Design of a lane departure driver-assist system under safety specifications: Theorems and Proofs

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Abstract

This note contains full proofs of all the results provided in the submission "On a semi-autonomous lane departure assist system" to the 2016 Conference on Decision and Control by the same authors, [2].

Introduction

This note is a complement to [2] and its main objective is to provide full proofs of the results presented in the mentioned paper. Unless stated otherwise all the notations are as in [2] and also the numbers of theorems refer to that paper. Additional material is provided in the Appendix.

Main results

Theorem 3.1. *Let* $x^0 = (U^0, V^0, r^0, \psi^0, d_l^0, d_r^0) \in \mathcal{W}_+ \cap \mathcal{W}_-$ *be such that*

(i) $(U^0, V^0, r^0) \in [U_{min}, U_{max}] \times [-\tilde{V}, \tilde{V}] \times [-\tilde{r}, \tilde{r}]$*;*

(ii) $\psi^0 \in]-\tilde{\psi}, \tilde{\psi}[$*.*

Define $\pi^s \colon X \rightsquigarrow [-\bar{\tau}_w, \bar{\tau}_w] \times [-\bar{\delta}_f, \bar{\delta}_f]$ *by*

$$\pi^s(x) := \begin{cases} (\tau_w^s, \bar{\delta}_f) & \text{if } x \in \overline{\mathcal{W}_+^c}, \\ (\tau_w^s, -\bar{\delta}_f) & \text{if } x \in \overline{\mathcal{W}_-^c} \setminus \overline{\mathcal{W}_+^c}, \\ [-\bar{\tau}_w, \bar{\tau}_w] \times [-\bar{\delta}_f, \bar{\delta}_f] & \text{otherwise.} \end{cases}$$

Then the corresponding flow satisfies for every $t \in \mathbb{R}_+$ *such that* $\mathbf{x}_1(s; \pi^s, x^0) \in [U_{min}, U_{max}]$ *for all* $s \in [0, t]$*,*

$$\mathbf{x}(t; \pi^s, x^0) \in \mathcal{S} \qquad \text{and} \qquad \mathbf{x}_4(t; \pi^s, x^0) \in]-\tilde{\psi}, \tilde{\psi}[.$$

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Proof. The proof requires some technical results that are provided in Appendix A. Fix an arbitrary x^0 satisfying the assumptions of Theorem 3.1. It suffices to show that $\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_+ \cap \mathcal{W}_-$ and $\hat{\mathbf{x}}_4(t;\pi^s,x^0) \in]-\tilde{\psi},\tilde{\psi}[$ for all $t \in \mathbb{R}_+$ such that

$$\hat{\mathbf{x}}_1(s;\pi^s,x^0) \in [U_{min}, U_{max}] \qquad \forall s \in [0,t]. \tag{1}$$

It will be convenient to denote the set of times $t \in \mathbb{R}_+$ for which (1) is satisfied by T. We start by showing that

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_+ \qquad \forall t \in T.$$

Assume to the contrary that there exists $t_1 \in T$ such that

$$x_1 \coloneqq \hat{\mathbf{x}}(t_1;\pi^s,x^0) \in \mathcal{W}_+^c. \tag{2}$$

Lemmas A.1 and A.2 state that in this case there exists $t_2 \in [0,t_1[\subset T$ satisfying

$$x_2 \coloneqq \hat{\mathbf{x}}(t_2;\pi^s,x^0) \in \partial\mathcal{W}_+ \subset \mathcal{W}_+, \tag{3}$$

and for all $t \in [t_2,t_1]$,

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \overline{\mathcal{W}_+^c}.$$

Thus from the very definition of π^s and (3) it follows that

$$x_1 = \hat{\mathbf{x}}(t_1 - t_2; \bar{\mathbf{d}}_f, x_2) \in \mathcal{W}_+,$$

contradicting (2). Next, let us define

$$t^* \coloneqq \inf\left\{ t \in T \;\middle|\; \hat{\mathbf{x}}_4(t;\pi^s,x^0) \in \{-\tilde{\psi},\tilde{\psi}\} \right\},$$

where we set $t^* = \infty$ if this set is empty. We show that for all $t \in [0,t^*[\cap T$,

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_-, \tag{4}$$

and that this in turn implies that $t^* = \infty$ which establishes the statement of the theorem.

Let us start by showing that (4) holds for all $t \in [0,t^*[\cap T$. We argue again by contradiction and assume that there exists $t_3 \in [0,t^*[$ such that $\hat{\mathbf{x}}(t_3;\pi^s,x^0) \in \mathcal{W}_-^c$. Applying as before Lemmas A.1 and A.2 there exists $t_4 \in [0,t_3[$ such that

$$x_4 \coloneqq \hat{\mathbf{x}}(t_4;\pi^s,x^0) \in \partial\mathcal{W}_- \subset \mathcal{W}_-,$$

and for all $t \in]t_4,t_3]$,

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_-^c. \tag{5}$$

Next we define

$$t_5 \coloneqq \min\left\{ t \in [t_4,t_3] \;\middle|\; \hat{\mathbf{x}}(t;\pi^s,x^0) \in \partial W_+ \right\},$$

where we use the convention that $t_5 = t_3$ if the set $\left\{ t \in [t_4,t_3] \;\middle|\; \hat{\mathbf{x}}(t;\pi^s,x^0) \in \partial W_+ \right\}$ is empty. Notice also that this set is compact, thus if it is not empty, then this minimum is well-defined. If $t_5 > t_4$, then for all $t \in [t_4,(t_5+t_4)/2]$

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \overline{\mathcal{W}_-^c} \setminus \overline{\mathcal{W}_+^c}.$$

A similar argument as above allows then to conclude that $\hat{\mathbf{x}}((t_5+t_4)/2;\pi^s,x^0) \in \mathcal{W}_-$, contradicting (5). It remains to consider the case $t_4 = t_5$, that is $x_4 \in \partial\mathcal{W}_+ \cap \partial\mathcal{W}_-$. We will show that in fact $x_4 \in \mathcal{W}_{0+} \cap \mathcal{W}_{0-}$ which then by Condition 2 establishes the contradiction.

Since $t_4 < t^*$, it is clear that

$$\hat{\mathbf{x}}_4(t_4;\pi^s,x^0) \in]-\tilde{\psi},\tilde{\psi}[. \tag{6}$$

Thus, by the characterization of the boundaries of $\mathcal{W}_+$ and $\mathcal{W}_-$ given in Proposition A.6 and Lemma A.4 there exist $t_+ \in \mathcal{T}_+(x_4)$ and $t_- \in \mathcal{T}_-(x_4)$ satisfying

$$\hat{\mathbf{x}}_6(t_+;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4) = \hat{\mathbf{x}}_5(t_-;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) = 0.$$

Hence, if $t_+ > 0$ or $t_- > 0$ then the maps $t \mapsto \hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)$ and $t \mapsto \hat{\mathbf{x}}_5(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)$ have a local minimum in t_+, respectively t_-. This establishes that in this case $\dot{\hat{\mathbf{x}}}_6(t_+;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4) = 0$, respectively $\dot{\hat{\mathbf{x}}}_5(t_-;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) = 0$. It remains to consider the cases when either t_+ or t_- equals zero. Let us start with the case when $t_+ = 0$. Since $x_4 \in \mathcal{W}_+$, $\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_+$ for all $t \in \mathbb{R}_+$ and (6) holds true, there exists $\epsilon > 0$ such that

$$\begin{aligned} \hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4) &\geq 0 \quad \forall t \in [0,\epsilon], \\ \hat{\mathbf{x}}_6(t;\pi^s,x^0) &\geq 0 \quad \forall t \in [t_4-\epsilon,t_4]. \end{aligned}$$

This yields

$$0 \leq \dot{\hat{\mathbf{x}}}_6(0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4) = \dot{\hat{\mathbf{x}}}_6(t_4;\pi^s,x^0) \leq 0.$$

Consequently, $\dot{\hat{\mathbf{x}}}_6(0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4) = 0$ as desired.

Consider now the case when $t_- = 0$, that is $\hat{\mathbf{x}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) = 0$. By (5), for every $n \in \mathbb{N}$ large enough,

$$x^n := \hat{\mathbf{x}}(1/n;\pi^s,x_4) \in \mathcal{W}_-^c.$$

That is, there exists $t^n \in \mathcal{T}_-(x^n)$ such that

$$\hat{\mathbf{x}}_5(t^n,\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n) < 0. \tag{7}$$

By continuity and Lemma A.3 it is clear that the sequence $(t^n,x^n)_{n\in\mathbb{N}}$ is contained in a compact set. Consequently there exists a convergent subsequence, for convenience still denoted by $(t^n,x^n)_n$. Its limit is denoted by $(\bar{t},\bar{x})$. Clearly $\bar{x} = x_4$ and by continuity of the flows

$$\hat{\mathbf{x}}_4(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) \leq \tilde{\psi} \quad \text{and} \quad \hat{\mathbf{x}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) \leq 0.$$

In fact, since $x_4 \in \mathcal{W}_-$ and $\hat{\mathbf{x}}_4(t_4;\pi^s,x^0) > -\tilde{\psi}$, it follows from Lemma A.4 and Corollary A.5 that in fact

$$\hat{\mathbf{x}}_4(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) < \tilde{\psi} \quad \text{and} \quad \hat{\mathbf{x}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) = 0. \tag{8}$$

We show that $\dot{\hat{\mathbf{x}}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) = 0$. This is clear if $\bar{t} > 0$ because $\hat{\mathbf{x}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)$ has a local minimum in $\bar{t}$ in this case. We can therefore assume that $\bar{t} = 0$. Moreover, since it follows readily from $x_4 \in \mathcal{W}_-$ and (8) that $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) \geq 0$, it suffices to exclude the case $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) > 0$. Indeed, if $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) > 0$ then there exist $\eta > 0$ and $\epsilon > 0$ such that

$$\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x) \geq \eta \qquad \forall x \in B(x_4,\epsilon).$$

By continuity of $t \mapsto \hat{\mathbf{x}}(t; \pi^s, x_4)$ there exists $\epsilon' > 0$ such that for all $t \in [0, \epsilon']$, $\hat{\mathbf{x}}(t; \pi^s, x^4) \in B(x_4, \epsilon/2)$. Similarly, the continuous function $(t, x) \mapsto \hat{\mathbf{x}}(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x)$ is uniformly continuous on the compact set $[0, 1] \times \overline{B(x_4, 1)}$. This implies in particular that for some $\epsilon'' > 0$ we have that for all $x \in \overline{B(x_4, 1)}$ and all $t_1, t_2 \in [0, 1]$ satisfying $|t_1 - t_2| \le \epsilon''$,

$$\left\| \hat{\mathbf{x}}(t_1; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) - \hat{\mathbf{x}}(t_2; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) \right\| \le \frac{\epsilon}{2}.$$

Using convergence of the sequence $t^n \to 0$ when $n \to \infty$, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} \le \epsilon' \quad \text{and} \quad t^n \le \epsilon''.$$

It is then easy to check that for all $s \in [0, 1/n]$ and all $t \in [0, t^n]$,

$$\hat{\mathbf{x}}(s; \pi^s, x^4) \in B(x_4, \epsilon/2) \quad \text{and} \quad \hat{\mathbf{x}}(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x^n) \in B(x_4, \epsilon).$$

Finally, noticing that $x^n = \hat{\mathbf{x}}(1/n; \pi^s, x^4)$ we find

$$\begin{aligned}
\hat{\mathbf{x}}_5(t^n; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x^n) &= \hat{\mathbf{x}}_5(0; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x^n) + \int_0^{t^n} \dot{\hat{\mathbf{x}}}_5(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x^n) dt \\
&= \hat{\mathbf{x}}_5(0; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x_4) + \int_0^{1/n} \hat{\mathbf{x}}(t; \pi^s, x^4) dt + \int_0^{t^n} \dot{\hat{\mathbf{x}}}_5(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x^n) dt \\
&\ge \left(\frac{1}{n} + t^n \right) \eta > 0.
\end{aligned}$$

This however contradicts with (7). Hence $\dot{\hat{\mathbf{x}}}_5(0; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x_4) = 0$ in the case when $\bar{t} = 0$. The proof of (4) is complete.

It remains to deduce that $t^* = \infty$. Indeed, if $t^* < \infty$, then by (4)

$$\min \left\{ \hat{\mathbf{x}}_6(t^*; \pi^s, x^0), \hat{\mathbf{x}}_5(t^*; \pi^s, x^0) \right\} \ge 0.$$

Furthermore, since by Lemma A.2 the sets $\mathcal{W}_+$ and $\mathcal{W}_-$ are closed, it is also clear that

$$\hat{\mathbf{x}}(t^*; \pi^s, x^0) \in \mathcal{W}_+ \cap \mathcal{W}_-,$$

which however by [2, Condition 3] contradicts with the fact that $\hat{\mathbf{x}}_4(t^*; \pi^s, x^0) \in \{-\tilde{\psi}, \tilde{\psi}\}$. The proof is complete. □

Proposition 3.2. *If* $\mathcal{V}_1 > 0$ *then* $\mathcal{W}_{0+} \cap \mathcal{W}_{0-} = \emptyset$.

Proof. Clearly it is equivalent to show that if $\mathcal{W}_{0+} \cap \mathcal{W}_{0-} \neq \emptyset$ then $\mathcal{V}_1 = 0$. For this let $x \in \mathcal{W}_{0+} \cap \mathcal{W}_{0-}$. By the very definition of these sets, see [2, Eq. (3)], it follows that there exist $s \in \overline{\mathcal{T}_+(x)}$ and $t \in \overline{\mathcal{T}_-(x)}$ such that

$$\begin{aligned}
\hat{\mathbf{x}}_6(s; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) &= \hat{\mathbf{x}}_5(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x)) = 0 \\
\dot{\hat{\mathbf{x}}}_6(s; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) &= \dot{\hat{\mathbf{x}}}_5(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) = 0.
\end{aligned}$$

Clearly, (s, t, x) is a feasible point of the optimization problem [2, Eq. (5)] achieving the lower bound 0. This allows to conclude that $\mathcal{V}_1 = 0$ and completes the proof. □

Proposition 3.3. *If $\mathcal{V}_2 > 0$ then for all $x \in X$ such that $\psi = -\tilde{\psi}$ and $d_r \leq W_\ell / \cos(\tilde{\psi})$ we have that $x \in \mathcal{W}_+^c$.*

Proof. It is equivalent to show that if there exits $x = (U, V, r, \psi, d_l, d_r) \in X$ such that $\psi = -\tilde{\psi}$ and $d_r \leq W_\ell \cos(\tilde{\psi})$ satisfying $x \in \mathcal{W}_+$ then $\mathcal{V}_2 \leq 0$. Let x be as above. As we discuss in more detail in Appendix A, see in particular the proof of Proposition A.6, the map $x \mapsto \hat{\mathbf{x}}_6(\cdot; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ is such that if $x_1 = (U, V, r, \psi)$ and $d_r^1 \leq d_r^2$ then for all $t \in \mathbb{R}_+$, we have

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, (x_1, d_l^1, d_r^1)) \leq \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, (x_1, d_l^2, d_r^2)) \qquad \forall t \in \mathbb{R}_+. \tag{9}$$

Since $x = (x_1, d_l, d_r) \in \mathcal{W}_+$, $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0$ for all $t \in \mathcal{T}_+(x)$. Furthermore, as $d_r \leq W_\ell / \cos(\tilde{\psi})$, using (9) and defining $\tilde{x} = (x_1, 0, W_\ell / \cos(\tilde{\psi}))$ it follows that

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \tilde{x})) \geq 0 \qquad \forall t \in \mathcal{T}_+(\tilde{x}) = \mathcal{T}_+(x).$$

Finally, using that by Lemma A.3 the set $\mathcal{T}_+(\tilde{x})$ is bounded, it follow from the intermediate value theorem that there exists $\bar{t} \in \mathbb{R}_+$ for which $\hat{\mathbf{x}}_4(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \tilde{x}) = 0$. Consequently $\mathcal{V}_2 \leq 0$. □

Appendix

A Some properties of the sets $\mathcal{W}_+$ and $\mathcal{W}_-$

In this section we provide some technical results used in the proof of the safety result Theorem 3.1.

Lemma A.1. *Let $A \subset \mathbb{R}^n$ be a closed set and $\phi \colon \mathbb{R}_+ \to \mathbb{R}^n$ be continuous. Let $\phi(0) \in A$ and assume there exists $\bar{t} \in \mathbb{R}_+$ such that $\phi(\bar{t}) \in A^c$. Then there exists $t^* \in [0, \bar{t}[$ such that $\phi(t^*) \in \partial A$ and $\phi(t) \in A^c$ for all $t \in]t^*, \bar{t}]$.*

Proof. The set $\mathcal{T} \coloneqq \phi^{-1}(A) \cap [0, \bar{t}]$ is compact by continuity of ϕ and does not contain $\bar{t}$. Defining $t^* \coloneqq \max \mathcal{T}$, we observe that $\phi(t^*) \in A$, $t^* < \bar{t}$ and $\phi(t) \in A^c$ for all $t \in]t^*, \bar{t}]$. It remains to show that $\phi(t^*) \in \partial A$.

For this we show that for every $\epsilon > 0$ there exists $y \in B(\phi(t^*), \epsilon) \cap A^c$. Let $\epsilon > 0$ be arbitrary. By continuity of ϕ there exists $\delta > 0$ such that for all $t \in]t^* - \delta, t^* + \delta[$ we have $|\phi(t^*) - \phi(t)| \leq \epsilon$. Thus it follows that $y \coloneqq \phi(t^* + \delta/2) \in B(\phi(t^*), \epsilon) \cap A^c$. The proof is complete. □

We use the statement of Lemma A.1 to prove properties of the sets $\mathcal{W}_+$ and $\mathcal{W}_-$. This, in particular can be done since these sets are closed by the following result.

Lemma A.2. *The sets $\mathcal{W}_+$ and $\mathcal{W}_-$ are closed.*

Proof. We show that $\mathcal{W}_+$ is closed, the proof for $\mathcal{W}_-$ is analogous. For all $t \in \mathbb{R}_+$ let us define

$$\mathcal{W}_+(t) = \left\{ x \;\middle|\; \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0 \wedge \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < \tilde{\psi} \right\} \cup \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi} \right\}. \tag{10}$$

It follows from the very definition that $\mathcal{W}_+ = \bigcap_{t \in \mathbb{R}_+} \mathcal{W}_+(t)$. Thus it suffices to show that for all $t \in \mathbb{R}_+$, $\mathcal{W}_+(t)$ is closed. This is however clear since (10) can be written as

$$\begin{aligned} \mathcal{W}_+(t) &= \left(\left\{ x \mid \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0 \right\} \cap \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < \tilde{\psi} \right\} \right) \cup \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi} \right\} \\ &= \left\{ x \mid \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)) \geq 0 \right\} \cup \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi} \right\}. \end{aligned}$$

□

The rest of the results are used to characterize the boundary of the sets $\mathcal{W}_+$ and $\mathcal{W}_-$. We start with some preliminary results.

Lemma A.3. *There exists a constant $K > 0$ such that for all $x = (U, V, r, \psi, d_l, d_r) \in X$ with $\psi \in [-\tilde{\psi}, \tilde{\psi}]$ we have*

$$\mathcal{T}_+(x) \subset B(0, K) \qquad \text{and} \qquad \mathcal{T}_-(x) \subset B(0, K).$$

Proof. Let $x = (U, V, r, \psi, d_l, d_r) \in X$ be as in the statement of the lemma. By the system dynamics [2, Eq. 1] it is clear that for all $t \in \mathbb{R}_+$ we have

$$\dot{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \mathbf{d}_f, x) = \mathbf{x}_3(t; \bar{\mathbf{t}}_w, \mathbf{d}_f, x) \qquad \forall \mathbf{d}_f \in S([-\bar{\delta}_f, \bar{\delta}_f]).$$

The first step consists of showing that there exist $\bar{t} \in \mathbb{R}_+$ and $\eta > 0$ such that for all $t \geq \bar{t}$,

$$\mathbf{x}_3(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \eta \qquad \text{and} \qquad \mathbf{x}_3(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) \leq -\eta. \tag{11}$$

We show only the former, the later is analogous. For this, let us introduce the notations

$$a_{11} = \frac{c_f + c_r}{m}, \qquad a_{12} = \frac{c_r l_r - c_f l_f}{m}, \qquad a_{21} = \frac{c_r l_r - c_f l_f}{J_z},$$
$$a_{22} = \frac{c_f l_f^2 + c_r l_r^2}{J_z}, \qquad b_1 = \frac{c_f}{m}, \qquad b_2 = \frac{c_f l_f}{J_z}.$$

By Assumption 2 all these constants are positive and for all $U \in [U_{min}, U_{max}]$ defining

$$A(U) = \begin{pmatrix} -\frac{a_{11}}{U} & \frac{a_{12}}{U} - U \\ \frac{a_{21}}{U} & -\frac{a_{22}}{U} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

the lateral dynamics can be written as

$$\begin{pmatrix} \dot{V} \\ \dot{r} \end{pmatrix} = A(U) \begin{pmatrix} V \\ r \end{pmatrix} + B\delta_f,$$

see [2, Eq. 1]. In the following we fix $U \in [U_{min}, U_{max}]$. Using Condition 1, a simple computation shows that the matrix $A(U)$ has conjugate complex eigenvalues with real part $\lambda_{Re}(U)$ and imaginary part $\lambda_{Im}(U)$ given by

$$\lambda_{Re}(U) = -\frac{a_{11} + a_{22}}{2U} < 0 \qquad \text{and} \qquad \lambda_{Im}(U) = \frac{\sqrt{4a_{21}U^2 - 4a_{12}a_{21} - (a_{11} - a_{22})^2}}{2U} > 0.$$

Notice that both $\lambda_{Re}(U)$ and $\lambda_{Im}(U)$ are increasing functions of U. Using Fulmer's method, see for instance [1, Section 9.4], we know that by setting

$$M(U) = (1/\lambda_{Im}(U))(A(U) - \lambda_{Re}(U)\text{Id}),$$

we get that

$$e^{A(U)t} = e^{\lambda_{Re}(U)t} \left(\text{Id} \cos(\lambda_{Im}(U)t) + M(U) \sin(\lambda_{Im}(U)t) \right).$$

Then, using that for all $t \in \mathbb{R}_+$,

$$\begin{pmatrix} \mathbf{x}_2(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \\ \mathbf{x}_3(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \end{pmatrix} = e^{A(U)t}\begin{pmatrix} V \\ r \end{pmatrix} + \bar{\delta}_f \int_0^t e^{A(U)(t-s)} B ds,$$

a simple computation shows that there exists a constant $C > 0$ such that

$$\begin{aligned} &\mathbf{x}_3(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \\ &\qquad \geq -e^{\lambda_{Re}(U_{max})t} C + \bar{\delta}_f \left(\frac{2a_{21}b_1 + (a_{11}-a_{22})b_2}{2U_{max}\left(\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2\right)} - \frac{\lambda_{Re}(U_{max})}{\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2} \right). \end{aligned}$$

Using Assumption 2, it is not difficult to check that $2a_{21}b_1 + (a_{11}-a_{22})b_2 > 0$. Thus setting,

$$\eta = \frac{\bar{\delta}_f}{2} \left(\frac{2a_{21}b_1 + (a_{11}-a_{22})b_2}{2U_{max}\left(\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2\right)} - \frac{\lambda_{Re}(U_{max})}{\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2} \right),$$

it follows from the fact that $e^{\lambda_{Re}(U_{max}t)} \to 0$ when $r \to \infty$ that there exists $\bar{t} \in \mathbb{R}_+$ such that $e^{\lambda_{Re}(U_{max})t} C \leq \eta$ for all $t \geq \bar{t}$. The first statement of (11) follows.

Next, we use (11) to deduce that there exists $K > 0$ such that for all $t \geq K$ and all $x \in X$ as in the Lemma, $\mathbf{x}_4(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \geq \tilde{\psi}$. First notice that by Condition 1 there exists a bounded over approximation $\mathcal{R}$ of the reachable set $\mathcal{R}(\tilde{V},\tilde{r})$ and hence $\bar{V}$ and $\bar{r}$ such that $\mathcal{R} \in [-\bar{V},\bar{V}] \times [-\bar{r},\bar{r}]$. Therefore $\mathbf{x}_3(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \in [-\bar{r},\bar{r}]$ for all $t \in \mathbb{R}_+$ and by assumption $\psi \in [-\tilde{\psi},\tilde{\psi}]$. This implies that

$$\mathbf{x}_4(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \geq \psi - \int_0^{\bar{t}} \bar{r} ds \geq -\tilde{\psi} - \bar{t}\bar{r}.$$

Thus for all $t \geq \bar{t}$ we have

$$\mathbf{x}_4(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = \mathbf{x}_4(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) + \int_{\bar{t}}^t \mathbf{x}_3(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \geq -\tilde{\psi} - \bar{t}\bar{r} + (t-\bar{t})\eta. \tag{12}$$

Setting $K = 2(\tilde{\psi} + \bar{t}\bar{r} + \eta\bar{t})/\eta$, it follows from (12) that for all $t \geq K$, $\mathbf{x}_4(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \geq \tilde{\psi}$. Notice that a similar argument allows to prove that for all $t \geq K$, and all $x \in X$ as in the Lemma, $\mathbf{x}_4(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x) \leq -\tilde{\psi}$.

Finally, since by definition of $\hat{\mathbf{x}}$ this implies that $\hat{\mathbf{x}}_4(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = \tilde{\psi}$ and $\hat{\mathbf{x}}_4(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x) = -\tilde{\psi}$ for all $t \geq K$ the assertion of the lemma follows. □

Lemma A.4. *Let $x = (U,V,r,\psi,d_l,d_r) \in \mathcal{W}_+$ be such that $\psi < \tilde{\psi}$ and there exists $\bar{t} \in \mathbb{R}_+$ such that $\hat{\mathbf{x}}_6(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = 0$. Then $\bar{t} \in \mathcal{T}_+(x)$. Moreover, an analogous statement holds for $x \in \mathcal{W}_-$.*

Proof. We argue by contradiction and assume that $\hat{\mathbf{x}}_4(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = \tilde{\psi}$. From the very definition of $\hat{\mathbf{x}}_6$ it follows then that

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_6(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) &= \left(\hat{\mathbf{x}}_6(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x)\hat{\mathbf{x}}_3(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) + U\right)\tan(\hat{\mathbf{x}}_4(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x)) + \hat{\mathbf{x}}_2(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \\ &= U\tan(\tilde{\psi}) + \hat{\mathbf{x}}_2(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \\ &\geq U_{min}\tan(\tilde{\psi}) - \bar{V}, \end{aligned}$$

where $\bar{V}$ is as in the proof of Lemma A.3. From Condition 3 it follows then that $\dot{\hat{\mathbf{x}}}_6(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) > 0$. By continuity of the map $t \mapsto \dot{\hat{\mathbf{x}}}_6(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x)$, there exists $\epsilon > 0$ such that

$$\hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) < 0 \quad \forall t \in [t-\epsilon,\bar{t}[.$$

Moreover, since $x \in \mathcal{W}_+$ and $\psi < \tilde{\psi}$, it is clear that $d_r \geq 0$. It follows therefore from Lemma A.1 that there exists $t_0 \in [0, t-\epsilon[$ for which

$$\hat{\mathbf{x}}_6(t_0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = 0 \quad \text{and} \quad \hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) < 0 \; \forall t \in \,]t_0, t-\epsilon]. \tag{13}$$

The fact that $x \in \mathcal{W}_+$ implies then also that $\hat{\mathbf{x}}_4(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = \tilde{\psi}$ for all $t \in \,]t_0, t-\epsilon]$ and hence by continuity, $\hat{\mathbf{x}}_4(t_0;\bar{\mathbf{d}}_f,x) = \tilde{\psi}$. The same arguments as above allow therefore to conclude that $\dot{\hat{\mathbf{x}}}_6(t_0;x) > 0$. However, by (13) we have also that

$$\dot{\hat{\mathbf{x}}}_6(t_0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = \lim_{s \to t_0+} \frac{\hat{\mathbf{x}}_6(s;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) - \hat{\mathbf{x}}_6(t_0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x)}{s - t_0} \leq 0,$$

which is impossible. □

The lemma has the following useful Corollary.

Corollary A.5. *Let $x = (U, V, r, \psi, d_l, d_r) \in \mathcal{W}_+$ such that $\psi < \tilde{\psi}$. Then*

$$\hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \geq 0 \qquad \forall t \in \mathbb{R}_+.$$

A similar statement holds true for $x \in \mathcal{W}_-$.

Proof. Assume to the contrary that there exists $\bar{t} \in \mathbb{R}_+$ such that $\hat{\mathbf{x}}_6(\bar{t};\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) < 0$. Since $x \in \mathcal{W}_+$ and $\psi < \tilde{\psi}$, $\hat{\mathbf{x}}_6(0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) \geq 0$. It follows then from Lemma A.1 that there exists $t_0 \in [0,\bar{t}[$ for which (13) holds true. As above in the proof of Lemma A.4 it then also follows that $\hat{\mathbf{x}}_4(t_0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = \tilde{\psi}$. This however contradicts with the fact that by Lemma A.4 $t_0 \in \mathcal{T}_+(x)$. □

Finally we are ready to provide a description of the boundary of $\mathcal{W}_+$ and $\mathcal{W}_-$ respectively.

Proposition A.6. *The following inclusions hold true:*

$$\begin{aligned} &\left\{x \in \mathcal{W}_+ \;\middle|\; \exists t \in \overline{\mathcal{T}_+(x)} \;\, s.t. \;\, \hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = 0\right\} \subset \partial\mathcal{W}_+, \\ &\left\{x \in \mathcal{W}_- \;\middle|\; \exists t \in \overline{\mathcal{T}_-(x)} \;\, s.t. \;\, \hat{\mathbf{x}}_5(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = 0\right\} \subset \partial\mathcal{W}_-. \end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned} \left\{x \in \partial\mathcal{W}_+ \;\middle|\; \psi < \tilde{\psi}\right\} &\subset \left\{x \in \mathcal{W}_+ \mid \exists t \in \mathcal{T}_+(x) \;\, s.t. \;\, \hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = 0\right\}, \\ \left\{x \in \partial\mathcal{W}_- \;\middle|\; \psi > -\tilde{\psi}\right\} &\subset \left\{x \in \mathcal{W}_- \mid \exists t \in \mathcal{T}_-(x) \;\, s.t. \;\, \hat{\mathbf{x}}_5(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) = 0\right\}. \end{aligned} \tag{15}$$

Remark A.7. Notice that the inclusion in (14) is strict. Indeed, consider $x \in X$ with $U \in [U_{min}, U_{max}]$ arbitrary and V and r be the steady state lateral velocity and yaw rate corresponding to U and the constant control input $\bar{\mathbf{d}}_f$. It is easy to check that in this case $r > 0$ and hence $t \mapsto \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ is an increasing function of time. Let $\psi = \tilde{\psi}$ and $d_r = 1$, i.e. $\hat{\mathbf{x}}_6(0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < -1$. Then it is clear that $x \in \mathcal{W}_+$ and $\overline{\mathcal{T}_+(x)} = \emptyset$. However, for all $\epsilon > 0$ small enough, $(U, V, r, \tilde{\psi} - \epsilon, d_l, d_r) \in \mathcal{W}_+^c$. Hence $x \in \partial \mathcal{W}_+$.

Proof of Proposition A.6. We show the result only in the case of $\mathcal{W}_+$. Similar arguments allow to prove the other case.

We start by showing (14). Let $\hat{x} \in \left\{ x \in \mathcal{W}_+ \,\middle|\, \exists t \in \overline{\mathcal{T}_+(x)} \text{ s.t. } \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \right\}$. Since $\hat{x} \in \mathcal{W}_+$ it suffices to show that for all $\epsilon > 0$ there exists $x \in B(\bar{x}, \epsilon)$ such that $x \notin \mathcal{W}_+$. Fix $\epsilon > 0$, recall that $\hat{x} = (\hat{U}, \hat{V}, \hat{r}, \hat{\psi}, d_l, \bar{d}_r)$ and set $x := (\hat{U}, \hat{V}, \hat{r}, \hat{\psi}, d_l, \bar{d}_r - \epsilon/2) \in B(\hat{x}, \epsilon)$. Consider next the linear time varying system with dynamics

$$f \colon (t, d_r) \mapsto \hat{\mathbf{x}}_2(t; \bar{\mathbf{d}}_f, \hat{U}, \hat{V}, \hat{r}) + \left(\hat{U} + d_r \hat{\mathbf{x}}_3(t; \bar{\mathbf{d}}_f, \hat{U}, \hat{V}, \hat{r}) \right) \tan(\hat{\mathbf{x}}_4(t; \bar{\mathbf{d}}_f, \hat{U}, \hat{V}, \hat{r}, \hat{\psi})), \tag{16}$$

where we omitted the arguments the flows do not depend on. By the definition of $\hat{\mathbf{x}}_6$ it is clear that for all $t \in \mathbb{R}_+$

$$\dot{\hat{\mathbf{x}}}_6(t; x) = f(t, \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)) \quad \text{and} \quad \dot{\hat{\mathbf{x}}}_6(t; \hat{x}) = f(t, \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x})).$$

Thus $t \mapsto \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ and $t \mapsto \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x})$ are the flows of the linear time varying system with dynamics given by (16). Since this is a 1-dimensional system, by uniqueness of solutions, it follows from $\bar{d}_r - \epsilon/2 < \bar{d}_r$ that for all $t \in \overline{\mathcal{T}_+(x)} = \overline{\mathcal{T}_+(\hat{x})}$,

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x}).$$

Finally, since there exists $\bar{t} \in \overline{\mathcal{T}_+(\hat{x})}$ such that

$$0 = \hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x}) > \hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x).$$

This shows that $x \notin \mathcal{W}_+$ and completes the proof of (14).

It remains to show (15). Let $\bar{x} = (U, V, r, \psi, d_l, d_r) \in \partial \mathcal{W}_+$ be such that $\psi < \tilde{\psi}$. Since by Lemma A.2 the set $\mathcal{W}_+$ is closed, it is clear that $\bar{x} \in \mathcal{W}_+$. We show that there exists $t \in \overline{\mathcal{T}_+(\bar{x})}$ such that $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = 0$. We argue by contradiction. Thus, assume that

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) > 0 \quad \forall t \in \overline{\mathcal{T}_+(\bar{x})}.$$

By Lemma A.3 there exists $K > 0$ such that

$$\mathcal{T}_+(x) \subset B(0, K) \qquad \forall x \in X_1 \times \mathbb{R}. \tag{17}$$

This in particular implies that $\overline{\mathcal{T}_+(\bar{x})}$ is compact. Consequently there exists $\eta_1 > 0$ such that

$$\min_{t \in \overline{\mathcal{T}_+(\bar{x})}} \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = \eta_1. \tag{18}$$

Next notice that by [2, Condition 3] there exists $\eta_2 > 0$ such that

$$\bar{V} / \tan(\tilde{\psi}) + \eta_2 = U_{min},$$

and therefore setting $\eta := \min\{\eta_1/2, \eta_2/(2\bar{r})\}$ we obtain

$$\bar{V}/\tan(\tilde{\psi}) + \eta\bar{r} < U_{min}. \tag{19}$$

We claim that for all $t \in \mathbb{R}_+$,

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) \geq \eta. \tag{20}$$

Assume to the contrary that there exists $\bar{t} \in \mathbb{R}_+$ such that $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) < \eta$. Since $\psi < \tilde{\psi}$ it follows from (18) that $\hat{\mathbf{x}}_6(0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) \geq \eta$. From Lemma A.1 we obtain then that there exists $t^* \in [0, \bar{t}[$ such that $\hat{\mathbf{x}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = \eta$ and

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) < \eta \qquad \forall t \in]t^*, \bar{t}]. \tag{21}$$

It follows readily from (21) that

$$\dot{\hat{\mathbf{x}}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) \leq 0.$$

On the other hand, since $\eta < \eta_1$ it follows from (18) that $t^* \in \mathcal{T}_+(\bar{x})^c$, that is, $\hat{\mathbf{x}}_4(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = \tilde{\psi}$. Furthermore by [2, Condition 3],

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) &= U\tan(\tilde{\psi}) + \hat{\mathbf{x}}_2(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) + \hat{\mathbf{x}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x})\hat{\mathbf{x}}_3(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x})\tan(\tilde{\psi}) \\ &\geq U_{min}\tan(\tilde{\psi}) - \bar{V} - \eta\bar{r}\tan(\tilde{\psi}) > 0,\end{aligned}$$

where the last inequality follows from (19). This establishes the desired contradiction and thus proves (20).

Next, the continuous function $(t, x) \mapsto \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ is uniformly continuous on the compact set $\overline{B(0,K)} \times \overline{B(\bar{x}, 1)}$. One can therefore find an $\epsilon > 0$ such that for all $x_1, x_2 \in B(\bar{x}, 1)$ with $\|x_1 - x_2\| \leq \epsilon$,

$$\left|\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x_1) - \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x_2)\right| \leq \frac{\eta}{2} \qquad \forall t \in \overline{B(0,K)}. \tag{22}$$

Take $x \in B(\bar{x}, \epsilon)$ arbitrary. By (17), (20) and (22),

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \frac{\eta}{2} \qquad \forall t \in \mathcal{T}_+(x).$$

We conclude that $B(\bar{x}, \epsilon) \subset \mathcal{W}_+$ which contradicts with the fact that $\bar{x} \in \partial\mathcal{W}_+$. This proves that there exists $t \in \overline{\mathcal{T}_+(\bar{x})}$ such that $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = 0$. Applying Lemma A.4 it is clear that actually $t \in \mathcal{T}_+(x)$ and the proof is complete. □

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