\mathcal{M}) = 0$ in Eq. (2.26), and the $\bar{F}$ and $\underline{F}$ equations set two types of projections of $E(\mathcal{M})$ equal to zero in Eq. (2.25), the remaining dynamical equation of the theory must be equivalent to

$$PE(\mathcal{M})\bar{P}^T = 0. \tag{2.30}$$

Using the above expression for $E(\mathcal{M})$ and the constraints of $\mathcal{H}$ and F, this gives

$$P\mathcal{V}(\mathcal{M})\bar{P}^T = 0, \tag{2.31}$$

and its transpose $\bar{P}\mathcal{V}(\mathcal{M})P^T = 0$. We will now show that variation with respect to $\mathcal{H}$ indeed yields equations that perturbatively in α' are equivalent to Eq. (2.30). To see this, we first note that, by the constraint (2.11) on F, a variation of $\mathcal{H}$ induces a variation of F,

$$\begin{aligned}\delta F &= \bar{P}\delta F\bar{P}^T + P\delta FP^T \\ &\quad + \delta\bar{P}F\bar{P}^T + \bar{P}F\delta\bar{P}^T + \delta PFP^T + PF\delta P^T.\end{aligned} \tag{2.32}$$

Using that the variation of the projectors in the second line is subject to Eq. (2.5), one may quickly verify that this can be written as

$$\begin{aligned}&\delta F = \bar{P}\delta F\bar{P}^T + P\delta FP^T + PX\bar{P}^T + \bar{P}X^TP^T, \\ &\quad \text{where } X = \underline{F}\delta P^T - \delta P\bar{F}.\end{aligned} \tag{2.33}$$

Note that the first two terms and the second two terms have complementary projections, which implies that it is self-consistent to set $\delta F = PX\bar{P}^T + \bar{P}X^TP^T$. Using this and the constrained variation (2.4) of $\mathcal{H}$ in the general variation (2.6) of the action, it is straightforward to verify that the equation of motion for $\mathcal{H}$ is

$$PE\bar{P}^T + \frac{1}{2}(PE\bar{P}^T)\eta\bar{F} - \frac{1}{2}\underline{F}\eta(PE\bar{P}^T) = 0. \tag{2.34}$$

This admits the solution $PE\bar{P}^T = 0$, which is the unique solution in perturbation theory in α'. Thus, we proved that Eq. (2.30) is the correct field equation for $\mathcal{H}$.

C. Deformed gauge transformations for generalized metric

Let us now determine the gauge transformations of $\mathcal{H}$. They result from those of $\mathcal{M}$ and the relation $\mathcal{M} = \mathcal{H} + F$ upon eliminating F by the above procedure. We first recall that

$$\begin{aligned}F &= \bar{P}F\bar{P}^T + PFP^T \\ &= \frac{1}{4}(1+\mathcal{H}\eta)F(1+\eta\mathcal{H}) + \frac{1}{4}(1-\mathcal{H}\eta)F(1-\eta\mathcal{H}),\end{aligned} \tag{2.35}$$

which simplifies to

$$F = \frac{1}{2}(F + \mathcal{H}\eta F\eta\mathcal{H}). \tag{2.36}$$

In order not to clutter the following computation, we will use a notation in which the explicit η's are suppressed, which is justified because the η's just make index contractions consistent. For instance, we then write $\mathcal{H}^2 = 1$ and similarly

$$\begin{aligned}F &= \frac{1}{2}(F + \mathcal{H}F\mathcal{H}) \\ &= \frac{1}{2}(\mathcal{H}(\mathcal{H}F) + (\mathcal{H}F)\mathcal{H}) \\ &= \frac{1}{2}(\mathcal{H}\mathcal{F} + \mathcal{F}\mathcal{H}), \quad \text{with} \quad \mathcal{F} \equiv \mathcal{H}F.\end{aligned} \tag{2.37}$$

We can thus write for the double metric

$$\mathcal{M} = \mathcal{H} + \frac{1}{2}(\mathcal{H}\mathcal{F} + \mathcal{F}\mathcal{H}). \tag{2.38}$$

Next, we write an expansion in orders of α' for the gauge transformations of the double metric $\mathcal{M}$. As the gauge transformations of $\mathcal{M}$ are exact with terms up to five derivatives (of order α'^2), we write the exact gauge variation as

$$\begin{aligned}\delta\mathcal{M} &= \delta^{(0)}\mathcal{M} + \delta^{(1)}\mathcal{M} + \delta^{(2)}\mathcal{M} \\ &= \delta^{(0)}\mathcal{M} + J^{(1)}(\mathcal{M}) + J^{(2)}(\mathcal{M}).\end{aligned} \tag{2.39}$$

Here, $J^{(1)}(\mathcal{M})$ and $J^{(2)}(\mathcal{M})$ are linear functions of their arguments, where superscripts in parentheses denote powers of α'. These functions can be read from Eq. (6.39) of Ref. [1], and they have no dilaton dependence. For general transformations, we also write

$$\delta = \delta^{(0)} + \hat{\delta}, \qquad \hat{\delta} = \delta^{(1)} + \delta^{(2)} + \cdots \tag{2.40}$$

For the following computation, it is convenient to define a projector $[\ldots]$ from general two index objects to mixed index projections:

$$[A] \equiv PA\bar{P} + \bar{P}AP \equiv \frac{1}{2}(A - \mathcal{H}A\mathcal{H}). \tag{2.41}$$

This projection satisfies

$$[\mathcal{H}B + B\mathcal{H}] = 0, \quad \text{for all } B. \tag{2.42}$$

Note also that variation of the constrained $\mathcal{H}$ then satisfies $[\delta\mathcal{H}] = \delta\mathcal{H}$ by Eq. (2.4) above.

Let us now derive relations for the gauge transformation of $\mathcal{H}$ by varying Eq. (2.38),

$$\begin{aligned}&\delta^{(0)}\mathcal{M} + J^{(1)}(\mathcal{M}) + J^{(2)}(\mathcal{M})\\ &= \delta^{(0)}\mathcal{H} + \hat{\delta}\mathcal{H} + \delta^{(0)}\frac{1}{2}(\mathcal{H}\mathcal{F} + \mathcal{F}\mathcal{H})\\ &\quad + \hat{\delta}\frac{1}{2}(\mathcal{H}\mathcal{F} + \mathcal{F}\mathcal{H}).\end{aligned} \tag{2.43}$$

The zeroth-order part $\delta^{(0)}$ is given by the generalized Lie derivative of double field theory, in the following denoted by $\hat{\mathcal{L}}_\xi$. Moreover, we use the notation $\Delta_\xi \equiv \delta_\xi - \hat{\mathcal{L}}_\xi$ to denote the noncovariant part of the variation of any structure. Note that by definition Δ_ξ leaves any generalized tensor invariant so that e.g. for the generalized metric $\Delta_\xi\mathcal{H} = 0$. Using this, we can write

$$\begin{aligned}&\hat{\mathcal{L}}_\xi\mathcal{M} + J^{(1)}(\mathcal{M}) + J^{(2)}(\mathcal{M})\\ &= \hat{\mathcal{L}}_\xi\mathcal{H} + \hat{\delta}\mathcal{H} + \hat{\mathcal{L}}_\xi\frac{1}{2}(\mathcal{H}\mathcal{F} + \mathcal{F}\mathcal{H}) + \frac{1}{2}(\mathcal{H}\Delta_\xi\mathcal{F} + \Delta_\xi\mathcal{F}\mathcal{H})\\ &\quad + \frac{1}{2}(\hat{\delta}\mathcal{H}\mathcal{F} + \mathcal{F}\hat{\delta}\mathcal{H}) + \frac{1}{2}(\mathcal{H}\hat{\delta}\mathcal{F} + \hat{\delta}\mathcal{F}\mathcal{H}).\end{aligned} \tag{2.44}$$

The terms with generalized Lie derivatives on the left-hand and right-hand sides cancel. Thus, we obtain

$$\begin{aligned}J^{(1)}(\mathcal{M}) + J^{(2)}(\mathcal{M}) = \hat{\delta}\mathcal{H} &+ \frac{1}{2}(\hat{\delta}\mathcal{H}\mathcal{F} + \mathcal{F}\hat{\delta}\mathcal{H})\\ &+ \frac{1}{2}(\mathcal{H}\Delta_\xi\mathcal{F} + \Delta_\xi\mathcal{F}\mathcal{H})\\ &+ \frac{1}{2}(\mathcal{H}\hat{\delta}\mathcal{F} + \hat{\delta}\mathcal{F}\mathcal{H}).\end{aligned} \tag{2.45}$$

Applying the $[\ldots]$ projector, the terms on the second and third line drop out by the property (2.42), and we get

$$[J^{(1)}(\mathcal{M}) + J^{(2)}(\mathcal{M})] = \hat{\delta}\mathcal{H} + \left[\frac{1}{2}(\hat{\delta}\mathcal{H}\mathcal{F} + \mathcal{F}\hat{\delta}\mathcal{H})\right]. \tag{2.46}$$

Recalling $\mathcal{F} = \mathcal{H}F$, this is more conveniently written as

$$\begin{aligned}\hat{\delta}\mathcal{H} &= [J^{(1)}(\mathcal{H})] + [J^{(2)}(\mathcal{H}) + J^{(1)}(F) + J^{(2)}(F)]\\ &\quad - \left[\frac{1}{2}(\hat{\delta}\mathcal{H}\mathcal{H}F + \mathcal{H}F\hat{\delta}\mathcal{H})\right].\end{aligned} \tag{2.47}$$

Using that by $\mathcal{H}^2 = 1$ we have for any variation $\delta\mathcal{H}\mathcal{H} = -\mathcal{H}\delta\mathcal{H}$, we can rewrite this as

$$\begin{aligned}\hat{\delta}\mathcal{H} &= [J^{(1)}(\mathcal{H})] + [J^{(2)}(\mathcal{H}) + J^{(1)}(F) + J^{(2)}(F)]\\ &\quad + \left[\frac{1}{2}\mathcal{H}(\hat{\delta}\mathcal{H}F - F\hat{\delta}\mathcal{H})\right].\end{aligned} \tag{2.48}$$

This is a recursion relation that can be solved iteratively for $\delta\mathcal{H}$. In order to make this explicit, let us expand the auxiliary field F in powers of α',

$$F = F^{(1)} + F^{(2)} + \cdots. \tag{2.49}$$

Inserting this expansion into Eq. (2.48), we read off

$$\begin{aligned}\delta^{(1)}\mathcal{H} &= [J^{(1)}(\mathcal{H})],\\ \delta^{(2)}\mathcal{H} &= [J^{(2)}(\mathcal{H}) + J^{(1)}(F^{(1)})]\\ &\quad + \frac{1}{2}[\mathcal{H}(\delta^{(1)}\mathcal{H}F^{(1)} - F^{(1)}\delta^{(1)}\mathcal{H})],\\ \delta^{(3)}\mathcal{H} &= [J^{(1)}(F^{(2)}) + J^{(2)}(F^{(1)})]\\ &\quad + \left[\frac{1}{2}\mathcal{H}(\delta^{(2)}\mathcal{H}F^{(1)} - F^{(1)}\delta^{(2)}\mathcal{H})\right]\\ &\quad + \left[\frac{1}{2}\mathcal{H}(\delta^{(1)}\mathcal{H}F^{(2)} - F^{(2)}\delta^{(1)}\mathcal{H})\right].\end{aligned} \tag{2.50}$$

Here, we have given the deformed gauge transformations of the generalized metric up to α'^3, but it is straightforward in principle to continue this recursion to arbitrary order in α'. In the following subsection, we investigate the first two nontrivial corrections.

D. Relation to Green-Schwarz–deformed gauge transformations

Our analysis of the gauge transformations in Ref. [8] was perturbative and restricted to the cubic part of the theory called DFT$^-$. We were led to the conclusion that, in conventional variables, the full gauge transformations are the Green-Schwarz–deformed diffeomorphisms written in form notation as [8]

$$\delta_\xi b = \mathcal{L}_\xi b + \frac{1}{2}\mathrm{tr}(\mathrm{d}(\partial\xi) \wedge \Gamma). \tag{2.51}$$

Our present result for the corrected gauge transformations of $\mathcal{H}$ in terms of $\mathcal{H}$, as opposed to fluctuations thereof, allows us to do a full analysis to order α' and thus establish directly the validity of Eq. (2.51).

Let us now explicitly work out the first-order correction to the gauge transformations of the generalized metric and thereby of the metric g and the b-field. We use Eq. (6.39) from Ref. [1] to read off the function $J^{(1)}$ introduced in Eq. (2.39),

$$J^{(1)}_{MN}(\mathcal{M}) = -\frac{1}{2}\partial_M \mathcal{M}^{PQ}\partial_P K_{QN} - \partial_P \mathcal{M}_{QM}\partial_N K^{QP} + (M \leftrightarrow N), \tag{2.52}$$

where $K_{MN} = 2\partial_{[M}\xi_{N]}$, with ξ^M the gauge parameter, and doubled derivatives $\partial_M = (\tilde{\partial}^i, \partial_i)$. From the first equations in Eqs. (2.50) and (2.41), we then infer that

$$\delta^{(1)}_\xi \mathcal{H}_{MN} = -\frac{1}{4}\partial_M \mathcal{H}^{PQ}\partial_P K_{QN} + \frac{1}{4}\mathcal{H}_M{}^K \mathcal{H}_N{}^L \partial_K \mathcal{H}^{PQ}\partial_P K_{QL} - \frac{1}{2}\partial_P \mathcal{H}_{QM}\partial_N K^{QP} + \frac{1}{2}\mathcal{H}_M{}^K \mathcal{H}_N{}^L \partial_P \mathcal{H}_{QK}\partial_L K^{QP} + (M \leftrightarrow N). \tag{2.53}$$

We compute the gauge transformation of g_{ij} by focusing on $\delta^{(1)}_\xi \mathcal{H}^{ij}$, using Eq. (1.1) for the generalized metric, and setting $\tilde{\partial}^i = 0$:

$$\begin{aligned}\delta^{(1)}_\xi g^{ij} &= \delta^{(1)}_\xi \mathcal{H}^{ij} \\ &= \frac{1}{4} g^{ik} g^{jl} \partial_k g^{pq} \partial_p (\partial_q \tilde{\xi}_l - \partial_l \tilde{\xi}_q) \\ &\quad + \frac{1}{4} g^{ik} g^{jl} \partial_k (g^{pr} b_{rq}) \partial_p \partial_l \xi^q \\ &\quad - \frac{1}{4} g^{ik} g^{jr} b_{rl} \partial_k g^{pq} \partial_p \partial_q \xi^l \\ &\quad - \frac{1}{2} g^{ik} g^{jl} \partial_p (g^{qr} b_{rk}) \partial_l \partial_q \xi^p \\ &\quad - \frac{1}{2} g^{ir} b_{rk} g^{jl} \partial_p g^{qk} \partial_l \partial_q \xi^p + (i \leftrightarrow j).\end{aligned} \tag{2.54}$$

We see that the gauge transformation of g_{ij} and its inverse g^{ij} has higher-derivative terms, none of which is present in the standard Lie derivative. Thus, g_{ij} cannot be identified with the conventional metric tensor. We will now show that g_{ij} is related by a noncovariant field redefinition to a metric g'_{ij} transforming conventionally under diffeomorphisms. To this end, we record the "noncovariant" variation of the partial derivatives of g and b, denoted by $\Delta_\xi \equiv \delta_\xi - \mathcal{L}_\xi$, under the zeroth-order gauge transformations,

$$\begin{aligned}\Delta_\xi(\partial_p b_{ql}) &= \partial_p(\partial_q \tilde{\xi}_l - \partial_l \tilde{\xi}_q) + \partial_p \partial_q \xi^r b_{rl} + \partial_p \partial_l \xi^r b_{qr}, \\ \Delta_\xi(\partial_k g^{pq}) &= -\partial_k \partial_r \xi^p g^{rq} - \partial_k \partial_r \xi^q g^{pr}, \\ \Delta_\xi(\partial_p g_{qi}) &= \partial_p \partial_q \xi^k g_{ki} + \partial_p \partial_i \xi^k g_{kq}.\end{aligned} \tag{2.55}$$

Here, we also included the noninvariance of ∂b under the b-field gauge transformation with parameter $\tilde{\xi}_i$. Consider now the field redefinition

$$g'^{ij} = g^{ij} - \frac{1}{4}(g^{ik} g^{jl} \partial_k g^{pq} \partial_p b_{ql} + (i \leftrightarrow j)) + \cdots, \tag{2.56}$$

where the missing terms indicated by dots will be determined momentarily. The higher-derivative terms, being written with partial derivatives, are noncovariant and therefore lead to extra terms in the $\delta^{(1)}$ variation of the metric. These are determined by acting with Δ_ξ on the $\mathcal{O}(\alpha')$ terms in Eq. (2.56). Using Eq. (2.55), a straightforward computation then shows that many of the $\mathcal{O}(\alpha')$ terms in Eq. (2.54) are cancelled, while the remaining terms organize into

$$\delta^{(1)}_\xi g'^{ij} = \frac{1}{4} g^{ik} g^{jl} g^{pr} \partial_p \partial_l \xi^q H_{krq} + (i \leftrightarrow j), \tag{2.57}$$

with the field strength $H_{ijk} = 3\partial_{[i} b_{jk]}$. Using that the latter is gauge invariant and that for the Christoffel symbols $\Delta_\xi \Gamma^k_{ij} = \partial_i \partial_j \xi^k$, we can remove this structure by taking the full field redefinition to be

$$g'^{ij} = g^{ij} - \frac{1}{4}(g^{ik} g^{jl} \partial_k g^{pq} \partial_p b_{ql} + g^{ik} g^{jl} g^{pr} \Gamma^q_{pl} H_{krq} + (i \leftrightarrow j)). \tag{2.58}$$

This then leads to a metric transforming conventionally under infinitesimal diffeomorphisms, $\delta_\xi g'_{ij} = \mathcal{L}_\xi g'_{ij}$, with the standard Lie derivative $\mathcal{L}_\xi$.

The gauge transformations of the b-field can be determined from $\delta^{(1)}_\xi \mathcal{H}^i{}_j$, see Eq. (1.1),

$$\delta^{(1)}_\xi \mathcal{H}^i{}_j = -(\delta^{(1)}_\xi g^{ik}) b_{kj} - g^{ik} \delta^{(1)}_\xi b_{kj}, \tag{2.59}$$

and using $\delta^{(1)} g$ from Eq. (2.54). In order to streamline the presentation, let us first consider the special case of the b-independent terms in δb, for which the first term in here can be omitted. From Eq. (2.53), we then read off, inserting the components (1.1) and setting $\tilde{\partial}^i = 0$,

$$\begin{aligned}-g^{ik} \delta^{(1)} b_{kj}|_{b=0} &= -\frac{1}{4}\partial_j g^{pq} \partial_p \partial_q \xi^i - \frac{1}{2}\partial_p g^{qi} \partial_j \partial_q \xi^p \\ &\quad + \frac{1}{4} g^{ik} g_{lj} \partial_k g^{pq} \partial_p \partial_q \xi^l \\ &\quad + \frac{1}{2} g_{jk} g^{il} \partial_p g^{qk} \partial_l \partial_q \xi^p.\end{aligned} \tag{2.60}$$

Multiplying with the inverse metric and relabeling indices, this yields

$$\delta^{(1)} b_{ij}|_{b=0} = \frac{1}{4}\partial_p \partial_q \xi^k g_{ik} \partial_j g^{pq} - \frac{1}{2}\partial_i \partial_p \xi^q g_{jk} \partial_q g^{pk} - (i \leftrightarrow j). \tag{2.61}$$

We now consider the field redefinition

$$b'_{ij} = b_{ij} - \frac{1}{4}(\partial_p g_{qi} \partial_j g^{pq} - (i \leftrightarrow j)). \tag{2.62}$$

As above, this leads to additional $\delta^{(1)}$ variations of b, which can be determined by computing the Δ_ξ variation of the higher-derivative terms in the redefinition. Using Eq. (2.55), one finds

$$\delta_\xi^{(1)} b'_{ij} = \frac{1}{2}\partial_i\partial_p\xi^q\left[\frac{1}{2}g^{pk}(\partial_j g_{qk} + \partial_q g_{jk} - \partial_k g_{qj})\right] - (i \leftrightarrow j)$$
$$= \partial_p\partial_{[i}\xi^q\Gamma^p_{j]q}, \tag{2.63}$$

with the Christoffel symbols Γ^k_{ij} associated to the Levi-Civitá connection.

We finally have to complete the analysis by returning to Eq. (2.59) and including all b-dependent terms in the gauge variation. A somewhat lengthy but straightforward computation using Eqs. (2.53) and (2.54), the details of which we do not display, then shows that all these terms in fact cancel. Thus, Eq. (2.63) is the complete result, and the total diffeomorphism gauge transformations are the Green-Schwarz–deformed diffeomorphisms in Eq. (2.51). This extends the perturbative, cubic analysis in Ref. [8] to the full nonlinear level in fields.

To summarize the above result, let us state it in an equivalent but perhaps instructive form. For this, we drop the primes from the fields that transform covariantly and add hats to the original fields that transform noncovariantly. The α'-deformed double field theory can be written in terms of a generalized metric parametrized canonically by a symmetric tensor $\hat{g}$ and an antisymmetric tensor $\hat{b}$,

$$\mathcal{H}_{MN} = \begin{pmatrix} \hat{g}^{ij} & -\hat{g}^{ik}\hat{b}_{kj} \\ \hat{b}_{ik}\hat{g}^{kj} & \hat{g}_{ij} - \hat{b}_{ik}\hat{g}^{kl}\hat{b}_{lj} \end{pmatrix}, \tag{2.64}$$

where our earlier relations imply that

$$\hat{g}_{ij} = g_{ij} - \frac{1}{4}(\partial_i g^{pq}\partial_p b_{qj} + g^{pq}\Gamma^r_{pi}H_{jqr} + (i \leftrightarrow j)),$$
$$\hat{b}_{ij} = b_{ij} + \frac{1}{4}(\partial_p g_{qi}\partial_j g^{pq} - (i \leftrightarrow j)). \tag{2.65}$$

Here, g_{ij} and b_{ij} transform conventionally under diffeomorphisms, up to the Green-Schwarz deformation on the b-field.

E. Dilaton dependence in the $\mathcal{H}$ gauge transformations

One may wonder if the gauge transformation of the generalized metric involves the dilaton. While the double-metric gauge transformation does not, the double-metric field equation does, and therefore the auxiliary field F is expected to depend on the dilaton. Such dependence would then be expected to appear in the gauge transformations of $\mathcal{H}$ due to the relations in Eq. (2.50). We have already seen explicitly in Eq. (2.53) that there is no dilaton dependence in $\delta^{(1)}\mathcal{H}$. In this subsection, we show that there is no dilaton dependence in $F^{(1)}$ and therefore no dilaton dependence in $\delta^{(2)}\mathcal{H}$, but we do expect dilaton dependence in $\delta^{(3)}\mathcal{H}$.

We will not compute the full gauge variation $\delta^{(2)}\mathcal{H}$ but rather confine ourselves to prove that the gauge variation $\delta^{(2)}\mathcal{H}$ is independent of the dilaton. While the dilaton dependence drops out in $\delta^{(2)}\mathcal{H}$, the proof below does not extend to higher order, and so these terms may depend on the dilaton.

Inspection of the second line in Eq. (2.50) shows that the dilaton dependence in $\delta^{(2)}\mathcal{H}$ could only arise through $F^{(1)}$. Using Eq. (2.29), we write

$$F^{(1)} = \underline{F}^{(1)} + \bar{F}^{(1)} = -P\mathcal{V}^{(2)}(\mathcal{H})P + \bar{P}\mathcal{V}^{(2)}(\mathcal{H})\bar{P}. \tag{2.66}$$

It was shown in Ref. [1] that the dilaton-dependent terms $\tilde{\mathcal{V}}^{(2)}$ in $\mathcal{V}^{(2)}$ appear through an $O(D,D)$ vector function $G^M(\mathcal{M},\phi)$. Specifically, one infers from Eq. (6.69) of Ref. [1] that

$$\tilde{\mathcal{V}}^{(2)}_{MN} = -\frac{1}{4}\hat{\mathcal{L}}_G\mathcal{H}_{MN}, \tag{2.67}$$

where $\hat{\mathcal{L}}_\xi$ is the generalized Lie derivative[4] and we can let $G^M \to \mathcal{H}^{MN}\partial_N\phi$ because all other terms in G are dilaton independent or have higher derivatives. Inserting this into Eq. (2.66), we infer that the ϕ-dependent terms in $F^{(1)}$ are contained in

$$F^{(1)}|_\phi \equiv \frac{1}{4}P\hat{\mathcal{L}}_G\mathcal{H}P - \frac{1}{4}\bar{P}\hat{\mathcal{L}}_G\mathcal{H}\bar{P} \equiv 0, \tag{2.68}$$

which is zero. This follows because any variation $\delta\mathcal{H}$ of a generalized metric, including $\hat{\mathcal{L}}_G\mathcal{H}$, satisfies $P\delta\mathcal{H}P = \bar{P}\delta\mathcal{H}\bar{P} = 0$, cf. Eq. (2.4). Since $F^{(1)}$ has no dilaton dependence, the gauge transformations of the generalized metric to order α'^2 are independent of the dilaton.

Since $F^{(1)}$ is dilaton independent, Eq. (2.28) implies that the dilaton-dependent terms of $F^{(2)}$ are given by

$$\underline{F}^{(2)}|_\phi = -P(\tilde{\mathcal{V}}^{(2)}(F^{(1)}) + \tilde{\mathcal{V}}^{(4)}(\mathcal{H}))P,$$
$$\bar{F}^{(2)}|_\phi = \bar{P}(\tilde{\mathcal{V}}^{(2)}(F^{(1)}) + \tilde{\mathcal{V}}^{(4)}(\mathcal{H}))\bar{P}. \tag{2.69}$$

Here, $\tilde{\mathcal{V}}^{(4)}$ denote those terms in $\mathcal{V}$ with four derivatives and containing dilatons. The above dilaton-dependent terms in $F^{(2)}$ would have to be inserted into Eq. (2.50) in order to determine the dilaton dependence of the $\mathcal{O}(\alpha'^3)$ gauge

[4]Here, we only need the zeroth-order part of the Lie derivative, carrying one derivative, but this relation is actually valid more generally for $\mathcal{M}$, with the α'-corrected Lie derivative determined from $\delta_\xi\mathcal{M}_{MN} = \mathbb{L}_\xi\mathcal{M}_{MN}$.

transformations of the generalized metric. We do not see any reason why this dilaton dependence would vanish.

III. CUBIC ACTION AT ORDER α' IN STANDARD FIELDS

A. Rewriting of cubic action

In this section, we aim to determine the double field theory action to order α' in terms of *conventional* physical fields. We will aim for the covariant action that yields the cubic action given in Ref. [9]. Thus, the order α' covariant action is uniquely determined only up to terms like H^4, that have the right number of derivatives but do not contribute to the cubic theory. This still allows us to address and clarify various issues related to T-duality and α' corrections.

The cubic action given in Ref. [9] was written in terms of the fluctuations $m_{\underline{M}\bar{N}}$ of the double metric $\mathcal{M}_{MN}$ after integrating out the auxiliary fields. For the comparison with standard actions, it is convenient to write it instead in terms of $e_{ij} \equiv h_{ij} + b_{ij}$, which is the sum of the symmetric metric fluctuation and the antisymmetric b-field fluctuation (modulo field redefinitions that we are about to determine). In Sec. 5.3 of Ref. [9], it is spelled out explicitly how to convert the fluctuations of $\mathcal{M}_{MN}$ into e_{ij}. Without discussing the details of this straightforward translation, in the following, we simply give the cubic theory in terms of e_{ij}.

The cubic DFT action is most easily written in terms of the (linearized) connections

$$\begin{aligned} \omega_{ijk} &\equiv D_j e_{ki} - D_k e_{ji}, \\ \bar{\omega}_{ijk} &\equiv \bar{D}_j e_{ik} - \bar{D}_k e_{ij}, \\ \omega_i &\equiv \bar{D}^j e_{ij} - 2D_i \phi, \\ \bar{\omega}_i &\equiv D^j e_{ji} - 2\bar{D}_i \phi, \end{aligned} \tag{3.1}$$

where the derivatives D and $\bar{D}$ are defined in terms of the doubled derivatives and the constant background $E_{ij} = G_{ij} + B_{ij}$ encoding the background metric and B-field,

$$D_i = \partial_i - E_{ij}\tilde{\partial}^j, \qquad \bar{D}_i = \partial_i + E_{ji}\tilde{\partial}^j. \tag{3.2}$$

For completeness, we give the inhomogeneous terms in the gauge transformation of e_{ij} and the associated transformations of the connections,

$$\begin{aligned} &\delta_\lambda e_{ij} = D_i \bar{\lambda}_j + \bar{D}_j \lambda_i, \\ &\delta_\lambda \omega_{ijk} = \bar{D}_i K_{jk}, \qquad \delta_\lambda \bar{\omega}_{ijk} = D_i \bar{K}_{jk}, \\ &\delta_\lambda \bar{\omega}_i = \bar{D}^j \bar{K}_{ji}, \qquad \delta_\lambda \omega_i = D^j K_{ji}, \end{aligned} \tag{3.3}$$

where

$$K_{ij} \equiv 2D_{[i}\lambda_{j]}, \qquad \bar{K}_{ij} \equiv 2\bar{D}_{[i}\bar{\lambda}_{j]}. \tag{3.4}$$

In the two-derivative DFT, the variation with respect to e_{ij} yields the generalized Ricci tensor, i.e.,

$$\delta_e S^{(2)} = \frac{1}{2}\int \delta e^{ij} \mathcal{R}_{ij}, \tag{3.5}$$

which can be written in terms of connections as

$$\mathcal{R}_{ij} \equiv \bar{D}^k \bar{\omega}_{ikj} - D_i \bar{\omega}_j \equiv D^k \omega_{jki} - \bar{D}_j \omega_i. \tag{3.6}$$

These two forms are equivalent as can be verified by the use of Eq. (3.1).

Let us now give the cubic, four-derivative DFT$^-$ Lagrangian, which we denote as $\mathcal{L}_-^{(3,4)}$. The result (from Eq. (6.27) in Ref. [9]) reads

$$\begin{aligned} \mathcal{L}_-^{(3,4)} = \frac{1}{32}(&\bar{\omega}^{pij}\omega_i{}^{kl} D_p \omega_{jkl} - \omega^{pij}\bar{\omega}_i{}^{kl}\bar{D}_p \bar{\omega}_{jkl} \\ &+ \bar{\omega}^i{}_{kl}\bar{\omega}^{jkl} D_i \omega_j - \omega^i{}_{kl}\omega^{jkl}\bar{D}_i \bar{\omega}_j). \end{aligned} \tag{3.7}$$

In order to relate this action to a conventional one, we have to set $D_i = \bar{D}_i = \partial_i$ and find the required field redefinition to standard fields. The gauge transformations that leave the quadratic action plus the above correction invariant have first-order corrections in α'. These gauge transformations are given in Eq. (5.25) of Ref. [9] and, upon setting $D_i = \bar{D}_i = \partial_i$, result in

$$\delta_\lambda^{(1)} e_{ij} = -\frac{1}{8}\partial_i K^{kl}\omega_{jkl} + \frac{1}{8}\partial_j \bar{K}^{kl}\bar{\omega}_{ikl}. \tag{3.8}$$

We now claim that the field redefinition to standard fields $\check{e}_{ij}$ is given by

$$\check{e}_{ij} = e_{ij} + \Delta e_{ij}, \tag{3.9}$$

where

$$\Delta e_{ij} = \frac{1}{16}[\omega_i{}^{kl}\omega_{jkl} - \bar{\omega}_i{}^{kl}\bar{\omega}_{jkl} - 2\partial_{[i}e^{kl}(\omega_{j]kl} - \bar{\omega}_{j]kl})]. \tag{3.10}$$

We first confirm that this redefinition leads to fields with the expected gauge transformations. The $\mathcal{O}(\alpha')$ transformation of $\check{e}$ is then corrected by the lowest-order gauge variation of Δe_{ij},

$$\delta^{(1)}\check{e}_{ij} \equiv \delta^{(1)} e_{ij} + \delta^{(0)}(\Delta e_{ij}). \tag{3.11}$$

A straightforward computation shows that many terms cancel, leaving

$$\delta^{(1)}\check{e}_{ij} = -\frac{1}{16}\partial_{[i}(K^{kl}+\bar{K}^{kl})(\omega_{j]kl}+\bar{\omega}_{j]kl}) - 2\partial_{[i}e^{kl}\partial_{j]}(K_{kl}-\bar{K}_{kl}). \tag{3.12}$$

Note that this result is manifestly antisymmetric in i, j, showing that, as expected, $\delta^{(1)}$ is trivialized on the metric fluctuation. The final term can be removed by a parameter redefinition and can hence be ignored. The remaining term can be further rewritten by using the relations (Eq. (5.51) in Ref. [9]) between the DFT gauge parameters λ and the diffeomorphism parameter ϵ^i:

$$K^{kl}+\bar{K}^{kl} = 2\partial^{[k}(\lambda^{l]}+\bar{\lambda}^{l]}) = 4\partial^{[k}\epsilon^{l]}. \tag{3.13}$$

Similarly, the sum of the DFT connections reads in conventional fields

$$\omega_{jkl}+\bar{\omega}_{jkl} = 4\partial_{[k}h_{l]j} \equiv -4\omega^{(1)}_{jkl}, \tag{3.14}$$

with $\omega^{(1)}_{jkl} \equiv -\partial_{[k}h_{l]j}$ the linearized spin connection. We finally obtain

$$\delta^{(1)}\check{e}_{ij} = \partial_{[i}\partial^k\epsilon^l\omega^{(1)}_{j]kl}, \tag{3.15}$$

the expected Green-Schwarz–deformed gauge transformation, recorded in Eq. (2.11) of Ref. [8].

We now perform the redefinition (3.10) in the quadratic two-derivative action, using Eq. (3.5),

$$\begin{aligned} S^{(2)}[e] &= S^{(2)}[\check{e}-\Delta e] \\ &= S^{(2)}[\check{e}] - \frac{1}{2}\int \Delta e_{ij}\mathcal{R}^{ij} \\ &\equiv S^{(2)}[\check{e}] + \int \Delta\mathcal{L}^{(2)}, \end{aligned} \tag{3.16}$$

giving

$$\begin{aligned} \Delta\mathcal{L}^{(2)} &= -\frac{1}{32}(\omega_i{}^{kl}\omega_{jkl}-\bar{\omega}_i{}^{kl}\bar{\omega}_{jkl})\mathcal{R}^{ij} \\ &\quad + \frac{1}{16}\partial_{[i}e^{kl}(\omega_{j]kl}-\bar{\omega}_{j]kl})\mathcal{R}^{ij} \\ &= -\frac{1}{32}[(\omega_i{}^{kl}\omega_{jkl}-\bar{\omega}_i{}^{kl}\bar{\omega}_{jkl})\mathcal{R}^{(ij)} \\ &\quad - 2\partial_{[i}e^{kl}(\omega_{j]kl}-\bar{\omega}_{j]kl})\mathcal{R}^{[ij]}]. \end{aligned} \tag{3.17}$$

The final cubic, four-derivative Lagrangian in terms of the physical fields e_{ij} (we now drop the check) is then given by $\Delta\mathcal{L}^{(2)}+\mathcal{L}^{(3,4)}$. Inserting the Ricci tensor into Eq. (3.17) and writing the action in terms of h_{ij}, b_{ij}, and ϕ, one finds that the terms involving the dilaton cancel in $\Delta\mathcal{L}^{(2)}+\mathcal{L}^{(3,4)}$. Moreover, it is relatively easy to see by inspection, using the connections (3.1) and the structure of the cubic action, that only terms with precisely one or three b-fields survive. The terms cubic in b turn out to combine into a total derivative. Up to total derivatives, the terms linear in b can be brought into the manifestly gauge invariant form

$$\Delta\mathcal{L}^{(2)}+\mathcal{L}^{(3,4)} = -\frac{1}{2}H^{ijk}\omega_i^{(1)pq}\partial_j\omega^{(1)}_{kqp}. \tag{3.18}$$

In order to verify this systematically, it is convenient to perform integrations by part so that the terms multiplying ∂b do not contain $\Box = \partial^i\partial_i$ or divergences. In this basis, the terms then organize into the above form, as may be verified by a somewhat lengthy but straightforward calculation. This form of the action linear in b is also fixed by gauge invariance.

We now want to identify the conventional covariant action that yields this $\mathcal{O}(\alpha')$ contribution upon expansion around flat space to cubic order. We will show that this action takes the form

$$S = \int d^Dx\sqrt{-g}e^{-2\phi}\left(R+4(\partial\phi)^2-\frac{1}{12}\hat{H}_{ijk}\hat{H}^{ijk}\right), \tag{3.19}$$

with the $\mathcal{O}(\alpha')$ corrections arising from the kinetic term for the Chern-Simons modified 3-form curvature:

$$\hat{H}_{ijk} = H_{ijk}+3\Omega_{ijk}(\Gamma). \tag{3.20}$$

Here,

$$\begin{aligned} H_{ijk} &= 3\partial_{[i}b_{jk]}, \\ \Omega_{ijk}(\Gamma) &= \Gamma^q_{[i|p|}\partial_j\Gamma^p_{k]q} + \frac{2}{3}\Gamma^q_{[i|p|}\Gamma^p_{[j|r|}\Gamma^r_{[k]q|}. \end{aligned} \tag{3.21}$$

Inserting this into the three-form kinetic term and expanding in the number of derivatives, one obtains

$$\begin{aligned} -\frac{1}{12}\hat{H}_{ijk}\hat{H}^{ijk} &= -\frac{1}{12}H_{ijk}H^{ijk} - \frac{1}{2}H^{ijk}\Omega_{ijk}(\Gamma) \\ &\quad - \frac{3}{4}\Omega^{ijk}(\Gamma)\Omega_{ijk}(\Gamma). \end{aligned} \tag{3.22}$$

In a perturbative expansion around the vacuum, the last term contains terms of quartic and higher power in fields, all with six derivatives, and will hence be ignored. Focusing on terms cubic in fields and with four derivatives, only the middle term contributes, via the quadratic part of the Chern-Simons term,

$$\mathcal{L}^{(3,4)} = -\frac{1}{2}H^{ijk}\Gamma^q_{ip}\partial_j\Gamma^p_{kq}. \tag{3.23}$$

This term agrees precisely with Eq. (3.18), as can be quickly verified using the relation

$$\Gamma_{ipq} = \partial_{[p}h_{q]i} + \frac{1}{2}\partial_i h_{pq} = -\omega^{(1)}_{ipq} + \frac{1}{2}\partial_i h_{pq}, \tag{3.24}$$

between the linearized spin and Christoffel connections. Thus, the DFT$^-$ action is entirely consistent with the covariant action (3.19). In particular, it is naturally written in terms of the *torsion-free* Levi-Civitá connection.

B. From torsionful to torsionless connections

We have shown that the covariant action that is equivalent to DFT$^-$ at order α' contains the Green-Schwarz deformation based on the torsion-free connection. At first sight, this seems to be in conflict with suggestions in the literature that T-duality requires a connection with torsion proportional to H, but we will discuss now that this question is in fact ambiguous since the theories written using different connections are related by field redefinitions, up to covariant terms.

We start by writing the Green-Schwarz modified curvature with a torsionful connection and the associated local Lorentz gauge transformation of the b-field, both in form notation,

$$\hat{H} = \mathrm{d}b + \frac{1}{2}\Omega\left(\omega - \frac{1}{2}\beta\underline{\hat{H}}\right),$$
$$\delta_\Lambda b = \frac{1}{2}\mathrm{tr}\left(\mathrm{d}\Lambda \wedge \left(\omega - \frac{1}{2}\beta\underline{\hat{H}}\right)\right), \tag{3.25}$$

where β is a constant and, as usual, $\delta_\Lambda\omega = \mathrm{d}\Lambda + [\omega, \Lambda]$. The underline on $\hat{H}$ denotes that it has been made into a matrix-valued one-form by converting curved into flat indices,

$$\underline{\hat{H}}^a{}_b = \hat{H}_i{}^a{}_b dx^i, \qquad \hat{H}_i{}^a{}_b \equiv \hat{H}_{ijk}e^{aj}e_b{}^k. \tag{3.26}$$

Note that the above $\hat{H}$ is iteratively defined; it is nonpolynomial in b and contains terms with an arbitrary number of derivatives. One can verify that $\hat{H}$ is gauge invariant: $\delta_\Lambda\hat{H} = 0$. Let us now consider the following field redefinition:

$$b' = b + \frac{1}{4}\beta\mathrm{tr}(\omega \wedge \underline{\hat{H}}). \tag{3.27}$$

Note that b' is nonpolynomial in b. We can quickly compute the new gauge transformation

$$\delta_\Lambda b' = \frac{1}{2}\mathrm{tr}\left(\mathrm{d}\Lambda \wedge \left(\omega - \frac{1}{2}\beta\underline{\hat{H}}\right)\right) + \frac{1}{4}\beta\mathrm{tr}(\mathrm{d}\Lambda \wedge \underline{\hat{H}})$$
$$= \frac{1}{2}\mathrm{tr}(\mathrm{d}\Lambda \wedge \omega), \tag{3.28}$$

where we used $\delta_\Lambda\hat{H} = 0$ and noted that, since Lorentz indices are fully contracted, we can ignore the transformation of the vielbeins in $\underline{\hat{H}}$ and use only the inhomogenous part $\mathrm{d}\Lambda$ of $\delta_\Lambda\omega$. Thus, we obtained a simple b'-independent b' transformation.

Next, we determine the redefined field strength. To this end, we need the behavior of the Chern-Simons three-form $\Omega(\omega)$ under a shift η of the one-form connection. One has

$$\Omega(\omega + \eta) = \Omega(\omega) + \mathrm{d}\mathrm{tr}(\eta \wedge \omega) + 2\mathrm{tr}(\eta \wedge R(\omega))$$
$$+ \mathrm{tr}\left(\eta \wedge D_\omega\eta + \frac{2}{3}\eta \wedge \eta \wedge \eta\right), \tag{3.29}$$

where $R(\omega)$ is the two-form curvature of ω and

$$D_\omega\eta = d\eta + \omega \wedge \eta + \eta \wedge \omega, \tag{3.30}$$

is the covariant derivative with connection ω. Writing $\eta = -\frac{1}{2}\beta\underline{\hat{H}}$, we get

$$\Omega\left(\omega - \frac{1}{2}\beta\underline{\hat{H}}\right) = \Omega(\omega) + \frac{1}{2}\beta\mathrm{d}\mathrm{tr}(\omega \wedge \underline{\hat{H}}) - \beta\mathrm{tr}(\underline{\hat{H}} \wedge R(\omega))$$
$$+ \frac{1}{4}\beta^2\mathrm{tr}\left(\underline{\hat{H}} \wedge D_\omega\underline{\hat{H}} - \frac{1}{3}\beta\underline{\hat{H}} \wedge \underline{\hat{H}} \wedge \underline{\hat{H}}\right). \tag{3.31}$$

Inserting this and the b-field redefinition (3.27) into the curvature in Eq. (3.25), one obtains

$$\hat{H} = \mathrm{d}b' + \frac{1}{2}\Omega(\omega) - \frac{1}{2}\beta\mathrm{tr}(\underline{\hat{H}} \wedge R(\omega))$$
$$+ \frac{1}{8}\beta^2\mathrm{tr}\left(\underline{\hat{H}} \wedge D_\omega\underline{\hat{H}} - \frac{1}{3}\beta\underline{\hat{H}} \wedge \underline{\hat{H}} \wedge \underline{\hat{H}}\right). \tag{3.32}$$

We identify $\hat{H}' \equiv \mathrm{d}b' + \frac{1}{2}\Omega(\omega)$ as the improved field strength that uses a torsion-free connection. This $\hat{H}'$ is gauge invariant under local Lorentz rotations due to Eq. (3.28). Therefore, we write

$$\hat{H} = \hat{H}' - \frac{1}{2}\beta\mathrm{tr}(\underline{\hat{H}} \wedge R(\omega))$$
$$+ \frac{1}{8}\beta^2\mathrm{tr}\left(\underline{\hat{H}} \wedge D_\omega\underline{\hat{H}} - \frac{1}{3}\beta\underline{\hat{H}} \wedge \underline{\hat{H}} \wedge \underline{\hat{H}}\right). \tag{3.33}$$

This equation determines $\hat{H}$ recursively in terms of $\hat{H}'$ and covariant objects based on ω. Thus, the field strength $\hat{H}$ differs from the "torsion-free" field strength $\hat{H}'$ by covariant terms. An action written with a Chern-Simons modified curvature with a torsionful connection can therefore be rewritten in terms of a curvature based on a torsion-free connection, up to further covariant terms, and vice versa. In particular, the Lagrangian (3.22) above that is most simply written in terms of the torsion-free connection could be rewritten in terms of torsionful connections and additional covariant terms. We conclude that asking which connection is preferred by T-duality is an ambiguous question. It may be, however, that writing the full theory to all orders in α' in

terms of conventional fields is easier with some particular choice of connection.

IV. DISCUSSION

In this paper, we have shown how to relate systematically the α'-deformed DFT constructed in Ref. [1] to conventional gravity actions as arising in string theory. The recursive procedure that expresses the double metric $\mathcal{M}$ in terms of the generalized metric $\mathcal{H}$ can, in principle, be applied to an arbitrary order in α'.

By restricting ourselves to first order in α', we have shown that the gauge transformations are precisely those in the Green-Schwarz mechanism, with Chern-Simons–type deformations of the gauge transformations. We have also shown that the action at order $\mathcal{O}(\alpha')$ is given by the terms following from $\hat{H}^2$, where $\hat{H}$ is the Chern-Simons improved curvature of the b-field. In particular, in the simplest form of the action, the Chern-Simons form is based on the (minimal) torsionless Levi-Civitá connection. DFT thus makes the prediction that switching on just the Green-Schwarz deformation (without the other corrections present in, say, heterotic string theory) is compatible with T-duality at $\mathcal{O}(\alpha')$, something that to our knowledge was not known.

It is tempting to believe that there should be some way to describe the full α'-deformed DFT, to all orders in α', using conventional fields. As an important first step, one could try to find out what the theory is at $\mathcal{O}(\alpha'^2)$. The Green-Schwarz deformation based on the torsion-free connection leads to pure metric terms with six derivatives and nothing else. Could this be equivalent to the full DFT? In a separate paper, we analyze the T-duality properties of the Green-Schwarz modification by conventional means, using dimensional reduction on a torus, elaborating on and generalizing the techniques developed by Meissner [3]. We find that the minimal Green-Schwarz modification is not compatible with T-duality at $\mathcal{O}(\alpha'^2)$ [21]. It then follows that starting at $\mathcal{O}(\alpha'^2)$ DFT describes more than just the Green-Schwarz deformation. We leave for future work the precise determination of these duality invariants which, for instance, could include Riemann-cubed terms.

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