Two Enumerative Results on Cycles of Permutations[1]

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Abstract

Answering a question of Bóna, it is shown that for $n \geq 2$ the probability that 1 and 2 are in the same cycle of a product of two n-cycles on the set $\{1, 2, \ldots, n\}$ is 1/2 if n is odd and $\frac{1}{2} - \frac{2}{(n-1)(n+2)}$ if n is even. Another result concerns the polynomial $P_\lambda(q) = \sum_w q^{\kappa((1,2,\ldots,n)\cdot w)}$, where w ranges over all permutations in the symmetric group $\mathfrak{S}_n$ of cycle type λ, $(1, 2, \ldots, n)$ denotes the n-cycle $1 \to 2 \to \cdots \to n \to 1$, and $\kappa(v)$ denotes the number of cycles of the permutation v. A formula is obtained for $P_\lambda(q)$ from which it is deduced that all zeros of $P_\lambda(q)$ have real part 0.

1 Introduction.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of n, denoted $\lambda \vdash n$. In general, we use notation and terminology involving partitions and symmetric functions from [12, Ch. 7]. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n] = \{1, 2, \ldots, n\}$. If $w \in \mathfrak{S}_n$ then write $\rho(w) = \lambda$ if w has cycle type λ, i.e., if the (nonzero) λ_i's are the lengths of the cycles of w. The conjugacy classes of $\mathfrak{S}_n$ are given by $K_\lambda = \{w \in \mathfrak{S}_n : \rho(w) = \lambda\}$.

The "class multiplication problem" for $\mathfrak{S}_n$ may be stated as follows. Given $\lambda, \mu, \nu \vdash n$, how many pairs $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ satisfy $u \in K_\lambda$, $v \in K_\mu$,

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$uv \in K_\nu$? The case when one of the partitions is (n) (i.e., one of the classes consists of the n-cycles) is particularly interesting and has received much attention. For a sample of some recent work, see [1][6] [9]. In this paper we make two contributions to this subject. For the first, we solve a problem of Bóna and Flynn [4] asking what is the probability that two fixed elements of $[n]$ lie in the same cycle of the product of two random n-cycles. In particular, we prove the conjecture of Bóna that this probability is $1/2$ when n is odd. Our method of proof is an ugly computation based on a formula of Boccara [2]. The technique can be generalized, and as an example we compute the probability that three fixed elements of $[n]$ lie in the same cycle of the product of two random n-cycles.

For our second result, let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$, and let $(1, 2, \ldots, n)$ denote the n-cycle $1 \to 2 \to \cdots \to n \to 1$. For $\lambda \vdash n$, define the polynomial

$$P_\lambda(q) = \sum_{\rho(w)=\lambda} q^{\kappa((1,2,\ldots,n)\cdot w)}. \tag{1}$$

In Theorem 3.1 we obtain a formula for $P_\lambda(q)$. We also prove from this formula (Corollary 3.3) that every zero of $P_\lambda(q)$ has real part 0.

2 A problem of Bóna.

Let π_n denote the probability that if two n-cycles u, v are chosen uniformly at random in $\mathfrak{S}_n$, then 1 and 2 (or any two elements i and j by symmetry) appear in the same cycle of the product uv. Miklós Bóna conjectured (private communication) that $\pi_n = 1/2$ if n is odd, and asked about the value when n is even. For the reason behind this conjecture, see Bóna and Flynn [4]. In this section we solve this problem. Let us note that it is easy to see (a straightforward generalization of [3, Prop. 6.18]) that the probability that $1, 2, \ldots, k$ appear in the same cycle of a random permutation in $\mathfrak{S}_n$ is $1/k$ for $k \leq n$.

Theorem 2.1. *For $n \geq 2$ we have*

$$\pi_n = \begin{cases} \frac{1}{2}, & n \text{ odd} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.} \end{cases}$$

Proof. First note that if $w \in \mathfrak{S}_n$ has cycle type λ, then the probability that 1 and 2 are in the same cycle of w is

$$q_\lambda = \frac{\sum \binom{\lambda_i}{2}}{\binom{n}{2}} = \frac{\sum \lambda_i(\lambda_i - 1)}{n(n-1)}.$$

Let a_λ be the number of pairs (u, v) of n-cycles in $\mathfrak{S}_n$ for which uv has type λ. Then

$$\pi_n = \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} a_\lambda q_\lambda.$$

By Boccara [2] the number of ways to write a fixed permutation $w \in \mathfrak{S}_n$ of type λ as a product of two n-cycles is

$$(n-1)! \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.$$

Let $n!/z_\lambda$ denote the number of permutations $w \in \mathfrak{S}_n$ of type λ. We get

$$\begin{aligned}
\pi_n &= \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \left(\sum_i \frac{\lambda_i(\lambda_i - 1)}{n(n-1)} \right) \\
&\quad \cdot (n-1)! \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx \\
&= \frac{1}{n-1} \sum_{\lambda \vdash n} z_\lambda^{-1} \left(\sum_i \lambda_i(\lambda_i - 1) \right) \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.
\end{aligned}$$

Now let $p_\lambda(a, b)$ denote the power sum symmetric function p_λ in the two variables a, b, and let $\ell(\lambda)$ denote the length (number of parts) of λ. It is easy to check that

$$2^{-\ell(\lambda)+1} \left(\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b} \right) p_\lambda(a, b)\Big|_{a=b=1} = \sum \lambda_i(\lambda_i - 1).$$

By the exponential formula (permutation version) [12, Cor. 5.1.9] or by [12, Prop. 7.7.4],

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} z_\lambda^{-1} 2^{-\ell(\lambda)} p_\lambda(a, b) \left(\prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) \right) t^n$$

$$= \exp \sum_{k\geq 1} \frac{1}{k}\left(\frac{a^k+b^k}{2}\right)(x^k-(x-1)^k)t^k.$$

It follows that $(n-1)\pi_n$ is the coefficient of t^n in

$$F(t) :=$$

$$2\int_0^1 \left(\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial a\partial b}\right)\exp\left[\sum_{k\geq 1}\frac{1}{k}\left(\frac{a^k+b^k}{2}\right)(x^k-(x-1)^k)t^k\right]\Bigg|_{a=b=1} dx.$$

We can easily perform this computation with Maple, giving

$$\begin{aligned} F(t) &= \int_0^1 \frac{t^2(1-2x-2tx+2tx^2)}{(1-t(x-1))(1-tx)^3}dx \\ &= \frac{1}{t^2}\log(1-t^2)+\frac{3}{2}+\frac{-\frac{1}{2}+t}{(1-t)^2}. \end{aligned}$$

Extract the coefficient of t^n and divide by $n-1$ to obtain π_n as claimed. □

It is clear that the argument used to prove Theorem 2 can be generalized. For instance, using the fact that

$$3^{-\ell(\lambda)+1}\left(\frac{\partial^3}{\partial a^3}-3\frac{\partial^3}{\partial a^2\partial b}+2\frac{\partial^3}{\partial a\partial b\partial c}\right)p_\lambda(a,b,c)\big|_{a=b=c=1}$$

$$=\sum \lambda_i(\lambda_i-1)(\lambda_i-2),$$

we can obtain the following result.

Theorem 2.2. *Let $\pi_n^{(3)}$ denote the probability that if two n-cycles u,v are chosen uniformly at random in $\mathfrak{S}_n$, then $1,2,$ and 3 appear in the same cycle of the product uv. Then for $n\geq 3$ we have*

$$\pi_n^{(3)} = \begin{cases} \frac{1}{3}+\frac{1}{(n-2)(n+3)}, & n \text{ odd} \\ \frac{1}{3}-\frac{3}{(n-1)(n+2)}, & n \text{ even.} \end{cases}$$

Are there simpler proofs of Theorems 2.1 and 2.2, especially Theorem 2.1 when n is odd?

3 A polynomial with purely imaginary zeros

Given $\lambda \vdash n$, let $P_\lambda(q)$ be defined by equation (1). Let $(a)_n$ denote the falling factorial $a(a-1)\cdots(a-n+1)$. Let E be the backward shift operator on polynomials in q, i.e., $Ef(q) = f(q-1)$.

Theorem 3.1. *Suppose that λ has length ℓ. Define the polynomial*

$$g_\lambda(t) = \frac{1}{1-t}\prod_{j=1}^{\ell}(1-t^{\lambda_j}).$$

Then

$$P_\lambda(q) = z_\lambda^{-1} g_\lambda(E)(q+n-1)_n. \tag{2}$$

Proof. Let $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$, and $z = (z_1, z_2, \dots)$ be three disjoint sets of variables. Let H_μ denote the product of the hook lengths of the partition μ (defined e.g. in [12, p. 373]). Write s_λ and p_λ for the Schur function and power sum symmetric function indexed by λ. The following identity is the case $k=3$ of [5, Prop. 2.2] and [12, Exer. 7.70]:

$$\sum_{\mu \vdash n} H_\mu s_\mu(x) s_\mu(y) s_\mu(z) = \frac{1}{n!}\sum_{uvw=1 \text{ in } \mathfrak{S}_n} p_{\rho(u)}(x) p_{\rho(v)}(y) p_{\rho(w)}(z). \tag{3}$$

For a symmetric function $f(x)$ let $f(1^q) = f(1,1,\dots,1,0,0,\dots)$ (q 1's). Thus $p_{\rho(w)}(1^q) = q^{\kappa(w)}$. Let $\chi^\lambda(\mu)$ denote the irreducible character of $\mathfrak{S}_n$ indexed by λ evaluated at a permutation of cycle type μ [12, §7.18]. Recall [12, Cor. 7.17.5 and Thm. 7.18.5] that

$$s_\mu = \sum_{\nu \vdash n} z_\nu^{-1} \chi^\mu(\nu) p_\nu,$$

where $\#K_\nu = n!/z_\nu$ as above. Take the coefficient of $p_n(x)p_\lambda(y)$ in equation (3) and set $z = 1^q$. Since there are $(n-1)!$ n-cycles u, the right-hand side becomes $\frac{1}{n}P_\lambda(q)$. Hence

$$P_\lambda(q) = n\sum_{\mu \vdash n} H_\mu z_n^{-1}\chi^\mu(n) z_\lambda^{-1}\chi^\mu(\lambda) s_\mu(1^q). \tag{4}$$

Write $\sigma(i) = \langle n-i, 1^i\rangle$, the "hook" with one part equal to $n-i$ and i parts equal to 1, for $0 \le i \le n-1$. Now $z_n = n$, and e.g. by [12, Exer. 7.67(a)] we have

$$\chi^{\mu}(n) = \begin{cases} (-1)^i, & \text{if } \mu = \sigma(i),\ 0 \le i \le n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $s_{\sigma(i)}(1^q) = (q+n-i-1)_n H^{-1}_{\sigma(i)}$ by the hook-content formula [12, Cor. 7.21.4]. Therefore we get from equation (4) that

$$P_\lambda(q) = z_\lambda^{-1} \sum_{i=0}^{n-1} (-1)^i \chi^{\sigma(i)}(\lambda)(q+n-i-1)_n. \tag{5}$$

The following identity is a simple consequence of Pieri's rule [12, Thm. 7.15.7] and appears in [7, I.3, Ex. 14]:

$$\prod_i \frac{1+tx_i}{1-ux_i} = 1 + (t+u)\sum_{i=0}^{n-1} s_{\sigma(i)} t^i u^{n-i-1}.$$

Substitute $-t$ for t, set $u=1$ and take the scalar product with p_λ. Since $\langle s_\mu, p_\lambda\rangle = \chi^\mu(\lambda)$ the right-hand side becomes $(1-t)\sum_{i=0}^{n-1}(-1)^i\chi^{\sigma(i)}(\lambda)t^i$. On the other hand, the left-hand side is given by

$$\begin{aligned} \left\langle \exp\left(\sum_{n\ge1}\frac{p_n}{n}\right)\cdot\exp\left(-\sum_{n\ge1}\frac{p_n}{n}t^n\right), p_\lambda\right\rangle &= \left\langle \exp\left(\sum_{n\ge1}\frac{p_n}{n}(1-t^n)\right), p_\lambda\right\rangle \\ &= \prod_{i=1}^{\ell}\left(1-t^{\lambda_i}\right), \end{aligned}$$

by standard properties of power sum symmetric functions [12, §7.7]. Hence

$$\sum_{i=0}^{n-1}(-1)^i\chi^{\sigma(i)}(\lambda)t^i = g_\lambda(t).$$

Comparing with equation (5) completes the proof. □

NOTE.

1. Since $(1-E)(q+n)_{n+1} = (n+1)(q+n-1)_n$, equation (2) can be rewritten as
$$P_\lambda(q) = \frac{1}{(n+1)z_\lambda} g'_\lambda(E)(q+n)_{n+1}, \tag{6}$$
where $g'_\lambda(t) = \prod_{j=1}^{\ell}(1-t^{\lambda_j})$.

2. A different kind of generating function for the coefficients of $P_\lambda(q)$ (though of course equivalent to Theorem 3.1) was obtained by D. Zagier [13, Thm. 1].

The zeros of the polynomial $P_\lambda(q)$ have an interesting property that will follow from the following result.

Theorem 3.2. *Let $g(t)$ be a complex polynomial of degree exactly d, such that every zero of $g(t)$ lies on the circle $|z| = 1$. Suppose that the multiplicity of 1 as a root of $g(t)$ is $m \geq 0$. Let $P(q) = g(E)(q+n-1)_n$.*

(a) If $d \leq n-1$, then

$$P(q) = (q+n-d-1)_{n-d}\, Q(q),$$

where $Q(q)$ is a polynomial of degree $d-m$ for which every zero has real part $(d-n+1)/2$.

(b) If $d \geq n-1$, then $P(q)$ is a polynomial of degree $n-m$ for which every zero has real part $(d-n+1)/2$.

Proof. First, the statements about the degrees of $Q(q)$ and $P(q)$ are clear; for we can write $g(t) = c\prod_u (t-u)$ and apply the factors $t-u$ consecutively. If $h(q)$ is any polynomial and $u \neq 1$ then $\deg\,(E-u)h(q) = \deg h(q)$, while $\deg\,(E-1)h(q) = \deg h(q) - 1$.

The remainder of the proof is by induction on d. The base case $d = 0$ is clear. Assume the statement for $d < n-1$. Thus for $\deg g(t) = d$ we have

$$\begin{aligned} g(E)(q+n-1)_n &= (q+n-d-1)_{n-d}\, Q(q) \\ &= (q+n-d-1)_{n-d} \prod_j \left(q - \frac{d-n+1}{2} - \delta_j i\right) \end{aligned}$$

for certain real numbers δ_j. Now

$$\begin{aligned} & (E-u)g(E)(q+n-1)_n \\ = {} & (q+n-d-1)_{n-d}\, Q(q) - u(q+n-d-2)_{n-d}\, Q(q-1) \\ = {} & (q+n-d-2)_{n-d-1}[(q+n-d-1)Q(q) - u(q-1)\, Q(q-1)] \\ = {} & (q+n-d-2)_{n-d-1} Q'(q), \end{aligned}$$

say. The proof now follows from a standard argument (e.g., [8, Lemma 9.13]), which we give for the sake of completeness. Let $Q'(\alpha+\beta i)=0$, where $\alpha,\beta\in\mathbb{R}$. Thus

$$(\alpha+\beta i+n-d-1)\prod_j\left(\alpha+\beta i-\frac{d-n+1}{2}-\delta_j i\right)$$

$$=u(\alpha+\beta i-1)\prod_j\left(\alpha-1+\beta i-\frac{d-n+1}{2}-\delta_j i\right).$$

Letting $|u|=1$ and taking the square modulus gives

$$\frac{(\alpha+n-d-1)^2+\beta^2}{(\alpha-1)^2+\beta^2}\prod_j\frac{\left(\alpha-\frac{d-n+1}{2}\right)^2+(\beta-\delta_j)^2}{\left(\alpha-1-\frac{d-n+1}{2}\right)^2+(\beta-\delta_j)^2}=1.$$

If $\alpha<(d-n+2)/2$ then

$$(\alpha+n-d-1)^2-(\alpha-1)^2<0$$

and

$$\left(\alpha-\frac{d-n+1}{2}\right)^2<\left(\alpha-1-\frac{d-n+1}{2}\right)^2.$$

The inequalities are reversed if $\alpha>(d-n+2)/2$. Hence $\alpha=(d-n+2)/2$, so the theorem is true for $d\le n-1$.

For $d\ge n-1$ we continue the induction, the base case now being $d=n-1$ which was proved above. The induction step is completely analogous to the case $d\le n-1$ above, so the proof is complete. ☐

Corollary 3.3. *The polynomial $P_\lambda(q)$ has degree $n-\ell(\lambda)+1$, and every zero of $P_\lambda(q)$ has real part 0.*

Proof. The proof is immediate from Theorem 3.1 and the special case $g(t)=g_\lambda(t)$ (as defined in Theorem 3.1) and $d=n-1$ of Theorem 3.2. ☐

It is easy to see from Corollary 3.3 (or from considerations of parity) that $P_\lambda(q)=(-1)^nP_\lambda(-q)$. Thus we can write

$$P_\lambda(q)=\begin{cases}R_\lambda(q^2), & n\text{ even}\\ qR_\lambda(q^2), & n\text{ odd,}\end{cases}$$

for some polynomial $R_\lambda(q)$. It follows from Corollary 3.3 that $R_\lambda(q)$ has (nonpositive) real zeros. In particular (e.g., [11, Thm. 2]) the coefficients of $R_\lambda(q)$ are log-concave with no external zeros, and hence unimodal.

The case $\lambda = (n)$ is especially interesting. Write $P_n(q)$ for $P_{(n)}(q)$. From equation (6) we have

$$P_n(q) = \frac{1}{n(n+1)}((q+n)_{n+1} - (q)_{n+1}).$$

Now

$$(q)_{n+1} = (-1)^{n+1}(-q+n)_{n+1}$$

and

$$(q+n)_{n+1} = \sum_{k=1}^{n+1} c(n+1,k)q^k,$$

where $c(n+1,k)$ is the signless Stirling number of the first kind (the number of permutations $w \in \mathfrak{S}_{n+1}$ with k cycles) [10, Prop. 1.3.4]. Hence

$$\frac{1}{n(n+1)}((q+n)_{n+1} - (q)_{n+1}) = \frac{1}{\binom{n+1}{2}} \sum_{k \equiv n \pmod 2} c(n+1,k)x^k.$$

We therefore get the following result, first obtained by Zagier [13, Application 3].

Corollary 3.4. *The number of n-cycles $w \in \mathfrak{S}_n$ for which $w \cdot (1,2,\ldots,n)$ has exactly k cycles is 0 if $n-k$ is odd, and is otherwise equal to $c(n+1,k)/\binom{n+1}{2}$.*

Is there a simple bijective proof of Corollary 3.4?

Let $\lambda, \mu \vdash n$. A natural generalization of $P_\lambda(q)$ is the polynomial

$$P_{\lambda,\mu}(q) = \sum_{\rho(w)=\lambda} q^{\kappa(w_\mu \cdot w)},$$

where w_μ is a fixed permutation in the conjugacy class K_μ. Let us point out that it is *false* in general that every zero of $P_{\lambda,\mu}(q)$ has real part 0. For instance,

$$P_{332,332}(q) = q^8 + 35q^6 + 424q^4 + 660q^2,$$

four of whose zeros are approximately $\pm 1.11366 \pm 4.22292i$.

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