On Wrapping Spheres and Cubes with Rectangular Paper

Alex Cole[1], Erik D. Demaine[1], and Eli Fox-Epstein[2*]

[1] MIT, Cambridge, MA, alexcole@csail.mit.edu, edemaine@mit.edu
[2] Brown University, Providence, RI, ef@cs.brown.edu

Abstract. What is the largest cube or sphere that a given rectangular piece of paper can wrap? This natural problem, which has plagued gift-wrappers everywhere, remains very much unsolved. Here we introduce new upper and lower bounds and consolidate previous results. Though these bounds rarely match, our results significantly reduce the gap.

1 Introduction

The problem of minimizing the amount of paper necessary to wrap a given 3-dimensional surface arises naturally from the economics of any factory packaging physical items. We study two closely related cases of this problem: given an $x \times 1/x$ unit-area rectangle of paper, what is the largest possible cube or sphere it can wrap?

Informally, we consider wrappings that do not stretch, cut, or intersect the paper with itself. We allow multiple layers of paper in the folding, and unlike [3], do not differentiate between the front and back of the paper. As in [5], to formally capture what it means to wrap a surface with non-zero curvature, we define a *wrapping* (a.k.a. *folding*) to be a *noncrossing, contractive mapping* from a 2-dimensional rectangle of paper to a subset of Euclidean 3-space. A contractive function ensures no pair of points move apart under their image. This definition captures the “crinkling” that you observe when, for example, you physically wrap a billiard ball with a sheet of paper. The noncrossing condition is difficult to formalize (see [7]); intuitively, it prohibits wrappings that cause surfaces to strictly intersect.

We present a variety of novel techniques that improve upper and lower bounds for wrapping both spheres and cubes. Figure 1 graphically summarizes these results. A *sphere wrapping* (respectively, *cube wrapping*) is a wrapping whose image is a sphere (cube). We denote cubes of sidelength S as S-cubes and spheres of radii R as R-spheres. Throughout, assume that $0 < x \leq 1$ so that x is the smaller side of our $x \times 1/x$ paper.

* Supported in part by NSF Grant CCF-0964037.

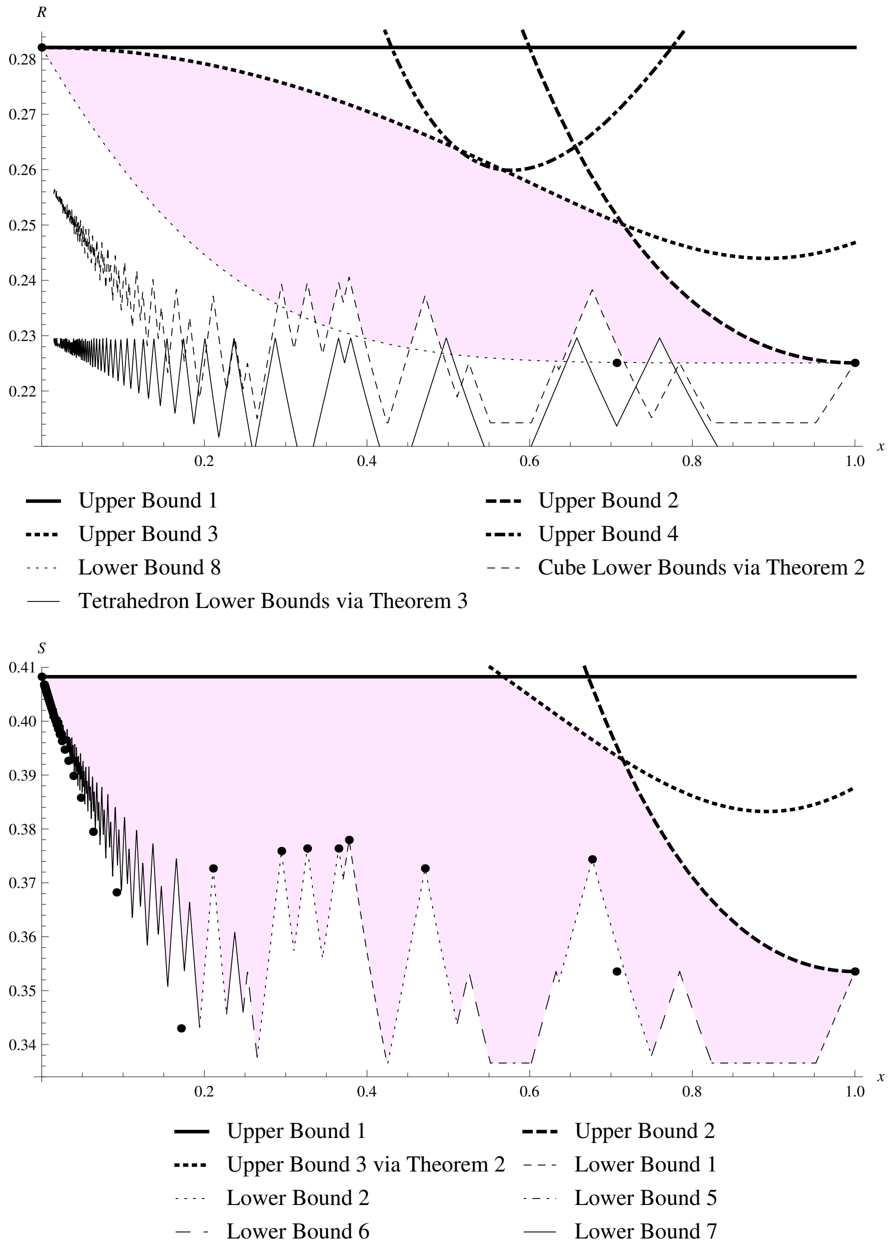

Fig. 1. A summary of the upper bounds (thick lines) and lower bounds (thin lines) on sphere (top) and cube (bottom) foldings. The horizontal axes indicate the smaller dimension of the unit-area paper. The vertical axes denote radii and side lengths, respectively. The shaded regions indicate the gaps between the bounds. See Section 6 for discussion.

2 Previous Results

2.1 Upper Bounds

Two techniques generated the previously known upper bounds on wrappings of spheres and cubes. The first is the surface area bound: the surface area of the image of a contractive mapping cannot exceed the surface area of the paper.

Upper Bound 1 (Folklore). *A unit-area rectangle of paper may wrap an S-sphere or an R-cube only if $S \leq 1/\sqrt{6}$ and $R \leq 1/(2\sqrt{\pi})$.*

Catalano-Johnson and Loeb [4] observe that every point on the S-cube has an antipodal point at least $2S$ away. Because wrappings are contractive, every point (particularly the center) on the original paper must also have another point that is $2S$ away, implying the paper's diagonal is at least $4S$. Demaine et al. [5] apply this argument to spheres.

Upper Bound 2 ([4, 5]). *An $x \times 1/x$ rectangle of paper may wrap an S-sphere or an R-cube only if $S \leq \sqrt{x^2 + x^{-2}}/4$ and $R \leq \sqrt{x^2 + x^{-2}}/(2\pi)$.*

The surface area bound becomes tight as x approaches 0 and the antipodal points bound is tight when $x = 1$. Between the endpoints, these bounds are likely far from optimal.

2.2 Lower Bounds

Numerous lower bounds for particular rectangles, some with unclear origins, exist in the form of physical foldings.

Lower Bound 1 ([9], [4], Folklore). *$1/\sqrt{7} \times \sqrt{7}$ paper wraps a $1/\sqrt{7}$-cube, and 1×1 and $1/\sqrt{2} \times \sqrt{2}$ papers each wrap a $1/(2\sqrt{2})$-cube.*

Akiyama, Ooya, and Segawa [3] produce a series of six efficient "symmetric-skew" wrappings which spiral the paper around the cube.

Lower Bound 2 ([3]). *$x \times 1/x$ paper wraps an S-cube for each (x, S) pair:*

$$\left(\sqrt{\tfrac{11}{24}}, \sqrt{\tfrac{37}{264}}\right), \left(\sqrt{\tfrac{2}{9}}, \sqrt{\tfrac{5}{36}}\right), \left(\sqrt{\tfrac{2}{15}}, \sqrt{\tfrac{17}{120}}\right),$$
$$\left(\sqrt{\tfrac{8}{75}}, \sqrt{\tfrac{17}{120}}\right), \left(\sqrt{\tfrac{2}{23}}, \sqrt{\tfrac{13}{92}}\right), \left(\sqrt{\tfrac{2}{45}}, \sqrt{\tfrac{5}{36}}\right).$$

Akiyama et al. [3] also invent a technique called *strip folding*, using extremely long, narrow rectangles to come arbitrarily close to the surface area bound.

Lower Bound 3 ([3]). *A strip of paper with $x = 1/\sqrt{24n^2 + 12n - 2}$ can wrap a $2n/\sqrt{24n^2 + 12n - 2}$-cube for integers $n \geq 1$.*

Demaine et al. [6] revisit strip folding, showing that any polyhedron can be wrapped by strip folding.

Sphere wrappings are less extensively studied than cube wrappings.

Lower Bound 4 ([5]). *1×1 and $1/\sqrt{2} \times \sqrt{2}$ rectangles wrap a $1/(\pi\sqrt{2})$-sphere.*

Demaine et al. [5] also apply strip folding to spheres but do not provide an explicit construction.

3 Upper Bounds

The following two techniques create new upper bounds on sphere wrapping and provide a substantial improvement over the previous upper bounds, as illustrated in Figure 1. Our general approach is to reduce the problem of bounding rectangular wrappings to simpler shapes like circles and stadiums.

3.1 Inscribed Stadiums on Spheres

As x approaches 1, Upper Bound 1 (surface area) becomes less effective, as it fails to account for necessary "crumpling" of the paper. This technique improves upon this by observing that a particular shape inscribed within paper must waste a certain amount of its surface area when mapped onto a sphere.

An $x \times y$ *stadium* is the Minkowski sum of a length-x line segment (called the *major path*) with a diameter-y disk. Refer to Figure 2.

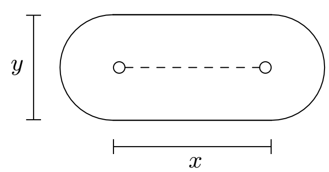

Fig. 2. $x \times y$ stadium, dashed major path.

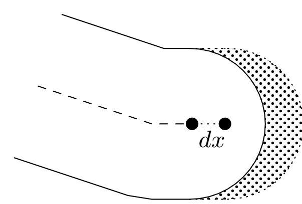

Fig. 3. Extension of a stadium by dx.

Proposition 1. *Given an $x \times y$ stadium S with major path P mapped onto a sphere by some contractive function f, let X be the points on the sphere within surface distance $y/2$ from $f(P)$. Then $f(S) \subset X$.*

Proof. On the flat paper, all of the points in S are within $y/2$ of the major path P. Because f is contractive, all of these distances can only decrease when S is mapped onto the sphere. $\square$

Proposition 2. *An $x \times y$ stadium of flat paper mapped onto an R-sphere may occupy no more surface area than*

$$A(x, y) = 2R\left(\pi R - \pi R \cos\frac{y}{2R} + x \sin\frac{y}{2R}\right).$$

Proof. To bound $A(x, y)$ we will first establish $A(0, y)$ and then bound the derivative dA/dx. This will allow us to bound the areas of all mapped stadiums.

First, consider an $0 \times y$ stadium: a radius-$y/2$ disk. By definition, the disk must fall within $y/2$ of its center on the sphere. A radius-$y/2$ spherical cap has area $2\pi R^2(1 - \cos\frac{y}{2R})$, proving the claim for $A(0, y)$.

Now consider an $x \times y$ stadium S with major path P. Let f be a contractive map to the sphere and A be the area of points within distance $y/2$ of $f(P)$ on the sphere. From Proposition 1, it suffices to bound A. Extend S by some length dx (see Figure 3). For sufficiently small dx, the extension of $f(P)$ runs along a geodesic. Call the added area dA. In Figure 3, this is the dotted region.

Now let $\theta = dx/R$. This is the central angle corresponding to a geodesic of length dx on the sphere. Extending P by dx will affect the latitudes within $y/2$ of our geodesic. Each latitude can be extended by at most an angle of θ. Let r_a be the radius of a circle of latitude at a spherical distance a from the equator. It is well-known that $r_a = R\cos\frac{a}{R}$. This yields:

$$dA \leq \int_{-y/2}^{y/2} \theta r_a da = \int_{-y/2}^{y/2} \frac{dx}{R} R\cos\frac{a}{R} da = 2R\sin\frac{y}{2R} dx$$

so $dA/dx \leq 2R\sin y/(2R)$. For a stadium of length x, the area on the sphere is bounded by $A(x, y)$ as desired. □

A $(1/x - x) \times x$ stadium can be inscribed within any $x \times 1/x$ paper rectangle. By Proposition 2, this stadium only occupies $A(1/x - x, x)$ area on the sphere. The remaining paper only has an area of $x^2 - \pi x^2/4$.

Upper Bound 3. *$x \times 1/x$ paper can wrap an R-sphere only if*

$$4\pi R^2 \;\leq\; x^2 - \pi x^2/4 \;+\; A(1/x - x, x).$$

3.2 n Circumscribed Circles on Spheres

Cutting the paper or adding more material can only increase the ability of the paper to wrap a sphere. With this as inspiration, we transform the paper into n congruent disks, and then relate upper bounds on spherical cap coverings back to rectangular sphere wrappings.

Proposition 3. *If an R-sphere can be wrapped by an $x \times 1/x$ paper, then it can also be wrapped by n congruent disks of diameter $\sqrt{x^2 + (nx)^{-2}}$.*

Proof. Consider a arbitrary wrapping from an $x \times 1/x$ paper to an R-sphere. Partition the flat paper into n small $x \times 1/(nx)$ rectangles. Circumscribe each $x \times 1/(nx)$ rectangle to get n disks of diameter $\sqrt{x^2 + (nx)^{-2}}$. These disks can contractively map into the original paper. □

Covering a sphere with n spherical caps is a well-studied problem (see e.g. [8, 11]) and for many values of n, bounds exist on how large the diameter d must be to admit a covering of a sphere of radius R. For $n = 1$, a disk of diameter d can wrap an R-sphere only if $d \geq 2\pi R$. This yields $\sqrt{x^2 + x^{-2}} \geq 2\pi R$, which is exactly Upper Bound 2! This generalization of the antipodal points bound is most useful when n is 1 or 3, where coverings are necessarily very wasteful.

Upper Bound 4. *An $x \times 1/x$ rectangle may wrap an R-sphere only if $R \leq \sqrt{x^2 + (3x)^{-2}}/\pi$.*

Proof. Three diameter-d disks can wrap an R-sphere only if $d \geq \pi R$ (see Table 2 of [11]). Composing with the contrapositive of Proposition 3 yields $R \leq \sqrt{x^2 + (3x)^{-2}}/\pi$. □

4 Lower Bounds

4.1 Rescaling Lower Bounds on Cubes

Most lower bounds on wrapping take the form of a construction for a specific x. Here we present a method to rescale particular foldings to produce a continuous set of lower bounds.

Theorem 1. *If $x \times 1/x$ paper wraps an S-cube, then there exists a folding of an $x' \times 1/x'$ rectangle into an $f(x')$-cube where $f(x') = S \min\{x'/x, x/x'\}$.*

Proof. Suppose $x' < x$. Uniformly scaling an $x \times 1/x$ rectangle by a factor of x'/x, yields an $x' \times x'/x^2$ rectangle, which wraps an Sx'/x-cube. An $x' \times 1/x'$ rectangle contracts to an $x' \times x'/x^2$ rectangle. A corresponding argument can be made for $x' > x$. □

4.2 Rectangle Conversions on Cubes

The rectangle-to-rectangle hinged dissection gadget of [1] inspires a technique to transform wrappings of a particular aspect ratio into general wrappings without any loss of efficiency.

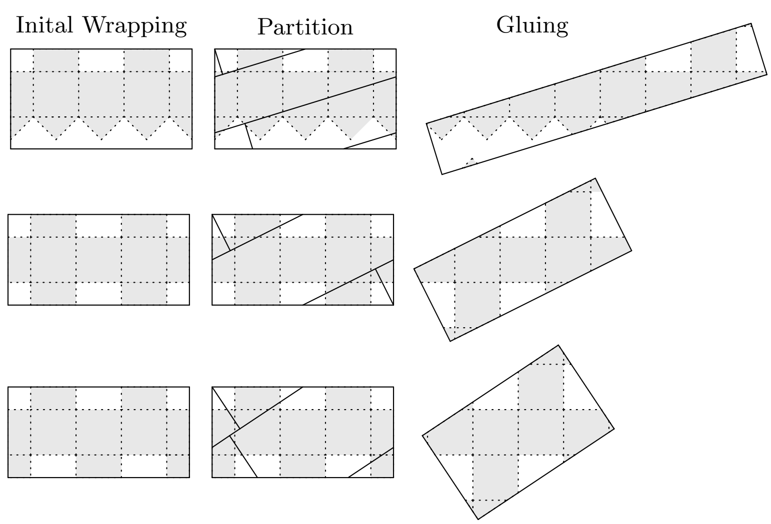

Fig. 4. The transformations used in Lower Bounds 5 (top) and 6 (middle and bottom).

Lower Bound 5. *Any unit-area rectangle wraps a $1/\sqrt{6+2\sqrt{2}}$-cube.*

Proof. The crease pattern in the top-left of Figure 4 shows a valid wrapping f of a $1/\sqrt{6+2\sqrt{2}}$-cube from an $x \times 1/x$ rectangle (fold each horizontal or vertical crease to a right angle) with these special properties:

1. Left and right edges of the paper map to same segment.
2. Left and right halves of the top edge map to the same segment.
3. The bottom edge maps to a point.

Wrapping f can be transformed into a wrapping from a different aspect ratio rectangle as follows. Partition an $x \times 1/x$ rectangle into some set of pieces P. Notice that one can still wrap a $1/\sqrt{6+2\sqrt{2}}$-cube with P by applying f to each part.

To create a new rectangular wrapping, we will "glue" P back together by identifying edges of elements of P such that they form an $x' \times 1/x'$ rectangle. To ensure this gluing produces a valid wrapping, identified points must map to the same point under f. Then f applied to each piece of P results in a contractive mapping f' from $x' \times 1/x'$ to the $1/\sqrt{6+2\sqrt{2}}$-cube.

To partition an $x \times 1/x$ rectangle and glue it into an $x' \times 1/x'$ rectangle, Montucla's dissection from [10] suffices. This is visualized in step 2 of Figure 4.

To glue the parts back together:

1. Identify the original left edge and the original right edge.
2. Identify the left half of the original top edge and the right half.
3. Identify the two small parts of the bottom edge with the large part.

Figure 4 illustrates this transformation. The first and second identifications correspond directly to special properties 1 and 2. The last one is valid because the entire bottom edge is mapped to a single point.

Varying the angle of the diagonal cut in the dissection creates rectangles of any aspect ratio. □

The efficiency of Lower Bound 5 can be increased by starting with a different wrapping. This causes the cuts to become more constrained, resulting in a discrete set of wrappings. This technique is similar to the tetrahedral wrappings in [2].

Lower Bound 6. *For any integer $n \geq 2$, a rectangle with $x = 2\sqrt{2}/\sqrt{n^2+4}$ wraps a $1/(2\sqrt{2})$-cube.*

The proof of this proposition is almost identical to that of Lower Bound 5. Figure 4 should give the reader the core ideas. The principal difference is that the bottom edge no longer maps to a single point, so the dissection is more constrained. Combining bounds on the middle and bottom wrappings in Figure 4 yields our lower bound. Interestingly, Lower Bound 6 for $n = 2$ reproduces the square folding by Catalano-Johnson and Loeb [4].

4.3 Strip Folding

Strip folding is a technique introduced in [3] that weaves a narrow strip of paper back and forth to cover a surface. This section sketches new strategies for strip folding that produce superior bounds on the sphere and the cube.

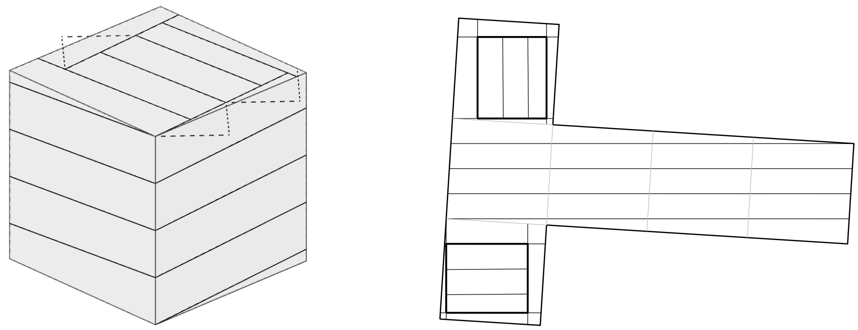

Fig. 5. Strip wrapping a cube. 3D diagram (left) and edge unfolding (right).

Cubes. Refer to Figure 5. Here we present a new technique for strip folding on the cube that is more efficient than that presented in [3]. The general strategy consists of 3 parts resembling an algorithm more than a function:

– Spiral around the 4 vertical faces of the cube (sides).
– Fold the excess over onto top and bottom faces.
– Try two different methods of doubling back and forth using turn gadgets (as seen in [3]) to cover the rest of the top and bottom faces.

We parameterize in terms of n, the number of times the top of the strip switches faces while covering the sides. For ease of computation, we require integral n.

The excess folded onto the top and bottom leaves a $w \times h$ rectangle uncovered on each, where $w = \frac{(n-5)S}{\sqrt{1+n^2}}$ and $h = \frac{(n-3)S}{\sqrt{1+n^2}}$. In Figure 5, the bold lines indicate these $w \times h$ rectangles.

Proposition 4. *A $w \times h$ rectangle can be covered by an $x \times f(w,h,x)$ rectangle of paper when one end of the paper starts outside the corner of the rectangle along the side of length h where*

$$f(w,h,x) = \min\left(2x\lceil h/x\rceil + w\lceil h/x\rceil - x, 2x\lceil w/x\rceil + h\lceil w/x\rceil\right).$$

Both parts of the minimum have the strip move straight in one direction until it would overlap some paper, then turning at a right angle twice to double back. Both terms start with an initial turn (or two) to enter the rectangle and orient to run parallel to a side. The second term is improved to $w + \left\lceil \frac{w}{x} \right\rceil (h+x)$ with a slightly more complicated construction.

After fixing n, we need two equations to solve for the width x and sidelength S. Combining the three lengths in the construction:

$$\underbrace{1/x}_{\text{total length}} = \underbrace{S\sqrt{1+n^2} + \sqrt{16S^2 - x^2}}_{\text{length to cover sides}} + \underbrace{2f\left(\frac{(n-5)S}{\sqrt{1+n^2}}, \frac{(n-3)S}{\sqrt{1+n^2}}, x\right)}_{\text{extra length for top and bottom}}. \quad (1)$$

Unrolling the spiraling portion of the strip and using similar triangles yields

$$x/S = 4/\sqrt{1+n^2}. \tag{2}$$

Lower Bound 7. *For any integer $n \geq 5$, $x \times 1/x$ paper wraps an S-cube where S satisfies Equations (1) and (2).*

When $n = 5$ and $n = 7$, we recover two of the foldings from [3].

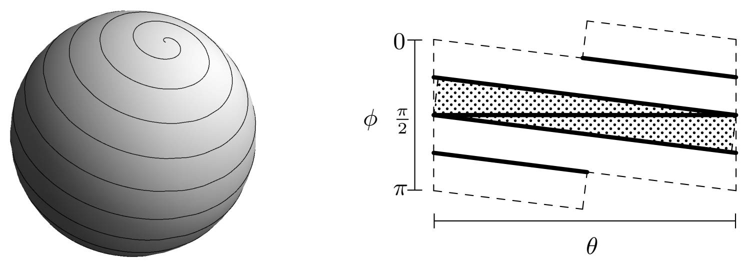

Fig. 6. Strip wrapping a sphere. Bold lines are mapped without any contraction.

Spheres. Any point on a sphere of fixed radius can be described by the two angles of spherical coordinates: θ (polar angle) and ϕ (azimuthal angle). The underlying strategy is to spiral with constant slope much in the way the 4 sides of the cube were wrapped. Figure 6 shows how the strip wraps a sphere by maintaining constant $d\theta/d\phi$. We focus on only the top hemisphere as the bottom follows by symmetry.

Start with an *initialization rectangle* of width x with the diagonal $2\pi R$ wrapped exactly around an equator. This rectangle is indicated by the dotted region in Figure 6. The top edge forms a line segment with $\frac{d\theta}{d\phi}$ held constant.

Now extend the initialization rectangle, spiraling up the sphere continuing to hold $\frac{d\theta}{d\phi}$ constant. Terminate when the center of the ends reaches the poles.

A "cut and paste" argument, similar to that used for Lower Bound 5, rearranges the ends of the strip to ensure the poles are covered. This is visualized in Figure 6 by moving the paper above $\phi = 0$ into the gap at the top and doing the same for $\phi = \pi$.

The length of the strip, $1/x$, is the length of the initialization rectangle plus the amount needed to spiral in each hemisphere: $1/x = L_{\text{init}} + 2L_{\text{spiral}}$. In calculating the lengths, care must be taken to only look at fully stretched paths because the rest of the strip is being contracted. These paths are bold in Figure 6.

The initialization rectangle has diagonal $2\pi R$ and height x, so its length is given by $L_{\text{init}} = \sqrt{(2\pi R)^2 - x^2}$.

Consider the upper hemisphere. By construction, in the upper hemisphere, the bottom of the strip incurs no contraction. Using the spherical arc length formula:

$$dl^2 = (R\sin(\phi)d\theta)^2 + (Rd\phi)^2 + dR^2 = Rd\phi\sqrt{\sin^2(\phi)\left(\frac{d\theta}{d\phi}\right)^2 + 1}$$

Integrating over ϕ gives the length:

$$L_{\text{spiral}} = \int dl = R\int_{\phi_0}^{\frac{\pi}{2}} \left(\sqrt{\sin^2(\phi)\left(\frac{d\theta}{d\phi}\right)^2 + 1}\right) d\phi$$

where ϕ_0 is the value of ϕ such that when the bottom of the strip is at an angle ϕ_0, the middle of the strip reaches the pole.

The slope $d\theta/d\phi$ is constant and thus equal to the ratio of the total change in θ to the total change in ϕ over the initialization rectangle. θ ranges from 0 to 2π. Similar triangles demonstrate that ϕ changes by $x/\sqrt{R^2 - (x/(2\pi))^2}$, so dividing gives

$$\frac{d\theta}{d\phi} = 2\pi\frac{\sqrt{R^2 - (x/(2\pi))^2}}{x} = \sqrt{(2\pi R/x)^2 - 1}.$$

The angular distance between the middle and the bottom of the strip is constant, so reasoning with similar right triangles in the initialization rectangle yields $\phi_0 = x/(2R)\sqrt{1 - (x/(2\pi R))^2}$.

Lower Bound 8. *An $x \times 1/x$ paper wraps an R-sphere if R satisfies*

$$\frac{1}{x} = \underbrace{\sqrt{(2\pi R)^2 - x^2}}_{\text{init}} + \underbrace{2\,R\int_{\frac{x}{2R}\sqrt{1-\left(\frac{x}{2\pi R}\right)^2}}^{\frac{\pi}{2}} \left(\sqrt{\sin^2(\phi)\left((2\pi R/x)^2 - 1\right) + 1}\right) d\phi}_{\text{each hemisphere}}$$

This bound is the first explicit sphere strip folding. It becomes optimally efficient as x tends to 0. In addition, because of how it handles the ends of the strip, it provides a powerful lower bound for larger values of x. When $x = 1$ we recover the optimal lower bound from [5].

5 Relating Cubes and Spheres

Given contractive mappings $f : A \to B$ and $g : B \to C$, $g \circ f$ constitutes a valid contractive mapping from A to C. With this as inspiration, we present mappings between spheres and cubes, allowing upper and lower bounds for one to be translated to the other. Upper Bound 3 for inscribed stadiums translates particularly well.

Theorem 2. *S-cubes can be contractively mapped to $(2S/\pi)$-spheres.*

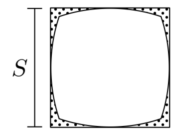

Fig. 7. One face of a cube with the area used by g drawn in.

Proof. Let f be our contractive mapping. Consider the Voronoi regions on a sphere induced by the six x-, y-, and z-extremal points. f will contractively map each face of the cube into one of these regions.

It suffices to examine one face F and the corresponding sixth of a sphere F'. Refer to Figure 7. Let the center of F be $(0, 0)$ and the center of F' be $(0, 0, R)$. Let $g : F' \to F$ be the map that sends a point with spherical coordinates $x = (R, \theta, \phi)$ to the polar point $g(x) = (R\phi, \theta)$ on the paper.

Now we will show g is expansive by looking at an infinitesimal neighborhood of an arbitrary x. Let $x = (R, \theta, \phi)$ and $\widetilde{x} = (R, \theta + d\theta, \phi + d\phi)$. Now let $dl_1 = \|x - \widetilde{x}\|$ and $dl_2 = \|g(x) - g(\widetilde{x})\|$. These are known as line elements. It is well-known that the sphere metric yields $dl_1^2 = (R \sin\phi d\theta)^2 + (Rd\phi)^2$. Doing the same about $g(x)$ with the metric on the paper gives us $dl_2^2 = (Rd\phi)^2 + (R\phi d\theta)^2$. Because $\sin^2\phi \leq \phi^2$, $dl_1 \leq dl_2$. These distances can be integrated into arclengths to show that all pairwise distances on the sphere are less than their images on the paper. Thus g is expansive, so $f = g^{-1}$ is contractive. The image of g is just a subset of F, but we can extend the domain of f to all of F by mapping the unused region to the boundary of F'.

The map f sends the line going through the centers of four faces of the cube to an equator of the sphere without any contraction. If S is the sidelength of the cube, then the resulting sphere will have radius $R = 2S/\pi$. This also shows f is optimal: no contractive mapping can produce larger spheres from a cube. $\square$

Theorem 3. *S-tetrahedra contractively map to $S/(2\sqrt{3} \arccos \frac{1}{\sqrt{3}})$-spheres.*

Theorem 3 is proved similarly to Theorem 2. Composing the tetrahedral wrappings in [2] with Theorem 3 improves the sphere lower bounds in some regions, as shown in Figure 1.

6 Conclusions

Figure 1 gives a complete picture of the upper and lower bounds on spheres and cubes, respectively. The horizontal axes denote the short paper dimension x. The vertical axes give radius R (for the sphere) and the sidelength S (for the cube). The shaded region is the area where the largest radius/sidelength could lie. To simplify presentation, lower bounds on cube foldings are only displayed when they are the best known. Previous lower bounds (Lower Bounds 1–4) are plotted as black dots.

Using Theorem 1, discrete constructions for cube lower bounds are transformed into a continuum. One surprise here is that the $1/\sqrt{2} \times \sqrt{2}$ wrapping of the $1/(2\sqrt{2})$-cube is less efficient than a rescaling of a construction from Lower Bound 2. Other results in Section 4 provided significant improvements over previous known bounds across a variety of aspect ratios.

The two new spherical upper bounds from Section 3 greatly improve upon previous bounds, especially for intermediate values of x. Upper Bound 3 (inscribed stadiums), in particular, is such an improvement that it transfers to the cube using contractive mappings, creating the first new cube upper bound since 2001. Upper Bound 4 is generally weaker but still provides an improvement for some aspect ratios. Quite surprisingly, composing Upper Bound 4 with Theorem 2 gives an upper bound tangent to Upper Bound 1 (surface area). Finally, the contractive mappings translate cube and tetrahedron wrappings to the sphere, elevating the lower bounds.

Acknowledgments. This research began in an open problem session and final project for MIT class 6.849: Geometric Folding Algorithms in Fall 2012. Thanks to Stephen Face for fruitful discussion and to Zachary Abel and Martin Demaine for their assistance with references.

References

[1] T. G. Abbott, Z. Abel, D. Charlton, E. D. Demaine, M. L. Demaine, and S. Kominers. Hinged dissections exist. *Discrete & Computational Geometry*, 47(1):150–186, 2012.

[2] J. Akiyama, K. Hirata, M. Kobayashi, and G. Nakamura. Convex developments of a regular tetrahedron. *Computational Geometry: Theory and Applications*, 34(1):2–10, 2006.

[3] J. Akiyama, T. Ooya, and Y. Segawa. Wrapping a cube. *Teaching Mathematics and its Applications*, 16(3):95–100, 1997.

[4] M. L. Catalano-Johnson, D. Loeb, and J. Beebee. Problem 10716: A cubical gift. *American Mathemetical Monthly*, 108(1):81–82, 2001.

[5] E. D. Demaine, M. L. Demaine, J. Iacono, and S. Langerman. Wrapping spheres with flat paper. *Computational Geometry: Theory and Applications*, 42(8):748–757, 2009.

[6] E. D. Demaine, M. L. Demaine, and J. S. B. Mitchell. Folding flat silhouettes and wrapping polyhedral packages: New results in computational origami. *Computational Geometry: Theory and Applications*, 16(1):3–21, 2000.

[7] E. D. Demaine and J O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*, pages 179–182. Cambridge University Press, 2007.

[8] P. W. Fowler and T. Tarnai. Transition from circle packing to covering on a sphere: the odd case of 13 circles. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 455:4131–4143, 1999.

[9] M. Gardner. *New Mathematical Diversions*, chapter 5: Paper Cutting, pages 58–69. Simon and Schuster, 1966.

[10] J. Ozanam. *Récréations Mathématiques et Physiques*, pages 297–302. C. A. Jombert, 1790.

[11] T. Tarnai and Zs. Gáspár. Covering a sphere by equal circles, and the rigidity of its graph. *Mathematical Proceedings of the Cambridge Philosophical Society*, 110:71–89, 7 1991.