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# On the Chvátal-Gomory Closure of a Compact Convex Set

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**Abstract** In this paper, we show that the Chvátal-Gomory closure of any compact convex set is a rational polytope. This resolves an open question of Schrijver [17] for irrational polytopes[1], and generalizes the same result for the case of rational polytopes [17], rational ellipsoids [8] and strictly convex bodies [7].

**Keywords** Chvátal-Gomory Closure · Compact Sets

## 1 Introduction

Gomory [12] introduced the Gomory fractional cuts, also known as Chvátal-Gomory (CG) cuts [5], to design the first finite cutting plane algorithm for Integer Linear Programs (ILP). Since then, many important classes of facet-defining inequalities for combinatorial optimization problems have been identified as CG cuts. For example, the classical Blossom inequalities for general

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[1] After the completion of this work, it has been brought to our notice that the polyhedrality of the Chvátal-Gomory Closure for irrational polytopes has recently been shown independently by J. Dunkel and A. S. Schulz in [9]. The proof presented in this paper has been obtained independently.

Matching [10] - which yield the integer hull - and Comb inequalities for the Traveling Salesman problem [13,14] are both CG cuts over the standard linear programming relaxations. CG cuts have also been effective from a computational perspective; see for example [2,11]. Although CG cuts have traditionally been defined with respect to rational polyhedra for ILP, they straightforwardly generalize to the nonlinear setting and hence can also be used for convex Integer Nonlinear Programming (INLP), i.e. the class of discrete optimization problems whose continuous relaxation is a general convex optimization problem. CG cuts for non-polyhedral sets were considered implicitly in [5,17] and more explicitly in [4,7,8]. Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let $h_K$ represent its support function, i.e. $h_K(a) = \sup\{\langle a, x\rangle : x \in K\}$. Given $a \in \mathbb{Z}^n$, we define the CG cut for $K$ derived from $a$ as the inequality

$$\langle a, x\rangle \leq \lfloor h_K(a) \rfloor \; . \tag{1}$$

The CG closure of $K$ is the convex set whose defining inequalities are exactly all the CG cuts for $K$. A classical result of Schrijver [17] is that the CG closure of a rational polyhedron is a rational polyhedron. Recently, we were able to verify that the CG closure of any strictly convex body[2] intersected with a rational polyhedron is a rational polyhedron [8,7]. We remark that the proof requires techniques significantly different from those described in [17].

While the intersections of strictly convex bodies with rational polyhedra yield a large and interesting class of bodies, they do not capture many natural examples that arise in convex INLP. For example, it is not unusual for the feasible region of a semi-definite or conic-quadratic program [1] to have infinitely many faces of different dimensions, where additionally a majority of these faces cannot be isolated by intersecting the feasible region with a rational supporting hyperplane (as is the case for standard ILP with rational data). Roughly speaking, the main barrier to progress in the general setting has been a lack of understanding of how CG cuts act on irrational affine subspaces (affine subspaces whose defining equations cannot be described with rational data).

As a starting point for this study, perhaps the simplest class of bodies where current techniques break down are polytopes defined by irrational data. Schrijver considers these bodies in [17], and in a discussion section at the end of the paper, he writes [3]:

> "We do not know whether the analogue of Theorem 1 is true in real spaces. We were able to show only that if $P$ is a bounded polyhedron in real space, and $P'$ has empty intersection with the boundary of $P$, then $P'$ is a (rational) polyhedron."

[2] A full dimensional compact convex set whose only non-trivial faces are vertices. It this paper, we call zero dimensional faces as vertices.

[3] Theorem 1 in [17] is the result that the CG closure is a polyhedron. $P'$ is the notation used for CG closure in [17]

In this paper, we prove that the CG closure of any compact convex set[4] is a rational polytope, thus also resolving the question raised in [17]. As seen by Schrijver [17], most of the "action" in building the CG closure will indeed take place on the boundary of $K$. While the proof presented in this paper has some high level similarities to the one in [7], a substantially more careful approach was required to handle the general facial structure of a compact convex set (potentially infinitely many faces of all dimensions) and completely new ideas were needed to deal with faces having irrational affine hulls (including the whole body itself).

This paper is organized as follows. In Section 2 we introduce some notation, formally state our main result and give an overview of the proof. We then proceed with the full proof which is presented in Sections 3–5.

## 2 Definitions, Main Result and Proof Idea

**Definition 1 (CG Closure)** For a convex set $K \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{Z}^n$ let $CC(K,S) := \bigcap_{v \in S} \{x \in \mathbb{R}^n : \langle v, x \rangle \leq \lfloor h_K(v) \rfloor\}$. The CG closure of $K$ is defined to be the set $CC(K) := CC(K, \mathbb{Z}^n)$.

The following theorem is the main result of this paper.

**Theorem 1** *If $K \subseteq \mathbb{R}^n$ is a non-empty compact convex set, then $CC(K)$ is finitely generated. That is, there exists $S \subseteq \mathbb{Z}^n$ such that $|S| < \infty$ and $CC(K) = CC(K,S)$. In particular $CC(K)$ is a rational polyhedron.*

Following are some definitions and notation we will use throughout the paper. For more details on definitions from convex analysis, we refer the reader to [15]. For a positive integer $n$, we let $[n]$ be the set $\{1, \ldots, n\}$. For $x, y \in \mathbb{R}^n$, let $[x,y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$ and $(x,y) = [x,y] \setminus \{x,y\}$. For $x \in \mathbb{R}^n$, we let $\|x\|_p = (\sum_{i=1}^n x_i^p)^{\frac{1}{p}}$, $p \geq 1$, denote the standard $l_p$ norms, where we let $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. For notational simplicity, we shall write $\|x\|$ to denote the standard euclidean norm. Let $B_p^n := \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$, the standard $l_p$ ball, and let $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$, the euclidean sphere. For $A \subseteq \mathbb{R}^n$, let $\mathrm{aff}(A)$ denote the smallest affine subspace containing $A$. Furthermore let $\mathrm{aff}_I(A) := \mathrm{aff}(\mathrm{aff}(A) \cap \mathbb{Z}^n)$, i.e. the largest integer subspace in $\mathrm{aff}(A)$. Let $\mathrm{int}(A), \mathrm{bd}(A)$ denote the interior and boundary of $A$ with respect to $\mathbb{R}^n$. Let $\mathrm{relint}(A), \mathrm{relbd}(A)$ denote the interior and boundary of $A$ with respect to $\mathrm{aff}(A)$ (under the subspace topology). For $A \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ we let $a + bA = \{a + bx : x \in A\}$. For sets $A, B \subseteq \mathbb{R}^n$, we define $d(A,B) = \inf\{\|y - x\| : y \in A, x \in B\}$ to be the distance between $A$ and $B$. If $B = \{x\}$ is a singleton, we shall write $d(A,x)$ for notational convenience. For a convex set $K$ and $v \in \mathbb{R}^n$, let $H_v(K) := \{x \in \mathbb{R}^n : \langle v, x \rangle \leq h_K(v)\}$ denote the supporting

[4] If the convex hull of integer points in a convex set is not polyhedral, then the CG closure cannot be expected to be polyhedral. Since we do not have a good understanding of when this holds for unbounded convex sets, we restrict our attention here to the CG closure of compact convex sets.

halfspace defined by $v$ for $K$, and let $H_v^=(K) := \{x \in \mathbb{R}^n : \langle v, x\rangle = h_K(v)\}$ denote the supporting hyperplane. A subset $F \subseteq K$ is a face of $K$ if for every line segment $[x, y] \subseteq K$, $(x, y) \cap F \neq \emptyset \Rightarrow [x, y] \subseteq F$. A face $F$ of $K$ is proper if $F \neq K$ and is exposed if $F = K \cap H_v^=(K)$ for some $v$. Let $F_v(K) := K \cap H_v^=(K)$ denote the face of $K$ exposed by $v$. If the context is clear, then we drop the $K$ and simply write $H_v$, $H_v^=$ and $F_v$. Finally, a vector $x \in K$ is an extreme point if $\{x\}$ is a face of $K$ (i.e. x is a zero dimensional face). We let $\text{ext}(K)$ denote the set of extreme points of $K$.

We present the outline of the proof for Theorem 1. The proof proceeds by induction on the dimension of $K$. The base case ($K$ is a single point) is trivial. By the induction hypothesis, we can assume that (†) every proper exposed face of $K$ has a finitely generated CG closure. We build the CG closure of $K$ in stages, proceeding as follows:

1. (Section 3) For a proper exposed face $F_v$, where $v \in \mathbb{R}^n$, show that $\exists\ S \subseteq \mathbb{Z}^n$, $|S| < \infty$ such that $CC(K, S) \cap H_v^= = CC(F_v)$ and $CC(K, S) \subseteq H_v$ using (†) and by proving the following:
   (a) (Section 3.1) A CG cut for $F_v$ can be rotated or "lifted" to a CG cut for $K$ such that points in $F_v \cap \text{aff}_I(H_v^=)$ separated by the original CG cut for $F_v$ are separated by the new "lifted" one.
   (b) (Section 3.2) A finite number of CG cuts for $K$ separate all points in $F_v \setminus \text{aff}_I(H_v^=)$ and all points in $\mathbb{R}^n \setminus H_v$.
2. (Section 4) Create an approximation $CC(K, S)$ of $CC(K)$ such that (i) $|S| < \infty$, (ii) $CC(K, S) \subseteq K \cap \text{aff}_I(K)$ (iii) $CC(K, S) \cap \text{relbd}(K) = CC(K) \cap \text{relbd}(K)$. This is done in two steps:
   (a) (Section 4.1) Using the lifted CG closures of $F_v$ from (1.) and a compactness argument on the sphere, create a first approximation $CC(K, S)$ satisfying (i) and (ii).
   (b) (Section 4.2) Noting that $CC(K, S) \cap \text{relbd}(K)$ is contained in the union of a finite number of proper exposed faces of $K$, add the lifted CG closures for each such face to $S$ to satisfy (iii).
3. (Section 5) We establish the final result by showing that there are only a finite number of CG cuts which separate at least one vertex of the approximation of the CG closure from (2).

## 3 $CC(K, S) \cap H_v^= = CC(F_v)$ and $CC(K, S) \subseteq H_v$

When $K$ is a rational polyhedron, a key property of the CG closure is that for every face $F$ of $K$, we have that $(*)$ $CC(F) = F \cap CC(K)$. In this setting, a relatively straightforward induction argument coupled with $(*)$ allows one to construct the approximation of the CG closure described above. In our setting, where $K$ is compact convex, the approach taken is similar in spirit, though we will encounter significant difficulties. First, since $K$ can have infinitely many faces, we must couple our induction with a careful compactness argument. Second and more significantly, establishing $(*)$ for compact convex sets is substantially more involved than for rational polyhedra. As we will see in the following sections, the standard lifting argument to prove $(*)$ for rational polyhedra cannot be used directly and must be replaced by a more involved two stage argument.

### 3.1 Lifting CG Cuts

To prove $CC(F) = F \cap CC(K)$ one generally uses a 'lifting approach', i.e., given a CG cut $CC(F, \{w\})$ for $F$, $w \in \mathbb{Z}^n$, we show that there exists a CG cut $CC(K, \{w'\})$ for $K$, $w' \in \mathbb{Z}^n$, such that

$$CC(K, \{w'\}) \cap \text{aff}(F) \subseteq CC(F, \{w\}) \cap \text{aff}(F). \tag{2}$$

To prove (2) when $K$ is a rational polyhedron, one proceeds as follows. For the face $F$ of $K$, we compute $v \in \mathbb{Z}^n$ such that $F_v(K) = F$ and $h_K(v) \in \mathbb{Z}$. For $w \in \mathbb{Z}^n$, we return the lifting $w' = w + lv$, $l \in \mathbb{Z}_{>0}$, where $l$ is chosen such that $h_K(w') = h_F(w')$. For general convex bodies though, neither of these steps may be achievable. When $K$ is strictly convex however, in [7] we show that the above procedure can be generalized. First, every proper face $F$ of $K$ is an exposed vertex, hence $\exists\ x \in K, v \in \mathbb{R}^n$ such that $F = F_v = \{x\}$. For $w \in \mathbb{Z}^n$, we show that setting $w' = w + v'$, where $v'$ is a fine enough Dirichlet approximation (see Theorem 2 below) to a scaling of $v$ is sufficient for (2). In the proof, we critically use that $F$ is simply a vertex. In the general setting, when $K$ is a compact convex set, we can still meaningfully lift CG cuts, but not from all faces and not with exact containment. First, we only guarantee lifting for an exposed face $F_v$ of $K$. Second, when lifting a CG cut for $F_v$ derived from $w \in \mathbb{Z}^n$, we only guarantee the containment on $\text{aff}_I(H_v^=)$, i.e. $CC(K, w') \cap \text{aff}_I(H_v^=) \subseteq CC(F, w) \cap \text{aff}_I(H_v^=)$. This lifting, Proposition 1 below, uses the same Dirichlet approximation technique as in [7] but with a more careful analysis. Since we only guarantee the behavior of the lifting $w'$ on $\text{aff}_I(H_v^=)$, we will have to deal with the points in $\text{aff}(F) \setminus \text{aff}_I(H_v^=)$ separately, which we discuss in the next section.
Lemmas 1–3 are technical results that are needed for proving Proposition 1.

**Lemma 1** *Let $K$ be a compact convex set in $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$, and let $(x_i)_{i=1}^\infty$, $x_i \in K$, be a sequence such that* $\lim_{i\to\infty} \langle v, x_i \rangle = h_K(v)$. *Then*

$$\lim_{i\to\infty} d(F_v(K), x_i) = 0.$$

*Proof* Let us assume that $\lim_{i\to\infty} d(F_v(K), x_i) \neq 0$. Then there exists an $\epsilon > 0$ such that for some subsequence $(x_{\alpha_i})_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ we have that $d(F_v(K), x_{\alpha_i}) \geq \epsilon$. Since $(x_{\alpha_i})_{i=1}^\infty$ is an infinite sequence on a compact set $K$, there exists a convergent subsequence $(x_{\beta_i})_{i=1}^\infty$ where $\lim_{i\to\infty} x_{\beta_i} = x$ and $x \in K$. We note that $d(F_v(K), x) = \lim_{i\to\infty} d(F_v(K), x_{\beta_i}) \geq \epsilon$, where the first equality follows from the continuity of $d(F_v(K), \cdot)$. Since $d(F_v(K), x) > 0$ we have that $x \notin F_v(K)$. On the other hand,

$$h_K(v) = \lim_{i \to \infty} \langle v, x_i \rangle = \lim_{i \to \infty} \langle v, x_{\beta_i} \rangle = \langle v, x \rangle$$

and hence $x \in F_v(K)$, a contradiction. □

**Lemma 2** *Let $K$ be a compact convex set in $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$, and let $(v_i)_{i=1}^{\infty}$, $v_i \in \mathbb{R}^n$, be a sequence such that $\lim_{i\to\infty} v_i = v$. Then for any sequence $(x_i)_{i=1}^{\infty}$, $x_i \in F_{v_i}(K)$, we have that*

$$\lim_{i \to \infty} d(F_v(K), x_i) = 0.$$

*Proof* We claim that $\lim_{i\to\infty} \langle v, x_i \rangle = h_K(v)$. Since $K$ is compact, there exists $R \geq 0$ such that $K \subseteq RB_2^n$. Hence we get that

$$\begin{aligned} h_K(v) &= \lim_{i \to \infty} h_K(v_i) = \lim_{i \to \infty} \langle v_i, x_i \rangle \\ &= \lim_{i \to \infty} \langle v, x_i \rangle + \langle v_i - v, x_i \rangle \leq \lim_{i \to \infty} \langle v, x_i \rangle + \|v_i - v\| R = \lim_{i \to \infty} \langle v, x_i \rangle, \end{aligned}$$

where the first equality follows by continuity of $h_K$ ($h_K$ is convex on $\mathbb{R}^n$ and finite valued). Since each $x_i \in K$, we get the opposite inequality $\lim_{i\to\infty} \langle v, x_i \rangle \leq h_K(v)$ and hence we get equality throughout. Finaly, by Lemma 1 we get that $\lim_{i\to\infty} d(F_v(K), x_i) = 0$ as needed. □

The next lemma describes the central mechanics of the lifting process explained above. The sequence $(w_i)_{i=1}^{\infty}$ will eventually denote the sequence of Dirichlet approximates of the scaling of $v$ added to $w$, where one of these will serve as the lifting $w'$.

**Lemma 3** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set. Take $v, w \in \mathbb{R}^n$, $v \neq 0$. Let $(w_i, t_i)_{i=1}^{\infty}$, $w_i \in \mathbb{R}^n, t_i \in \mathbb{R}_+$ be a sequence such that*

$$a.\ \lim_{i \to \infty} t_i = \infty, \quad b.\ \lim_{i \to \infty} w_i - t_i v = w. \tag{3}$$

*Then for every $\epsilon > 0$ there exists $N_\epsilon \geq 0$ such that for all $i \geq N_\epsilon$*

$$h_K(w_i) + \epsilon \geq t_i h_K(v) + h_{F_v(K)}(w) \geq h_K(w_i) - \epsilon. \tag{4}$$

*Proof* By (3) we have that

$$\lim_{i \to \infty} \frac{w_i}{t_i} = v \tag{5}$$

and that we may pick $N_1 \geq 0$ such that

$$\|w_i - t_i v\| \leq \|w\| + 1 \leq C \quad \text{for } i \geq N_1. \tag{6}$$

Let $(x_i)_{i=1}^{\infty}$ be any sequence such that $x_i \in F_{w_i}(K) = F_{w_i/t_i}(K)$. For each $i \geq 1$, let $\tilde{x}_i = \operatorname{argmin}_{y \in F_v(K)} \|x_i - y\|$. By (5) and Lemma 2, we may pick $N_2 \geq 0$ such that

$$d(F_v(K), x_i) = \|x_i - \tilde{x}_i\| \leq \frac{\epsilon}{2C} \quad \text{for } i \geq N_2. \tag{7}$$

Since $h_{F_v(K)}$ is a continuous function, we may pick $N_3 \geq 0$ such that

$$|h_{F_v(K)}(w_i - t_i v) - h_{F_v(K)}(w)| \leq \frac{\epsilon}{2} \quad \text{for } i \geq N_3. \tag{8}$$

Let $N_\epsilon = \max\{N_1, N_2, N_3\}$. Since $x_i \in F_{w_i}(K)$ and $\tilde{x}_i \in F_v(K)$ we have that

$$\langle w_i, x_i \rangle \geq \langle w_i, \tilde{x}_i \rangle \quad \text{and } \langle t_i v, \tilde{x}_i \rangle \geq \langle t_i v, x_i \rangle . \tag{9}$$

From (6), (7), (9) we get that for $i \geq N_\epsilon$

$$\begin{aligned} \langle w_i, x_i \rangle - \langle w_i, \tilde{x}_i \rangle &\leq \langle w_i, x_i \rangle - \langle w_i, \tilde{x}_i \rangle + \langle t_i v, \tilde{x}_i \rangle - \langle t_i v, x_i \rangle = \langle w_i - t_i v, x_i - \tilde{x}_i \rangle \\ &\leq \|w_i - t_i v\| \|x_i - \tilde{x}_i\| \leq C\left(\frac{\epsilon}{2C}\right) = \frac{\epsilon}{2}. \end{aligned} \tag{10}$$

From (10) we see that for $i \geq N_\epsilon$

$$h_K(w_i) \geq h_{F_v(K)}(w_i) \geq \langle w_i, \tilde{x}_i \rangle \geq \langle w_i, x_i \rangle - \frac{\epsilon}{2} = h_K(w_i) - \frac{\epsilon}{2}. \tag{11}$$

Since $\langle v, \cdot \rangle$ is constant on $F_v(K)$, we have that

$$\begin{aligned} h_{F_v(K)}(w_i) = h_{F_v(K)}(w_i - t_i v + t_i v) &= h_{F_v(K)}(w_i - t_i v) + t_i h_{F_v(K)}(v) \\ &= h_{F_v(K)}(w_i - t_i v) + t_i h_K(v) \end{aligned} \tag{12}$$

Combining (8), (11) and (12) we get that for $i \geq N_\epsilon$,

$$h_K(w_i) + \epsilon \geq t_i h_K(v) + h_{F_v(K)}(w) \geq h_K(w_i) - \epsilon$$

as needed. □

**Theorem 2 (Dirichlet's Approximation Theorem)** *Let $(\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l$. Then for every positive integer $N$, there exists $1 \leq n \leq N$ such that $\max_{1 \leq i \leq l} |n\alpha_i - \lfloor n\alpha_i \rceil| \leq 1/N^{1/l}$.*

**Proposition 1** *Let $K \subseteq \mathbb{R}^n$ be a compact and convex set, $v \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n$. Then $\exists w' \in \mathbb{Z}^n$ such that $CC(K, w') \cap \operatorname{aff}_I(H_v^=(K)) \subseteq CC(F_v(K), w) \cap \operatorname{aff}_I(H_v^=(K))$.*

*Proof* First, by possibly multiplying $v$ by a positive scalar we may assume that $h_K(v) \in \mathbb{Z}$. Let $S = \text{aff}_I(H_v^=(K))$. We may assume that $S \neq \emptyset$, since otherwise the statement is trivially true.

From Theorem 2 for any $v \in \mathbb{R}^n$ there exists $(s_i, t_i)_{i=1}^\infty$, $s_i \in \mathbb{Z}^n$, $t_i \in \mathbb{N}$ such that (a.) $t_i \to \infty$ and (b.) $\|s_i - t_i v\| \to 0$. Define the sequence $(w_i, t_i)_{i=1}^\infty$, where $w_i = w + s_i$, $i \geq 1$. Note that the sequence $(w_i, t_i)$ satisfies (3) and hence by Lemma 3 for any $\epsilon > 0$, there exists $N_\epsilon$ such that (4) holds. Let $\epsilon = \frac{1}{2}\big(1 - (h_{F_v(K)}(w) - \lfloor h_{F_v(K)}(w) \rfloor)\big)$, and let $N_1 = N_\epsilon$. Note that $\lfloor h_{F_v(K)}(w) + \epsilon \rfloor = \lfloor h_{F_v(K)}(w) \rfloor$. Hence, since $h_K(v) \in \mathbb{Z}$ by assumption, for all $i \geq N_1$ we have that $\lfloor h_K(w_i) \rfloor \leq \lfloor t_i h_K(v) + h_{F_v(K)}(w) + \epsilon \rfloor = t_i h_K(v) + \lfloor h_{F_v(K)}(w) + \epsilon \rfloor = t_i h_K(v) + \lfloor h_{F_v(K)}(w) \rfloor$.

Pick $z_1, \ldots, z_k \in S \cap \mathbb{Z}^n$ such that $\text{aff}(z_1, \ldots, z_k) = S$ and let $R = \max\{\|z_j\| : 1 \leq j \leq k\}$. Choose $N_2$ such that $\|w_i - t_i v - w\| \leq \frac{1}{2R}$ for $i \geq N_2$. Note that for $i \geq N_2$, $|\langle w_i, z_j \rangle - \langle t_i v + w, z_j \rangle| = |\langle w_i - t_i v - w, z_j \rangle| \leq \|w_i - t_i v - w\|\|z_j\| \leq R\frac{1}{2R} = \frac{1}{2} \quad \forall j \in \{1, \ldots, k\}$.

Next note that since $z_j, w_i \in \mathbb{Z}^n$, $\langle w_i, z_j \rangle \in \mathbb{Z}$. Furthermore, $t_i \in \mathbb{N}$, $\langle v, z_j \rangle = h_K(v) \in \mathbb{Z}$ and $w \in \mathbb{Z}^n$ implies that $\langle t_i v + w, z_j \rangle \in \mathbb{Z}$. Given this, we must have $\langle w_i, z_j \rangle = \langle t_i v + w, z_j \rangle \quad \forall j \in \{1, \ldots, k\}$, $i \geq N_2$ and hence we get $\langle w_i, x \rangle = \langle t_i v + w, x \rangle \quad \forall x \in S$, $i \geq N_2$.

Let $w' = w_i$ where $i = \max\{N_1, N_2\}$. Let $L = \{x : \langle w', x \rangle \leq \lfloor h_K(w') \rfloor\} \cap S$. Here we get that $\langle w_i, x \rangle \leq t_i h_K(v) + \lfloor h_{F_v(K)}(w) \rfloor$ and $\langle v, x \rangle = h_K(v)$ for all $x \in L$. Hence, we see that $\langle w_i - t_i v, x \rangle \leq \lfloor h_{F_v(K)}(w) \rfloor$ for all $x \in L$. Furthermore, since $\langle w_i - t_i v, x \rangle = \langle w, x \rangle$ for all $x \in L \subseteq S$, we have that $\langle w, x \rangle \leq \lfloor h_{F_v(K)}(w) \rfloor$ for all $x \in L$, as needed. □

### 3.2 Separating All Points in $F_v \setminus \text{aff}_I(H_v^=)$

Since the guarantees on the lifted CG cuts produced in the previous section are restricted to $\text{aff}_I(H_v^=)$, we must still deal with the points in $F_v \setminus \text{aff}_I(H_v^=)$. In this section, we show that points in $F_v \setminus \text{aff}_I(H_v^=)$ can be separated by using a finite number of CG cuts in Proposition 2. To prove this, we will need Kronecker's theorem on simultaneous diophantine approximation which is stated next. See Niven [16] or Cassels [3] for a proof.

**Theorem 3 (Kronecker's Approximation Theorem)** *Let $(x_1, \ldots, x_d) \in \mathbb{R}^d$ be such that the numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$. Then the set $\{(\text{fr}(nx_1), \ldots, \text{fr}(nx_n)) : n \in \mathbb{N}\}$ is dense in $[0,1)^d$, where $\text{fr}(x) = x - \lfloor x \rfloor$ denotes the fractional part of $x$.*

We give the following simple corollary.

**Corollary 1** *Let $v = (z_1, \ldots, z_{d-r}, x_1, \ldots, x_r) \in \mathbb{R}^d$, where $z_1, \ldots, z_{d-r} \in \mathbb{Z}$, and $1, \ldots, x_1, \ldots, x_r$ are linearly independent over $\mathbb{Q}$. Then for any $m, N_0 \in \mathbb{N}$, and $k \in [m]$, the set $\{w + nv : w \in \mathbb{Z}^d, n \equiv k \pmod{m}, n \geq N_0\}$ is dense in $\mathbb{Z}^{d-r} \times \mathbb{R}^r$.*

*Proof* Let $\xi = (x_1, \ldots, x_r) \in \mathbb{R}^r$, and let $S = \mathbb{Z}^r + \{mn\xi : n \geq N_0\}$.

*Claim 1: $S$ is dense in $\mathbb{R}^r$:* Take $y \in \mathbb{R}^r$, and note that $y = \lfloor y \rfloor + \mathrm{fr}(y)$ (here $\lfloor \cdot \rfloor$ and $\mathrm{fr}(\cdot)$ are applied coordinate wise) where $\lfloor y \rfloor \in \mathbb{Z}^n$ and $\mathrm{fr}(y) \in [0,1)^r$. For any $0 < \epsilon < 1/2$ and $i \in [r]$, let $I_i^\epsilon = (\mathrm{fr}(y_i), \mathrm{fr}(y_i) + \epsilon)$ if $\mathrm{fr}(y_i) < \frac{1}{2}$ and $I_i^\epsilon = (\mathrm{fr}(y_i) - \epsilon, \mathrm{fr}(y_i))$ if $\mathrm{fr}(y_i) \geq \frac{1}{2}$. Letting $I^\epsilon = I_1^\epsilon \times \cdots \times I_r^\epsilon$, we have that by construction $I^\epsilon$ is an open subset of $[0,1)^r$. Since $m \in \mathbb{N}$, note that $1, mx_1, \ldots, mx_r$ are linearly independent over $\mathbb{Q}$. By Theorem 3, we have that the set $\{\mathrm{fr}(nm\xi) : n \geq 1\}$ is dense in $[0,1)^r$. Since $\{\mathrm{fr}(nm\xi) : n \geq N_0\}$ contains all but finitely many of the elements in $\{\mathrm{fr}(nm\xi) : n \geq 1\}$, we must have that $\{\mathrm{fr}(nm\xi) : n \geq N_0\}$ is also dense in $[0,1)^r$. Therefore there exists $n_0 \geq N_0$ such that $n_0 m\xi - \lfloor n_0 m\xi \rfloor \in I^\epsilon$. Let $\bar{y} = n_0(m\xi) - \lfloor n_0 m\xi \rfloor + \lfloor y \rfloor$. Since $\lfloor y \rfloor - \lfloor n_0 m\xi \rfloor \in \mathbb{Z}^r$, we see that $\bar{y} \in S$. Next note that

$$\|\bar{y} - y\|_\infty = \max_{i=1}^{r} |\mathrm{fr}(n_0 m\xi_i) - \mathrm{fr}(y_i)| \leq \epsilon$$

Since $\epsilon$ can be made arbitrarily small, we have that $S$ is dense in $\mathbb{R}^r$, as needed.

Since $(z_1, \ldots, z_{d-r}) \in \mathbb{Z}^{d-r}$ note that for any $n \in \mathbb{N}$, $\mathbb{Z}^d + nv = \mathbb{Z}^{d-r} \times (\mathbb{Z}^r + n\xi)$. Take $k \in [m]$. From the previous remark we see that

$$\left\{w + nv : w \in \mathbb{Z}^d, n \equiv k \pmod{m}, n \geq N_0\right\} \supseteq \mathbb{Z}^{d-r} \times (\mathbb{Z}^r + S + k\xi)$$

Since $S$ is dense in $\mathbb{R}^r$, we clearly also get that $S + k\xi$ is dense in $\mathbb{R}^r$ (indeed this holds for any translation of $S$). Therefore the set $\mathbb{Z}^{d-r} \times (S + k\xi)$ is dense in $\mathbb{Z}^{d-r} \times \mathbb{R}^r$ as needed. □

The following lemmas will allow us to normalize the vector $v$ defining $F_v$ and $H_v^=$ and simplify the analysis that follows.

**Lemma 4** *Let $K \subseteq \mathbb{R}^n$ be a closed convex set, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then $h_K(v) = h_{TK}(T^{-t}v)$ and $TF_v(K) = (F_{T^{-t}v}(TK))$ for all $v \in \mathbb{R}^n$. If $T$ is a unimodular transformation, then for $S \subseteq \mathbb{Z}^n$, $TCC(K,S) = (CC(TK, T^{-t}S))$. Furthemore, $TCC(K) = CC(TK)$ and $T\,\mathrm{aff}_I(K) = \mathrm{aff}_I(TK)$.*

*Proof* Observe that

$$h_{TK}(T^{-t}v) = \sup_{x \in TK} \left\langle T^{-t}v, x \right\rangle = \sup_{x \in K} \left\langle T^{-t}v, Tx \right\rangle = \sup_{x \in K} \langle v, x \rangle = h_K(v).$$

Note that

$$\begin{aligned} T^{-1}(F_{T^{-t}v}(TK)) &= T^{-1}\left( \ \{x : \ x \in TK, \ h_{TK}(T^{-t}v) = \left\langle T^{-t}v, x \right\rangle\} \ \right) \\ &= \left\{x : \ Tx \in TK, \ h_{TK}(T^{-t}v) = \left\langle T^{-t}v, Tx \right\rangle\right\} \\ &= \{x : \ x \in K, \ h_K(v) = \langle v, x \rangle\} = F_v(K). \end{aligned}$$

Next for $S \subseteq \mathbb{Z}^n$, note that since $T$ is unimodular, we have that $T^{-t}S \subseteq \mathbb{Z}^n$. Therefore

$$\begin{aligned} T^{-1}(CC(TK, T^{-t}S)) &= T^{-1}\left(\left\{x : x \in TK, \ \langle v, x \rangle \leq \lfloor h_{TK}(v) \rfloor \ \forall \ v \in T^{-t}S\right\}\right) \\ &= T^{-1}\left\{x : x \in TK, \ \left\langle T^{-t}v, x \right\rangle \leq \left\lfloor h_{TK}(T^{-t}v) \right\rfloor \ \forall \ v \in S\right\} \\ &= \{x : Tx \in TK, \ \langle v, x \rangle \leq \lfloor h_K(v) \rfloor \ \forall \ v \in S\} \\ &= \{x : x \in K, \ \langle v, x \rangle \leq \lfloor h_K(v) \rfloor \ \forall v \in S\} = CC(K, S). \end{aligned}$$

Furthermore, by unimodularity of $T$, we have that $T^{-t}\mathbb{Z}^n = \mathbb{Z}^n$ and hence $CC(K) = T^{-1}CC(TK, T^{-t}\mathbb{Z}^n) = T^{-1}CC(TK)$, as needed. Lastly, note that $T(\mathrm{aff}(K) \cap \mathbb{Z}^n) = \mathrm{aff}(TK) \cap \mathbb{Z}^n$, and hence $T(\mathrm{aff}_I(K)) = \mathrm{aff}_I(TK)$ as needed. $\square$

An affine subspace $A \subseteq \mathbb{R}^n$ is rational if $A = \{x \in \mathbb{R}^n : Cx = d\}$, where $C, d$ define a rational linear system (i.e. the entries of $C, d$ are in $\mathbb{Q}$). Equivalently, $A$ is rational if $A = \mathrm{span}(x_1, \ldots, x_k) + x_{k+1}$, where $x_i \in \mathbb{Q}^n$ for $i \in [k+1]$. We will need the following standard theorem.

**Theorem 4** *Let $V \subseteq \mathbb{R}^n$ denote a rational linear subspace of dimension $r \geq 1$. Then there exists vectors $b_1, \ldots, b_n \in \mathbb{Z}^n$ such that*

1. $\sum_{i=1}^{r} \mathbb{Z}b_i = V \cap \mathbb{Z}^n$.
2. $\sum_{i=1}^{n} \mathbb{Z}b_i = \mathbb{Z}^n$.

**Lemma 5** *For $v \in \mathbb{R}^n$, let*

$$\mathrm{rdim}(v) = \min\left\{\dim(A) : v \in A, A \subseteq \mathbb{R}^n \text{ a rational affine subspace}\right\}.$$

*Then there exists a unimodular transformation $T \in \mathbb{Z}^{n\times n}$, such that*

$$Tv = (0, \ldots, 0, \lambda, \alpha_1, \ldots, \alpha_r)\,, \tag{13}$$

*where $\lambda \in \mathbb{Q}_{\geq 0}$, $r = \mathrm{rdim}(v)$, and the numbers $1, \alpha_1, \ldots, \alpha_r \geq 0$ are linearly independent over $\mathbb{Q}$.*

*Proof* Let $A = \{x \in \mathbb{R}^n : Cx = d\}$ denote the smallest rational affine subspace containing $v$, and let $r = \dim(A)$. Clearly, we may assume that the rows of $C$ are linearly independent, and hence that $C \in \mathbb{Q}^{n-r\times n}$ and $d \in \mathbb{Q}^{n-r}$. Let $c_1, \ldots, c_{n-r}$ denote the rows of $C$. Since $v \in A$, note that $d_i = \langle c_i, v\rangle$ for all $i \in [n-r]$. Let $H = \mathrm{span}(c_1, \ldots, c_{n-r})$. Since $H$ is a rational linear space, note that $\mathrm{span}(H \cap Z^n) = \mathrm{span}(H)$. Therefore, by Theorem 4, there exists $b_1, \ldots, b_n \in \mathbb{Z}^n$ such that

$$\sum_{i=1}^{n-r} \mathbb{Z}b_i = \mathrm{span}(H) \cap \mathbb{Z}^n \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{Z}b_i = \mathbb{Z}^n. \tag{14}$$

Let $f \in \mathbb{R}^n$ denote the $n$-dimensional vector satisfying $f_i = \langle b_i, v\rangle$ for $i \in [n]$. Note that by possibly negating the vectors in $b_1, \ldots, b_n$, we may assume that $f \geq 0$. Since for each $i \in [n-r]$, the vector $b_i$ can be obtained as a rational combination of $c_1, \ldots, c_{n-r}$, we also have that $\langle b_i, v\rangle = f_i \in \mathbb{Q}$. Furthermore, since $\mathrm{span}(b_1, \ldots, b_{n-r}) = H$, we see that $A = \{x \in \mathbb{R}^n : Cx = d\} = \{x \in \mathbb{R}^n : \langle b_i, x\rangle = f_i, i \in [n-r]\}$.

If $f_1 = \cdots = f_{n-r} = 0$, let $\sigma = 1$. Otherwise, there exists $\sigma \in \mathbb{Q}_{>0}$, such that $\sigma f_1, \ldots, \sigma f_{n-r} \in \mathbb{Z}_{\geq 0}$ and that $\gcd(\sigma f_1, \ldots, \sigma f_{n-r}) = 1$. Here we note that $\sigma f_i = \langle b_i, \sigma v\rangle$. Since $\sigma \in \mathbb{Q}$, we also have that $\mathrm{rdim}(v) = \mathrm{rdim}(\sigma v)$ (just replace the system $Cx = d$ by $Cx = \sigma d$). Since it suffices to prove the statement for $\sigma v$, we shall assume that $\sigma = 1$ satisfies the requirements for both cases.

Let us assume that we are still in the latter case, i.e. that $f_1, \ldots, f_{n-r} \in \mathbb{Z}_{\geq 0}$ and that $\gcd(f_1, \ldots, f_{n-r}) = 1$. Then from the euclidean algorithm, we may construct a unimodular matrix $U \in \mathbb{Z}^{n-r \times n-r}$ (corresponding to the performed sequence of elementary integral column operations) such that $(f_1, \ldots, f_{n-r})U = (0, \ldots, 0, 1)$. Let $(\bar{b}_1, \ldots, \bar{b}_{n-r}) = (b_1, b_2, \ldots, b_{n-r})U$. Since $U$ is unimodular we have that $\sum_{i=1}^{n-r} \mathbb{Z}\bar{b}_i = \sum_{i=1}^{n-r} \mathbb{Z}b_i = H \cap \mathbb{Z}^n$, and hence the basis $\bar{b}_1, \ldots, \bar{b}_{n-r}, b_{n-r+1}, \ldots, b_n$ satisfies (14). Lastly, by construction of $\bar{b}_1, \ldots, \bar{b}_{n-r}$ the following holds

$$v^t(\bar{b}_1, \ldots, \bar{b}_{n-r}) = v^t(b_1, \ldots, b_{n-r})U = (f_1, \ldots, f_{n-r})U = (0, \ldots, 0, 1).$$

From the above, we may assume that we have an integral basis $b_1, \ldots, b_n$ satisfying (14), such that $(f_1, \ldots, f_{n-r})$ equals either $(0, \ldots, 0, 1)$ or $(0, \ldots, 0)$.

Let $\alpha_i = f_{n-r+i}$ for $i \in [r]$. Note that $\alpha_1, \ldots, \alpha_r \geq 0$ since $f \geq 0$. We claim that $1, \alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Q}$. Assume not, then there exists a non-trivial combination $a_1, \ldots, a_r \in \mathbb{Q}$ such that $\sum_{i=1}^r a_i \alpha_i \in \mathbb{Q}$. Note that $w = \sum_{i=1}^r a_i b_{n-r+i} \in \mathbb{Q}^n$ and that $\langle w, v \rangle = \sum_i^r a_i \langle b_{n-r+i}, v \rangle = \sum_i^r a_i \alpha_i \in \mathbb{Q}$. Furthermore, $w$ is linearly independent from $b_1, \ldots, b_{n-r}$, and hence the subspace $A' = \{x \in \mathbb{R}^n : \langle w, x \rangle = \langle w, v \rangle, \langle b_i, x \rangle = f_i, i \in [n-r]\} \subseteq A$ is rational, contains $v$, and $\dim(A') < \dim(A)$. However, this contradicts our assumption on $A$, and therefore such a combination cannot exist.

Let $T \in \mathbb{Z}^{n \times n}$ denote the matrix with rows $b_1, \ldots, b_n$, and note that $Tv = f$. Since the rows of $T$ generate $\mathbb{Z}^n$, we note that $T$ is unimodular. Lastly, by the previous arguments, we have that $f$ satisfies (13) as needed. □

We show that the points in $F_v \setminus \mathrm{aff}_I(H_v^=)$ can be separated using a finite number of CG cuts. We first give a rough sketch of the proof. We restrict to the case where $\mathrm{aff}_I(H_v^=) \neq \emptyset$. From here one can verify that any rational affine subspace contained in $\mathrm{aff}(H_v^=)$ must also lie in $\mathrm{aff}_I(H_v^=)$. Next we use Kronecker's theorem to build a finite set $C \subseteq \mathbb{Z}^n$, where each vector in $C$ is at distance at most $\epsilon$ from some scaling of $v$, and where $v$ can be expressed as a non-negative combination of the vectors in $C$. By choosing $\epsilon$ and the scalings of $v$ appropriately, we can ensure that the CG cuts derived from $C$ dominate the inequality $\langle v, x \rangle \leq h_K(v)$, i.e. $CC(K, C) \subseteq H_v$. If $CC(K, C)$ lies in the interior of $H_v(K)$, we have separated all of $H_v^=$ (including $F_v \setminus \mathrm{aff}_I(H_v^=)$) and hence are done. Otherwise, $T := CC(K, C) \cap H_v^=$ is a face of a rational polyhedron, and therefore $\mathrm{aff}(T)$ is a rational affine subspace. Since $\mathrm{aff}(T) \subseteq \mathrm{aff}(H_v^=)$, as discussed above we obtain $T \subseteq \mathrm{aff}(T) \subseteq \mathrm{aff}_I(H_v^=)$ as required.

The following lemma will be needed in the proof of Proposition 2.

**Lemma 6** *Let $e_1, \ldots, e_n \in \mathbb{R}^n$ denote the standard unit vectors. For $s > 0$, let $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$, be vectors such that*

$$\|v_i - se_i\|_\infty \leq \frac{s}{2(2n+1)}, i \in [n], \qquad \left\| v_{n+1} - s\sum_{i=1}^n e_i \right\|_\infty \leq \frac{s}{2(2n+1)}.$$

*Then $0 \in \mathrm{int}(\mathrm{conv}(v_1, \ldots, v_{n+1}))$ and $\mathrm{span}(v_1, \ldots, v_{n+1}) = \mathbb{R}^n$.*

*Proof* Since the statement is invariant under positive scalings of $v_1, \dots, v_{n+1}$, we may assume that $s = 1$. Let $S_0 = \mathrm{conv}(e_1, \dots, e_n, -\sum_{i=1}^n e_i)$ and $S = \mathrm{conv}(v_1, \dots, v_{n+1})$. We first show that for any $w \in \mathbb{R}^n$, such that $\|w\|_1 = 1$, we have that

$$h_{S_0}(w) = \max\left\{\max_{1\le i\le n} w_i, -\sum_{i=1}^n w_i\right\} \ge \frac{1}{2n+1}$$

Assume that $-\sum_{i=1}^n w_i < \frac{1}{2n+1}$ (†). Let $I_-, I_+ \subseteq [n]$ denote the indices where $w_i \le 0$ and $w_i > 0$ respectively. Then note that

$$\|w\|_1 = \sum_{i=1}^r |w_i| = \sum_{i\in I_+} w_i - \sum_{i \in I_-} w_i = 1$$

Combining the above with the (†) yields

$$2\sum_{i\in I_+} w_i > 1 - \frac{1}{2n+1} \Rightarrow 2n \max_{1\le i\le n} w_i > 1 - \frac{1}{2n+1} \Rightarrow \max_{1\le i \le n} w_i > \frac{1}{2n+1}$$

as needed.

From here, we see that

$$\begin{aligned} h_S(w) &= \max\left\{\max_{1\le i\le n} \langle w, v_i\rangle, \langle w, v_{n+1}\rangle\right\} \\ &= \max\left\{\max_{1\le i\le n} w_i + \langle w, v_i - e_i\rangle, -\sum_{i=1}^n w_i + \left\langle w, v_{n+1} + \sum_{i=1}^n e_i\right\rangle\right\} \\ &\ge \max\left\{\max_{1\le i\le n} w_i - \|w\|_1\|v_i - e_i\|_\infty, -\sum_{i=1}^n w_i - \|w\|_1\left\|v_{n+1} + \sum_{i=1}^n e_i\right\|_\infty\right\} \\ &\ge \max\left\{\max_{1\le i\le n} w_i, -\sum_{i=1}^n w_i\right\} - \frac{1}{2(2n+1)} \ge \frac{1}{2(2n+1)}. \end{aligned} \tag{15}$$

We claim that $\frac{1}{2(2n+1)}B_\infty^n \subseteq S$. Take $x \in \mathbb{R}^n$, $\|x\|_\infty \le \frac{1}{2(2n+1)}$. Assume that $x \notin S$, then by the separator theorem, there exists $w \in \mathbb{R}^n$, $\|w\|_1 = 1$, such that

$$h_S(w) < \langle w, x\rangle \le \|x\|_\infty \|w\|_1 = \frac{1}{2(2n+1)},$$

a clear contradiction to (15). Since $0 \in \mathrm{int}\left(\frac{1}{2(2n+1)}B_\infty\right)$, we clearly have that $0 \in \mathrm{int}(S)$. Assume that $\mathrm{span}(v_1, \dots, v_{n+1}) \ne \mathbb{R}^n$. Then there exists $w \in \mathbb{R}^n$, $\|w\|_1$, such that $\langle v_1, w\rangle = \cdots = \langle v_{n+1}, w\rangle = 0$. Again, this contradicts (15), and hence $\mathrm{span}(v_1, \dots, v_{n+1}) = \mathbb{R}^n$ as needed. □

**Lemma 7** *Let $P \subseteq \mathbb{R}^n$ be a polytope with extreme points $\mathrm{ext}(P) = \{w_1, \dots, w_k\}$. If $x \in \mathrm{relint}(P)$, then there exists a convex combination $\lambda_1, \dots, \lambda_k > 0$, $\sum_{i=1}^k \lambda_i = 1$, such that $\sum_{i=1}^k \lambda_i w_i = x$.*

*Proof* Let $\bar{w} = \sum_{i=1}^{k} \frac{1}{k} w_i$. By convexity $\bar{w} \in P$. Since $x \in \text{relint}(P)$ and $\bar{w} \in P$, there exists $\epsilon > 0$ such that $x_\epsilon = (1+\epsilon)x - \epsilon\bar{w} \in P$. Since $x_\epsilon \in P$, we may write $x_\epsilon = \sum_{i=1}^{k} \alpha_i w_i$ for a convex combination $\alpha_1, \ldots, \alpha_k \geq 0$. Let $\lambda_i = \frac{1}{1+\epsilon}\alpha_i + \frac{\epsilon}{k(1+\epsilon)}$ for $i \in [k]$. Note that the $\lambda_i$s are strictly positive and form a convex combination. Lastly, we have that

$$\sum_{i=1}^{k} \lambda_i w_i = \frac{1}{1+\epsilon}\left(\sum_{i=1}^{k} \alpha_i w_i\right) + \frac{\epsilon}{1+\epsilon}\left(\sum_{i=1}^{k} \frac{1}{k} w_i\right)$$
$$= \frac{1}{1+\epsilon}((1+\epsilon)x - \epsilon\bar{w}) + \frac{\epsilon}{1+\epsilon}\bar{w} = x,$$

as needed. □

**Proposition 2** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set and $v \in \mathbb{R}^n$. Then there exists $C \subseteq \mathbb{Z}^n$, $|C| = \text{rdim}(v) + 1$, such that*

$$CC(K,C) \subseteq H_v(K) \quad \text{and} \quad CC(K,C) \cap H_v^=(K) \subseteq \text{aff}_I(H_v^=(K)).$$

*Proof* By scaling $v$ by a positive scalar if necessary, we may assume that $h_K(v) \in \{0, 1, -1\}$. Let $T \in \mathbb{Z}^{n \times n}$ be the unimodular transformation provided by Lemma 5 for $v$. By Lemma 4, since $T$ is unimodular, it suffices to prove the statement for the convex body $T^{-t}K$ and the vector $Tv$, and hence we may assume that $T$ is the identity.

From Lemma 5, the vector $v$ is of the form $v = (0, \ldots, 0, \lambda, \alpha_1, \ldots, \alpha_r)$, where $\lambda \in \mathbb{Q}_{\geq 0}$, $r = \text{rdim}(v)$, and $1, \alpha_1, \ldots, \alpha_r \geq 0$ are linearly independent over $\mathbb{Q}$. Since $h_K(v) \in \{0, 1, -1\}$ and $\lambda \in \mathbb{Q}_{\geq 0}$, we may scale $v$ by a positive rational number to achieve $h_K(v) = \gamma$, $\gamma \in \mathbb{Q}$, such that $v$ is of the form $(0, \ldots, 0, b, \alpha_1, \ldots, \alpha_r)$ for $b \in \{0, 1\}$.

We shall distinguish two cases, either $\text{aff}_I(H_v^=) \neq \emptyset$ or $\text{aff}_I(H_v^=) = \emptyset$. In the the former case, we will define CG cuts that yield a subset of $\text{aff}_I(H_v^=)$ when restricted to $H_v^=$. In the latter case, we will define CG cuts that imply the inequality $\langle v, x\rangle < h_K(v)$, separating the CG closure from $H_v^=$ entirely. Since $K$ is compact, we may choose a radius $R \geq 0$ such that $K \subseteq RB_1^n$. Let $w_i = e_{n-r+i}$, $i \in [r]$, and $w_{r+1} = -\sum_{i=1}^{r} e_{n-r+i}$, where the $e_i$'s denote the standard unit vectors. Here we note that $\|w_i\|_\infty = 1$ for all $i \in [r+1]$.

*Case 1:* $\text{aff}_I(H_v^=) \neq \emptyset$. We first claim that $\gamma \in \mathbb{Z}$ and that $\text{aff}_I(H_v^=) = H_v^= \cap \{x \in \mathbb{R}^n : x_{n-r+i} = 0, i \in [r]\}$. Pick $z \in \mathbb{Z}^n \cap H_v^=$. Then by definition, $\langle v, z\rangle = bz_{n-r} + \sum_{i=1}^{r} \alpha_i z_{n-r+i} = h_K(v) = \gamma$. Since $bz_{n-r} \in \mathbb{Z}$ and $\gamma \in \mathbb{Q}$, we must have that $\sum_{i=1}^{r} \alpha_i z_{n-r+1} \in \mathbb{Q}$. However, since $1, \alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Q}$, we must have that $z_{n-r+1} = \cdots = z_n = 0$. Therefore we must have that $bz_{n-r} = \gamma$. Therefore $\gamma \in \mathbb{Z}$ as needed. If $b = 1$, we have that $z_{n-r} = \gamma$, and therefore $H_v^= \cap \mathbb{Z}^n = \mathbb{Z}^{n-r-1} \times \gamma \times 0^r$. Hence, $\text{aff}_I(H_v^=) = \mathbb{R}^{n-r-1} \times \gamma \times 0^r$ as required. Otherwise, if $b = 0$, then $\gamma = 0$ and $H_v^= \cap \mathbb{Z}^n = \mathbb{Z}^{n-r} \times 0^r$. Therefore $\text{aff}_I(H_v^=) = \mathbb{R}^{n-r} \times 0^r$ as required.

From the above remarks, note that we need only find CG cuts that imply that $x_{n-r+i} = 0$, $\forall\ i \in [r]$, when restricted to $H_v^=$. Let $\epsilon = \frac{1}{4(R+1)}$. From

Corollary 1, we know that the set $\mathbb{Z}^n + \{nv : n \geq N_0\}$ is dense in $\mathbb{Z}^{n-r} \times \mathbb{R}^r$ for any $N_0 \in \mathbb{N}$. Therefore, for each $i \in [r+1]$, there exists an infinite sequence $x_j = n_j v - y_j \in 0^{n-r} \times \mathbb{R}^r$, $y_j \in \mathbb{Z}^n$, such that $n_j \to \infty$ and $\lim_{j\to\infty} x_j \to \epsilon w_i$. By Lemma 3, there exists $j' \geq 1$ such $h_K(y_{j'}) \leq n_{j'} h_K(v) + h_{F_v(K)}(-\epsilon w_i) + \epsilon$, and where $\|x_{j'} - \epsilon w_i\|_\infty \leq \frac{\epsilon}{2(2R+1)}$ (since $\lim_{i\to\infty} x_j = \epsilon w_i$). Let us denote $\bar{w}_i = x_{j'}$, $\bar{z}_i = y_{j'}$, and $\bar{n}_i = n_{j'}$, noting that $\bar{z}_i = \bar{n}_i v - \bar{w}_i$.

Let $C = \{\bar{z}_1, \ldots, \bar{z}_{r+1}\} \subseteq \mathbb{Z}^n$. We will show that $CC(K, C)$ satisfies the requirements of the theorem. For each $i \in [r+1]$, by construction of $\bar{z}_i$ we have that

$$\begin{aligned}\lfloor h_K(\bar{z}_i) \rfloor &\leq \lfloor \bar{n}_i h_K(v) + h_{F_v(K)}(-\epsilon w_i) + \epsilon \rfloor \\ &\leq \lfloor \bar{n}_i h_K(v) + \epsilon(h_{RB_1^n}(w_i) + 1)) \rfloor \\ &= \bar{n}_i h_K(v) + \lfloor \epsilon(R\|w_i\|_\infty + 1) \rfloor \leq \bar{n}_i h_K(v) + \left\lfloor \frac{1}{4} \right\rfloor = \bar{n}_i h_K(v).\end{aligned} \tag{16}$$

Here note that the last two inequalities follow since $\bar{n}_i h_K(v) = \bar{n}_i \gamma \in \mathbb{Z}$ (we proved earlier that $\gamma \in \mathbb{Z}$). From (16), we see that

$$CC(K, C) \subseteq \{x \in \mathbb{R}^n : \langle \bar{z}_i, x \rangle \leq \bar{n}_i h_K(v) \ \forall i \in [r+1]\}. \tag{17}$$

Again by construction, for each $i \in [r+1]$, the first $n - r$ coordinates of $\bar{w}_i$ are zero and $\|\bar{w}_i - \epsilon w_i\|_\infty < \frac{\epsilon}{2(2R+1)}$. Therefore $\bar{w}_1, \ldots, \bar{w}_{r+1}$ satisfy the conditions of Lemma 6 (restricting to the last $r$ coordinates), and hence $0 \in \mathrm{relint}(\mathrm{conv}(\bar{w}_1, \ldots, \bar{w}_{r+1}))$ and $\mathrm{span}(\bar{w}_1, \ldots, \bar{w}_{r+1}) = 0^{n-r} \times \mathbb{R}^r$ (since the first $n - r$ coordinates are all 0). By Lemma 7, there exists a convex combination $\lambda_1, \ldots, \lambda_{r+1} > 0$, $\sum_{i=1}^{r+1} \lambda_i = 1$, satisfying $\sum_{i=1}^{r+1} \lambda_i \bar{w}_i = 0$. From here, note that

$$\sum_{i=1}^{r+1} \lambda_i \bar{z}_i = \sum_{i=1}^{r+1} \lambda_i (\bar{n}_i v - \bar{w}_i) = \sum_{i=1}^{r+1} \lambda_i \bar{n}_i v \tag{18}$$

Given the above, and that $\lambda_1, \ldots, \lambda_{r+1}$ form a convex combination, we get that any $x \in CC(K, C)$ satisfies the inequality

$$\begin{aligned}&\sum_{i=1}^{r+1} \lambda_i \langle \bar{z}_i, x \rangle \leq \sum_{i=1}^{r+1} \lambda_i \bar{n}_i h_K(v) \Rightarrow \\ &\left(\sum_{i=1}^{r+1} \lambda_i \bar{n}_i\right) \langle v, x \rangle \leq \left(\sum_{i=1}^{r+1} \lambda_i \bar{n}_i\right) h_K(v) \Rightarrow \langle v, x \rangle \leq h_K(v).\end{aligned} \tag{19}$$

Therefore $CC(K, C) \subseteq H_v(K)$ as needed. Assume that $x \in CC(K, C) \cap H_v^=$. Since $\langle v, x \rangle = h_K(v)$, combining with (16) we get that

$$\langle \bar{w}_i, x \rangle = \bar{n}_i \langle v, x \rangle - \langle \bar{z}_i, x \rangle \geq \bar{n}_i h_K(v) - \bar{n}_i h_K(v) = 0 \quad \forall i \in [r+1]. \tag{20}$$

Furthermore, by construction of the $\bar{w}_i$s

$$\sum_{i=1}^{r+1} \lambda_i \langle \bar{w}_i, x \rangle = \left\langle \sum_{i=1}^{r+1} \lambda_i \bar{w}_i, x \right\rangle = \langle 0, x \rangle = 0. \tag{21}$$

Since the $\lambda_1, \ldots, \lambda_{r+1} > 0$, (21) implies that all the inequalities in (20) must hold at equality, i.e. that $\langle \bar{w}_i, x \rangle = 0$ for all $i \in [r+1]$. Since $\text{span}(\bar{w}_1, \ldots, \bar{w}_{r+1}) = 0^{n-r} \times \mathbb{R}^r$, the previous statment immediately implies that $x_{n-r+1} = \cdots = x_n = 0$. Therefore $x \in H_v^= \cap \{z \in \mathbb{R}^n : z_{n-r+i} = 0 \ \forall\ i \in [r]\} = \text{aff}_I(H_v^=)$, as needed.

*Case 2:* $\text{aff}_I(H) = \emptyset$. As the analysis here closely mirrors that of case 1, we will refer to arguments in the previous case whenever appropriate. In the previous case, we saw that $\langle v, x \rangle = h_K(v)$ has an integer solution (i.e. where $x \in \mathbb{Z}^n$) if and only if $bx_{n-r} = h_K(v)$ admits an integer solution. Since $\text{aff}_I(H) = \emptyset$, we can assume that either $b = 1$ and $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$, or that $b = 0$ and $\gamma \in \mathbb{Q} \setminus \{0\}$.

We claim that there exists a scaling $\eta \in \mathbb{Q}$ such that $\eta b \in \mathbb{Z}$ and $\frac{1}{3} \leq \text{fr}(\eta\gamma) \leq \frac{2}{3}$. If we are in the case where $b = 0$ and $\gamma \in \mathbb{Q} \setminus \{0\}$, we may set $\eta = \frac{1}{2\gamma}$. This satisfies the requirements since $b\eta = 0 \in \mathbb{Z}$ and $\eta\gamma = \frac{1}{2}$. Otherwise, if $b = 1$ and $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$, we may write $\gamma = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \geq 2$, and $p, q$ are relatively prime. Therefore, we may choose $\eta \in \mathbb{N}$, such that $\eta p \equiv \lfloor \frac{q}{2} \rfloor \pmod q$. Since $\eta \in \mathbb{N}$, $b\eta \in \mathbb{Z}$, and $\text{fr}(\eta\gamma) = \frac{\lfloor \frac{q}{2} \rfloor}{q} \in [\frac{1}{3}, \frac{2}{3}]$ (since $q \geq 2$).

After scaling $v$ by $\eta$, we may assume that $v = (0, \ldots, 0, \bar{b}, \alpha_1, \ldots, \alpha_r)$, where $\bar{b} \in \mathbb{N}$, $h_K(v) \in \mathbb{Q}$, and $\text{fr}(h_K(v)) = \text{fr}(\gamma) \in [\frac{1}{3}, \frac{2}{3}]$. Let $q \in \mathbb{N}$ denote the least integer such that $q\gamma \in \mathbb{Z}$. By Corollary 1, we know that the set $\mathbb{Z}^n + \{nv : n \geq N_0, n \equiv 1 \pmod q\}$, for any $N_0 \in \mathbb{N}$, is dense in $\mathbb{Z}^{n-r} \times \mathbb{R}^r$. Using the identical construction as case 1, for $\epsilon = \frac{1}{4(R+1)}$ and $i \in [r+1]$, we can find $\bar{z}_i = \bar{n}_i v - \bar{w}_i$ satisfying $\bar{n}_i \equiv 1 \pmod q$, $\bar{z}_i \in \mathbb{Z}^n$, $\bar{w}_i \in 0^{n-r} \times \mathbb{R}^r$ and $\|\bar{w}_i - \epsilon w_i\|_\infty \leq \frac{\epsilon}{2(2r+1)}$, and $h_K(\bar{z}_i) \leq \bar{n}_i h_K(v) + h_{F_v(K)}(-\epsilon w_i) + \epsilon$. Let $C = \{\bar{z}_1, \ldots, \bar{z}_{r+1}\}$.

Since $\bar{n}_i \equiv 1 \pmod q$, note that $\text{fr}(\bar{n}_i h_K(v)) = \text{fr}(\bar{n}_i\gamma) = \text{fr}(\gamma) \in [\frac{1}{3}, \frac{2}{3}]$. As before, we have that $h_{F_v(K)}(-\epsilon w_i) + \epsilon \leq \epsilon(R+1) = \frac{1}{4}$ for all $i \in [r+1]$. Therefore

$$\begin{aligned} \lfloor h_K(\bar{z}_i) \rfloor &\leq \left\lfloor \bar{n}_i h_K(v) + \frac{1}{4} \right\rfloor = \lfloor \bar{n}_i h_K(v) \rfloor + \left\lfloor \text{fr}(h_K(v)) + \frac{1}{4} \right\rfloor \\ &\leq \lfloor \bar{n}_i h_K(v) \rfloor + \left\lfloor \frac{2}{3} + \frac{1}{4} \right\rfloor = \lfloor \bar{n}_i h_K(v) \rfloor < \bar{n}_i h_K(v). \end{aligned} \tag{22}$$

As before, by construction of $\bar{w}_i$, $i \in [r+1]$, there exists $\lambda_1, \ldots, \lambda_{r+1} > 0$, $\sum_{i=1}^{r+1} \lambda_i = 1$, satisfying $\sum_{i=1}^{r+1} \lambda_i \bar{w}_i = 0$. Therefore by (22), any $x \in CC(K, C)$ satisfies the inequality

$$\begin{aligned} \left( \sum_{i=1}^{r+1} \lambda_i \bar{n}_i \right) \langle v, x \rangle = \sum_{i=1}^{r+1} \lambda_i \langle \bar{z}_i, x \rangle < \left( \sum_{i=1}^{r+1} \lambda_i \bar{n}_i \right) h_K(v) \Rightarrow \\ \langle v, x \rangle < h_K(v). \end{aligned}$$

Therefore the CG cuts $CC(K, C)$ separate the CG closure from $H_v^=$ (and in particular $\text{aff}_I(H_v^=)$) as needed. □

### 3.3 Lifting the CG Closure of an Exposed Face of $K$

**Proposition 3** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set. Take $v \in \mathbb{R}^n$. Assume that $CC(F_v(K))$ is finitely generated. Then $\exists\ S \subseteq \mathbb{Z}^n$, $|S| < \infty$, such that $CC(K,S)$ is a polytope and*

$$CC(K,S) \cap H_v^{=}(K) = CC(F_v(K)) \tag{23}$$
$$CC(K,S) \subseteq H_v. \tag{24}$$

*Proof* The right to left containment in (23) is direct from $CC(F_v(K)) \subseteq CC(K,S)$ as every CG cut for $K$ is a CG cut for $F_v(K)$. For the reverse containment and for (24) we proceed as follows.

Using Proposition 2 there exists $S_1 \subseteq \mathbb{Z}^n$ such that $CC(K,S_1) \cap H_v^{=}(K) \subseteq \mathrm{aff}_I(H_v^{=}(K))$ and $CC(K,S_1) \subseteq \{x \in \mathbb{R}^n : \langle v,x\rangle \le h_K(v)\}$. Next let $G \subseteq \mathbb{Z}^n$ be such that $CC(F_v(K),G) = CC(F_v(K))$. For each $w \in G$, by Proposition 1 there exists $w' \in \mathbb{Z}^n$ such that $CC(K,w') \cap \mathrm{aff}_I(H_v^{=}(K)) \subseteq CC(F_v(K),w) \cap \mathrm{aff}_I(H_v^{=}(K))$. For each $w \in G$, add $w'$ above to $S_2$. Note that

$$\begin{aligned} CC(K, S_1 \cup S_2) \cap H_v^{=}(K) &= CC(K,S_1) \cap CC(K,S_2) \cap H_v^{=}(K) \\ &\subseteq CC(K,S_2) \cap \mathrm{aff}_I(H_v^{=}(K)) \\ &\subseteq CC(F_v(K),G) \cap \mathrm{aff}(H_v^{=}(K)) \subseteq CC(F_v(K)). \end{aligned}$$

Let $S_3 = \{\pm e_i : 1 \le i \le n\}$. Note that since $K$ is compact $CC(K,S_3)$ is a cuboid with bounded side lengths, and hence is a polytope. Letting $S = S_1 \cup S_2 \cup S_3$, yields the desired result. □

Before we proceed further, we remark here that Proposition 3 taken together with the main result of the paper, i.e. Theorem 1, implies a generalization of another classcial result about CG closures of rational polytopes that is presented as Corollary 2 below. (As clarification, we note that Corollary 2 below is not used in the proof ofTheorem 1.)

**Corollary 2** *Let $K$ be a compact convex set and let $F$ be an exposed face of $K$, then we have that $CC(F) = CC(K) \cap F$.*

## 4 Approximation of the CG Closure

### 4.1 Approximation 1 of the CG Closure

In this section, we construct a first approximation of the CG closure of $K$. Under the assumption that the CG closure of every proper exposed face is finitely generated, we use a compactness argument to construct a finite set of CG cuts $S \subseteq \mathbb{Z}^n$ such that $CC(K,S) \subseteq K \cap \mathrm{aff}_I(K)$. We use the following lemma to simplify the analysis of integral affine subspaces.

**Lemma 8** *Take $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then there exists $\lambda \in \mathbb{R}^m$ such that for $a' = \lambda A$, $b' = \lambda b$, we have that $\{x \in \mathbb{Z}^n : Ax = b\} = \{x \in \mathbb{Z}^n : a'x = b'\}$.*

*Proof* If $\{x \in \mathbb{R}^n : Ax = b\} = \emptyset$, then by Farkas' Lemma there exists $\lambda \in \mathbb{R}^m$ such that $\lambda A = 0$ and $\lambda b = 1$. Hence $\{x \in \mathbb{R}^n : Ax = b\} = \{x \in \mathbb{R}^n : 0x = 1\} = \emptyset$ as needed. We may therefore assume that $\{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$. Therefore we may also assume that the rows of the augmented matrix $[A \,|\, b]$ are linearly independent.

Let $T = \text{span}(a_1, \ldots, a_m)$, where $a_1, \ldots, a_m$ are the rows of $A$. Define $r : T \to \mathbb{R}$ as $r(w) = \lambda b$ where $\lambda \in \mathbb{R}^m$ such that $\lambda A = w$. Since the rows of $A$ are linearly independent we obtain that $r$ is well defined and is a linear operator. Let $S = \{x \in \mathbb{Z}^n : Ax = b\}$. For $z \in \mathbb{Z}^n$, examine $T_z = \{w \in T : \langle w, z \rangle = r(w)\}$. By linearity of $r$, we see that $T_z$ is a linear subspace of $T$. Note that for $z \in \mathbb{Z}^n$, $T_z = T$ iff $z \in S$. Therefore $\forall\ z \in \mathbb{Z}^n \setminus S$, we must have that $T_z \neq T$, and hence $\dim(T_z) \leq \dim(T) - 1$. Let $m_T$ denote the Lebesgue measure on $T$. Since $\dim(T_z) < \dim(T)$, we see that $m_T(T_z) = 0$. Let $T' = \bigcup_{z \in \mathbb{Z}^n \setminus S} T_z$. Since $\mathbb{Z}^n \setminus S$ is countable, by the countable subadditivity of $m_T$ we have that $m_T(T') \leq \sum_{z \in \mathbb{Z}^n \setminus S} m_T(T_z) = 0$. Since $m_T(T) = \infty$, we must have that $T \setminus T' \neq \emptyset$. Hence we may pick $a' \in T \setminus T'$. Letting $b' = r(a')$, we note that by construction there $\exists\ \lambda \in \mathbb{R}^m$ such that $\lambda A = a'$ and $\lambda b = b'$. Hence for all $z \in S$, $\lambda A z = \lambda b \Rightarrow a'z = b'$. Take $z \in \mathbb{Z}^n \setminus S$. Since $a' \in T \setminus T'$, we have that $a' \notin T_z$. Hence $a'z \neq b'$. Therefore we see that $\{x \in \mathbb{Z}^n : a'x = b'\} = \{x \in \mathbb{Z}^n : Ax = b\}$ as needed. □

**Proposition 4** *Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be a compact convex set. If $CC(F_v(K))$ is finitely generated for any proper exposed face $F_v(K)$ then $\exists\ S \subseteq \mathbb{Z}^n$, $|S| < \infty$, such that $CC(K, S) \subseteq K \cap \text{aff}_I(K)$ and $CC(K, S)$ is a polytope.*

*Proof* Let us express $\text{aff}(K)$ as $\{x \in \mathbb{R}^n : Ax = b\}$. Note that $\text{aff}(K) \neq \emptyset$ since $K \neq \emptyset$. By Lemma 8 there exists $\lambda$, $c = \lambda A$ and $d = \lambda b$, and such that $\text{aff}(K) \cap \mathbb{Z}^n = \{x \in \mathbb{Z}^n : \langle c, x \rangle = b\}$. Since $h_K(c) = b$ and $h_K(-c) = -b$, using Proposition 2 on $c$ and $-c$, we can find $S_A \subseteq \mathbb{Z}^n$ such that $CC(K, S_A) \subseteq \text{aff}(\{x \in \mathbb{Z}^n : \langle c, x \rangle = b\}) = \text{aff}_I(K)$.

Express $\text{aff}(K)$ as $W + a$, where $W \subseteq \mathbb{R}^n$ is a linear subspace and $a \in \mathbb{R}^n$. Take $v \in W \cap S^{n-1}$. Note that $F_v(K)$ is a proper exposed face and hence, by assumption, $CC(F_v(K))$ is finitely generated. Hence by Proposition 3 there exists $S_v \subseteq \mathbb{Z}^n$ such that $CC(K, S_v)$ is a polytope, $CC(K, S_v) \cap H_v^{=}(K) = CC(F_v(K))$ and $CC(K, S_v) \subseteq H_v$. Let $K_v = CC(K, S_v)$, then we have the following claim.

**Claim:** There exists an open neighborhood $N_v$ of $v$ in $W \cap S^{n-1}$ such that $v' \in N_v \Rightarrow h_{K_v}(v') \leq h_K(v')$.

Since $K_v$ is a polytope, there exists $Z \subseteq \mathbb{R}^n$, $|Z| < \infty$, such that $K_v = \text{conv}(Z)$. Then note that $h_{K_v}(w) = \sup_{z \in Z} \langle z, w \rangle$. Let $H = \{z : h_K(v) = \langle v, z \rangle, z \in Z\}$. By construction, we have that $\text{conv}(H) = CC(F_v(K))$.

First assume that $CC(F_v(K)) = \emptyset$. Then $H = \emptyset$, and hence $h_{K_v}(v) < h_K(v)$. Since $K_v, K$ are compact convex sets, we have that $h_{K_v}, h_K$ are both continuous functions on $\mathbb{R}^n$ and hence $h_K - h_{K_v}$ is continuous. Therefore there exists $\epsilon > 0$ such that $h_{K_v}(v') < h_K(v')$ for $\|v - v'\| \leq \epsilon$ as needed.

Assume that $CC(F_v(K)) \neq \emptyset$. Let $R = \max_{z \in Z} \|z\|$, and let

$$\delta = h_K(v) - \sup\{\langle v, z\rangle : z \in Z \setminus H\}.$$

Let $\epsilon = \frac{\delta}{2R}$ and take any $v'$ such that $\|v' - v\| < \epsilon$. Then for all $z \in H$, we have that

$$\begin{aligned}\langle z, v'\rangle = \langle z, v\rangle + \langle z, v' - v\rangle = h_K(v) + \langle z, v' - v\rangle \geq h_K(v) - \|z\|\|v' - v\| \\ > h_K(v) - R\frac{\delta}{2R} = h_K(v) - \frac{\delta}{2},\end{aligned}$$

and that for all $z \in Z \setminus H$, we have that

$$\begin{aligned}\langle z, v'\rangle = \langle z, v\rangle + \langle z, v' - v\rangle \leq h_K(v) - \delta + \langle z, v' - v\rangle \leq h_K(v) - \delta + \|z\|\|v' - v\| \\ < h_K(v) - \delta + \frac{\delta}{2} = h_K(v) - \frac{\delta}{2}.\end{aligned}$$

Therefore we have that $\langle z, v'\rangle > \langle z', v'\rangle$ for all $z \in H$, $z' \in Z \setminus H$ and hence

$$h_{K_v}(v') = \sup_{z \in Z} \langle z, v'\rangle = \sup_{z \in H} \langle z, v'\rangle = h_{CC(F_v(K))}(v') \leq h_K(v'), \tag{25}$$

since $CC(F_v(K)) \subseteq F_v(K) \subseteq K$. The statement thus holds by letting $N_v = \{v' \in W \cap S^{n-1} : \|v' - v\| \leq \epsilon\}$.

Note that $\{N_v : v \in W \cap S^{n-1}\}$ forms an open cover of $W \cap S^{n-1}$, and since $W \cap S^{n-1}$ is compact, there exists a finite subcover $N_{v_1}, \ldots, N_{v_k}$ such that $\bigcup_{i=1}^k N_{v_i} = W \cap S^{n-1}$. Let $S = S_A \cup \bigcup_{i=1}^k S_{v_i}$. We claim that $CC(K, S) \subseteq K$. Assume not, then there exists $x \in CC(K, S) \setminus K$. Since $CC(K, S) \subseteq CC(K, S_A) \subseteq W + a$ and $K \subseteq W + a$, by the separator theorem there exists $w \in W \cap S^{n-1}$ such that $h_K(w) = \sup_{y \in K} \langle w, y\rangle < \langle w, x\rangle \leq h_{CC(K,S)}(w)$. Since $w \in W \cap S^{n-1}$, there exists $i$, $1 \leq i \leq k$, such that $w \in N_{v_i}$. Note then we obtain that $h_{CC(K,S)}(w) \leq h_{CC(K,S_{v_i})}(w) = h_{K_{v_i}}(w) \leq h_K(w)$, a contradiction. Hence $CC(K, S) \subseteq K$ as claimed. $CC(K, S)$ is a polytope because it is the intersection of polyhedra of which at least one is a polytope. □

### 4.2 Approximation 2 of the CG Closure

In this section, we augment the first approximation of the $CC(K)$ with a finite number of extra CG cuts so that this second approximation matches $CC(K)$ on the relative boundary of $K$.

To achieve this, we observe that our first approximation of $CC(K)$ is polyhedral and contained in $K$, and hence its intersection with the relative boundary of $K$ is contained in the union of a finite number of proper exposed faces of $K$. Therefore, by applying Proposition 3 to each such face (i.e. adding their lifted CG closure), we can match $CC(K)$ on the relative boundary as required. The following lemma makes precise the previous statements.

**Lemma 9** *Let $K \subseteq \mathbb{R}^n$ be a convex set and $P \subseteq K$ be a polytope. Then there exists $F_{v_1}, \dots, F_{v_k} \subseteq K$, proper exposed faces of $K$, such that $P \cap \mathrm{relbd}(K) \subseteq \bigcup_{i=1}^k F_{v_i}$*

*Proof* Let $\mathcal{F} = \{F : F \subseteq P, F \text{ a face of P}, \mathrm{relint}(F) \cap \mathrm{relbd}(K) \neq \emptyset\}$. Since $P$ is polytope, note that the total number of faces of $P$ is finite, and hence $|\mathcal{F}| < \infty$. We claim that

$$P \cap \mathrm{relbd}(K) \subseteq \bigcup_{F \in \mathcal{F}} F. \tag{26}$$

Take $x \in P \cap \mathrm{relbd}(K)$. Let $F_x$ denote the minimal face of $P$ containing $x$ (note that $P$ is a face of itself). By minimality of $F_x$, we have that $x \in \mathrm{relint}(F_x)$. Since $x \in \mathrm{relbd}(K)$, we have that $F_x \in \mathcal{F}$, as needed.

Take $F \in \mathcal{F}$. We claim that there exists $E_F \subseteq K$, $E_F$ a proper exposed face of $K$, such that $F \subseteq E_F$. Take $x \in \mathrm{relint}(F) \cap \mathrm{relbd}(K)$. Let $\mathrm{aff}(K) = W + a$, where $W$ is a linear subspace and $a \in \mathbb{R}^n$. Since $x \notin \mathrm{relint}(K)$, by the separator theorem, there exists $v \in W \cap S^{n-1}$ such that $h_K(v) = \langle v, x \rangle$. Let $E_F = F_v(K)$. Note that since $v \in W \cap S^{n-1}$, $F_v(K)$ is a proper exposed face of $K$. We claim that $F \subseteq E_F$. Since $F$ is a polytope, we have that $F = \mathrm{conv}(\mathrm{ext}(F))$. Write $\mathrm{ext}(F) = \{c_1, \dots, c_k\}$. Since $x \in \mathrm{relint}(F)$, there exists $\lambda_1, \dots, \lambda_k > 0$, $\sum_{i=1}^k \lambda_i = 1$, such that $\sum_{i=1}^k \lambda_i c_i = x$. Since $c_i \in K$, we have that $\langle c_i, v \rangle \leq h_K(v)$. Therefore, we note that

$$\langle v, x \rangle = \left\langle \sum_{i=1}^k \lambda_i c_i, v \right\rangle = \sum_{i=1}^k \lambda_i \langle c_i, v \rangle \leq \sum_{i=1}^k \lambda_i h_K(v) = h_K(v) \tag{27}$$

Since $\langle v, x \rangle = h_K(v)$, we must have equality throughout. To maintain equality, since $\lambda_i > 0$ for all $1 \leq i \leq k$, we must have that $\langle c_i, v \rangle = h_K(v)$ for all $1 \leq i \leq k$. Therefore $c_i \in E_F$ for all $1 \leq i \leq k$, and hence $F = \mathrm{conv}(c_1, \dots, c_k) \subseteq E_F$, as needed.

To conclude the proof, we note that the set $\{E_F : F \in \mathcal{F}\}$ satisfies the conditions of the lemma. □

**Proposition 5** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set. If $CC(F_v)$ is finitely generated for any proper exposed face $F_v$ then $\exists\ S \subseteq \mathbb{Z}^n$, $|S| < \infty$, such that*

$$CC(K, S) \subseteq K \cap \mathrm{aff}_I(K) \tag{28}$$

$$CC(K, S) \cap \mathrm{relbd}(K) = CC(K) \cap \mathrm{relbd}(K) \tag{29}$$

*Proof* By Proposition 4, there exists $S_I \subseteq \mathbb{Z}^n$, $|S_I| < \infty$, such that $CC(K, S_I) \subseteq K \cap \mathrm{aff}_I(K)$ and $CC(K, S_I)$ is a polytope. Since $CC(K, S_I) \subseteq K$ is a polytope, let $F_{v_1}, \dots, F_{v_k}$ be the proper exposed faces of $K$ given by Lemma 9. By Proposition 3, there exists $S_i \subseteq \mathbb{Z}^n$, $|S_i| < \infty$, such that $CC(K, S_i) \cap H_{v_i} = CC(F_{v_i})$. Let $S = S_I \cup \bigcup_{i=1}^k S_i$. We claim that $CC(K, S) \cap \mathrm{relbd}(K) \subseteq CC(K) \cap \mathrm{relbd}(K)$. For this note that $x \in CC(K, S) \cap \mathrm{relbd}(K)$ implies $x \in CC(K, S_I) \cap \mathrm{relbd}(K)$, and hence there exists $i$, $1 \leq i \leq k$, such that $x \in F_{v_i}$. Then $x \in CC(K, S) \cap H_{v_i} \subseteq CC(K, S_i) \cap H_{v_i} = CC(F_{v_i}) \subseteq CC(K) \cap \mathrm{relbd}(K)$. The reverse inclusion is direct. □

## 5 Proof of Theorem

Finally, we have all the ingredients to prove the main result of this paper. The proof is by induction on the dimension of $K$. Trivially, the result holds for zero dimensional convex bodies. Using the induction hypothesis, we can construct the second approximation of $CC(K)$ using Proposition 5 (since it assumes that the CG closure of every exposed face is finitely generated). Lastly, we observe that any CG cut for $K$ not dominated by those already considered in the second approximation of $CC(K)$ must separate a vertex of this approximation lying in the relative interior of $K$. From here, it is not difficult to show that only a finite number of such cuts exists, thereby proving the polyhedrality of $CC(K)$. The proof here is similar to the one used for strictly convex sets, with the additional technicality that here $\mathrm{aff}(K)$ may be irrational.

**Theorem 5** *Let $K \subseteq \mathbb{R}^n$ be a non-empty compact convex set. Then $CC(K)$ is finitely generated.*

*Proof* We proceed by induction on the affine dimension of $K$. For the base case, $\dim(\mathrm{aff}(K)) = 0$, i.e. $K = \{x\}$ is a single point. Here it is easy to see that setting $S = \{\pm e_i : i \in \{1, \dots, n\}\}$, we get that $CC(K, S) = CC(K)$. The base case thus holds.

For the inductive step let $0 \leq k < n$. Let $K$ be a compact convex set where $\dim(\mathrm{aff}(K)) = k+1$ and assume the result holds for sets of lower dimension. By the induction hypothesis, we know that $CC(F_v)$ is finitely generated for every proper exposed face $F_v$ of $K$, since $\dim(F_v) \leq k$. By Proposition 5, there exists a set $S \subseteq \mathbb{Z}^n$, $|S| < \infty$, such that (28) and (29) hold. If $CC(K, S) = \emptyset$, then we are done. So assume that $CC(K, S) \neq \emptyset$. Let $A = \mathrm{aff}_I(K)$. Since $CC(K, S) \neq \emptyset$, we have that $A \neq \emptyset$ (by (28)), and so we may pick $t \in A \cap \mathbb{Z}^n$. Note that $A - t = W$, where $W$ is a linear subspace of $\mathbb{R}^n$ satisfying $W = \mathrm{span}(W \cap \mathbb{Z}^n)$. Let $L = W \cap \mathbb{Z}^n$. Since $t \in \mathbb{Z}^n$, we easily see that $CC(K-t, T) = CC(K, T) - t$ for all $T \subseteq \mathbb{Z}^n$. Therefore $CC(K)$ is finitely generated iff $CC(K - t)$ is. Hence replacing $K$ by $K - t$, we may assume that $\mathrm{aff}_I(K) = W$.

Let $\pi_W$ denote the orthogonal projection onto $W$. Note that for all $x \in W$, and $z \in \mathbb{Z}^n$, we have that $\langle z, x\rangle = \langle \pi_W(z), x\rangle$. Since $CC(K, S) \subseteq K \cap W$, we see that for all $z \in \mathbb{Z}^n$, $CC(K, S \cup \{z\}) = CC(K, S) \cap \{x : \langle z, x\rangle \leq \lfloor h_K(z)\rfloor\} = CC(K, S) \cap \{x : \langle \pi_W(z), x\rangle \leq \lfloor h_K(z)\rfloor\}$. Let $L^* = \pi_W(\mathbb{Z}^n)$. Since $W$ is a rational subspace, we have that $L^*$ is full dimensional lattice in $W$. Fix an element of $w \in L^*$ and examine $V_w := \{\lfloor h_K(z)\rfloor : \pi_W(z) = w, z \in \mathbb{Z}^n\}$. Note that $V_w \subseteq \mathbb{Z}$. We claim that $\inf(V_w) > -\infty$. To see this, note that

$$\begin{aligned}\inf\{\lfloor h_K(z)\rfloor : \pi_W(z) = w, z \in \mathbb{Z}^n\} &\geq \inf\{\lfloor h_{K\cap W}(z)\rfloor : \pi_W(z) = w, z \in \mathbb{Z}^n\} \\ &= \inf\{\lfloor h_{K\cap W}(\pi_W(z))\rfloor : \pi_W(z) = w, z \in \mathbb{Z}^n\} \\ &= \lfloor h_{K\cap W}(w)\rfloor > -\infty.\end{aligned}$$

Since $V_w$ is a lower bounded set of integers, there exists $z_w \in \pi_W^{-1}(w) \cap \mathbb{Z}^n$ such that $\inf(V_w) = \lfloor h_K(z_w)\rfloor$. From the above reasoning, we see that

$CC\left(K, S \cup \left(\pi_W^{-1}(z) \cap \mathbb{Z}^n\right)\right) = CC(K, S \cup \{z_w\})$. Let

$$C = \{w : w \in L^*, CC(K, S \cup \{z_w\}) \subsetneq CC(K, S)\} .$$

Here we get that

$$CC(K) = CC(K, S \cup \mathbb{Z}^n) = CC(K, S \cup \{z_w : w \in L^*\}) = CC(K, S \cup \{z_w : w \in C\}).$$

From the above equation, if we show that $|C| < \infty$, then $CC(K)$ is finitely generated. To do this, we will show that there exists $R > 0$, such that $C \subseteq RB_2^n$, and hence $C \subseteq L^* \cap RB_2^n$. Since $L^*$ is a lattice, $|L^* \cap RB_2^n| < \infty$ for any fixed $R$, and so we are done.

Let $P = CC(K, S)$. Since $P$ is a polytope, we have that $P = \text{conv}(\text{ext}(P))$. Let $I = \text{ext}(P) \cap \text{relint}(K)$, and let $B = \text{ext}(P) \cap \text{relbd}(K)$. Hence $\text{ext}(P) = I \cup B$. By assumption on $CC(K, S)$, we know that for all $v \in B$, we have that $v \in CC(K)$. Hence for all $z \in \mathbb{Z}^n$, we must have that $\langle z, v\rangle \leq \lfloor h_K(z) \rfloor$ for all $v \in B$. Assume that for some $z \in \mathbb{Z}^n$, $CC(K, S \cup \{z\}) \subsetneq CC(K, S) = P$. We claim that $\langle z, v\rangle > \lfloor h_K(z) \rfloor$ for some $v \in I$. If not, then $\langle v, z\rangle \leq \lfloor h_K(z) \rfloor$ for all $v \in \text{ext}(P)$, and hence $CC(K, S \cup \{z\}) = CC(K, S)$, a contradiction. Hence such a $v \in I$ must exist.

For $z \in \mathbb{Z}^n$, note that $h_K(z) \geq h_{K\cap W}(z) = h_{K\cap W}(\pi_W(z))$. Hence $\langle z, v\rangle > \lfloor h_K(z) \rfloor$ for $v \in I$ only if $\langle \pi_W(z), v\rangle = \langle z, v\rangle > \lfloor h_{K\cap W}(\pi_W(z)) \rfloor$. Let $C' := \{w \in L^* : \exists v \in I, \langle v, w\rangle > \lfloor h_{K\cap W} \rfloor (w)\}$. From the previous discussion, we see that $C \subseteq C'$.

Since $I \subseteq \text{relint}(K) \cap W = \text{relint}(K \cap W)$ we have

$$\delta_v = \sup\{r \geq 0 : rB_2^n \cap W + v \subseteq K \cap W\} > 0$$

for all $v \in I$. Let $\delta = \inf_{v\in I} \delta_v$. Since $|I| < \infty$, we see that $\delta > 0$. Let $R = \frac{1}{\delta}$. Take $w \in L^*$, $\|w\| \geq R$. Note that $\forall v \in I$,

$$\lfloor h_{K\cap W}(w) \rfloor \geq h_{K\cap W}(w) - 1 \geq h_{(v+\delta B_2^n)\cap W}(w) - 1 = \langle v, w\rangle + \delta\|w\| - 1 \geq \langle v, w\rangle .$$

Hence $w \notin C'$. Therefore $C \subseteq C' \subseteq RB_2^n$ and $CC(K)$ is finitely generated. □

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