Erdős-Hajnal-type theorems in hypergraphs

David Conlon[*]   Jacob Fox[†]   Benny Sudakov[‡]

Abstract

The Erdős-Hajnal conjecture states that if a graph on n vertices is H-free, that is, it does not contain an induced copy of a given graph H, then it must contain either a clique or an independent set of size $n^{\delta(H)}$, where $\delta(H) > 0$ depends only on the graph H. Except for a few special cases, this conjecture remains wide open. However, it is known that a H-free graph must contain a complete or empty bipartite graph with parts of polynomial size.

We prove an analogue of this result for 3-uniform hypergraphs, showing that if a 3-uniform hypergraph on n vertices is $\mathcal{H}$-free, for any given $\mathcal{H}$, then it must contain a complete or empty tripartite subgraph with parts of order $c(\log n)^{\frac{1}{2}+\delta(\mathcal{H})}$, where $\delta(\mathcal{H}) > 0$ depends only on $\mathcal{H}$. This improves on the bound of $c(\log n)^{\frac{1}{2}}$, which holds in all 3-uniform hypergraphs, and, up to the value of the constant $\delta(\mathcal{H})$, is best possible.

We also prove that, for $k \geq 4$, no analogue of the standard Erdős-Hajnal conjecture can hold in k-uniform hypergraphs. That is, there are k-uniform hypergraphs $\mathcal{H}$ and sequences of $\mathcal{H}$-free hypergraphs which do not contain cliques or independent sets of size appreciably larger than one would normally expect.

1   Introduction

One version of Ramsey's theorem states that, for any s and t, there is a natural number n such that, in any 2-colouring of the edges of the complete graph K_n, there is either a red K_s or a blue K_t. The smallest number n for which this holds is the Ramsey number $r(s,t)$. A classical result of Erdős and Szekeres [14] implies that, for $s \geq t$, we have

$$r(s,t) \leq \binom{s+t-2}{t-1} \leq s^{t-1}.$$

Phrasing this differently, we see that every graph on n vertices which contains no copy of K_t must have an independent set of size $n^{1/(t-1)}$.

We call a graph H-free if it contains no induced copy of a given graph H. The famous conjecture of Erdős and Hajnal [11] states that if a graph on n vertices is H-free then it must contain either a clique or an independent set of size $n^{\delta(H)}$, where $\delta(H) > 0$ depends only on the graph H. The observations

of the last paragraph imply that this conjecture is true when H is a complete graph. Suppose now that we know the conjecture to be true for two graphs H_1 and H_2. Erdős and Hajnal showed that the conjecture will then also hold for a graph H formed by connecting the vertices of H_1 and H_2 by either a complete or an empty bipartite graph. This easily allows one to prove the conjecture for all graphs with at most 4 vertices except P_4, the path with 3 edges.

A more general result, due to Alon, Pach and Solymosi [2], says that if H_1 and H_2 have the Erdős-Hajnal property then so does the graph H formed by blowing up a vertex of H_1 and replacing it with a copy of H_2. The conjecture is also known for some graphs which do not fall naturally into these inductively defined classes (but which may be added as base cases to widen the range of these classes). For example, Erdős and Hajnal noted that since graphs with no induced P_4 are perfect the conjecture is true for P_4. More recently, Chudnovsky and Safra [3] have shown that the conjecture holds for the bull, a self-complementary graph on 5 vertices consisting of a triangle with two pendant edges. The smallest cases that now remain open are the cycle C_5 and the path P_5.

For general H, Erdős and Hajnal proved that if a graph on n vertices is H-free then it must contain a clique or an independent set of size $e^{c(H)\sqrt{\log n}}$. This is a significant improvement over the bound of $c \log n$ which, by Ramsey's theorem, holds in all graphs, but it is still quite far from the conjecture. However, as observed in [12], their method does allow one to find complete or empty bipartite subgraphs each side of which are of polynomial size. Recently, Fox and Sudakov [17] went one step further by proving that there is either a complete bipartite graph or an independent set of polynomial size. For further related results, see [1, 4, 15, 16, 26, 28].

For 3-uniform hypergraphs, Erdős and Rado [13] proved that, in any 2-colouring of the edges of the complete graph $K_n^{(3)}$ on n vertices, there is a monochromatic clique of size $c \log \log n$. Phrased differently, this says that any 3-uniform hypergraph on n vertices contains either a clique or an independent set of size $c \log \log n$. Given the situation for graphs, it is tempting to conjecture that if a 3-uniform hypergraph on n vertices is $\mathcal{H}$-free, for some given $\mathcal{H}$, then there should be a clique or an independent set of size much larger than $\log \log n$. We feel, but have been unable to prove, that for general $\mathcal{H}$ this may be too much to expect.

However, at least for some excluded hypergraphs, an Erdős-Hajnal-type estimate does appear to hold. Let $r_3(s, t)$ be the smallest number n such that, in any 2-colouring of the edges of $K_n^{(3)}$, there is a red $K_s^{(3)}$ or a blue $K_t^{(3)}$. A recent result of the authors [6], improving on the work of Erdős and Rado, implies that, for t fixed,
$$r_3(s, t) \leq 2^{c_t s^{t-2} \log s}.$$

It follows that any 3-uniform hypergraph with n vertices which does not contain a copy of $K_t^{(3)}$ must contain an independent set of size roughly $(\log n / \log \log n)^{1/(t-2)}$.

Even this is only progress in a weak sense. It is not yet known whether $\log \log n$ is the correct order of magnitude for Ramsey's theorem in 3-uniform hypergraphs. Indeed, this question has become rather notorious, Erdős offering $500 dollars for its resolution. The best construction, given by a random hypergraph, only tells us that there are hypergraphs on n vertices with no clique or independent set of size $c(\log n)^{\frac{1}{2}}$. So we cannot say with certainty whether the estimate for $K_t^{(3)}$-free graphs really does improve on the general bound.

Given this state of affairs, we follow a different route, suggested by Rödl and Schacht, attempting to

extend the bipartite counterpart of the Erdős-Hajnal theorem to tripartite 3-uniform hypergraphs. In any given 3-uniform hypergraph on n vertices, one may always find a complete or empty tripartite subgraph with parts of order at least $c(\log n)^{\frac{1}{2}}$. This follows from a standard extremal result due to Erdős [10]. We improve this result for $\mathcal{H}$-free graphs.

Theorem 1.1 *Let $\mathcal{H}$ be a 3-uniform hypergraph. Then there exists a constant $\delta(\mathcal{H}) > 0$ such that, for n sufficiently large, any $\mathcal{H}$-free 3-uniform hypergraph on n vertices contains a complete or empty tripartite subgraph each part of which has order at least $(\log n)^{\frac{1}{2}+\delta(\mathcal{H})}$.*

This improves upon a result of Rödl and Schacht [29], who used the regularity method for hypergraphs to show that the size of the largest complete or empty tripartite subgraph grows arbitrarily faster than the function $(\log n)^{\frac{1}{2}}$. However, because their results depend upon the regularity lemma, their result does not provide good bounds on the multiplying function.

Our result, on the other hand, is not far from best possible, since, for many $\mathcal{H}$, one cannot do better than $c \log n$. To see this, consider the random graph on the vertex set $\{1, 2, \ldots, n\}$ where each edge is chosen with probability $\frac{1}{2}$. For c sufficiently large, with high probability, this graph contains no complete or empty bipartite graph with parts of order $c \log n$. Fix such a graph and call it G_n. Let $\mathcal{G}_n$ be the 3-uniform hypergraph on the same vertex set whose edge set consists of all those triples (i_1, i_2, i_3) with $i_1 < i_2 < i_3$ such that (i_1, i_2) is an edge in G_n.

It is easy to see that $\mathcal{G}_n$ contains no complete or empty tripartite subgraph with sets of size $c \log n$. Suppose otherwise and let U, V and W be subsets of size $c \log n$ which define such a tripartite graph. Without loss of generality, assume that the largest vertex w, in the ordering inherited from the integers, lies in W. Then, by construction, there must be a complete or empty bipartite subgraph in G_n between U and V, a contradiction. Now, for any subset X of the vertices of $\mathcal{G}_n$, let x_1 and x_2 be the two smallest vertices in X. Then, again by construction, for every $x \in X \backslash \{x_1, x_2\}$, either all edges of the form (x_1, x_2, x) are in $\mathcal{G}_n$ or none of them are. Choose a small hypergraph $\mathcal{H}$ containing no vertex pair (x_1, x_2) with this property. For example, one may take $\mathcal{H}$ to be a tight cycle on five vertices, that is, with vertices $\{1, 2, 3, 4, 5\}$ and edge set $\{123, 234, 345, 451, 512\}$. Then $\mathcal{G}_n$ is $\mathcal{H}$-free. Since it also contains no tripartite subgraph of size $c \log n$, this completes our claim.

The proof of Theorem 1.1 relies upon a new embedding lemma which says that if the edges of $\mathcal{G}$ are fairly well-distributed, in the sense that in any graph containing many triangles a positive proportion of these triangles form edges both of $\mathcal{G}$ and of its complement $\overline{\mathcal{G}}$, then one may embed an induced copy of any particular small hypergraph $\mathcal{H}$. If the hypergraph is not well-distributed in the sense described above, then it turns out that it must contain a complete or empty tripartite subgraph which is much larger than one would normally expect.

Despite this description being a reasonable one for any uniformity, the proof does not extend to the k-uniform case for any $k \geq 4$. We will say more about this in the concluding remarks. For now, we will return to considering the analogue of the usual Erdős-Hajnal problem in hypergraphs. That is, given a $\mathcal{H}$-free k-uniform hypergraph, how large of a clique or independent set must it contain?

Let $r_k(\ell)$ be the diagonal Ramsey function, that is, the minimum n such that in any 2-colouring of the edges of $K_n^{(k)}$ there is a monochromatic copy of $K_\ell^{(k)}$. The tower function $t_k(x)$ is defined by $t_1(x) = x$

and $t_{i+1}(x) = 2^{t_i(x)}$. A result of Erdős and Rado implies that, for $k \geq 4$,

$$r_k(\ell) \leq t_{k-2}((r_3(c_k\ell))^3).$$

On the other hand, a result of Erdős and Hajnal (see [9, 23]), referred to as the stepping-up lemma, allows one, for $k \geq 4$, to take a colouring of the $(k-1)$-uniform hypergraph on n vertices containing no monochromatic cliques of size ℓ and to show that there is a colouring of the k-uniform hypergraph on 2^n vertices containing no monochromatic clique of size $2\ell + k - 5$. In particular, this can be used to show that, for ℓ sufficiently large,

$$r_k(\ell) \geq t_{k-2}(r_3(c_k'\ell)).$$

Therefore, once the asymptotic behaviour of $r_3(\ell)$ is understood, so is that of $r_k(\ell)$. To be more precise, let r_k^{-1} be the inverse function of r_k. Restating the results quoted above and using the fact that $r_3(\ell)$ is at least exponential in ℓ tells us that, for $k \geq 3$ and n sufficiently large depending on k, $r_k^{-1}(n)$ has upper and lower bounds of the form

$$a_k'r_3^{-1}(\log_{(k-3)} n) \leq r_k^{-1}(n) \leq a_k r_3^{-1}(\log_{(k-3)} n),$$

where $\log_{(0)}(x) = x$ and, for $i \geq 1$, $\log_{(i)}(x) = \log(\log_{(i-1)}(x))$ is the iterated logarithm.

For an Erdős-Hajnal-type theorem to hold, we would therefore need that whenever a hypergraph on n vertices is $\mathcal{H}$-free there is a clique or independent set of size much larger than $r_3^{-1}(\log_{(k-3)} n)$. We will disprove this by showing that there are already simple examples of hypergraphs $\mathcal{H}$ which are not contained in step-up colourings. This implies the following theorem.

Theorem 1.2 *For $k \geq 4$, there exists a constant c_k, a k-uniform hypergraph $\mathcal{H}$ and a sequence $\mathcal{G}_n$ of $\mathcal{H}$-free k-uniform hypergraphs with n vertices such that the size of the largest clique or independent set in $\mathcal{G}_n$ is at most $c_k r_k^{-1}(n)$.*

In the next two sections, we will prove Theorem 1.1. In section 4, we will study properties of stepping-up lemmas in order to prove Theorem 1.2. We conclude with several remarks in section 5. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs. All logarithms, unless otherwise stated, are taken to the base e. As is quite customary in Ramsey theory, our approach to the problems in this paper is asymptotic in nature. We thus assume that the underlying parameter (normally the order of the graph/hypergraph) is sufficiently large whenever necessary.

2   The embedding lemma

The *edge density* $d(X, Y)$ between two disjoint vertex subsets X, Y of a graph G is the fraction of pairs $(x, y) \in X \times Y$ that are edges of G. That is, $d(X, Y) = \frac{e(X,Y)}{|X||Y|}$, where $e(X, Y)$ is the number of edges with one endpoint in X and the other in Y.

Definition A graph G on n vertices is said to be bi-(ϵ, ρ)-dense if, for all X and Y with $X \cap Y = \emptyset$ and $|X|, |Y| \geq \epsilon n$, the density of edges $d(X, Y)$ between X and Y satisfies $d(X, Y) \geq \rho$.

This definition, introduced by Graham, Rödl and Ruciński [21], has proved very useful in Ramsey theory (see, for example, [5, 7, 18, 22, 31]). The reason for this is that there is an embedding lemma which says that for any given H and ρ there exists an ϵ such that if G is a sufficiently large bi-(ϵ, ρ)-dense graph then G contains a copy of H. More generally, if both G and its complement $\overline{G}$ are bi-(ϵ, ρ)-dense, for ϵ sufficiently small, then G will contain an induced copy of H. Moreover, the dependency of ϵ on ρ is polynomial, that is, $\epsilon = \rho^{c(H)}$ (up to a constant factor, which we ignore for convenience). To see why this is helpful, by taking $\rho = n^{-1/2c(H)}$, we have that if G and $\overline{G}$ are both bi-$(n^{-1/2}, \rho)$-dense, then there is an induced copy of H in G. Suppose, on the other hand, that either G or $\overline{G}$ is not bi-$(n^{-1/2}, \rho)$-dense. Without loss of generality, we may assume that G is the bad graph. This implies that there are two disjoint subsets X and Y with $|X|, |Y| \geq n^{1/2}$ and density less than ρ in G. In the complement $\overline{G}$, this says that the density between X and Y is at least $1 - \rho$. By the choice of ρ, this in turn implies that there is an empty bipartite graph with parts each of polynomial size.

We would like to imitate this argument for 3-uniform hypergraphs. There are several steps. Firstly, we need to find the correct notion of tri-density. We then need to show that this yields an appropriate embedding lemma. Finally, we need to show that if a hypergraph is not tri-dense, then it contains a large empty tripartite subgraph. We will deal with the first two of these in this section.

Roughly speaking, a graph is bi-dense if, between any two large vertex sets, there are many edges. It is tempting to define tri-density in a similar fashion, by saying that between any three large vertex sets there should be many edges. Unfortunately, there are examples of 3-uniform hypergraphs which are tri-dense in this sense but do not contain any copies of $K_4^{(3)}$ or even $K_4^{(3)}$ with one edge removed (see, for example, [25]). Instead, it is necessary to define tri-density with respect to edge sets. This is similar to the sort of quasirandomness which arises in the study of hypergraph regularity [19, 20, 27, 30]. Formally, we have the following definition.

Definition A 3-uniform hypergraph $\mathcal{G}$ on n vertices is said to be tri-(ϵ, ρ)-dense if for any three disjoint vertex subsets V_1, V_2 and V_3 and any triple of graphs G_{12}, G_{23} and G_{31}, with G_{ij} between V_i and V_j, for which there are at least ϵn^3 triangles with one edge in each of the G_{ij}, a ρ-proportion of these triangles form edges in $\mathcal{G}$.

We now prove an embedding lemma to complement this definition. We will need a notion of bi-density defined for bipartite graphs.

Definition A bipartite graph G between sets U and V is said to be bi-(ϵ, ρ)-dense if, for all X and Y with $X \subseteq U$, $Y \subseteq V$ and $|X| \geq \epsilon |U|$, $|Y| \geq \epsilon |V|$, the density of edges $d(X, Y)$ between X and Y satisfies $d(X, Y) \geq \rho$.

Lemma 2.1 *Let $\mathcal{H}$ be a complete 3-uniform hypergraph on $t \geq 3$ vertices $v_1, \ldots, v_t$, where each edge has been assigned a colour from the set $\{1, 2, \ldots, \ell\}$, and let $\rho > 0$ be a positive constant. Let $\epsilon = (2t)^{-10} \rho^{3t^2}$. Then, if $\mathcal{G}$ is an ℓ-coloured 3-uniform hypergraph on $n \geq (2t)^{10} \rho^{-3t^2}$ vertices which is tri-(ϵ, ρ)-dense in each of the ℓ colours, $\mathcal{G}$ contains a copy of $\mathcal{H}$.*

Proof: Let the vertices of $\mathcal{H}$ be $v_1, v_2, \ldots, v_t$ and let $\chi : \binom{V(\mathcal{H})}{3} \to \{1, 2, \ldots, \ell\}$ be the colouring of the edges of $\mathcal{H}$. Split the vertex set $V(\mathcal{G})$ into t vertex sets $U_1, \ldots, U_t$ each of size n/t. We will embed the graph $\mathcal{H}$ one vertex at a time, embedding v_i into $f(v_i) \in U_i$. We will prove by induction on i that when vertices $v_1, v_2, \ldots, v_i$ have been embedded, there are vertex sets U_j^i, for $j > i$, and graphs G_{jk}^i, for $i < j < k \le t$, such that the following conditions hold.

1. $|U_j^i| \ge c_i n$ with $c_i = \frac{1}{t} \rho^{i^2/2}$;

2. G_{jk}^i is a graph between U_j^i and U_k^i which is bi-(ϵ_i, ρ^i)-dense with $\epsilon_i = \frac{1}{2t^2} \rho^{t^2 - i^2/2}$;

3. for every $h \le i$, every edge in G_{jk}^i forms an edge of $\mathcal{G}$ of colour $\chi(h, j, k)$ with $f(v_h)$ and, for all $h_1 < h_2 \le i$, every vertex in U_j^i forms an edge of $\mathcal{G}$ of colour $\chi(h_1, h_2, j)$ with $f(v_{h_1})$ and $f(v_{h_2})$.

For $i = 0$, we let $U_j^0 = U_j$ and G_{jk}^0 be the complete graph between U_j and U_k. The three conditions then hold trivially. Suppose, therefore, that we have embedded vertices $v_1, v_2, \ldots, v_i$ while maintaining conditions $1, 2$ and 3 and that we now wish to embed v_{i+1}.

Let W_{i+1} be the set of vertices in U_{i+1}^i such that the neighborhood $U_j^i(w)$ of w in the graph $G_{i+1,j}^i$, for each $i + 2 \le j \le t$, has size at least $\rho^i |U_j^i|$. By the bi-(ϵ_i, ρ^i)-density condition on $G_{i+1,j}^i$, there are at most $\epsilon_i |U_{i+1}^i| \le \epsilon_i n$ vertices in U_{i+1}^i with fewer than $\rho^i |U_j^i|$ neighbours in U_j^i. Adding over all $i + 2 \le j \le t$, we have, since $c_i \ge 2t\epsilon_i$, that

$$|W_{i+1}| \ge |U_{i+1}^i| - t\epsilon_i n \ge \frac{c_i}{2} n.$$

Given a vertex $w \in W_{i+1}$ and $i + 2 \le j < k \le t$, let $H_{jk}(w)$ be the subgraph of G_{jk}^i between $U_j^i(w)$ and $U_k^i(w)$ consisting of those edges xy with $\chi(w, x, y) = \chi(i+1, j, k)$. Let W_{jk}^{i+1} be the set of vertices $w \in W_{i+1}$ such that the graph $H_{jk}(w)$ is not bi-$(\epsilon_{i+1}, \rho^{i+1})$-dense between $U_j^i(w)$ and $U_k^i(w)$. Note that for any $w \in W_{jk}^{i+1}$ there exist subsets $Y_j(w) \subseteq U_j^i(w)$ and $Y_k(w) \subseteq U_k^i(w)$ such that $|Y_j(w)| \ge \epsilon_{i+1} |U_j^i(w)|$, $|Y_k(w)| \ge \epsilon_{i+1} |U_k^i(w)|$ and the density of $H_{jk}(w)$ between them is less than ρ^{i+1}.

Let $J_{i+1,j}$ be the graph between W_{jk}^{i+1} and U_j^i connecting w to $Y_j(w)$ and let $J_{i+1,k}$ be defined similarly. Note that

$$|Y_j(w)| \ge \epsilon_{i+1} |U_j^i(w)| \ge \epsilon_{i+1} \rho^i |U_j^i| \ge \epsilon_i |U_j^i| \tag{1}$$

and, similarly, $|Y_k(w)| \ge \epsilon_i |U_k^i|$. Since the graph G_{jk}^i is bi-(ϵ_i, ρ^i)-dense between U_j^i and U_k^i, the tripartite graph between W_{jk}^{i+1}, U_j^i and U_k^i where the subgraphs are $J_{i+1,j}$, $J_{i+1,k}$ and G_{jk}^i has at least $\rho^i \sum_{w \in W_{jk}^{i+1}} |Y_j(w)||Y_k(w)|$ triangles. If $|W_{jk}^{i+1}| \ge \frac{|W_{i+1}|}{2t^2} \ge \frac{c_i}{4t^2} n$, by (1), this is at least

$$\rho^i \epsilon_i^2 |W_{jk}^{i+1}||U_j^i||U_k^i| \ge \rho^i \epsilon_i^2 c_i^2 |W_{jk}^{i+1}| n^2 \ge \frac{\rho^i \epsilon_i^2 c_i^3}{4t^2} n^3 \ge \epsilon n^3$$

triangles. Therefore, by the definition of tri-density, at least a ρ-proportion of these triangles, that is, at least $\rho^{i+1} \sum_{w \in W_{jk}^{i+1}} |Y_j(w)||Y_k(w)|$ triangles will be in the colour $\chi(i+1, j, k)$.

On the other hand, the number of 3-uniform edges in colour $\chi(i+1, j, k)$ which contain w and have one edge in each of $J_{i+1,j}$, $J_{i+1,k}$ and G_{jk}^i is the number of edges in $H_{jk}(w)$ between $Y_j(w)$ and $Y_k(w)$. By

definition of $Y_j(w)$ and $Y_k(w)$, this is less than $\rho^{i+1}|Y_j(w)||Y_k(w)|$. Therefore, the total number of 3-uniform edges with colour $\chi(i+1,j,k)$ in the tripartite graph is less than $\rho^{i+1}\sum_{w\in W^{i+1}_{jk}}|Y_j(w)||Y_k(w)|$. This is a contradiction. We must therefore have that $|W^{i+1}_{jk}| < \frac{|W_{i+1}|}{2t^2}$.

Note therefore that the number of vertices w in W_{i+1} which are not in W^{i+1}_{jk} for any $i+2\le j<k\le t$ is at least $|W_{i+1}| - \binom{t-i-1}{2}\frac{|W_{i+1}|}{2t^2} \ge \frac{|W_{i+1}|}{2} \ge \frac{c_i}{4}n$. Let $f(v_{i+1}) = w_{i+1}$ be any vertex from this set. For $i+2\le j\le t$, let $U^{i+1}_j = U^i_j(w_{i+1})$ and, for $i+2\le j<k\le t^2$, let $G^{i+1}_{jk} = H_{jk}(w_{i+1})$. Note that $|U^{i+1}_j| \ge \rho^i|U^i_j| \ge \rho^i c_i n \ge c_{i+1}n$ and G^{i+1}_{jk} is bi-$(\epsilon_{i+1},\rho^{i+1})$-dense between U^{i+1}_j and U^{i+1}_k.

Finally, by definition, for every edge xy in $G^{i+1}_{jk} = H_{jk}(w_{i+1})$, $\chi(w_{i+1},x,y) = \chi(i+1,j,k)$ and, for every $h\le i$ and every $x\in U^{i+1}_j$, $\chi(f(v_h),w_{i+1},x) = \chi(h,i+1,j)$. Therefore, all 3 conditions are satisfied and the result follows by induction. $\qquad\square$

The particular case where $\ell = 2$ implies that for any given $\mathcal{H}$ and ρ there is a constant ϵ, polynomial in ρ, such that if $\mathcal{G}$ and its complement $\overline{\mathcal{G}}$ are tri-(ϵ,ρ)-dense then $\mathcal{G}$ contains an induced copy of $\mathcal{H}$. This is all we will require to prove Theorem 1.1.

3 A tripartite Erdős-Hajnal theorem

The problem of Zarankiewicz [32] asks for the maximum number $z(m,n;s,t)$ of edges in a bipartite graph G which has m vertices in the first class, n vertices in the second and does not contain a complete bipartite subgraph $K_{s,t}$ with s vertices in the first class and t in the second. In their celebrated paper, Kővári, Sós and Turán [24] used double counting together with the pigeonhole principle to give a general upper bound on $z(m,n;s,t)$. Using this technique, we obtain the following simple lemmas which we will need to analyse 3-uniform hypergraphs which are not tri-dense. The degree $d(v)$ of a vertex v is the number of vertices adjacent to v.

Lemma 3.1 *If G is a bipartite graph between sets A and B with at least $\epsilon|A||B|$ edges and such that $s^{3/2} \le \frac{\epsilon}{2}|A|$, then it contains a copy of $K_{s,t}$ with $t = e^{-s^{1/2}}\epsilon^s|B|$ for which the set of s vertices is in A and the set of t vertices is in B.*

Proof: The number of pairs (U,v) with U being a subset of A of size s and v being a vertex in B adjacent to every vertex of U is at least

$$\sum_{v\in B}\binom{d(v)}{s} \ge |B|\binom{\epsilon|A|}{s} \ge |B|\frac{(\epsilon|A|)^s}{s!}e^{-s^{1/2}},$$

where the inequalities follow from the convexity of $f(x) = \binom{x}{s}$ and the fact that, for $s^{3/2} \le \frac{\epsilon}{2}|A|$ and $x = \epsilon|A|$,

$$x(x-1)\dots(x-s+1) \ge x^s\prod_{i=1}^{s-1}\left(1-\frac{i}{x}\right) \ge x^s e^{-2\sum_{i=1}^{s-1}\frac{i}{x}} \ge x^s e^{-s^2/x} \ge x^s e^{-s^{1/2}}.$$

7

Here we used that $1 - z \geq e^{-2z}$ for $0 \leq z \leq \frac{1}{2}$. If G does not contain $K_{s,t}$ then we know that every subset of A of size s has at most $t - 1$ common neighbours. Therefore,

$$e^{-s^{1/2}}|B|\frac{(\epsilon|A|)^s}{s!} \leq (t-1)\binom{|A|}{s} < t\frac{|A|^s}{s!} = e^{-s^{1/2}}|B|\frac{(\epsilon|A|)^s}{s!},$$

a contradiction. $\qquad\square$

Lemma 3.2 *Let G be a bipartite graph with parts A and B and at least $\epsilon|A||B|$ edges. Then G contains a complete bipartite subgraph $K_{s,t}$ with $s = \epsilon|A|$ vertices from A and $t = 2^{-|A|}|B|$ vertices from B.*

Proof: By the convexity of the function $f(x) = \binom{x}{s}$ and the fact that the average degree of A is at least s, we conclude that the number of pairs (U, v) with U a subset of A of size s and v a vertex in B connected to every element of U is at least

$$\sum_{v \in B} \binom{d(v)}{s} \geq |B|\binom{\frac{1}{|B|}\sum_{v \in B} d(v)}{s} \geq |B|.$$

Since A has at most $2^{|A|}$ subsets, the pigeonhole principle implies that for some $U \subset A$ of size s there are at least $t = 2^{-|A|}|B|$ elements b of B which are connected to every element of U. This yields the required copy of $K_{s,t}$. $\qquad\square$

The following lemma, which we believe to be of independent interest, says that if a graph G contains many triangles, a 3-uniform hypergraph $\mathcal{G}$ whose edges form a dense subset of the set of triangles in G contains a larger copy of $K_{s,s,s}^{(3)}$ than one could normally expect in a 3-uniform hypergraph of the same density.

Lemma 3.3 *Suppose that V_1, V_2 and V_3 are disjoint vertex sets of size at most n and, for $1 \leq i < j \leq 3$, there is a bipartite graph G_{ij} between V_i and V_j. Suppose that there are at least δn^3 triangles in this tripartite graph. Suppose further that $\mathcal{G}$ is a 3-uniform hypergraph which contains a $(1 - \eta)$-proportion of the triangles in the tripartite graph, where $0 < \eta \leq \frac{1}{8}$. Then $\mathcal{G}$ contains a copy of $K_{s,s,s}^{(3)}$, provided that*

$$e^{2^{10}\delta^{-2}s^{3/2}}(1 - 4\eta)^{-4s^2}\left(\frac{16}{\delta}\right)^{4s} \leq n.$$

Proof: We suppose without loss of generality that $|V_1| = |V_2| = |V_3| = n$. Indeed, we can otherwise add isolated vertices to make each part have size n, and the hypothesis and conclusion of the lemma are not affected. For every edge e in G_{12}, let Δ_e be the number of triangles containing e. We will say a particular edge e is good if the number of edges of $\mathcal{G}$ containing e is a $(1 - 2\eta)$-proportion of the total number of triangles containing e. Otherwise, we say an edge is bad. Note that

$$\sum_{e \text{ good}} \Delta_e + \sum_{e \text{ bad}} (1 - 2\eta)\Delta_e \geq (1 - \eta)\Delta,$$

8

where Δ is the total number of triangles in the graph. Therefore, since $\sum_e \Delta_e = \Delta$,

$$\sum_{e \ bad} \Delta_e \leq \frac{1}{2}\Delta,$$

that is, at least half the triangles in the tripartite graph contain a good edge. Since there are at least δn^3 triangles in the tripartite graph, this tells us that there are at least $\frac{\delta}{2}n^3$ triangles containing good edges. Calling the set of good edges G'_{12}, we see that G'_{12} has at least $\frac{\delta}{2}n^2$ edges.

Let H_{12} be the subgraph of G'_{12} consisting of edges which are contained in at least $\frac{\delta}{4}n$ triangles from the tripartite graph. Then H_{12} has at least $\frac{\delta}{4}n^2$ edges. Otherwise, there would be fewer than $\frac{\delta}{4}n \cdot n^2 + \frac{\delta}{4}n^2 \cdot n = \frac{\delta}{2}n^3$ triangles containing good edges, which would be a contradiction. Let $s_1 = \frac{\delta}{16}\log n$ and $t_2 = n^{\frac{1}{4}}$. Then, since $s_1^{3/2} \leq \frac{\delta}{8}n$ and

$$e^{-s_1^{1/2}}\left(\frac{\delta}{4}\right)^{s_1} n \geq e^{-\frac{1}{4}\log n}n^{-\frac{\delta}{16}\log(4/\delta)}n \geq n^{-\frac{1}{4}}n^{-\frac{1}{4}}n \geq n^{\frac{1}{4}} = t_2,$$

we may apply Lemma 3.1 to the graph H_{12} to find a set S_1 of size s_1 in V_1 the elements of which have at least t_2 common neighbors in V_2. Call this set of neighbors T_2.

We will now restrict our attention to the tripartite graph between the vertex sets S_1, T_2 and V_3. By the choice of H_{12}, there are at least $\frac{\delta}{4}|S_1||T_2||V_3|$ triangles in this graph. Moreover, since each of the edges between S_1 and T_2 is good, at least a $(1-2\eta)$-proportion of these triangles are triples in the hypergraph $\mathcal{G}$.

We may now repeat the same process within this new tripartite graph. We say that an edge of the graph between S_1 and V_3 is good if, within the new graph, the number of triples in $\mathcal{G}$ containing e is a $(1-4\eta)$-proportion of the total number of triangles containing e. Then, by the same argument as before, at least half the triangles in the new tripartite graph contain good edges. If we let G'_{13} be the set of good edges, we see that this set has at least $\frac{\delta}{8}|S_1||V_3|$ edges. Moreover, letting H_{13} be the subgraph of G'_{13} consisting of edges which are contained in at least $\frac{\delta}{16}|T_2|$ triangles from the tripartite graph, we see that H_{13} contains at least $\frac{\delta}{16}|S_1||V_3|$ edges. Let $s_2 = \left(\frac{\delta}{16}\right)^2 \log n$ and $t_3 = n^{\frac{1}{4}}$. Since $s_2 = \frac{\delta}{16}s_1$ and $2^{-s_2}n \geq t_3$, we may apply Lemma 3.2 to the graph H_{13} to find a set S_2 of size s_2 in S_1 the elements of which have at least t_3 common neighbors in V_3. Call this set of neighbors T_3.

We now have a tripartite graph between sets S_2, T_2 and T_3 such that the graphs between S_2 and T_2 and S_2 and T_3 are complete. Moreover, the graph contains at least $\frac{\delta}{16}|S_2||T_2||T_3|$ triangles, a $(1-4\eta)$-proportion of which are triples in the hypergraph $\mathcal{G}$.

Consider the induced subgraph of G_{23} between T_2 and T_3. Let the set of edges in this graph be E_{23}. Note that this graph has density at least $\frac{\delta}{16}$. We consider the bipartite graph K between S_2 and E_{23}, where v and e are connected if together they span a triple in $\mathcal{G}$. Since a $(1-4\eta)$-proportion of the triangles between S_2, T_2 and T_3 are triples, the graph K must have at least $(1-4\eta)|S_2||E_{23}|$ edges. Note that, since

$$e^{2^{10}\delta^{-2}s^{3/2}} \leq n,$$

we have $s^{3/2} \leq \frac{1}{4}\left(\frac{\delta}{16}\right)^2 \log n \leq \frac{1}{2}(1-4\eta)s_2$. Therefore, an application of Lemma 3.1 implies that there is a set S of s vertices in S_2 which are connected to $e^{-s^{1/2}}(1-4\eta)^s|E_{23}|$ edges in E_{23}. This yields a subgraph H_{23} of G_{23} between T_2 and T_3 of density at least $\epsilon_0 = e^{-s^{1/2}}(1-4\eta)^s\frac{\delta}{16}$ such that every vertex in S is connected to every edge in H_{23}.

9

Note that
$$s^{3/2}e^{s^{1/2}}\epsilon_0^{-s} \leq e^{3s^{3/2}}(1-4\eta)^{-s^2}\left(\frac{16}{\delta}\right)^s \leq n^{\frac{1}{4}}.$$

Therefore, $s^{3/2} \leq \frac{\epsilon_0}{2}t_2$ and $e^{-s^{1/2}}\epsilon_0^s t_3 \geq s$. Applying Lemma 3.1 to the graph H_{23} yields a complete subgraph between two subsets S' and S'', each of size s, of V_2 and V_3. The 3-uniform hypergraph between S, S' and S'' is the required $K_{s,s,s}^{(3)}$. □

We are now ready to put Lemmas 2.1 and 3.3 together to prove Theorem 1.1 in the following precise form.

Theorem 3.1 *Let $\mathcal{H}$ be a fixed 3-uniform hypergraph with t vertices. Then any $\mathcal{H}$-free 3-uniform hypergraph on n vertices with n sufficiently large contains a complete or empty tripartite subgraph each part of which has order at least $(\log n)^{\frac{1}{2}+\delta(\mathcal{H})}$, where $\delta(\mathcal{H}) = 1/(55t^2)$.*

Proof: Let $\mathcal{G}$ be a 3-uniform hypergraph on n vertices. Let $\rho = (\log n)^{1/(27t^2)}$ and $\epsilon = (2t)^{-10}\rho^{3t^2} = \Omega\left((\log n)^{-\frac{1}{9}}\right)$. If both the graph $\mathcal{G}$ and its complement $\overline{\mathcal{G}}$ are tri-(ϵ, ρ)-dense, then we may apply Lemma 2.1 to conclude that $\mathcal{G}$ contains an induced copy of $\mathcal{H}$.

We may therefore assume that either $\mathcal{G}$ or its complement is not tri-(ϵ, ρ)-dense. Without loss of generality, we will assume that $\overline{\mathcal{G}}$ is not tri-(ϵ, ρ)-dense, that is, that there exist three disjoint vertex sets V_1, V_2 and V_3 and bipartite graphs G_{12}, G_{23} and G_{31}, with G_{ij} between V_i and V_j, such that the number of triangles with one edge in each of the G_{ij} is at least ϵn^3 but the number of triples of $\overline{\mathcal{G}}$ contained within these triangles is less than a ρ-fraction of these triangles. Taking the complement, we see that at least a $(1-\rho)$-fraction of the triangles are edges of $\mathcal{G}$.

Let $s = (\log n)^{\frac{1}{2}+\delta(\mathcal{H})}$, where $\delta(\mathcal{H}) = 1/(55t^2)$. Note that, since $\rho \leq \frac{1}{8}$ and $1 - z \geq e^{-2z}$ for $0 \leq z \leq \frac{1}{2}$,

$$e^{2^{10}\epsilon^{-2}s^{3/2}}(1-4\rho)^{-4s^2}\left(\frac{16}{\epsilon}\right)^{4s} \leq e^{2^{10}\epsilon^{-2}s^{3/2}}e^{32\rho s^2}e^{64s/\epsilon}. \tag{2}$$

This expression has three terms, which we consider in turn. The exponent of the first is

$$2^{10}\epsilon^{-2}s^{3/2} = O\left((\log n)^{2\cdot\frac{1}{9}+\frac{3}{2}\cdot\left(\frac{1}{2}+1/(55t^2)\right)}\right) = o(\log n).$$

The exponent of the second term is

$$32\rho s^2 = O\left((\log n)^{-\frac{1}{27t^2}+2\left(\frac{1}{2}+1/(55t^2)\right)}\right) = o(\log n).$$

The exponent of the third term is

$$\frac{64s}{\epsilon} = O\left((\log n)^{\frac{1}{2}+1/(55t^2)+\frac{1}{9}}\right) = o(\log n).$$

Hence, as n is sufficiently large, overall the expression in (2) is $n^{o(1)} \leq n$. Since the number of triangles with one vertex in each of the parts V_1, V_2, V_3 is at least ϵn^3, and each of the parts has order at most n, an application of Lemma 3.3 with $\eta = \rho$ and $\delta = \epsilon$ tells us that the graph contains a copy of $K_{s,s,s}^{(3)}$ with $s = (\log n)^{\frac{1}{2}+\delta(\mathcal{H})}$, where $\delta(\mathcal{H}) = 1/(55t^2)$. □

Erdős and Hajnal [11] also considered the case where the edges of a complete graph K_n have been ℓ-coloured and some fixed coloured subgraph H is banned. It is now too much to hope that there might be a large clique in one particular colour. Instead, the natural object to look for is a large clique which avoids one particular colour. In this case, Erdős and Hajnal showed that there is a clique of size $e^{c\sqrt{\log n}}$, where c depends only on H and ℓ. Moreover, their methods also allow one to find a bipartite graph with polynomial sized parts which avoids a particular colour. The following 3-uniform analogue has essentially the same proof as Theorem 3.1. We omit the details.

Theorem 3.2 *Let $\mathcal{H}$ be a complete 3-uniform hypergraph such that each edge has been assigned a colour from the set $\{1, 2, \ldots, \ell\}$. Then there exists a constant $\delta(\mathcal{H}) > 0$ such that, for n sufficiently large, any ℓ-coloured 3-uniform hypergraph on n vertices which does not contain a coloured copy of $\mathcal{H}$ must contain a complete tripartite subgraph which avoids a particular colour class such that each part has order at least $(\log n)^{\frac{1}{2}+\delta(\mathcal{H})}$.*

4   Some properties of step-up colourings

In order to show that step-up colourings do not contain certain subgraphs, we must first know what these colourings look like. Assume, therefore, that the edges of the k-uniform hypergraph $K_n^{(k)}$ have been red/blue-coloured. Let

$$T = \{(\gamma_1, \ldots, \gamma_n) : \gamma_i = 0 \text{ or } 1\}.$$

If $\epsilon = (\gamma_1, \cdots, \gamma_n)$, $\epsilon' = (\gamma'_1, \cdots, \gamma'_n)$ and $\epsilon \neq \epsilon'$, define

$$\delta(\epsilon, \epsilon') = \max\{i : \gamma_i \neq \gamma'_i\},$$

that is, $\delta(\epsilon, \epsilon')$ is the largest coordinate at which they differ. Given this, we can define an ordering on T, saying that

$$\epsilon < \epsilon' \text{ if } \gamma_i = 0, \gamma'_i = 1,$$

$$\epsilon' < \epsilon \text{ if } \gamma_i = 1, \gamma'_i = 0,$$

where $i = \delta(\epsilon, \epsilon')$. Equivalently, associate to any ϵ the number $b(\epsilon) = \sum_{i=1}^n \gamma_i 2^{i-1}$. The ordering then says simply that $\epsilon < \epsilon'$ if and only if $b(\epsilon) < b(\epsilon')$.

We colour the complete $(k+1)$-uniform hypergraph on the set T as follows. If $\epsilon_1 < \ldots < \epsilon_{k+1}$, for $1 \leq i \leq k$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. A fundamental property of step-up colourings is that δ_i is different from δ_{i+1}.

If $\delta_1, \ldots, \delta_k$ form a monotone sequence (increasing or decreasing), then let the colour of $\{\epsilon_1, \ldots, \epsilon_{k+1}\}$ be given by the colour of $\{\delta_1, \ldots, \delta_k\}$.

If $\delta_1, \ldots, \delta_k$ is not monotone, it must contain a local maximum or local minimum, that is, an i for which $\delta_{i-1} < \delta_i > \delta_{i+1}$ or $\delta_{i-1} > \delta_i < \delta_{i+1}$ respectively. Let δ_j be the first such local extremum, that is, the one with the smallest subscript. We colour $\{\epsilon_1, \ldots, \epsilon_{k+1}\}$ blue or red, respectively, depending on whether the sequence $\delta_1, \ldots, \delta_k$ has a local maximum or minimum at δ_j.

This colouring is exactly that used by Erdős and Hajnal to prove their stepping-up lemma. We will refer to such a colouring as a step-up colouring. In order to prove Theorem 1.2, we will show that there are certain 2-coloured hypergraphs $\mathcal{H}$ which do not occur within step-up colourings.

Lemma 4.1 *For $k \geq 3$, there is a fixed 2-coloured complete $(k+1)$-uniform hypergraph $\mathcal{H}$ which never occurs within a step-up colouring T.*

Proof: We will in fact show that for every $h \geq k + 5$ there is a 2-coloured complete $(k + 1)$-uniform hypergraph on h vertices which never occurs within a step-up colouring. We accomplish this by a simple counting argument which shows that the number of colourings on h vertices which occur within a step-up colouring is considerably less than the total number of colourings on h vertices. We first bound the number of colourings on h vertices which occur within a step-up colouring.

A total preorder $\leq$ is a binary relation that is transitive (i.e., for all x, y, and z, if $x \leq y$ and $y \leq z$, then $x \leq z$) and total (i.e., for all x and y, $x \leq y$ or $y \leq x$). For example, the elements of a sequence of integers has the total preorder $\leq$. In particular, we can associate to each sequence $\delta_1, \ldots, \delta_{h-1}$ its total preorder $\leq$. The ordered Bell number H_n is the number of total preorders on a sequence of n elements, and also counts the number of ordered partitions of the set $[n] := \{1, \ldots, n\}$. For each ordered partition $P : [n] = I_1 \cup \ldots \cup I_t$, consider the mapping $f_P : [n] \to [n]$ defined by, for $i \in I_j$, $f_P(i) = j$. This gives an injective mapping from the set of ordered partitions of $[n]$ to the set of mappings from $[n]$ to $[n]$. Hence, $H_n \leq n^n$.

For a sequence $\epsilon_1 < \ldots < \epsilon_h$, a useful property is that, for $1 \leq i < j \leq h$, we have

$$\delta_{i,j} := \delta(\epsilon_i, \epsilon_j) = \max_{i \leq l \leq j-1} \delta_l, \tag{3}$$

where $\delta_l = \delta(\epsilon_l, \epsilon_{l+1})$. In particular, the $\binom{h}{2}$ integers $\delta_{i,j}$ are completely determined by the $h - 1$ integers δ_i.

We claim that the colours of the $(k+1)$-tuples of $\epsilon_1, \ldots, \epsilon_h$ in a step-up colouring are completely determined by the colours of the k-tuples of $\delta_1, \ldots, \delta_{h-1}$ and the total preorder $\leq$ on these numbers. In particular, the colouring of the $(k+1)$-tuples does not depend on the δs, but only on the colouring of the edges between them and their order. To verify this claim, consider an edge $e = (\epsilon_{i_1}, \ldots, \epsilon_{i_{k+1}})$ with $\epsilon_{i_1} < \ldots < \epsilon_{i_{k+1}}$. For $1 \leq i \leq k$, consider the associated k-tuple $d = (\delta_{i_t, i_{t+1}})_{t=1}^k$, where the terms of this sequence are defined in (3). If the sequence d is not monotone, then the colour of e is determined by whether or not the first local extrema of d is a maxima or minima. This is determined by the order of the elements of d, and it follows from (3) that this is determined by the total preorder $\leq$ on the $h - 1$ numbers δ_i. Otherwise, d is monotone, and the colour of e is the colour of d, which only depends on the colouring of the k-tuples.

The number of 2-colourings of the k-tuples of a set of size $h - 1$ is $2^{\binom{h-1}{k}}$. Hence, the number of 2-coloured complete $(k+1)$-uniform hypergraphs on h vertices which occur within a step-up colouring is at most $H_{h-1} 2^{\binom{h-1}{k}} \leq (h-1)^{h-1} 2^{\binom{h-1}{k}} := A$.

The number of 2-edge-coloured complete $(k + 1)$-uniform hypergraphs on h labeled vertices is $2^{\binom{h}{k+1}}$, and hence the number of distinct (up to isomorphism) 2-edge-coloured complete $(k + 1)$-uniform hypergraphs on h vertices is at least $\frac{1}{h!} 2^{\binom{h}{k+1}} := B$. As for $h \geq k + 5$, we have

$$A/B = h!(h-1)^{h-1} 2^{-\left(\frac{h-k-1}{k+1}\right)\binom{h-1}{k}} < 1,$$

there is a 2-coloured complete $(k + 1)$-uniform hypergraphs on h vertices which does not appear in a step-up colouring. $\square$

One can deduce Theorem 1.2 from Lemma 4.1 as follows. Start with a 2-edge-colouring of the complete $(k-1)$-uniform hypergraph on n vertices whose largest monochromatic clique is of order $O(r_{k-1}^{-1}(n))$. Apply the stepping-up construction described above. By the Erdős-Hajnal result mentioned in the introduction, this gives a 2-edge-coloring of the complete k-uniform hypergraph on $N = 2^n$ vertices whose largest monochromatic clique is still of order $O(r_{k-1}^{-1}(n)) = O(r_k^{-1}(N))$. On, the other hand by Lemma 4.1, this colouring does not contain a fixed 2-coloured complete k-uniform hypergraph $\mathcal{H}$.

5 Concluding remarks

A simple extension of the example given after Theorem 1.1 allows one to show that there are k-uniform hypergraphs $\mathcal{H}$ and $\mathcal{H}$-free hypergraphs $\mathcal{G}$ on n vertices within which the largest complete or empty k-partite subgraph, with all parts of the same order, is of order at most $c(\log n)^{1/(k-2)}$. Indeed, this can be established by considering a random $(k-1)$-uniform hypergraph $\mathcal{G}_0$ on the vertex set $\{1, \ldots, n\}$, and forming a k-uniform hypergraph $\mathcal{G}$ on the same vertex set whose edges are all k-tuples $(i_1, \ldots, i_k)$ with $i_1 < \ldots < i_k$ such that $(i_1, \ldots, i_{k-1})$ is an edge of $\mathcal{G}_0$. On the other hand, a standard density argument shows that any hypergraph on n vertices must contain a complete or empty k-partite subgraph with parts of order $c(\log n)^{1/(k-1)}$. We make the following conjecture.

Conjecture 1 *Let $\mathcal{H}$ be a k-uniform hypergraph, where $k \geq 3$. Then any $\mathcal{H}$-free k-uniform hypergraph on n vertices contains a complete or empty k-partite subgraph each part of which has order at least $(\log n)^{1/(k-2)-o(1)}$.*

It is possible to show that the notion of tri-density and the embedding lemma, Lemma 2.1, both extend to the k-uniform case. Indeed, if one is not particularly concerned with bounds, an application of hypergraph regularity and counting immediately allows one to prove such an embedding lemma. The real stumbling block in extending our result to the k-uniform case is Lemma 3.3, which does not seem to generalise easily. Understanding and resolving this difficulty could also be useful in extending the results of [8], on finding almost monochromatic subsets within colourings of $K_n^{(3)}$, to the k-uniform case.

Another direction which one could take would be to consider random $\mathcal{H}$-free hypergraphs. In a recent paper, Loebl, Reed, Scott, Thomason and Thomassé [26] showed that almost every H-free graph on n vertices contains a clique or independent set of size $n^{\delta(H)}$. Perhaps this result could be extended to hypergraphs in some interesting fashion.

Finally, it would be of great interest to show that Theorem 1.2 also holds for 3-uniform hypergraphs. It seems likely that in order to do this one must first resolve the central Ramsey problem for 3-uniform hypergraphs. That is, one would need to show that there are 2-colourings of the edges of $K_n^{(3)}$ which contain no cliques or independent sets of size $c \log \log n$. Such a colouring, if it does exist, is likely to avoid some class of subgraphs.

References

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, Crossing patterns of semi-algebraic sets, *J. Combin. Theory Ser. A* **111** (2005), 310–326.

[2] N. Alon, J. Pach and J. Solymosi, Ramsey-type theorems with forbidden subgraphs, Paul Erdős and his mathematics, *Combinatorica* **21** (2001), 155–170.

[3] M. Chudnovsky and S. Safra, The Erdős-Hajnal conjecture for bull-free graphs, *J. Combin. Theory Ser. B* **98** (2008), 1301–1310.

[4] M. Chudnosky and Y. Zwols, Large cliques and stable sets in graphs with no four-edge path and no five-edge path in the complement, *submitted.*

[5] D. Conlon, The Ramsey number of dense graphs, to appear in *Bull. London Math. Soc.*

[6] D. Conlon, J. Fox and B. Sudakov, Hypergraph Ramsey numbers, *J. Amer. Math. Soc.* **23** (2010), 247–266.

[7] D. Conlon, J. Fox and B. Sudakov, On two problems in graph Ramsey theory, *submitted.*

[8] D. Conlon, J. Fox and B. Sudakov, Large almost monochromatic subsets in hypergraphs, *Israel J. Math.* **181** (2011), 423–432.

[9] D. Conlon, J. Fox and B. Sudakov, An improved bound for the stepping-up lemma, to appear in *Discrete Appl. Math.*

[10] P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183–190.

[11] P. Erdős and A. Hajnal, Ramsey-type theorems, *Discrete Appl. Math.* **25** (1989), 37–52.

[12] P. Erdős, A. Hajnal and J. Pach, A Ramsey-type theorem for bipartite graphs, *Geombinatorics* **10** (2000), 64–68.

[13] P. Erdős and R. Rado, Combinatorial theorems on classifications of subsets of a given set, *Proc. London Math. Soc.* **3** (1952), 417–439.

[14] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[15] J. Fox, J. Pach and Cs. D. Tóth, Intersection patterns of curves, *J. London Math. Soc.* **83** (2011), 389–406.

[16] J. Fox and B. Sudakov, Induced Ramsey-type theorems, *Adv. Math.* **219** (2008), 1771–1800.

[17] J. Fox and B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results, *Combinatorica* **29** (2009), 153–196.

[18] J. Fox and B. Sudakov, Two remarks on the Burr-Erdős conjecture, *European J. Combin.* **30** (2009), 1630–1645.

[19] W.T. Gowers, Quasirandomness, Counting and Regularity for 3-Uniform Hypergraphs, *Combin. Probab. Comput.* **15** (2006), 143–184.

[20] W.T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, *Ann. of Math.* **166** (2007), 897–946.

[21] R.L. Graham, V. Rödl and A. Ruciński, On graphs with linear Ramsey numbers, *J. Graph Theory* **35** (2000), 176–192.

[22] R.L. Graham, V. Rödl and A. Ruciński, On bipartite graphs with linear Ramsey numbers, *Combinatorica* **21** (2001), 199–209.

[23] R. L. Graham, B. L. Rothschild, and J. H. Spencer, **Ramsey theory**, 2nd edition, John Wiley & Sons (1980).

[24] T. Kővári, V. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50–57.

[25] Y. Kohayakawa, B. Nagle, V. Rödl and M. Schacht, Weak regularity and linear hypergraphs, *J. Combin. Theory Ser. B* **100** (2010), 151–160.

[26] M. Loebl, B. Reed, A. Scott, A. Thomason and S. Thomassé, Almost all H-free graphs have the Erdős-Hajnal property, in *An Irregular Mind, Szemerédi is 70,* Bolyai Society Mathematical Studies **21** (2010), 405–414.

[27] B. Nagle, V. Rödl and M. Schacht, The counting lemma for regular k-uniform hypergraphs, *Random Structures Algorithms* **28** (2006), 113–179.

[28] H. Prömel and V. Rödl, Non-Ramsey graphs are $c \log n$-universal, *J. Combin. Theory Ser. A* **88** (1999), 379–384.

[29] V. Rödl and M. Schacht, *personal communication.*

[30] V. Rödl and J. Skokan, Regularity lemma for uniform hypergraphs, *Random Structures Algorithms* **25** (2004), 1–42.

[31] B. Sudakov, A conjecture of Erdős on graph Ramsey numbers, *Adv. Math.* **227** (2011), 601–609.

[32] K. Zarankiewicz, Problem P101, *Colloq. Math.* **2** (1951), 301.