ON ELLIPTIC CALOGERO-MOSER SYSTEMS FOR COMPLEX CRYSTALLOGRAPHIC REFLECTION GROUPS

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Abstract. To every irreducible finite crystallographic reflection group (i.e., an irreducible finite reflection group G acting faithfully on an abelian variety X), we attach a family of classical and quantum integrable systems on X (with meromorphic coefficients). These families are parametrized by G-invariant functions of pairs (T, s), where T is a hypertorus in X (of codimension 1), and $s \in G$ is a reflection acting trivially on T. If G is a real reflection group, these families reduce to the known generalizations of elliptic Calogero-Moser systems, but in the non-real case they appear to be new. We give two constructions of the integrals of these systems - an explicit construction as limits of classical Calogero-Moser Hamiltonians of elliptic Dunkl operators as the dynamical parameter goes to 0 (implementing an idea of [BFV]), and a geometric construction as global sections of sheaves of elliptic Cherednik algebras for the critical value of the twisting parameter. We also prove algebraic integrability of these systems for values of parameters satisfying certain integrality conditions.

To Corrado De Concini on his 60-th birthday with admiration

1. Introduction

Classical and quantum integrable many-particle systems on the line have been a hot topic since 1970s. Among these, especial attention has been paid to Calogero-Moser systems with rational, trigonometric, and elliptic potential. A review of the history of these systems and a variety of important applications with references can be found in [CMS]. In particular, in [OP1, OP2], Calogero-Moser systems were generalized to the case of any root system, so that the many-particle systems of [C1, C2, S, M] correspond to type A.

There are a number of ways to construct Calogero-Moser systems and to prove their integrability. One is the Lax matrix method ([M, C1, CMR, K1, OP1, OP2, BCS]). Another, related method is Hamiltonian reduction ([KKS], [F], [E1] (see in particular the remark at the

end of section III), [GN]). The third method is based on computing radial parts of Laplace operators on symmetric spaces ([BPF],[OP4]); this method produces quantum Calogero-Moser systems only for some special values of coupling constants. The fourth method is based on the analytic study of hypergeometric functions associated to root systems and is due to Heckman and Opdam ([HO, He3, O1, O2]); it yielded the first proof of integrability of rational and trigonometric Calogero-Moser systems for any root system. Finally, the fifth method, most relevant to this paper, is due to G. Heckman [He1, He2], and is based on considering invariant polynomials of Dunkl operators [D]. Namely, Heckman managed to use this method to give a simple algebraic proof of the integrability of Calogero-Moser systems for any root system and any values of coupling constants in the rational and trigonometric cases; his method was further improved by Cherednik [Ch1]. Later Cherednik [Ch2] settled the elliptic case, by introducing Dunkl operators for affine root systems.

An alternative approach to proving the integrability of the quantum elliptic Calogero-Moser system was proposed in the paper [BFV], which introduces the elliptic counterparts of Dunkl operators. However, this approach did not quite succeed, because of the following difficulty: elliptic Dunkl operators depend on a "dynamical" parameter λ (lying in the reflection representation of the Weyl group W), and are not W-invariant, but rather W-equivariant (i.e., λ is also transformed by W); so to get W-invariant Hamiltonians from invariant polynomials of elliptic Dunkl operators, one would have to set λ to 0, which is impossible since the elliptic Dunkl operators have poles on the root hyperplanes. It was suggested in [BFV] that these poles should be cancelled by some kind of a subtraction procedure (namely, the calculation in the A_2 case made in [BFV], p. 909, indicated that the classical integrals in the dynamical variables may be used here), but it was unclear what exactly this procedure should be.

Since the paper [BFV], it has seemed certain to the authors that elliptic Dunkl operators are "the right" objects. For example, in [EM1], they were generalized to the case of finite crystallographic complex reflection groups (following the generalization of usual Dunkl operators to finite complex reflection groups in [DO]), and linked to double affine Hecke algebras and Cherednik algebras on complex tori. Yet, the problem of providing a precise connection between elliptic Dunkl operators and elliptic Calogero-Moser systems remained open.

The goal of this paper is to finally solve this problem, and use the approach of [BFV] to give a new proof of the integrability of quantum

elliptic Calogero-Moser systems. In fact, our main result is more general: we use the elliptic Dunkl operators of [EM1] to attach a family of classical and quantum integrable systems to every finite irreducible crystallographic complex reflection group G, i.e. a finite irreducible complex reflection group acting faithfully on a complex torus (preserving 0) [1]. When G is a real reflection group (i.e., a Weyl group), our construction reproduces the elliptic Calogero-Moser system attached to G (in fact, in the BC_n case it reproduces the full 5-parameter Inozemtsev system [I]). On the other hand, when G is not real, we obtain new examples of integrable systems with elliptic coefficients. We will call these systems *crystallographic elliptic Calogero-Moser systems.*[2] The simplest example is given by (4.1) below.

The main idea of our construction is to consider the *classical* Calogero-Moser Hamiltonians (in the rational case constructed by Heckman's method [He2] as G-invariant polynomials of the usual Dunkl operators [D]), and substitute the elliptic Dunkl operators for momentum variables, and the dynamical parameters λ for the position variables. Our main result is that the resulting operators are regular in λ near $\lambda = 0$ (i.e., this construction provides the cancelation of poles asked for in [BFV]). Thus we can now set $\lambda = 0$ and obtain a collection of G-invariant commuting operators. If we restrict these operators to the space of G-invariant functions, they become differential operators, and thus yield the desired integrable system.

We also give a geometric construction of crystallographic elliptic Calogero-Moser systems, as global sections of sheaves of elliptic Cherednik algebras for the critical value of the twisting parameter. This is a construction in the style of the Beilinson-Drinfeld construction of the quantum Hitchin system, [BD], as global sections of the sheaf of twisted differential operators on the moduli stack of principal bundles over a curve, for critical twisting.

Finally, we establish algebraic integrability of the quantum crystallographic elliptic Calogero-Moser systems for parameters satisfying certain integrality conditions.

The paper is organized as follows. In Section 2, we recall the basics on complex reflection groups, rational Dunkl operators, rational Calogero-Moser Hamiltonians, and elliptic Dunkl operators. In Section 3, we state the main theorem and give some examples. In Section 4, we

[1] Such groups were classified by Popov [Po] (see also [Ma]).

[2] We note that these new integrable systems may not have a direct physical meaning, since their Hamiltonians are polynomials in momenta of degree higher than 2 and in general have complex coefficients.

describe the main new example, attached to the groups $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$, where $m = 3, 4$ or 6. In Section 5, we give two different proofs of the main theorem, and explain the relation between our arguments and the ones of [Ch2]. In Section 6, we give the geometric construction of the crystallographic elliptic Calogero-Moser systems. In Section 7, we establish their algebraic quantum integrability of our quantum integrable systems at integer points.

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2. Preliminaries

2.1. **Complex tori and line bundles on them.** Let $\mathfrak{h}$ be a finite dimensional complex vector space, and $\mathfrak{h}^\vee$ be the Hermitian dual of $\mathfrak{h}$ (i.e. the dual $\mathfrak{h}^*$ with the conjugate complex structure). Suppose that $\Gamma \subset \mathfrak{h}$ is a cocompact lattice. Then $X = \mathfrak{h}/\Gamma$ is a complex torus.

Let $X^\vee$ be the dual torus to X, i.e. $X^\vee = \mathfrak{h}^\vee/\Gamma^\vee$, where $\Gamma^\vee$ is the dual lattice to Γ under the form $\mathrm{Im}\lambda(v)$, $v \in \mathfrak{h}$, $\lambda \in \mathfrak{h}^\vee$.

It is well known that $X^\vee$ is naturally identified with $\mathrm{Pic}_0(X)$, the set of classes of topologically trivial holomorphic line bundles on X. For any $\lambda \in \mathfrak{h}^\vee$, let $\mathcal{L}_\lambda \in \mathrm{Pic}_0(X)$ denote the corresponding holomorphic line bundle; it is obtained by taking the quotient of the trivial line bundle on $\mathfrak{h}$ by the Γ-action given by the formula

$$\gamma(\mathbf{x}, z) = (\mathbf{x} + \gamma, e^{2\pi \mathrm{i} \mathrm{Im}\lambda(\gamma)} z), \gamma \in \Gamma.$$

Note that the line bundle $\mathcal{L}_\lambda$ comes with a natural Hermitian structure and flat unitary connection (coming from the constant ones on $\mathfrak{h}$). We will denote this connection by ∇.

If X is 1-dimensional (an elliptic curve), then we have a natural identification $X \cong X^\vee$, sending $\mathbf{x} \in X$ to the bundle $\mathcal{O}(\mathbf{x}) \otimes \mathcal{O}(0)^*$. This identification yields a natural positive Hermitian form $\langle\ ,\ \rangle$ on the line $T_0 X^\vee$. Hence, for every hypertorus $T \subset X$ passing through 0 (of codimension 1), there is a natural positive Hermitian form $\langle\ ,\ \rangle$ on the line $T_0(X/T)^\vee = T_0((X/T)^\vee)$.

2.2. **Complex reflection groups.** Let $\mathfrak{h}$ be a finite dimensional complex vector space. A semisimple element $s \in \mathrm{GL}(\mathfrak{h})$ is called a complex reflection if $\mathrm{Im}(1 - s)$ is 1-dimensional. For a complex reflection s, let $\zeta_s = \det(s|_{\mathfrak{h}^*})$, and let α_s be a nonzero linear function on $\mathfrak{h}$ vanishing on the fixed hyperplane of s.

Let $\mathcal{S}$ be the set of complex reflections in G. For any $s \in \mathcal{S}$, we have a decomposition:

$$\mathfrak{h} = \mathfrak{h}^s \oplus \mathfrak{h}_s,$$

where $\mathfrak{h}^s$ is the codimension 1 subspace of $\mathfrak{h}$ with the trivial action of s, and $\mathfrak{h}_s = ((\mathfrak{h}^*)^s)^\perp$, which is an s-invariant 1-dimensional space. We also have a similar decomposition on the dual space: $\mathfrak{h}^* = (\mathfrak{h}^*)^s \oplus \mathfrak{h}_s^*$.

A finite subgroup $G \subset GL(\mathfrak{h})$ is called a *complex reflection group* if it is generated by complex reflections. The representation $\mathfrak{h}$ is then called the reflection representation of G. A complex reflection group G is called irreducible if $\mathfrak{h}$ is an irreducible representation of G.

Let G be a complex reflection group with reflection representation $\mathfrak{h}$. For a hyperplane $H \subset \mathfrak{h}$, denote by $G_H \subset G$ the stabilizer of a generic point in H. We call H a *reflection hyperplane* if G_H is nontrivial; in this case, G_H is a cyclic group. Let $\mathfrak{h}_{\mathrm{reg}}$ be the complement of the reflection hyperplanes in $\mathfrak{h}$.

By the Shephard-Todd-Chevalley theorem (see [Che]), if G is a complex reflection group, then $(S\mathfrak{h})^G$ is a polynomial algebra. Let $P_i, i = 1, \dots, n$, denote homogeneous generators of $(S\mathfrak{h})^G$.

2.3. **Complex tori with an action of a complex reflection group.** Let G be an irreducible complex reflection group with reflection representation $\mathfrak{h}$, and let $\Gamma \subset \mathfrak{h}$ be a cocompact lattice preserved by G, i.e., G is a crystallographic complex reflection group. Then we get a G-action on the complex torus $X = \mathfrak{h}/\Gamma$ preserving 0.

The action of G on X induces a G-action on the dual torus $X^\vee \cong \mathrm{Pic}_0(X)$. For a line bundle $\mathcal{L} \in \mathrm{Pic}_0(X)$, denote the image of $\mathcal{L}$ under g by $\mathcal{L}^g$.

For any complex reflection $s \in G$, let X^s be the set of $x \in X$ s.t. $sx = x$. Connected components of X^s (which all have codimension 1) are called *reflection hypertori*. Among them, there is one passing through 0, which we denote by T_s. Let m_s be the order of s (note that $m_s = 2, 3, 4$ or 6), and let $j(s) \in \{1, \dots, m_s - 1\}$ be such that $\zeta_s = e^{-2\pi \mathrm{i} j(s)/m_s}$.

Let X_{reg} be the complement of the reflection hypertori in X. For a reflection hypertorus $T \subset X$, denote by $G_T \subset G$ the stabilizer of a generic point in T. It is a cyclic group. Denote its order by m_T (so $m_{T_s} = m_s$), and let s_T be the generator of G_T which acts on the normal bundle of T by multiplication by $e^{2\pi \mathrm{i}/m_T}$.

Denote by $\mathcal{A}$ the set

$$\{(T, j) | T \text{ is a reflection hypertorus }, j = 1, \dots, m_T - 1\}.$$

For any reflection hypertorus T, $\mathfrak{h}_{s_T^j}$ is independent on j, so we will denote it by $\mathfrak{h}_T$.

Remark 2.1. Note that the complex torus X in our situation is always an abelian variety, which is isogenous to a product of elliptic curves. Indeed, let $s_1, \dots, s_n \in G$ be a collection of reflections such that $\{\alpha_{s_1}, \dots, \alpha_{s_n}\}$ is a basis of $\mathfrak{h}^*$. Then the natural map

$$X \to X/T_{s_1} \times \cdots \times X/T_{s_n}$$

is an isogeny.

2.4. **Dunkl operators for complex reflection groups.** Let us recall the basic theory of Dunkl operators for complex reflection groups (see [DO], [EM2]).

Let $c : \mathcal{S} \to \mathbb{C}$ be a G-invariant function. The (rational) *Dunkl operators* for G are the following family of pairwise commuting linear operators acting on the space of rational functions on $\mathfrak{h}$:

$$\mathbf{D}_{v,c} = \partial_v + \sum_{s \in \mathcal{S}} \frac{2c(s)\alpha_s(v)}{(1-\zeta_s)\alpha_s} s, \tag{2.1}$$

where $v \in \mathfrak{h}$, and ∂_v is the derivation associated to the vector v.[3] Thus, the Dunkl operators are elements in $\mathbb{C}G \ltimes D(\mathfrak{h}_{\mathrm{reg}})$, where $D(\mathfrak{h}_{\mathrm{reg}})$ denotes the algebra of differential operators on $\mathfrak{h}_{\mathrm{reg}}$.

Similarly, one defines the quasiclassical limits of Dunkl operators, called the *classical Dunkl operators*, which are elements of $\mathbb{C}G \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. Namely, for $v \in \mathfrak{h}$, let p_v be the corresponding momentum coordinate in $\mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. Then the classical Dunkl operators are defined by the formula

$$\mathbf{D}^0_{v,c} = p_v + \sum_{s \in \mathcal{S}} \frac{2c(s)\alpha_s(v)}{(1-\zeta_s)\alpha_s} s,$$

which is obtained by replacing the derivative ∂_v by its symbol p_v in (2.1).

2.5. **Calogero-Moser Hamiltonians.** Let $\mathbf{m} : \mathbb{C}G \ltimes D(\mathfrak{h}_{\mathrm{reg}}) \to D(\mathfrak{h}_{\mathrm{reg}})$ be the map defined by the formula $\mathbf{m}(Lg) = L$, where $L \in D(\mathfrak{h}_{\mathrm{reg}})$. Define the G-invariant differential operators $\widehat{P}_i^c$ on $\mathfrak{h}_{\mathrm{reg}}$ by the formula

$$\widehat{P}_i^c := \mathbf{m}(P_i(\mathbf{D}_{\bullet,c})).$$

[3]This definition of Dunkl operators is slightly different from the one in [E3], [EM2], namely we have replaced $s-1$ by s. This has no significant effect on the considerations below, since our Dunkl operators are conjugate to the ones in [E3], [EM2].

In other words, $\mathbf{D}_{\bullet,c}$ is a linear map $\mathfrak{h} \to \mathbb{C}G \ltimes D(\mathfrak{h}_{\mathrm{reg}})$ whose image is commuting, so it defines an algebra homomorphism $S\mathfrak{h} \to \mathbb{C}G \ltimes D(\mathfrak{h}_{\mathrm{reg}})$, and $P_i(\mathbf{D}_{\bullet,c})$ is the image of P_i under this homomorphism. Note that $\widehat{P}_i^0 = P_i(\partial)$. It is known (see [He2], [EM2], [BC]) that these operators are pairwise commuting (i.e., form a quantum integrable system). They are called the *rational Calogero-Moser operators*.

Similarly, one can define the quasiclassical limits of $\widehat{P}_i^c$. Namely, let $\mathbf{m} : \mathbb{C}G \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}}) \to \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$ be the map defined by the formula $\mathbf{m}(Pg) = P$, where $P \in \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. Define the G-invariant functions $P_i^c \in \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$ by the formula

$$P_i^c(\mathbf{p}, \mathbf{q}) := \mathbf{m}(P_i(\mathbf{D}_{\bullet,c}^0)).$$

(Here $\mathbf{q} \in \mathfrak{h}$ is the position variable, and $\mathbf{p} \in \mathfrak{h}^*$ is the momentum variable). Note that $P_i^0 = P_i(\mathbf{p})$. It is known (see [He2], [EM2]) that these functions are pairwise Poisson commuting (i.e., form a classical integrable system). They are called the *rational classical Calogero-Moser Hamiltonians*.

The following important lemma will be used below.

Lemma 2.2. *$P_i(\mathbf{D}_{\bullet,c})$ is a function on $T^*\mathfrak{h}_{\mathrm{reg}}$, i.e., it does not involve nontrivial elements of G. Thus, $P_i^c = P_i(\mathbf{D}_{\bullet,c}^0)$, i.e. the application of $\mathbf{m}$ is not necessary.*

Note that this lemma does not hold in the quantum setting.

Proof. Consider the classical rational Cherednik algebra for G, $H_{0,c}(G, \mathfrak{h})$, generated inside $\mathbb{C}G \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$ by G, $S\mathfrak{h}^*$ (the algebra of polynomials on $\mathfrak{h}$), and the classical Dunkl operators (see [E3], Section 7, and [EM2], Section 3).

It is easy to see that $(S\mathfrak{h}^*)^G$ is contained in the center of $H_{0,c}(G, \mathfrak{h})$. On the other hand, there is an isomorphism $H_{0,c}(G, \mathfrak{h}^*) \to H_{0,c}(G, \mathfrak{h})$ which maps linear functions on $\mathfrak{h}^*$ to classical Dunkl operators on $\mathfrak{h}$ (see [EM2], proof of Prop. 3.16). Thus, for any $P \in (S\mathfrak{h})^G$, $P(\mathbf{D}_{\bullet,c}^0)$ is also in the center of $H_{0,c}(G, \mathfrak{h})$, and thus, in the center of $\mathbb{C}G \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. So $P(\mathbf{D}_{\bullet,c}^0)$ commutes with functions of $\mathbf{p}$ and $\mathbf{q}$, and hence is itself a function. $\square$

2.6. **Elliptic Dunkl operators.** Let G, X be as above. Fix a generic line bundle $\mathcal{L} \in \mathrm{Pic}_0(X)$ (i.e., such that $\mathcal{L}^g \neq \mathcal{L}$ for any reflection g). From [EM1], we know that for any $(T, j) \in \mathcal{A}$, there is a unique global meromorphic section $f_{T,j}^{\mathcal{L}}$ of the bundle $(\mathcal{L}^{s_T^j})^* \otimes \mathcal{L} \otimes \mathfrak{h}_T^*$ which has a simple pole along T with residue 1 and no other singularities.

Let C be a G-invariant function on $\mathcal{A}$. Recall from [EM1] that the *elliptic Dunkl operator* corresponding to $\mathcal{L}, \nabla, C$, and a vector $v \in \mathfrak{h}$ is the following operator acting on the local meromorphic sections of $\mathcal{L}$:

$$\mathcal{D}^{\mathcal{L}}_{v,C} = \nabla_v + \sum_{(T,j)\in\mathcal{A}} C(T,j)(f^{\mathcal{L}}_{T,j}, v)s^j_T.$$

(Here we regard $\mathfrak{h}^*_T$ as a subspace of $\mathfrak{h}^*$ in a natural way, using that $\mathfrak{h}_T$ has a distinguished complement in $\mathfrak{h}$).

Example 2.3. In the Weyl group case the elliptic Dunkl operators are the operators from [BFV]:

$$\mathcal{D}^{\lambda}_{v,C} = \nabla_v + \sum_{\alpha\in R_+} C_\alpha(\alpha, v)\sigma_{(\alpha^\vee,\lambda)}(\alpha)s_\alpha,$$

where

$$\sigma_\mu(z) = \frac{\theta(z - z_0 - \mu)\theta'(0)}{\theta(z - z_0)\theta(-\mu)},$$

and

$$\theta(z) = \theta_1(z,\tau) = -\sum_{n=-\infty}^{\infty} e^{2\pi\mathrm{i}(z+1/2)(n+1/2)+\pi\mathrm{i}\tau(n+1/2)^2}$$

is the first Jacobi theta-function [WW].

Remark 2.4. This differs from the definition of [EM1] by the sign of C. We choose this sign convention to reconcile the notation with texts on rational Cherednik algebras, e.g. [E3] and [EM2].

Proposition 2.5. *(*[BFV, EM1]*) The elliptic Dunkl operators have the following properties.*

(1) *commutativity:* $[\mathcal{D}^{\mathcal{L}}_{v,C}, \mathcal{D}^{\mathcal{L}}_{v',C}] = 0$, *for any* $v, v' \in \mathfrak{h}$.
(2) *equivariance:* $g \circ \mathcal{D}^{\mathcal{L}}_{v,C} \circ g^{-1} = \mathcal{D}^{\mathcal{L}^g}_{gv,C}$, *where* $g \in G$.

It is also useful to consider *classical elliptic Dunkl operators*, which are quasiclassical limits of elliptic Dunkl operators. These operators are parametrized by $v, C, \mathcal{L}$, and are given by the formula

$$\mathcal{D}^{0,\mathcal{L}}_{v,C} = p_v + \sum_{(T,j)\in\mathcal{A}} C(T,j)(f^{\mathcal{L}}_{T,j}, v)s^j_T.$$

The properties of these operators are similar to those of the quantum elliptic Dunkl operators.

2.7. **Behavior of elliptic Dunkl operators near** $0 \in \mathrm{Pic}_0(X)$**.** Let $\alpha_T := \alpha_{s_T} \in \mathfrak{h}_T^*$. Then we have

$$f_{T,j}^{\mathcal{L}_\lambda} = \varphi_{T,j}(\lambda)\alpha_T,$$

where $\varphi_{T,j}(\lambda)$ is a section of $(\mathcal{L}_\lambda^{s_T^j})^* \otimes \mathcal{L}_\lambda$. We are going to study the behavior of this section near $\lambda = 0$.

Fix a G-invariant positive definite Hermitian form[4] $B(\ ,\ )$ on $\mathfrak{h}^\vee$ (which is unique up to a positive factor), and use it to identify $\mathfrak{h}$ with $\mathfrak{h}^\vee$; so the element of $\mathfrak{h}$ corresponding to $\lambda \in \mathfrak{h}^\vee$ will be denoted by $B(\lambda)$.

For a reflection $s \in G$, set

$$a_B(s) = \frac{B(u,u)}{\langle u,u\rangle}$$

for $0 \neq u \in T_0(X/T_s)^\vee$ (where $\langle\ ,\ \rangle$ is defined in Subsection 2.1).

Proposition 2.6. *The section* $\widetilde{\varphi}_{T,j}(\lambda) := B(\lambda, \alpha_T)\varphi_{T,j}(\lambda)$ *is regular in* λ *near* $\lambda = 0$*, and if* $B(\lambda, \alpha_T) = 0$ *(i.e.,* $s_T\lambda = \lambda$*), we have*

$$\widetilde{\varphi}_{T,j}(\lambda) = -\frac{a_B(s_T)}{1 - e^{2\pi \mathrm{i} j/m_T}}.$$

Proof. Suppose $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$, $\mu \in E$, and $\mathcal{E} = \mathcal{O}(\mu) \otimes \mathcal{O}(0)^*$ is a degree zero holomorphic line bundle on E. Let σ_μ be a section of $\mathcal{E}$ with a first order pole at a point z_0 and no other singularities. Then, up to scaling, we have

$$\sigma_\mu(z) = \frac{\theta(z - z_0 - \mu)\theta'(0)}{\theta(z - z_0)\theta(-\mu)},$$

where, as before, θ is the first Jacobi theta-function. Near $\mu = 0$, this function has the expansion

$$\sigma_\mu(z) = -\frac{1}{\mu} + O(1). \tag{2.2}$$

Now let $E = X/T_s$ (an elliptic curve). It is clear that the bundle $(\mathcal{L}_\lambda^{s_T^j})^* \otimes \mathcal{L}_\lambda$ is pulled back from E, namely it is the pullback of the line bundle $\mathcal{E}$ corresponding to the point $(1 - e^{2\pi \mathrm{i} j/m_T})\lambda(\alpha_T^*)\alpha_T$, where $\alpha_T^* \in \mathfrak{h}_T$ is such that $\alpha_T(\alpha_T^*) = 1$. This together with formula (2.2) implies the statement. □

[4]We agree that Hermitian forms are linear in the first argument and antilinear in the second one.

Let $c_B : \mathcal{S} \to \mathbb{C}$ be the function given by the formula

$$c_B(s) = -\frac{1}{2}\zeta_s a_B(s) \sum_{T \subset X^s} C(T, j(s)).$$

(the summation is over the connected components of X^s).

Corollary 2.7. *Near $\lambda = 0$, the elliptic Dunkl operators have the form:*

$$\mathcal{D}^{\mathcal{L}_\lambda}_{v,C} = -\sum_{s \in \mathcal{S}} \frac{2c_B(s)\alpha_s(v)}{(1-\zeta_s)\alpha_s(B(\lambda))} s + \text{regular terms}.$$

Remark 2.8. Here we realize sections of line bundles on X as functions on $\mathfrak{h}$ with prescribed periodicity properties under Γ.

Remark 2.9. Clearly, the same result applies to classical elliptic Dunkl operators.

Proof. The Corollary follows directly from Proposition 2.6 and the definition of $c_B(s)$. □

3. The main theorem

3.1. **The statement of the main theorem.** Define the operators

$$\overline{L}_i^{C,\lambda} := P_i^{c_B}(\mathcal{D}^{\mathcal{L}_\lambda}_{\bullet,C}, B(\lambda)),$$

acting on local meromorphic sections of $\mathcal{L}_\lambda$ (where $P_i^c(\mathbf{p}, \mathbf{q})$ are the classical Calogero-Moser Hamiltonians, defined in Subsection 2.5). It is easy to see that these operators are independent on the choice of B and commute with each other.

Our main result is the following theorem.

Theorem 3.1. *(i) For any fixed C, the operators $\overline{L}_i^{C,\lambda}$ are regular in λ near $\lambda = 0$, and in particular have limits $\overline{L}_i^C$ as $\mathcal{L}_\lambda$ tends to the trivial bundle (i.e., λ tends to 0).*

(ii) The operators $\overline{L}_i^C$ are G-invariant and pairwise commuting elements of $\mathbb{C}G \ltimes D(X_{\mathrm{reg}})$.

(iii) The restrictions L_i^C of $\overline{L}_i^C$ to G-invariant meromorphic functions on X are commuting differential operators on X_{reg}, whose symbols are the polynomials P_i.

Definition 3.2. The commutative algebra generated by the collection of operators $\{L_i^C\}$ is called the quantum crystallographic elliptic Calogero-Moser system attached to G, X, C.

Note that only part (i) of Theorem 3.1 requires proof; once it is proved, parts (ii) and (iii) follow immediately. We will give two proofs of Theorem 3.1(i). The first proof, given in Subsection 5.1, is based on Lemma 2.2. The second proof, given in Subsection 5.3, is based on the techniques of [BE] and reduction to rank 1 (where the result can be proved by a direct calculation).

Remark 3.3. Theorem 3.1(iii) can be generalized as follows: for any character χ of G, the restrictions $L_i^{C,\chi}$ of $\overline{L}_i^C$ to G-equivariant meromorphic functions on X which change according to χ under the G-action are commuting differential operators on X_{reg}, whose symbols are the polynomials P_i. Moreover, similarly to the results of [BC] in the rational case, one can show that $L_i^{C,\chi} = L_i^{C+\delta_\chi}$, where δ_χ is a certain shift of parameters.

3.2. **The classical version of the main theorem.** The quantum system of Theorem 3.1 can be easily degenerated to a classical integrable system, by replacing elliptic Dunkl operators with their classical counterparts. Namely, define

$$\overline{L}_i^{0,C,\lambda} := P_i^{c_B}(\mathcal{D}_{\bullet,C}^{0,\mathcal{L}_\lambda}, B(\lambda)),$$

Theorem 3.4. *(i) For any fixed C, the elements $\overline{L}_i^{0,C,\lambda}$ are regular in λ near $\lambda = 0$, and in particular have limits $\overline{L}_i^{0,C}$ as $\mathcal{L}_\lambda$ tends to the trivial bundle (i.e., λ tends to 0).*

(ii) The elements $\overline{L}_i^{0,C}$ are G-invariant and belong to $\mathbb{C}G \ltimes \mathcal{O}(T^*X_{\mathrm{reg}})$.

(iii) The functions $L_i^{0,C} := \mathbf{m}(\overline{L}_i^{0,C})$ are Poisson commuting regular functions on T^*X_{reg}, whose leading terms in momentum variables are the polynomials $P_i(\mathbf{p})$.

Theorem 3.4 is proved analogously to Theorem 3.1, and can also be deduced from it by taking the quasiclassical limit.

Definition 3.5. The algebra generated by the collection of functions $\{L_i^{0,C}\}$ is called the classical crystallographic elliptic Calogero-Moser system attached to G, X, C.

3.3. **Examples and remarks.**

Example 3.6. Let $\Gamma_\tau \subset \mathbb{C}$ be a lattice generated by 1 and $\tau \in \mathbb{C}_+$. Let $E_\tau = \mathbb{C}/\Gamma_\tau$ be the corresponding elliptic curve. Let R be a reduced irreducible root system, and $P^\vee$ be the coweight lattice of R. Let $G = W$ be the Weyl group of R. Let $X = E_\tau \otimes P^\vee$. In this case, the reflections s_α correspond to positive roots $\alpha \in R_+$, and we will write

T_α for T_{s_α}. It is easy to see that the elliptic curve X/T_α is naturally identified with E_τ via the map $\alpha : X/T_\alpha \to E_\tau$.

Let $(\ ,\ )$ be the W-invariant inner product on $\mathfrak{h}^*$, normalized by the condition that the long roots have squared length 2. It is easy to see from the above that one can uniquely choose B so that

$$a_B(s_\alpha) = (\alpha, \alpha).$$

Assume first that $C(T, 1) = 0$ unless T passes through the origin (e.g., this happens automatically if X^{s_α} is connected for all roots α). Let $C_\alpha = C(T_\alpha, 1)$. Then we have $c_\alpha := c_B(s_\alpha) = C_\alpha(\alpha, \alpha)/2$ (so in the simply laced case, $c_\alpha = C_\alpha$). In this case, $P_1(\mathbf{p}) = (\mathbf{p}, \mathbf{p})$, and the corresponding differential operator L_1^C is the elliptic Calogero-Moser operator

$$L_1^C = \Delta_\mathfrak{h} - \sum_{\alpha>0} C_\alpha(C_\alpha + 1)(\alpha, \alpha)\wp((\alpha, \mathbf{x}), \tau), \tag{3.1}$$

where $\Delta_\mathfrak{h}$ is the Laplace operator defined by $(\ ,\ )$, and $\wp$ is the Weierstrass function.

It remains to consider the case when X^{s_α} is disconnected for some α, and $C(T, 1)$ can be nonzero for T not necessarily passing through 0. This happens only in type $B_n, n \geq 1$, for short roots α. (Here $B_1 = A_1$, but we use the normalization of the form given by $(\alpha, \alpha) = 1$.) In this case, $X = E_\tau^n$, and s_α negates the i-th coordinate for some $i = 1, \dots, n$, so there are 4 components of X^{s_α}: $\alpha(\mathbf{x}) = \xi_l$, $l = 1, 2, 3, 4$, where $\xi_1 = 0, \xi_2 = 1/2, \xi_3 = \tau/2, \xi_4 = (1 + \tau)/2$ are the points of order 2 on E_τ. Let us denote the values of C corresponding to these components by C_l. Then $c_\alpha = (C_1 + C_2 + C_3 + C_4)/2$, and denoting by k the value of C for the long roots, we get

$$\begin{aligned} L_1^C &= \sum_{i=1}^n \partial_i^2 - \sum_{i\neq j} k(k+1)(\wp(x_i - x_j, \tau) + \wp(x_i + x_j, \tau)) \\ &\quad - \sum_{l=1}^4 \sum_{j=1}^n C_l(C_l + 1)\wp(x_j - \xi_l, \tau), \end{aligned}$$

which is the Hamiltonian of the 5-parameter Inozemtsev system [I]. For $n = 1$ we have a 4-parameter generalization of the Lamé operator,

$$L = D - \sum_{l=1}^4 C_l(C_l + 1)\wp(z - \xi_l, \tau), D := \frac{d}{dz},$$

which was first considered by Darboux (see [Da]).

Remark 3.7. The integrable systems $\{L_i^C\}$, $\{L_i^{0,C}\}$ have rational limits, which are obtained as the lattice Γ is rescaled by a parameter that goes to infinity. Namely, it is easy to see that these limits are the rational Calogero-Moser systems $\widehat{P}_i^{c'}$, $P_i^{c'}$, respectively, where

$$c'(s) = (1 - \zeta_s)C(T_s, j(s))/2$$

(so for real reflection groups $c'(s) = C(T_s, j(s))$).

However, the systems $\{L_i^C\}$, $\{L_i^{0,C}\}$ do not admit a trigonometric degeneration unless G is a Weyl group.

Remark 3.8. If G is a real reflection group, then instead of rational classical Calogero-Moser Hamiltonians, P_i^c, we could have used their *trigonometric deformations*, and all the statements and proofs would carry over with obvious changes. Furthermore, for any G, instead of P_i^c we could have used classical *elliptic* Calogero-Moser Hamiltonians associated to G and the dual abelian variety $X^\vee$. In this case, the arguments of Section 5 show that if the parameters of these classical Hamiltonians are chosen appropriately, then the resulting operators $\overline{L}_i^{C,\lambda}$ are regular *for all* λ, not only for $\lambda = 0$. This was conjectured by the authors of [BFV] in 1994 (unpublished); for types A_1 and A_2, this conjecture was confirmed by an explicit computation (see [BFV], pp. 908-909).[5]

Example 3.9. Consider the type BC_n case (the Inozemtsev system, Example 3.6). It is easy to check that in this case the appropriate choice of parameters for the "dual" classical system is as follows:

$$\begin{aligned} k' &= k, \\ C_1' &= \tfrac{1}{2}(C_1 + C_2 + C_3 + C_4) \\ C_2' &= \tfrac{1}{2}(C_1 + C_2 - C_3 - C_4) \\ C_3' &= \tfrac{1}{2}(C_1 - C_2 + C_3 - C_4) \\ C_4' &= \tfrac{1}{2}(C_1 - C_2 - C_3 + C_4) \end{aligned}$$

(i.e., the function C' is the Fourier transform of the function C on the group of points of order 2 on the elliptic curve).

[5]This construction of the crystallographic elliptic Calogero-Moser Hamiltonians is, in a sense, more natural than the one using the classical rational Hamiltonians, but unfortunately we could not have used it as the basic construction, since this would lead to a "vicious circle" (initially we don't have the elliptic Calogero-Moser Hamiltonians available even at the classical level).

4. The main example

4.1. **The systems attached to groups $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$.** Let n be a positive integer, and $m = 1, 2, 3, 4$ or 6. Then $G = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$ is a complex crystallographic reflection group. Namely, G acts on the torus $X = E_\tau^n$, where $E_\tau := \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ is an elliptic curve, and τ is any point in $\mathbb{C}_+$ for $m = 1, 2$, $\tau = e^{2\pi \mathrm{i}/3}$ for $m = 3, 6$, and $\tau = \mathrm{i}$ for $m = 4$. In this case, the above construction produces a quantum integrable system with Hamiltonians $L_1^C, \dots, L_n^C$ (G-invariant differential operators on E_τ^n with meromorphic coefficients) such that

$$L_j^C = \sum_{i=1}^n \partial_i^{mj} + \text{l.o.t.},$$

where l.o.t. stands for lower order terms. A similar construction involving classical counterparts of elliptic Dunkl operators yields a classical integrable system with Hamiltonians

$$L_j^{0,C} = \sum_{i=1}^n p_i^{mj} + \text{l.o.t.}.$$

In the case $m = 1$, this system essentially reduces to the previous example (the Calogero-Moser system of type A_{n-1}). In the case $m = 2$, it reduces to the 5-parameter Inozemtsev system, described above. However, for $m = 3, 4, 6$, we get new integrable systems with elliptic coefficients with cubic, quartic, and sextic lowest Hamiltonian, respectively.

The parameters of these systems are attached to the hypertori $x_i = x_j$ (a single parameter k) and to the hypertori $x_i = \xi$, where $\xi \in E_\tau$ is a point with a nontrivial stabilizer in $\mathbb{Z}/m\mathbb{Z}$ (the number of such parameters is the order of the stabilizer minus 1). For $m = 3$, we have three fixed points ξ of order 3, for $m = 4$ – two fixed points of order 4 and a fixed point of order 2, and for $m = 6$ – fixed points of orders 2, 3, 6, one of each (up to the action of $\mathbb{Z}/m\mathbb{Z}$). Therefore, for $m = 3$ this system has 7 parameters, for $m = 4$ it has 8 parameters, and for $m = 6$ it has 9 parameters (if $n = 1$, the number of parameters drops by 1, since the parameter k is not present).

Let us emphasize that the new crystallographic elliptic Calogero-Moser systems for $m > 2$ exist only for special elliptic curves with additional $\mathbb{Z}/m\mathbb{Z}$-symmetry, which means that the corresponding $\wp$-function satisfies the equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

with either $g_3 = 0$ (the lemniscatic case, $\mathbb{Z}/4\mathbb{Z}$-symmetry) or $g_2 = 0$ (the equianharmonic case, $\mathbb{Z}/3\mathbb{Z}$-symmetry).

4.2. **The equianharmonic case with $m = 3$.** In the equianharmonic case with $m = 3$, $\tau = e^{2\pi \mathrm{i}/3}$, we have the following proposition.

Proposition 4.1. *The quantum Hamiltonian L_1^C has the form*

$$\begin{aligned} L_1^C &= \sum_{i=1}^n \partial_i^3 + \sum_{i=1}^n (a_0 \wp(x_i) + a_1 \wp(x_i - \eta_1) + a_2 \wp(x_i - \eta_2))\partial_i \\ &\quad -3k(k+1) \sum_{i<j} \sum_{p=0}^{2} \wp(x_i - \varepsilon^p x_j)(\partial_i + \varepsilon^{-p}\partial_j) \\ &\quad + \sum_{i=1}^n (b_0 \wp'(x_i) + b_1 \wp'(x_i - \eta_1) + b_2 \wp'(x_i - \eta_2)), \end{aligned} \tag{4.1}$$

where $\tau = \varepsilon := e^{2\pi \mathrm{i}/3}$, $\wp(x) := \wp(x, \tau)$, $\eta_1 = \mathrm{i}\sqrt{3}/3$, $\eta_2 = -\mathrm{i}\sqrt{3}/3$, and a_l, b_l, k are parameters.

Proof. L_1^C must be a differential operator with meromorphic coefficients on E_τ^n which satisfies the following conditions:

1) The symbol of L_1^C is $\sum \partial_i^3$;

2) L_1^C is invariant under $S_n \ltimes \mathbb{Z}_3^n$;

3) the coefficients of order $3 - r$ in L_1^C are sums of meromorphic functions on E_τ^n with poles on the hypertori $x_i = 0, x_i = \eta_1, x_i = \eta_2$, $x_i = \varepsilon^p x_j$ $(p = 0, 1, 2)$, and the sum of orders of all the poles being $\leq r$.

It is easy to see that the only operators with this property are those of the form (4.1). □

Thus, the one-dimensional operator corresponding to the case $m = 3$ has the form

$$\begin{aligned} L &= D^3 + \sum_{i=1}^n (a_0 \wp(z) + a_1 \wp(z - \eta_1) + a_2 \wp(z - \eta_2))D \\ &\quad + \sum_{i=1}^n (b_0 \wp'(z) + b_1 \wp'(z - \eta_1) + b_2 \wp'(z - \eta_2)), \end{aligned} \tag{4.2}$$

where $D = \frac{d}{dz}$.

4.3. **The lemniscatic case.** In the lemniscatic case, with $m = 4$ and $\tau = \mathrm{i}$, the corresponding one-dimensional operator has the following explicit form:

$$\begin{aligned} L = D^4 &+ [a_0 \wp(z) + a_1 \wp(z - \omega_1) - 2k(k+1)(\wp(z - \omega_2) + \wp(z - \omega_3))]D^2 \\ &+ [b_0 \wp'(z) + b_1 \wp'(z - \omega_1) - 2k(k+1)(\wp'(z - \omega_2) + \wp'(z - \omega_3))]D \\ &+ [k(k+1)(k+3)(k-2)(\wp^2(z - \omega_2) + \wp^2(z - \omega_3)) \\ &+ k(k+1)\wp(\omega_3)(\wp(z - \omega_2) - \wp(z - \omega_3)) \end{aligned}$$

$$+ c_0\wp^2(z) + c_1\wp^2(z-\omega_1)], \tag{4.3}$$

where $\omega_1 = (1+\mathrm{i})/2, \omega_2 = \mathrm{i}/2, \omega_3 = 1/2$ and $k, a_0, a_1, b_0, b_1, c_0, c_1$ are arbitrary parameters.

5. Proofs of the Main Theorem

In this section, we give two different proofs of Theorem 3.1 (i).

5.1. **The first proof of Theorem 3.1.** For simplicity of exposition, we will work in a neighborhood U of 0 in X (or, equivalently, in $\mathfrak{h}$), which allows us to naturally trivialize the bundles $\mathcal{L}_\lambda$, and regard sections of all line bundles as ordinary functions.

For $v \in \mathfrak{h}$, define an operator on the space of meromorphic functions of $\mathbf{x}$ and λ by the formula

$$(\mathcal{E}_{v,C}F)(\mathbf{x},\lambda) = (\mathcal{D}^{\mathcal{L}_\lambda}_{v,C} + \sum_{s\in\mathcal{S}} \frac{2c_B(s)\alpha_s(v)}{(1-\zeta_s)\alpha_s(B(\lambda))}(s\otimes s^\vee))F(\mathbf{x},\lambda),$$

where $(s^\vee F)(\mathbf{x},\lambda) := F(\mathbf{x}, s^{-1}\lambda)$.

Proposition 5.1. *The operators $\mathcal{E}_{v,C}$ commute, i.e. $[\mathcal{E}_{v,C}, \mathcal{E}_{v',C}] = 0$ for all $v, v' \in \mathfrak{h}$.*

Proof. By Proposition 2.5, the operators $\mathcal{D}^\bullet_{v,C}, v \in \mathfrak{h}$, linear functions $\psi(B(\lambda))$, $\psi \in \mathfrak{h}^*$, and the operators $s\otimes s^\vee$ satisfy the defining relations of the algebra $\mathbb{C}G \ltimes S(\mathfrak{h}\oplus\mathfrak{h}^*)$. This implies the desired statement, since the operators $\mathcal{E}_{v,C}$ are exactly the classical Dunkl operators $\mathbf{D}^0_{\bullet,c_B}$ on these generators. □

Set $\widetilde{L}^C_i := P_i(\mathcal{E}_{\bullet,C})$ (these operators make sense and are pairwise commuting by Proposition 5.1).

Proposition 5.2. *One has $\widetilde{L}^C_i = \overline{L}^{C,\lambda}_i$.*

Proof. By Lemma 2.2, $P_i(\mathbf{D}^0_{\bullet,c}) = P^c_i(\mathbf{p},\mathbf{q})$. Substituting $\mathcal{D}^{\mathcal{L}_\lambda}_{\bullet,C}$ instead of $\mathbf{p}$ (which we can do by Proposition 2.5) and replacing $\mathbf{q}$ by $B(\lambda)$, we get the desired equality. □

Corollary 5.3. *The operators $\widetilde{L}^C_i$ are linear over functions of λ.*

Proof. Follows immediately from Proposition 5.2. □

Proposition 5.4. *The operators $\mathcal{E}_{v,C}$ map the space of functions which are regular in λ near $\lambda = 0$ to itself.*

Proof. By Corollary 2.7, near $\lambda = 0$, the operator $\mathcal{E}_{v,C}$ has the form

$$\sum_{s\in\mathcal{S}} \frac{2c_B(s)\alpha_s(v)}{(1-\zeta_s)\alpha_s(B(\lambda))} s\otimes(s^\vee - 1) + \text{regular terms}\ .$$

Since the operator $\frac{1}{\alpha_s(B(\lambda))}(s^\vee - 1)$ preserves regularity in λ, the statement follows. □

By Proposition 5.4, the operators $\widetilde{L}_i^C$ preserve the space of functions which are regular in λ near $\lambda = 0$. By Corollary 5.3, this means that $\widetilde{L}_i^C$ are themselves regular in λ for λ near 0. Hence, by Proposition 5.2, the operators $\overline{L}_i^{C,\lambda}$ are regular in λ near $\lambda = 0$, as desired.

5.2. **Relation to Cherednik's proof.** In this subsection we would like to explain the connection between the construction of Subsection 5.1 and Cherednik's proof of the integrability of elliptic Calogero-Moser systems attached to Weyl groups ([Ch2]).

Recall that to obtain the operators $\mathcal{E}_{v,C}$ used in Subsection 5.1 from the elliptic Dunkl operators $\mathcal{D}_{v,C}^{\mathcal{L}_\lambda}$, we "subtract" the pole in λ by adding the reflection part of the rational Dunkl operator with respect to λ.

As we mentioned in Remark 3.8, in the real reflection group case, instead of the rational Dunkl operator we could have used the trigonometric one. Let us denote the corresponding operators by $\mathcal{E}_{v,C}^{\mathrm{trig}}$.

For $\lambda \in \mathrm{Hom}(P^\vee, \mathbb{C}^*) = \mathfrak{h}^\vee/Q$, denote by $\mathcal{F}_\lambda$ the space of meromorphic functions on $\mathfrak{h}$ which are periodic under $P^\vee$ and transform by a character under $\tau P^\vee$, representing sections of $\mathcal{L}_\lambda$. Let $\mathcal{F} = \oplus_{\lambda \text{ regular}} \mathcal{F}_\lambda$. It is easy to check that the operator $\mathcal{E}_{v,C}^{\mathrm{trig}}$ acts naturally on $\mathcal{F}$.

On the other hand, in [Ch2], Cherednik defined affine Dunkl operators, $\mathcal{D}_{v,C}^{\mathrm{aff}}$ ([Ch2], formula (3.4) after specialization of the central element). These are differential-difference operators on functions on $\mathfrak{h}/P^\vee$ (involving shifts by elements of $\tau P^\vee$ composed with reflections), which preserve the space $\mathcal{F}$.

It turns out that the operators $\mathcal{E}_{v,C}^{\mathrm{trig}}$ and $\mathcal{D}_{v,C}^{\mathrm{aff}}$ on the space $\mathcal{F}$ coincide. This shows that in the real reflection group case, the construction of Subsection 5.1 is, essentially, a modification of the construction of [Ch2].

5.3. **The second proof of Theorem 3.1.**

Proposition 5.5. *Theorem 3.1(i) holds in rank 1, i.e., if* $\dim X = 1$.

Proof. In the rank 1 case, $G = \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{C}$ be the reflection representation of G with coordinate function x. Let g be the generator of

G acting on $\mathbb{C}$ by multiplication by $\xi = e^{2\pi \mathrm{i}/m}$ (i.e., $gx = \xi^{-1}x$). The primitive idempotents of $\mathbb{C}G$ are defined by

$$e_i = \frac{1}{m}\sum_{j=0}^{m-1} \xi^{ij} g^j,\ i = 0, ..., m-1;$$

they satisfy the relations

$$e_i e_j = \delta_{ij} e_i, \quad \sum_{i=0}^{m-1} e_i = 1.$$

We also have the following cross relations (the indexing is modulo m):

$$e_i x = x e_{i-1}, e_i \partial_x = \partial_x e_{i+1}, e_i p = p e_{i+1} (p \text{ is the symbol of } \partial_x).$$

For brevity, we will abuse notation and write λ instead of $B(\lambda)$. From Corollary 2.7, we know that near $\lambda = 0$ the elliptic Dunkl operator can be written as

$$\mathcal{D}_{v,C}^{\mathcal{L}_\lambda} = \partial_x + \frac{1}{\lambda}\sum_{i=0}^{m-1} b_i e_i + \sum_{j=0}^{m-1} R_j \sum_{i=0}^{m-1} b_i e_i,$$

where $\sum b_i = 0$, $\mathbf{b} = (b_0, \ldots, b_{m-1})$ is related to c_B by a certain invertible linear transformation, and R_j has the form

$$R_j = \sum_{\substack{s \geq -1, t \geq 0 \\ s \equiv j \bmod m \\ s+t \equiv -1 \bmod m}} a_{st} x^s \lambda^t, \quad \text{where } a_{st} \text{ are constants.}$$

So we have $R_j e_i = e_{i+j} R_j$. Here all indices are modulo m.

We have $P_1 = P = p^m$, and

$$P^{c_B}(p, q) = (p - \frac{b_0}{q}) \cdots (p - \frac{b_{m-1}}{q}). \tag{5.1}$$

Define $\Phi_i(p, q) = p - \frac{b_i}{q}$ and $\Phi_i = \Phi_i(\mathcal{D}_{v,C}^{\mathcal{L}_\lambda}, \lambda)$.

Lemma 5.6. *For any integer r, s with $1 \leq s \leq m$, the expression*

$$\Phi_{r+1} \cdots \Phi_{r+s} e_{r+s}$$

is regular in λ at $\lambda = 0$.

Proof. We prove the statement by induction on s. By a direct computation, one can see that the statement is true when $s = 1$. Now for the induction step suppose the statement is true for $s < k$, where $k \geq 2$, and let us prove it holds for $s = k$. We have

$$\Phi_{r+1} \cdots \Phi_{r+k} e_{r+k}$$

$$= \Phi_{r+1}\cdots\Phi_{r+k-1}e_{r+k-1}(\partial_x+b_{r+k}R_{m-1})+b_{r+k}\sum_{j=2}^{m}\Phi_{r+1}\cdots\Phi_{r+k-1}e_{r+k-j}R_{m-j}.$$

Notice that $R_{m-j}=\lambda^{j-1}R'_{m-j}$ for $j=1,...,m$, where R'_j is regular at $\lambda=0$, and Φ_i has only a simple pole in λ. So $j=2,...,k-1$ we have

$$\begin{aligned}
&\Phi_{r+1}\cdots\Phi_{r+k-1}e_{r+k-j}R_{m-j}\\
=\;&\Phi_{r+1}\cdots\Phi_{r+k-j}(\Phi_{r+k-j+1}\lambda)\cdots(\Phi_{r+k-1}\lambda)e_{r+k-j}R'_{m-j}\\
=\;&(\Phi_{r+k-j+1}\lambda)\cdots(\Phi_{r+k-1}\lambda)\Phi_{r+1}\cdots\Phi_{r+k-j}e_{r+k-j}R'_{m-j},
\end{aligned}$$

which is regular in λ by the induction hypothesis, while for $k\le j\le m$ the above expression is regular since R_{m-j} is divisible by λ^{j-1}. Also, the expression

$$\Phi_{r+1}\cdots\Phi_{r+k-1}e_{r+k-1}(\partial_x+b_{r+k}R_{m-1})$$

is regular by the induction hypothesis. The induction step is thus completed. □

Now since

$$\overline{L}_i^{C,\lambda}=\prod_{i=0}^{m-1}\Phi_i=\sum_{j=0}^{m-1}\Phi_{j-m+1}\cdots\Phi_j e_j,$$

Lemma 5.6 implies that the operators $\overline{L}_i^{C,\lambda}$ are regular in λ near $\lambda=0$. □

Now we proceed to prove Theorem 3.1(i) in rank $n>1$. By Hartogs' theorem, it suffices to check the regularity of $\overline{L}_i^C$ at a generic point of a reflection hyperplane $H\subset\mathfrak{h}$. To this end, we will use the following proposition.

Let H be a reflection hyperplane in $\mathfrak{h}$. Let $s\in\mathcal{S}$ be a generator of $G_H\cong\mathbb{Z}_m$. Let $p\in\mathfrak{h}_s$, $q\in\mathfrak{h}_s^*$ be such that $(p,q)=1$, and let $p_1,\dots,p_{n-1}$, $q_1,\dots,q_{n-1}$ be bases of $\mathfrak{h}^s$, $(\mathfrak{h}^s)^*$. Also, since the 1-dimensional space $\mathfrak{h}_s$ carries a G_H-action, we can define the classical Calogero-Moser Hamiltonian $P^c(p,q)$ (given by formula (5.1)).

Let $\mathbf{x}_0$ be a generic point of H, and let $q_i^0:=q_i(\mathbf{x}_0)$. (Note that $q(\mathbf{x}_0)=0$.)

Proposition 5.7. *Near a generic point $\mathbf{x}_0$ of H, for any $i=1,\dots,n$, the function P_i^c can be written as a polynomial of the functions $p_1,\dots,p_{n-1}$, pq, and $P^c(p,q)$, whose coefficients are power series in the functions $q_1-q_1^0,\dots,q_{n-1}-q_{n-1}^0$, q^m.*

Proof. Let

$$\mathbf{e} = \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{C}G \tag{5.2}$$

be the symmetrizing idempotent. The function P_i^c belongs to the spherical subalgebra $B_{0,c}(G,\mathfrak{h}) := \mathbf{e}H_{0,c}(G,\mathfrak{h})\mathbf{e}$ of the rational Cherednik algebra $H_{0,c}(G,\mathfrak{h})$ (sitting inside $\mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})^G$). By the classical version of Theorem 3.2 of [BE] (see also [B]), the completion at $\mathbf{x}_0$ of the algebra $B_{0,c}(G,\mathfrak{h})$ is isomorphic to the completion at 0 of the algebra $\mathbb{C}[q_1,\dots,q_{n-1},p_1,\dots,p_{n-1}]\otimes B_{0,c}(G_H,\mathfrak{h}_s)$. However, the algebra $B_{0,c}(G_H,\mathfrak{h}_s)$ is generated by q^m, pq and $P^c(p,q)$. This implies the desired statement. □

Theorem 3.1(i) follows immediately from Proposition 5.5 and Proposition 5.7.

6. A geometric construction of quantum crystallographic elliptic Calogero-Moser systems

In this section we will give a geometric construction of the quantum crystallographic elliptic Calogero-Moser systems described above, in the style of the Beilinson-Drinfeld construction of the quantum Hitchin system, [BD]. Namely, we construct these systems as algebras of global sections of sheaves of spherical elliptic Cherednik algebras, for the critical value of the twisting parameter. On the other hand, if the twisting parameter is not critical, we show that the algebra of global sections reduces to $\mathbb{C}$.

6.1. **Cherednik algebras of varieties with a finite group action.** Let us recall the basics on the Cherednik algebras of varieties with a finite group action, introduced in [E2] (see also [EM2], Section 7).

Let X be a smooth affine algebraic variety over $\mathbb{C}$. For a closed hypersurface $Y\subset X$, let $\mathcal{O}_X(Y)$ be the space of regular functions on $X\setminus Y$ with a pole of at most first order on Y. Let $\xi_Y : \mathrm{Vect}(X)\to \mathcal{O}_X(Y)/\mathcal{O}_X$ be the natural map.

Let G be a finite group of automorphisms of X. Let $\mathcal{Y}$ be the set of pairs (Y,s), where $s\in G$, and Y is a connected component of the set of fixed points X^s such that $\mathrm{codim}Y = 1$ (called a *reflection hypersurface*). Let $\lambda_{Y,s}$ be the eigenvalue of s on the conormal bundle of Y. Let X_{reg} be the complement of reflection hypersurfaces in X.

Fix $\omega\in \mathrm{H}^2(X)^G$, and let $D_\omega(X)$ be the algebra of twisted differential operators on X with twisting ω.

Let $c : \mathcal{Y} \to \mathbb{C}$ be a G-invariant function. Let v be a vector field on X, and let $f_Y \in \mathcal{O}_X(Y)$ be an element of the coset $\xi_Y(v) \in \mathcal{O}_X(Y)/\mathcal{O}_X$.

A *Dunkl-Opdam operator* for G, X is an operator given by the formula

$$\mathcal{D} := \mathbf{L}_v + \sum_{(Y,s)\in\mathcal{Y}} f_Y \cdot \frac{2c_{Y,s}}{1-\lambda_{Y,s}}(s-1),$$

where $\mathbf{L}_v \in D_\omega(X)$ is the ω-twisted Lie derivative along v (here we pick a closed 2-form representing ω).

The *Cherednik algebra of* G, X, $H_{1,c,\omega}(G,X)$, is generated inside $\mathbb{C}G \ltimes D_\omega(X_{\mathrm{reg}})$ by the function algebra $\mathcal{O}_X$, the group G, and the Dunkl-Opdam operators $\mathcal{D}$.

Now let X be any smooth algebraic variety (not necessarily affine), and let G be a finite group acting on X. Assume that X has a G-invariant affine open covering, so that X/G is also a variety. Recall that twistings of differential operators on X are parametrized by $\mathrm{H}^2(X, \Omega_X^{\geq 1})$; in particular, if X is projective, they are parametrized by $\mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X)$ (see [BB]). So for $\psi \in \mathrm{H}^2(X, \Omega_X^{\geq 1})^G$, we can define the sheaf of Cherednik algebras $H_{1,c,\psi,G,X}$ (a quasicoherent sheaf on X/G), by gluing the above constructions on G-invariant affine open sets. Namely, for an affine open set $U \subset X/G$, we set

$$H_{1,c,\psi,G,X}(U) := H_{1,c,\psi}(G, \overline{U}),$$

where $\overline{U}$ is the preimage of U in X. We can also define the sheaf of spherical Cherednik algebras, $B_{1,c,\psi,G,X}$, given by

$$B_{1,c,\psi,G,X}(U) = \mathbf{e}H_{1,c,\psi}(G, \overline{U})\mathbf{e}$$

where $\mathbf{e}$ is the symmetrizing idempotent of G, defined by (5.2).

Finally, let us define the sheaves of modified Cherednik algebras, $H_{1,c,\psi,\eta,G,X}$ and modified spherical Cherednik algebras $B_{1,c,\psi,\eta,G,X}$. Let η be a G-invariant function on the set of reflection hypersurfaces in X. Define a modified Dunkl-Opdam operator for G, X (when X is affine) by the formula

$$\mathcal{D} := \mathbf{L}_v + \sum_{(Y,s)\in\mathcal{Y}} \frac{2c_{Y,s}}{1-\lambda_{Y,s}} f_Y \cdot (s-1) + \sum_Y \eta(Y) f_Y,$$

(where the summation in the second sum is over all reflection hypersurfaces), and define the sheaf of algebras $H_{1,c,\psi,\eta,G,X}$ to be locally generated by $\mathcal{O}_X$, G, and modified Dunkl-Opdam operators (so, we have $H_{1,c,\psi,0,G,X} = H_{1,c,\psi,G,X}$). Also, set $B_{1,c,\psi,\eta,G,X} := \mathbf{e}H_{1,c,\psi,\eta,G,X}\mathbf{e}$.

Note that according to the PBW theorem, the sheaf $H_{1,c,\psi,\eta,G,X}$ has an increasing filtration $F^\bullet$, such that $\mathrm{gr}(H_{1,c,\psi,\eta,G,X}) = G \ltimes \mathcal{O}_{T^*X}$.

Note also that the modified Cherednik algebras can be expressed via the usual ones (see [E2], [EM2]). Namely, let ψ_Y be the twisting of differential operators on X by the line bundle $\mathcal{O}_X(Y)^*$. Then one has

$$H_{1,c,\psi,\eta,G,X} \cong H_{1,c,\psi+\sum_Y \eta(Y)\psi_Y,G,X}.$$

Finally, note that we have a canonical isomorphism of sheaves

$$H_{1,c,0,\eta,G,X}|_{X_{\mathrm{reg}}} \cong \mathbb{C}G \ltimes D_{X_{\mathrm{reg}}}.$$

6.2. **Elliptic Cherednik algebras and crystallographic elliptic Calogero-Moser systems.** Now let X be an abelian variety, and G an irreducible complex reflection group acting on X, as in Section 2. It is easy to see that $(\wedge^2\mathfrak{h}^*)^G = 0$, so X does not admit nonzero global 2-forms. This implies that the space of G-invariant twistings of differential operators on X is $\mathrm{H}^{1,1}(X)^G$, which is 1-dimensional, and spanned by the Kähler form on X defined by the Hermitian form B. So we can make the identification $\mathrm{H}^{1,1}(X)^G \cong \mathbb{C}$.

It is well known that X admits a G-invariant affine open covering, so X/G is an algebraic variety, and we can consider the sheaves $H_{1,c,\psi,\eta,G,X}$ and $B_{1,c,\psi,\eta,G,X}$ on X/G.

Notice that we have an isomorphism $\mathcal{Y} \cong \mathcal{A}$. Thus we can substitute for c the function

$$c_{T,s} = (1 - e^{-2\pi \mathrm{i} j(s)/m_s})C(T, j(s))/2.$$

Also, define a function η_C on the set of reflection hypertori by the formula

$$\eta_C(T) := \sum_{j=1}^{m_T-1} C(T, j).$$

The main result of this section is the following theorem, which gives a geometric construction of the quantum elliptic integrable systems.

Theorem 6.1. *(i) Restriction to X_{reg} defines an isomorphism*

$$\Gamma(X/G, B_{1,c,0,\eta_C,G,X}) \cong \mathbb{C}[L_1^C, \dots, L_n^C].$$

(ii) The algebra of global sections $\Gamma(X/G, B_{1,c,\psi,G,X})$ is nontrivial (i.e. not isomorphic to $\mathbb{C}$) if and only if

$$\psi = \sum_{(T,j)\in\mathcal{A}} C(T, j)\psi_T. \tag{6.1}$$

If (6.1) *holds, $\Gamma(X/G, B_{1,c,\psi,G,X})$ is a polynomial algebra in generators L_i whose symbols are P_i.*

Example 6.2. If $C = 0$, Theorem 6.1 states that for $\psi \in \mathbb{C}$, there exist nontrivial G-invariant ψ-twisted global differential operators on X if and only if $\psi = 0$, in which case the algebra of such operators is $(S\mathfrak{h})^G$. This is, of course, easy to check directly.

6.3. **Proof of Theorem 6.1.** We first prove (i). The sheaf of algebras $H_{1,c,0,\eta_C,G,X}$ is locally generated by regular functions on X, elements of G, and Dunkl-Opdam operators without a "pure function" term:

$$\mathcal{D} = \mathbf{L}_v + \sum_{(Y,s)\in\mathcal{Y}} f_Y \cdot \frac{2c_{Y,s}}{1-\lambda_{Y,s}} s.$$

This implies that for a generic $\mathcal{L}$, the elliptic Dunkl operators $\mathcal{D}^{\mathcal{L}}_{v,C}$ are sections of the sheaf $H_{1,c,0,\eta_C,G,X}$ on the formal neighborhood of any point in X/G. Thus, the same applies to the operators $\overline{L}_i^{C,\lambda}$, and hence to their limits at $\lambda = 0$, $\overline{L}_i^C$ (which exist by Theorem 3.1). But since the coefficients of $\overline{L}_i^C$ are periodic, $\overline{L}_i^C$ are actually *global* sections of the sheaf $H_{1,c,0,\eta_C,G,X}$. Thus, L_i^C are global sections of $B_{1,c,0,\eta_C,G,X}$, i.e., $\mathbb{C}[L_1^C,\dots,L_n^C] \subset \Gamma(X/G, B_{1,c,0,\eta_C,G,X})$. To see that this inclusion is an isomorphism, it suffices to show that it is an isomorphism for the corresponding graded algebras, which is obvious, since $\Gamma(X/G, \mathrm{gr}(B_{1,c,0,\eta_C,G,X})) = (S\mathfrak{h})^G$.

Now we prove (ii). As explained above, we have an isomorphism

$$H_{1,c,0,\eta_C,G,X} \cong H_{1,c,\sum_{(T,j)\in\mathcal{A}} C(T,j)\psi_T,G,X},$$

which proves the "if" part of (ii). It remains to prove the "only if" part, i.e. that if equation (6.1) does not hold then the algebra of global sections is trivial. To this end, for $r \geq 1$ consider the vector bundle

$$\mathcal{E} := F^r H\mathbf{e}/F^{r-2}H\mathbf{e},$$

on X, where $H = H_{1,c,\psi,G,X}$. We have an exact sequence of vector bundles on X:

$$0 \to S^{r-1}\mathfrak{h} \to \mathcal{E} \to S^r\mathfrak{h} \to 0,$$

where the bundles $S^k\mathfrak{h}$ are trivial. Such an extension is determined by an extension class β in

$$\begin{aligned}\mathrm{Ext}^1(S^r\mathfrak{h}, S^{r-1}\mathfrak{h}) &= \mathrm{Hom}_{\mathbb{C}}(S^r\mathfrak{h}, S^{r-1}\mathfrak{h}) \otimes \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \\ &= \mathrm{Hom}_{\mathbb{C}}(S^r\mathfrak{h}, S^{r-1}\mathfrak{h} \otimes \mathfrak{h}).\end{aligned}$$

(since $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = \mathfrak{h}$). A direct calculation shows that (up to a nonzero constant) β is the canonical inclusion multiplied by the number

$$\psi - \sum_{(T,j)\in\mathcal{A}} C(T,j)\psi_T.$$

So if (6.1) does not hold, β is injective, and thus no nonzero section of $S^r\mathfrak{h}$ can be lifted to a section of $\mathcal{E}$. This implies the "only if" part of (i).

6.4. **The classical analog of Theorem 6.1.** In this subsection we give a geometric construction of the classical crystallographic elliptic Calogero-Moser systems.

Define a modified classical Dunkl-Opdam operator for G, X (when X is affine) by the formula

$$\mathcal{D}^0 := p_v + \sum_{(Y,s)\in\mathcal{Y}} \frac{2c_{Y,s}}{1-\lambda_{Y,s}} f_Y \cdot (s-1) + \sum_Y \eta(Y) f_Y.$$

Let $T^*_\psi X$ denote the ψ-twisted cotangent bundle of X (see [BB], Section 2), and define the sheaf of modified classical elliptic Cherednik algebras $H_{0,c,\psi,\eta,G,X}$ to be locally generated inside $\mathbb{C}G \ltimes \mathcal{O}(T^*_\psi X_{\mathrm{reg}})$ by $\mathcal{O}_X$, G, and modified classical Dunkl-Opdam operators ([E2]). The "unmodified" version $H_{0,c,\psi,0,G,X}$ will be shortly denoted by $H_{0,c,\psi,G,X}$. Also, set $B_{0,c,\psi,\eta,G,X} := \mathbf{e}H_{0,c,\psi,\eta,G,X}\mathbf{e}$.

Theorem 6.3. *(i) Restriction to X_{reg} defines an isomorphism*

$$\Gamma(X/G, B_{0,c,0,\eta_C,G,X}) \cong \mathbb{C}[L_1^{0,C}, \dots, L_n^{0,C}].$$

(ii) The algebra of global sections $\Gamma(X/G, B_{0,c,\psi,G,X})$ is nontrivial (i.e. not isomorphic to $\mathbb{C}$) if and only if

$$\psi = \sum_{(T,j)\in\mathcal{A}} C(T,j)\psi_T. \tag{6.2}$$

If (6.2) holds, $\Gamma(X/G, B_{0,c,\psi,G,X})$ is a polynomial algebra in generators $L_i^{(0)}$ whose leading terms in momentum variables are P_i.

Proof. The proof is parallel to the proof of Theorem 6.1, using Theorem 3.4. □

7. Algebraic integrability of quantum crystallographic elliptic Calogero-Moser systems

Let $\{L_1, \dots, L_n\}$ be a quantum integrable system (i.e., a commuting system of differential operators) on an open set $U \subset \mathbb{C}^n$. Assume that the symbols P_i of L_i have constant coefficients, and $\mathbb{C}[p_1, \dots, p_n]$ is a finitely generated module (of some rank r) over $\mathbb{C}[P_1, \dots, P_n]$. Consider the joint eigenvalue problem:

$$L_i\Psi = \Lambda_i\Psi. \tag{7.1}$$

Clearly, the space of local holomorphic solutions of this system near a generic point $\mathbf{x}_0 \in U$ has dimension r. Recall [K2, CV1, CV2, BEG] that the system $\{L_i\}$ is said to be *algebraically integrable* if there exists a differential operator L on U which commutes with L_i and acts with distinct eigenvalues on the space of local solutions of (7.1) for generic Λ_i. In this case, the system of differential equations

$$L_i\Psi = \Lambda_i\Psi, L\Psi = \Lambda\Psi$$

(where Λ is a certain algebraic function of the Λ_i) can be reduced to a first order scalar system, and thus the solutions of system (7.1) can (in principle) be written explicitly in quadratures.

It was proved in [CV1, VSC] that the rational and trigonometric quantum Calogero-Moser systems are algebraically integrable for any Weyl group if the parameters c_α are integers. The same result in the elliptic case was conjectured in [CV1][6] and proved in [CEO] (for type A, it was proved earlier in [BEG]). It was also proved in [CEO] that algebraic integrability holds for the Inozemtsev system with integer parameters. Finally, algebraic integrability of the rational quantum Calogero-Moser systems of complex reflection groups was established recently in [BC].

The following theorem establishes algebraic integrability of the crystallographic elliptic Calogero-Moser system attached to any complex crystallographic reflection group, under an integrality assumption on the parameters. Namely, for any reflection hypertorus $T \subset X$ and any $l = 0, 1, \dots, m_T - 1$ define the number

$$m_l(T) = l + \sum_{j=1}^{m_T-1} C(T,j)e^{2\pi \mathrm{i} jl/m_T}.$$

Theorem 7.1. *If for all l and T the numbers $m_l(T)$ are integers which are pairwise distinct modulo m_T, then the quantum integrable system $\{L_i^C\}$ is algebraically integrable.*

Proof. The proof is similar to the proof in the real reflection case, given in [CEO].

Namely, we first show that the holonomic system of differential equations

$$L_i^C\Psi = \Lambda_i\Psi \tag{7.2}$$

[6]It is interesting that this conjecture was inspired by a remarkable result of J. Ritt, who classified in dimension one all commuting rational maps in terms of the symmetry groups of elliptic curves (see [Ve] and references therein)

has regular singularities. This follows from the fact that (7.2) is a limit as $\lambda \to 0$ of the eigenvalue problem for elliptic Dunkl operators

$$\mathcal{D}_{v,C}^{\mathcal{L}_\lambda,\nabla}\Psi = \Lambda(v)\Psi, \tag{7.3}$$

([EM1]) which obviously has regular singularities. [7]

Thus, by Remark 3.10 of [CEO], it suffices to show that the monodromy of (7.2) around the reflection hypertori is trivial. For the system (7.3), this property follows from the fact that this monodromy representation factors through the orbifold Hecke algebra (see [EM1], Section 6.2); indeed, since the parameters are integral and distinct modulo m_T, the orbifold Hecke algebra reduces to the group algebra of the orbifold fundamental group, implying the triviality of the monodromy. Now the required statement for (7.2) follows by taking the limit $\lambda \to 0$. □

Corollary 7.2. *The quantum integrable system defined by the operator (4.1) is algebraically integrable if $k \in \mathbb{Z}$, and there exist integers m_{ij}, $i,j \in \{0,1,2\}$, pairwise non-congruent modulo 3, with $m_{i0}+m_{i1}+m_{i2} = 3$, such that*

$$a_i = m_{i0}m_{i1} + m_{i0}m_{i2} + m_{i1}m_{i2} - 2,$$

and

$$b_i = \frac{1}{2}\prod_{j=0}^{2} m_{ij}.$$

for $i = 0,1,2$.

Proof. The last condition means that the indices of the corresponding 1-variable operator $\partial^3 + \frac{a_i}{x^2}\partial - \frac{2b_i}{x^3}$ (which we obtain by looking at the neighborhood of a generic point of the hypertori $x_j = 0$, $x_j = \eta_1$, $x_j = \eta_2$ in E_τ^n) are m_{i0}, m_{i1}, m_{i2}, and thus are all integers. Now the absence of logarithmic terms (and hence, the triviality of monodromy) follows from the symmetry $x \to \varepsilon x, \varepsilon^3 = 1$ and the fact that the indices m_{i0}, m_{i1}, m_{i2} have different residues modulo 3. □

Remark 7.3. The integrality and non-congruence assumptions in Theorem 7.1 (and in particular in Corollary 7.2) are necessary. Here is a sketch of a proof. Suppose the system is algebraically integrable. Let us translate the origin in X to a generic point $\mathbf{x}$ of some reflection hypertorus $T \subset X$ with $m_T = m$, and then go to the rational limit by multiplying the lattice Γ by a factor K going to infinity. Then we will

[7] Here it is important that we don't have moving poles. Otherwise (if poles are allowed to move and collide), a system with regular singularities can be degenerated to a system with irregular ones.

get that the single-variable rational Calogero-Moser operator L of order m with the appropriate parameters is algebraically integrable (this is seen by looking at what happens to the Dunkl-Opdam operators in the limit). But in the single-variable rational case, it is known (and easy to prove) that the integrality and non-congruence conditions are necessary (see e.g. [BC]). Namely, in this case the operator L is homogeneous of degree $-m$, and in the algebraically integrable case it should have eigenfunctions (with eigenvalue μ^m) of the form $F(\mu x)$, where $F(x) = e^x Q(1/x)$, Q being a polynomial, and it is easy to compute when there are such solutions using the power series method. Since this argument can be applied to all reflection hypertori, it gives the integrality and non-congruence conditions for all the parameters.

Example 7.4. Consider the case $n = 1$, and $a_0 = a_1 = a_2 = a$, $b_0 = b_1 = b_2 = b$. Then Corollary 7.2 implies that the operator

$$L = D^3 + a\wp(z)D + b\wp'(z).$$

(where $\wp(z) = \wp(z, \tau)$, $\tau = e^{2\pi \mathrm{i}/3}$) is algebraically integrable if there exists an triple of integers $\mathbf{m} = \{m_0, m_1, m_2\}$ $(m_0 < m_1 < m_2)$, pairwise non-congruent modulo 3, with $m_0 + m_1 + m_2 = 3$, such that

$$a = m_0m_1 + m_0m_2 + m_1m_2 - 2,$$

and

$$b = \frac{1}{2}m_0m_1m_2.$$

Remark 7.5. In the case $m_2 - m_1 = m_1 - m_0 = n \in \mathbb{N}$, the operator L has the form

$$L = D^3 + (1 - n^2)\wp(z)D + \frac{1 - n^2}{2}\wp'(z).$$

A proof of algebraic integrability of this operator (i.e., of the existence of meromorphic eigenfunctions) in the equianharmonic case is given by Halphen, [Ha], p.571; this proof easily extends to general values of m_i.

Remark 7.6. Note that if $\mathbf{m} = \{0, 1, 2\}$, then $L = D^3$, so the algebraic integrability of L is obvious, and if $\mathbf{m} = \{-1, 1, 3\}$, the algebraic integrability of L follows from the fact that L commutes with the Lamé operator $D^2 - 2\wp(z)$. The case $\mathbf{m} = (-1, 0, 4)$ is a special case of the algebraically integrable operator

$$L = D^3 - (6\wp(z) + c)D, c \in \mathbb{C}$$

considered by Picard in 1881 ([Pi]; see also [For], page 464, Ex. 13).

Observe that these examples are algebraically integrable for any elliptic curve. On the other hand, as explained in [U], if $\mathbf{m} = (-3, 1, 5)$,

then the operator L is algebraically integrable only in the equianharmonic case. [8]

Example 7.7. Similarly, the operator (4.3) is algebraically integrable when the parameter k is integer and for $i = 0, 1$ there exist integers $m_{1i}, m_{2i}, m_{3i}, m_{4i}$ with $m_{1i} + m_{2i} + m_{3i} + m_{4i} = 6$, which are distinct modulo 4, such that

$$a_i = \sum_{1\le k<l\le 4} m_{ki}m_{li} - 11, \quad b_i = \frac{1}{2}\Big(\sum_{1\le k<l<j\le 4} m_{ki}m_{li}m_{ji} - 6 - a_i\Big)$$

and

$$c_i = m_{1i}m_{2i}m_{3i}m_{4i}.$$

Remark 7.8. When $a_0 = a_1 = b_0 = b_1 = c_0 = c_1 = 0$, the operator (4.3) specializes to

$$L = D^4 - 2k(k+1)\wp(z)D^2 - 2k(k+1)\wp'(z)D + k(k+1)(k+3)(k-2)\wp^2(z),$$

(after the change of variable $z = (1+i)w$ and multiplication by -4), which is the square of the Lamé operator $D^2 - k(k+1)\wp$ plus a constant.

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