Localization from Superselection Rules in Translationally Invariant Systems

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(Received 9 October 2015; published 15 January 2016)

The cubic code model is studied in the presence of arbitrary extensive perturbations. Below a critical perturbation strength, we show that most states with finite energy are localized; the overwhelming majority of such states have energy concentrated around a finite number of defects, and remain so for a time that is near exponential in the distance between the defects. This phenomenon is due to an emergent superselection rule and does not require any disorder. Local integrals of motion for these finite energy sectors are identified as well. Our analysis extends more generally to systems with immobile topological excitations.

Recently there has been significant interest in the mechanism behind how isolated quantum systems thermalize. While the eigenstate thermalization hypothesis asserts that quantum many-body systems thermalize at the level of the eigenstates [1,2], there are well-known counterexamples to this proposal as well. In particular, a seminal work of Anderson implies that disorder can inhibit thermalization for noninteracting systems, as particles are localized indefinitely at some fixed location [3]. Recent studies indicate that the effect of localization can persist in the presence of interaction, even at finite energy densities [4–7]. These results suggest that disorder can cause the system to "freeze" in time, and hinders equilibration. There are also some translation-invariant systems which were proposed to be localized under generic disordered initial conditions [8–13]. However, currently there does not seem to be a decisive consensus as to whether such effects persist in the thermodynamic limit; see Ref. [14].

In this Letter, we point out that interacting quantum many-body systems can be localized by a completely different mechanism. This is due to an emergent superselection rule, and does not rely on disorder. We explicitly consider a locally interacting spin model, and find a set of states with manifestly localized spatial energy profile. These states remain almost invariant under Hamiltonian evolution, and constitute the overwhelming majority of the states at intensive finite energy. We also show that these properties remain stable against arbitrary perturbations, so long as the perturbation strength is smaller than some finite critical value. Therefore, this phenomenon is a robust property of the phase.

To be more precise, we consider an exactly solvable and translation-invariant spin Hamiltonian H_0. Given a weak perturbation, denoted as Y, we construct a set of states $|\psi_Y\rangle$ such that their energy is concentrated around a finite number of points with respect to the perturbed Hamiltonian. These states are quasieigenstates in the sense that

$$|\langle\psi_Y|e^{-it(H_0+Y)}|\psi_Y\rangle| \geq 1 - tL^\alpha \exp(-cL^\eta), \qquad (1)$$

where $\alpha, c, \eta > 0$ are constants and L is the smallest distance between the defects. Furthermore, these states span an overwhelming portion of the low energy subspace; the dimension of the span of these localized states is a $1-\epsilon$ fraction of the total low energy subspace dimension where ϵ vanishes in the thermodynamic limit. Note that localized states satisfying Eq. (1) are unlikely to exist in a system with a nontrivial dispersion relation; a localized state would spread out quickly.

While we only consider a system with a finite number of defects, we emphasize that our system is fundamentally different from noninteracting fermionic systems. The transport of the low-energy excitations are hindered by a novel form of emergent superselection rule, which does not arise in free systems. Interestingly, localization originates from strong interaction within our setup, contrary to what is observed in the context of disordered systems.

In the rest of this Letter, we shall introduce our model and explain how Eq. (1) is derived. We begin by explaining the central concept, the *locally gapped state.* Roughly speaking, the notion of a locally gapped state formalizes the intuition that two eigenstates do not mix with each other with respect to a perturbation if they are either (i) separated from each other with a large energy gap or (ii) the corresponding off-diagonal matrix element of the perturbation is sufficiently small. We then proceed by identifying all the low-energy states of our model that are locally gapped. They span the majority of the low-energy subspace. Last, we rigorously prove that the aforementioned properties remain stable in the presence of weak perturbations, and find a large set of approximate local integrals of motion. We shall adopt the following technical terminology: A function $f(x)$ *rapidly decays* if it scales like $x^\alpha \exp(-cx^\beta)$ for large x, where $c, \beta > 0$ and α are constants.

We remark that, though our analysis focuses on one particular model, a similar phenomenon is also observed in other models such as Chamon's spin model [15,16], other "cubic code" models [17–19], and Majorana models [20]. There exists an isolated excitation such that no operator can move it without creating extra excitations. A sparse configuration of these isolated excitations is locally gapped, and our analysis carries over.

Locally gapped states.—With respect to a Hamiltonian H, a state $|\psi\rangle$ is said to be *locally gapped* with a diameter d and an energy Δ, if for any state $|\phi\rangle$ orthogonal to $|\psi\rangle$, one of the following holds: (1) $|\langle\phi|H|\phi\rangle - \langle\psi|H|\psi\rangle| \geq \Delta > 0$, (2) $\langle\phi|O|\psi\rangle = 0$, whenever O is supported on a ball of diameter d. This means that $|\psi\rangle$ is separated by an energy gap of Δ from all the other states that are reachable by local operators from $|\psi\rangle$. Unless specified otherwise, we will be interested in locally gapped states with an extensive diameter and finite energy gap. Let us mention some examples. A nondegenerate ground state of a Hamiltonian with an energy gap is locally gapped. A less trivial example is the ground state sector of topologically ordered systems. The ground states can be only connected to each other by an operator whose support is extended across the system.

For clarity, we mention states that are *not* locally gapped. Consider a Hamiltonian $H = +\sum_j \sigma_j^z$ on a chain where σ^z is the Pauli-z matrix. Its ground state is locally gapped, but a first excited state $|\sigma_{j_0} = +1\rangle$, is not, because $\sigma_{j_0}^x \sigma_{j_0+1}^x |\sigma_{j_0} = +1\rangle = |\sigma_{j_0+1} = +1\rangle$ is an orthogonal first excited state with the same energy. These states are not locally gapped with diameter $d > 1$. This exemplifies a general observation: If the hopping of a quasiparticle is realized by a local operator, states with localized excitations are not locally gapped. This intuition is true for many translation-invariant systems, and it is thus tempting to conjecture that the only way to have a locally gapped excited state is through some strong disorder. However, we shall show that there exists a translation-invariant system with many locally gapped excited states.

Unperturbed cubic code Hamiltonian.—The cubic code model is defined on a simple cubic lattice with two qubits per site, where the Hamiltonian is a translation-invariant sum of the two terms [17].

$$H_0 = -J\sum_i G_i^Z + G_i^X \quad (J > 0),$$

$$G_i^Z = \sigma_{i,1}^z \sigma_{i,2}^z \sigma_{i-\hat{x},1}^z \sigma_{i-\hat{y},1}^z \sigma_{i-\hat{z},1}^z \sigma_{i-\hat{y}-\hat{z},2}^z \sigma_{i-\hat{z}-\hat{x},2}^z \sigma_{i-\hat{x}-\hat{y},2}^z,$$

$$G_i^X = \sigma_{i,1}^x \sigma_{i,2}^x \sigma_{i+\hat{x},2}^x \sigma_{i+\hat{y},2}^x \sigma_{i+\hat{z},2}^x \sigma_{i+\hat{y}+\hat{z},1}^x \sigma_{i+\hat{z}+\hat{x},1}^x \sigma_{i+\hat{x}+\hat{y},1}^x. \quad (2)$$

See Fig. 1 for the arrangement of the Pauli marices. Every term in the Hamiltonian commutes with each other, and thus the energy spectrum is discrete. For the ground state, both G_i^X and G_i^Z take eigenvalues of $+1$. The ground state subspace is degenerate and is topologically ordered in the
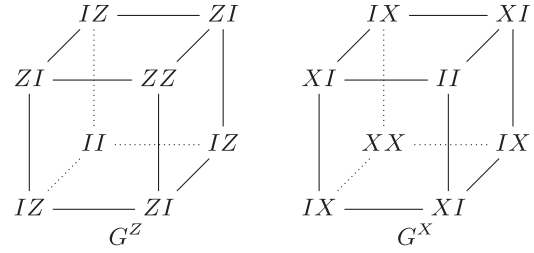

FIG. 1. Cubic code model. X and Z represent Pauli matrices σ^x and σ^z, respectively. A term in the Hamiltonian is the product of 8 Pauli matrices arranged as in the diagram.

sense that any local operator has the same expectation value for all the ground states. According to our definition, the ground state is locally gapped by an energy $\Delta = 2J$ and diameter $L - 1$. The excited states can be described by *defects*, which are the violated local terms, e.g., $G_i^X = -1$ or $G_i^Z = -1$. We refer to the types of the violated terms as the X type and Z type. Since the X-type term and Z-type term are lattice inversions of each other, it suffices to analyze just one of the two types.

The most general energy eigenstate is a superposition of valid configurations with the same number of defects. Note that the configuration of defects does not uniquely determine the state; e.g., the ground state has no defect, but is degenerate. Given the defect configuration, this is the only residual degeneracy, which we call the *topological degeneracy*.

Not every configuration of defects is physically allowed (*valid*), but it is known that under periodic boundary conditions there is an infinite family of system sizes (e.g., linear size $L = 2^n + 1$ for any integer $n \geq 1$ [21]) such that any defect configuration with an even number of X-type defects and an even number of Z-type defects is valid, and the topological degeneracy is 4. This is the family that we study here.

No-strings rule.—The most important property, called the *no-strings* rule [17,22], is a formalization of the fact that isolated defects are immobile. More precisely, suppose $|\psi\rangle$ is a state with a defect at site i and no other defects within distance d from i. If $|\psi'\rangle$ is another state with a defect at $i' \neq i$ and no other defect within distance d from i, then the no-strings rule asserts that $\langle\psi'|O|\psi\rangle = 0$ for any operator O supported on a ball of diameter d. This is quite nontrivial since i' can be even in the vicinity of i. See Ref. [23].

The no-strings rule implies the existence of low-energy excited states which are locally gapped. To see this, consider an excited state $|\psi\rangle$ describing a configuration of defects that are separated from one another by a distance d. Let e be the total number of defects, so that the energy of the state is $2Je$. If another state of the same energy, denoted as $|\phi\rangle$, has a configuration of defects different from that of $|\psi\rangle$, there must be a defect at i that is present in $|\phi\rangle$ but not in $|\psi\rangle$. By the construction of $|\psi\rangle$, the no-strings rule implies that the matrix element $\langle\phi|O|\psi\rangle$ is zero for any

operator O whose support is contained in a ball of diameter d. More generally, any state orthogonal to $|\psi\rangle$ with the same energy has two components: One represents different defect configurations with the same energy, and the other represents topological degeneracy. The transition from $|\psi\rangle$ to any orthogonal and topologically degenerate state requires an operator whose support is comparable to the system size. By the no-strings rule, the transition from $|\psi\rangle$ to any other state with a different defect configuration requires an operator whose support is at least d. Other states have energy that is different from $2Je$ by at least $2J$. Therefore, the no-strings rule implies that a configuration of separated defects by distance d is locally gapped with diameter d and energy $2J$.

We once again emphasize that the existence of the locally gapped excited states is not due to any disorder. It follows from the no-strings rule, a consequence of the special interaction. A correct interpretation should be that defects at different locations represent distinct *superselection* sectors. A similar yet weaker version of this phenomenon is observed in Wen's plaquette model [24]. There, a defect can be transported to next-nearest neighboring site by a local operator, but not to the nearest neighboring site. In other words, the superselection sector is changed under a unit translation, although it is not changed under two units of translations. In the cubic code model, there is no such finite periodicity, and, consequently, there are infinitely many superselection sectors.

Typicality of locally gapped states.—Now we show that almost all defect configurations are locally gapped. Without loss of generality, let us assume that m defects are all X type; the analysis for the Z-type defects is exactly the same. Any configuration is valid as long as m is even, and there is no further restriction on the defects' positions. The total number of states with a fixed energy will be given by the number of distinct configurations, multiplied by the topological degeneracy 4. A simple counting shows that the fraction of sparse configurations among all configurations is at least

$$\frac{V(V-v)\cdots[V-(m-1)v]}{V(V-1)\cdots(V-m+1)} \geq \left(1 - \frac{mv}{V}\right)^m, \quad (3)$$

where $v = (2d)^3$ is the volume of the box in which there is a single defect. The fraction approaches 1 algebraically in the system size V, provided that $d \sim L^{1-\epsilon}$, where $0 < \epsilon < 1$. We conclude that the majority of the sparse configurations is locally gapped. In the *unperturbed* system, the energy window where this holds can be as large as $O(V^{\epsilon'})$ from the ground, where $\epsilon' < \epsilon/2$.

There are excited states that are not locally gapped, though they form a vanishing fraction of the low-energy subspace. These are the excited states with locally created (topologically neutral) clusters of defects. They behave just as a single spin flip in the trivial Hamiltonian $\sum_i \sigma_i^z$, and

may hybridize to become a momentum eigenstate upon perturbations. Such states are inevitable in any translation-invariant system.

Perturbations.—Consider a one-parameter family of Hamiltonians of the following form:

$$H_s = H_0 + sY, \qquad s \in [0, 1],$$

where H_0 is the original cubic code Hamiltonian, Eq. (2), and Y is some perturbation. We assume that the perturbation consists of local terms of bounded operator norm $\leq J$.

The energy spectrum of H_0 is $E_n = 4Jn$ ($n = 0, 1, 2, \ldots$) up to overall shift. Since H_0 obeys local indistinguish-ability, the theorem of gap stability [25,26] implies that the spectrum of H_s is contained in the union of intervals $[E_n(1-cs), E_n(1+cs)]$ up to a small correction that vanishes in the thermodynamic limit. Here, the number $c > 0$ is a constant independent of the system size. Therefore, at energies below (J/cs), every energy band remains separated from the rest of the spectrum by a uniform gap $\Delta = O(J)$, even in the presence of a pertur-bation of strength s. Hence, we can unambiguously define projectors $P_s^{(n)}$ onto the *band subspace* corresponding to the energy window $[E_n(1-cs), E_n(1+cs)]$. Below, we will only consider states in these low-energy bands. By the machinery of quasiadiabatic continuation [27,28], this further implies that there exists a *locality-preserving* unitary U_s such that

$$P_s^{(n)} = U_s P_0^{(n)} U_s^\dagger, \qquad (4)$$

$$U_s^\dagger O U_s = \sum_r O_r', \qquad \|O_r'\| \leq f(r) \qquad (5)$$

for any operator O supported on a region M, where O_r' is some operator supported on a distance-r neighborhood of M and $f(r)$ is a rapidly decaying function. This is based on a well-known decomposition of a quasilocal operator into a telescopic sum of terms with bounded support [29,30]; see Ref. [23] for more detail.

Approximately locally gapped states.—By making use of these results, we now construct a large set of (approx-imately) locally gapped states,

$$|\psi_s\rangle = U_s|\psi_0\rangle, \qquad (6)$$

where $|\psi_0\rangle$ is any locally gapped stated with a diameter d. These states are approximately locally gapped in the following sense. Consider a transition to an orthogonal state $|\phi_s\rangle$. If $|\phi_s\rangle$ is not in the same band as $|\psi_s\rangle$, there is an energy gap Δ. If it is in the same band, these two states cannot be mapped into each other by a local operator. To see this, choose an operator O supported on a ball of diameter $d/2$ such that $\langle\phi_s|O|\psi_s\rangle = \delta$. This is equivalent to

the condition that $\langle \phi_0 | U_s^\dagger O U_s | \psi_0 \rangle = \delta$, where $|\phi_0\rangle = U_s^\dagger |\phi_s\rangle$ is a state that belongs to $P_0^{(m)}$. Using the decomposition Eq. (5), the operator $U_s^\dagger O U_s$ is quasilocal and hence can be approximated by an operator O' supported on the ball of diameter d such that $\|U_s^\dagger O U_s - O'\|$ decays rapidly in d. Since $|\psi_0\rangle$ is locally gapped with radius d, $\langle \phi_0 | O' | \psi_0 \rangle = 0$. It follows that δ decays rapidly with d. This establishes that $|\psi_s\rangle$ is approximately locally gapped with a diameter of $d/2$, up to an "error" that decays rapidly with d.

Energy profile.—Here, we show that the spatial energy profile of $|\psi_s\rangle$ is localized. In order to see this, recall that the local reduced density matrix is completely determined by the expectation values of local observables:

$$\langle \psi_s | O | \psi_s \rangle = \langle \psi_0 | U_s^\dagger O U_s | \psi_0 \rangle.$$

Since U_s preserves locality, the "dressed" operator $U_s^\dagger O U_s$ can be well approximated by some local operator, say O'. If this local approximation acts away from the defects, its expectation value with respect to $|\psi_0\rangle$ reproduces the ground state expectation value of the unperturbed system. If the ground states of the unperturbed and perturbed Hamiltonians are $|\Omega_0\rangle$ and $|\Omega_s\rangle$, respectively, the preceding discussion implies that $\langle \psi_s | O | \psi_s \rangle \simeq \langle \Omega_0 | O' | \Omega_0 \rangle = \langle \Omega_s | U_s O' U_s^\dagger | \Omega_s \rangle \simeq \langle \Omega_s | O | \Omega_s \rangle$. Therefore, any observable acting far from the defects cannot distinguish $|\psi_s\rangle$ from $|\Omega_s\rangle$. The error term decays rapidly in the distance from O to the defect's location. In contrast, the difference between the local reduced density matrices of a momentum eigenstate and that of the ground state can only be algebraically small ($x^{-\gamma}$ for some $\gamma > 0$) in the system size.

Dynamic properties.—Now we derive Eq. (1). The key insight is that the local Hamiltonian has small matrix elements between the locally gapped state $|\psi_s\rangle$ and the other states. To formalize this, let us fix a basis of the band subspace $P_s^{(m)}$ consisting of

$$|\tilde{1}\rangle = |\psi_s\rangle = U_s|\psi_0\rangle, \qquad |\tilde{q}\rangle = U_s|q\rangle \, (q > 1),$$

where $|q\rangle$ are energy eigenstates of H_0 with definite defect configurations. Here, $|\tilde{1}\rangle$ is approximately locally gapped, but $|\tilde{q}\rangle$ may not be so. In the Supplemental Material [23], we show that

$$\sum_{q \neq 1} |\langle \tilde{q} | H_s | \psi_s \rangle| \leq g(d), \qquad (7)$$

where g is a rapidly decaying function. We approximate the time evolution by neglecting these small matrix elements as follows. Let Q be a Hermitian matrix acting within $P_s^{(s)}$, defined as

$$\langle \tilde{u} | Q | \tilde{v} \rangle = \begin{cases} 0 & \text{if } u = 1 \neq v \quad \text{or} \quad u \neq 1 = v, \\ \langle \tilde{u} | H_s | \tilde{v} \rangle & \text{otherwise} \end{cases}.$$

$$(8)$$

Then, Eq. (7) implies that $\|P_s(Q - H_s)P_s\| \leq g(d)$, where we used the fact that the operator norm is at most the sum of absolute values of the matrix elements. This in turn implies that $\|e^{-itA} - e^{-itB}\| = \|1 - e^{itA}e^{-itB}\| = \|\int_0^t dw \partial_w \times (e^{iwA}e^{-iwB})\| \leq \int_0^{|t|} dw \|e^{iwA}(A-B)e^{-iwB}\| \leq |t|g(d)$, where $A = P_s H_s P_s$ and $B = P_s Q P_s$, from which we conclude that $|\langle \psi_s | e^{-itH_s} | \psi_s \rangle| \geq 1 - |t|g(d)$. This is Eq. (1).

Approximate local integrals of motion.—The definition of local integrals of motion in our context is a bit more relaxed than the ones that are discussed in the context of many-body localization. Since our system can be translation invariant including perturbations, we should not expect a local observable that commutes with the full Hamiltonian in general; however, there are local operators that almost commute with our Hamiltonian *within* the localized subspace. More concretely, let us define P_{loc} be the projector onto the localized subspace. This refers to the linear span of $U_s|q\rangle$, where $|q\rangle$ describes a state with defects which are separated by a distance d. Define

$$\mathcal{I}_j(s) = U_s h_j^{(0)} U_s^\dagger, \qquad (9)$$

where $h_j^{(0)}$ is the local term of the Hamiltonian H_0. This operator is quasilocal due to the locality-preserving property, Eq. (5).

Now let us see whether this operator commutes with the Hamiltonian within the localized subspace.

$$\|P_{\text{loc}}[H_s, \mathcal{I}_j(s)]P_{\text{loc}}\|$$
$$\leq \sum_{a,b} |\langle \tilde{a}|[H_s, \mathcal{I}_j(s)]|\tilde{b}\rangle| = \sum_{a,b} |(\ell_a - \ell_b)\langle \tilde{a}|H_s|\tilde{b}\rangle|$$
$$\leq \sum_{a \neq b} 2|\langle \tilde{a}|H_s|\tilde{b}\rangle| \leq 2(\dim P_{\text{loc}})g(d) \leq 2L^{3m}g(d),$$

$$(10)$$

where L is the linear system size. The first inequality is the triangle inequality, the second one follows from $\|\mathcal{I}_j(s)\| = 1$, and the third one is from Eq. (7). If we choose $d \sim L^{1-\epsilon}$ for a small $\epsilon \in (0,1)$, then the upper bound decays rapidly in the system size. Thus, $\mathcal{I}_j(s)$ is almost an integral of motion, for a large portion of the band by Eq. (3).

Discussion.—There is a considerable amount of work in the literature regarding the fate of the disorder-driven localized system in the presence of interaction; see, e.g., reviews [31,32]. Here we have taken a different route, and identified a mechanism for localization which is driven by a strong interaction. By exploiting the fact that the ground

state of our model is topologically ordered, we rigorously showed that the localized energy profile of the most of the low energy states remains unchanged under arbitrary perturbations. Our work strongly suggests that the prevalent dichotomous view on the role of disorder and interaction needs to be modified. That is, certain strong enough interactions can drive the system into an exotic topologically ordered phase, in which localization occurs at low energy.

An outstanding open problem is whether a locally gapped state can exist at finite energy density (extensive energy), and remain stable against perturbations. At such high energy, the gap between the bands would collapse upon perturbation, and the machinery of quasiadiabatic continuation in its present form cannot be applied.

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