A note on light geometric graphs

Eyal Ackerman* Jacob Fox† Rom Pinchasi‡

March 19, 2012

Abstract

Let G be a geometric graph on n vertices in general position in the plane. We say that G is *k-light* if no edge e of G has the property that each of the two open half-planes bounded by the line through e contains more than k edges of G. We extend the previous result in [1] and with a shorter argument show that every k-light geometric graph on n vertices has at most $O(n\sqrt{k})$ edges. This bound is best possible.

Keywords: Geometric graphs, k-near bipartite.

1 Introduction

Let G be an n-vertex *geometric graph*. That is, G is a graph drawn in the plane such that its vertices are distinct points and its edges are straight-line segments connecting corresponding vertices. It is usually assumed, as we will assume in this paper, that the set of vertices of G is in general position in the sense that no three of them lie on a line.

A typical question in geometric graph theory asks for the maximum number of edges that a geometric graph on n vertices can have assuming a forbidden configuration in that graph. This is a popular area of study extending classical extremal graph theory, utilizing diverse tools from both geometry and combinatorics. For example, an old result of Hopf and Pannwitz [3] and independently Sutherland [7] states that any geometric graph on n vertices with no pair of disjoint edges has at most n edges. This is a special case of Conway's thrackle conjecture.

Let e be an edge of G. We say that G has a *k-light side* with respect to e, if one of the two open half-planes bounded by the line through e contains at most k edges of G. If G has a *k-light side* with respect to *every* edge e, then we say that G is *k-light*. In other words, G is k-light if no edge of G has the property that each of the two open half-planes bounded by the line through e contains more than k edges of G.

The notion of a k-light graph is a weakening of the notion of a k-near bipartite graph defined in [1]. A graph G is k-near bipartite if every line in the plane bounds an open half plane containing at most k edges of G. Therefore, every k-near bipartite graph is also a k-light graph. It is shown in [1] that k-near bipartite graphs on n vertices contain $O(\sqrt{k}n)$ edges. In this paper we prove the same result for k-light graphs, thus strengthening the result in [1]. Moreover, our proof is much shorter but on the other hand relies on other results about geometric graphs.

*Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel. ackerman@sci.haifa.ac.il.

†Department of Mathematics, MIT, Cambridge, MA 02139-4307. Research supported by a Simons Fellowship and NSF grant DMS-1069197. fox@math.mit.edu.

‡Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. room@math.technion.ac.il. Supported by BSF grant (grant No. 2008290).

2 The maximum number of edges in k-light geometric graphs

We are interested in the maximum number of edges of an n-vertex k-light geometric graph. A simple construction from [1] shows an $n\sqrt{k}$ lower bound for $k \leq \left(\frac{n}{2}-1\right)^2$, even for k-near bipartite graphs. In this construction every line contains at most k edges of G in one of the two open half-planes bounded by it. Another construction of a k-light graph with $n\sqrt{k}$ edges is obtained by taking the vertices of a regular n-gon and connecting by edges vertices whose cyclic distance is at most $\sqrt{k}$. In this construction, however, it is no longer true that every line bounds an open half-plane containing at most k edges of G.

Our main result shows that these constructions are essentially best possible.

Theorem 1. *Let n and k be positive integers. Every n-vertex k-light geometric graph has at most $O(n\sqrt{k})$ edges.*

Proof. Let G be an n-vertex k-light geometric graph with m edges. We orient every edge e of G in such a way that the open half-plane bounded to the left of e contains at most k edges of G. Because G is k-light such an orientation exists.

We will need the following two lemmas.

Lemma 2.1. *Let G be an oriented geometric graph on n vertices. There exists an absolute constant c_3 such that if G has more than c_3n edges, then it contains an edge e such that the open half-plane bounded to the left of e contains an edge of G.*

Proof. It is enough to show that in any (unoriented) geometric graph G with n vertices and sufficiently many (that is, at least c_3n) edges there is an edge e such that each of the two open half-planes bounded by the line through e contains an edge of G. This is in fact the case $k = 1$ in Theorem 1 that we wish to prove. The reader is encouraged to find a simple proof of this fact. Here we will rely on a rather elaborate argument of Valtr [8] that proves a much stronger statement than what we need.

We refer the reader to [5, 4, 8]. Two edges of a geometric graph are called *avoiding* or sometimes *parallel* if no line passing through one edge meets the other edge. Equivalently, two edges are avoiding if they are opposite edges in a convex quadrilateral.

The notion of avoiding edges was first defined by Kupitz [5], who conjectured that any geometric graph on n vertices with more than $2n-2$ edges must contain a pair of avoiding edges. In [4] it is shown that if a graph G on n vertices does not contain a pair of avoiding edges, then the number of edges in G is at most $2n-1$. In [8] Valtr improved this bound by one, completing the proof of Kupitz' conjecture. He further generalized this result, showing that for any fixed k, every geometric graph with more than c_kn edges contains k pairwise avoiding edges. Here c_k is an absolute constant that depends only on k.

In fact, Valtr's result is a bit stronger. Looking into the proof in [8] reveals that he actually shows that a geometric graph with more than c_kn edges contains k edges $e_1, \ldots, e_k$ that are pairwise avoiding, but what is more important to our needs is that the line through e_i separates $e_1, \ldots e_{i-1}$ from $e_{i+1}, \ldots, e_k$. More specifically, Valtr defines three partial orders on a set of edges in G and any chain with respect to any of the partial orders is a collection of such edges. It is then shown that if the number of edges in G is large enough, then there exists a chain of length k in one of the partial orders.

Thus, for the case $k = 3$ it follows that if G contains more than c_3n edges, then there are three pairwise avoiding edges e, f, g such that the line through f separates e and g. This immediately implies Lemma 2.1, as in any orientation of f the half-plane bounded to the left of f will contain an edge of G. ∎

Lemma 2.2. *Let G be an oriented geometric graph on n vertices with m edges. There exists a positive absolute constant d with the following property. If the number of edges in G is greater than $2c_3n$ (where c_3 is the constant from Lemma 2.1), then G contains at least dm^3/n^2 pairs of edges (e, f) such that the open half-plane bounded to the left of e contains f.*

Proof. This is by now a quite standard consequence of the result in Lemma 2.1 and is carried out by a similar probabilistic technique used to derive a similar bound for the number of pairs of crossing edges in a geometric graph (see p. 55 in [6], also p. 45 in [2]).

Denote by $x(G)$ the number of pairs of edges (e, f) in G such that the open half-plane bounded to the left of e contains f. Pick every vertex of G independently with probability p, and denote by $G' = (V', E')$ the subgraph of G that is induced by the chosen vertices. Clearly, $\mathbb{E}[|V'|] = pn$, $\mathbb{E}[|E'|] = p^2m$, and $\mathbb{E}[x(G')] = p^4x(G)$. On the other hand, it follows from Lemma 2.1 that $x(G') \geq |E'| - c_3|V'|$, and this holds also for the expected values: $\mathbb{E}[x(G')] \geq \mathbb{E}[|E'|] - c_3\mathbb{E}[|V'|]$. Plugging in the expected values and setting $p = 2c_3n/m < 1$ we get that $x(G) \geq \frac{1}{8c_3^2}\frac{m^3}{n^2}$. ∎

Let c_3 and d be the constants from Lemmas 2.1 and 2.2. Clearly we may assume that G contains at least $2c_3n$ edges or else we are done. By Lemma 2.2, G contains at least dm^3/n^2 pairs (e, f) of edges such that the open half-plane bounded to the left of e contains f. However, by the choice of orientation of the edges in G, an edge e can belong to at most k such pairs (e, f). We conclude that $dm^3/n^2 \leq km$. This now easily implies that $m \leq \frac{1}{\sqrt{d}}n\sqrt{k}$ as desired. ∎

References

[1] E. Ackerman and R. Pinchasi, On the light side of geometric graphs, *Discrete Math.*, in press.

[2] S. Felsner, Geometric graphs and arrangements. Some chapters from combinatorial geometry. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Wiesbaden, 2004.

[3] H. Hopf and E. Pannwitz, Aufgabe Nr. 167, *Jahresbericht. Deutsch. Math.-Verein.* **43** (1934), 114.

[4] M. Katchalski and H. Last, On geometric graphs with no two edges in convex position, *Discrete Comput. Geom.* **19** (1998), no. 3, Special Issue, 399–404.

[5] Y. Kupitz, On pairs of disjoint segments in convex position in the plane, *Ann. Discrete Math.* **20** (1984), 203–208.

[6] J. Matoušek, Lectures on discrete geometry. Graduate Texts in Mathematics, 212. Springer-Verlag, New York, 2002.

[7] J. W. Sutherland, Lösung der Aufgabe 167, *Jahresbericht Deutsch. Math.-Verein.* **45** (1935), 33–35.

[8] P. Valtr, On geometric graphs with no k pairwise parallel edges, *Discrete Comput. Geom.* **19** (1998), no. 3, 461–469.