String graphs and incomparability graphs

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Dedicated to the memory of Paul Turán on his 100th birthday

Abstract

Given a collection C of curves in the plane, its *string graph* is defined as the graph with vertex set C, in which two curves in C are adjacent if and only if they intersect. Given a partially ordered set $(P, <)$, its *incomparability graph* is the graph with vertex set P, in which two elements of P are adjacent if and only if they are incomparable.

It is known that every incomparability graph is a string graph. For "dense" string graphs, we establish a partial converse of this statement. We prove that for every $\varepsilon > 0$ there exists $\delta > 0$ with the property that if C is a collection of curves whose string graph has at least $\varepsilon|C|^2$ edges, then one can select a subcurve γ' of each $\gamma \in C$ such that the string graph of the collection $\{\gamma' : \gamma \in C\}$ has at least $\delta|C|^2$ edges and is an incomparability graph. We also discuss applications of this result to extremal problems for string graphs and edge intersection patterns in topological graphs.

1 Introduction

The *intersection graph* of a collection C of sets has vertex set C and two sets in C are adjacent if and only if they have nonempty intersection. A *curve* is a subset of the plane which is homeomorphic to the interval $[0, 1]$. A *string graph* is an intersection graph of a collection of curves. It is straightforward to show the intersection graph of any collection of arcwise connected sets in the plane is a string graph.

String graphs have been intensely studied both for practical applications and theoretical interest. Benzer [4] was the first to introduce these graphs in 1959, while exploring the topology of genetic structures. In 1966, interested in electrical networks realizable by printed circuits, Sinden [41] considered the same constructs at Bell Labs. He showed that not every graph is a string graph but all planar graphs are. He also raised the question whether there exists any algorithm for recognizing string graphs.

In 1976, reporting on Sinden's work, Graham [20] introduced string graphs to the mathematics community. Later that year, Ehrlich, Even, and Tarjan [8] proved that computing the chromatic number of a string graph is NP-hard. A decade later, Kratochvíl, Goljan, and Kučera [28] wrote a tract devoted to string graphs. They showed that every string graph can be realized by a family of polygonal arcs with a finite number of intersections. Kratochvíl [27] proved that the recognition

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of string graphs is NP-hard. Kratochvíl and Matoušek [29] constructed string graphs on n vertices that require at least 2^{cn} intersection points in any realization, where c is a positive constant. They conjectured that every string graph on n vertices can be realized with at most 2^{cn^k} intersection points, for some constants c and k.

Sinden's question remained a challenging open problem for 35 years, until the conjecture of Kratochvíl and Matoušek was confirmed independently in [36] and [40], implying that the string graph problem is decidable. A short time later, Schaefer, Sedgewick, and Štefankovič [39] proved that recognizing string graphs is NP-complete. Despite these results, understanding the *structure* of string graphs has remained a wide open problem.

Given a partially ordered set (poset, for short) $(P, <)$, its *incomparability graph* is the graph with vertex set P, in which two elements are adjacent if and only if they are incomparable. Unlike string graphs, incomparability graphs are fairly well understood. In 1950, Dilworth [7] proved that every incomparability graph is a perfect graph, so the chromatic number of an incomparability graph is equal to its clique number (the analogous result for comparability graphs was earlier proved by Erdős and Szekeres [10]). In 1967, Gallai [16] gave a characterization of incomparability graphs in terms of minimal forbidden induced subgraphs. It is known that incomparability graphs can be recognized in polynomial time [17].

In 1983, Golumbic, Rotem, and Urrutia [18] and Lovász [33] proved that every incomparability graph is a string graph (see Proposition 2.1). There are many string graphs, such as odd cycles of length at least *five*, which are not incomparability graphs. In fact, the number of string graphs on n vertices is $2^{(3/4+o(1))\binom{n}{2}}$ [38], while the number of incomparability graphs on n vertices is only $2^{(1/2+o(1))\binom{n}{2}}$ [23]. Nevertheless, as the main result of this paper demonstrates, most string graphs contain huge subgraphs that are incomparability graphs.

Theorem 1.1 *For every $\varepsilon > 0$ there exists $\delta > 0$ with the property that if C is a collection of curves whose string graph has at least $\varepsilon|C|^2$ edges, then one can select a subcurve γ' of each $\gamma \in C$ such that the string graph of the collection $\{\gamma' : \gamma \in C\}$ has at least $\delta|C|^2$ edges and is an incomparability graph.*

It follows from our proof that δ can be chosen to be a polynomial in ε, that is, we can choose $\delta = \varepsilon^c$ for an appropriate absolute constant c.

We say that a graph with n vertices is *dense* if, for some $\varepsilon > 0$, its number of edges is at least εn^2. Our theorem immediately implies that every dense string graph contains a dense spanning subgraph (with a different ε) which is an incomparability graph.

Theorem 1.1 cannot be strengthened to say that every dense string graph contains a dense *induced* subgraph with a linear number of vertices that is an incomparability graph. Indeed, a construction of Kynčl [30] (improving on earlier constructions [31] and [22]) shows that there is a dense intersection graph of n segments in the plane whose largest clique or independent set is of size $O(n^{\frac{\log 8}{\log 169}})$. Since incomparability graphs are perfect graphs, then there are dense string graphs on n vertices whose largest induced subgraph which is an incomparability graph has $O(n^{2\frac{\log 8}{\log 169}})$ vertices, where $2\frac{\log 8}{\log 169} < .811$.

A *bi-clique* is a complete bipartite graph whose two parts are of equal size. It follows from a result of Kővári, Sós, and Turán [25] that every graph on n vertices or its complement contains a (not necessarily induced) subgraph which is a bi-clique with $\log n - \log\log n$ vertices in each of its parts.[1] Considering a random graph on n vertices, we obtain that this bound is tight apart from a constant factor. Fox [11] proved that every incomparability graph or its complement contains a bi-clique whose parts are of size $\frac{n}{4\log n}$, and that this bound is also tight up to a constant factor. This result was applied by Fox, Pach, and Cs. Tóth [14] to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that every incomparability graph with n vertices and εn^2 edges contains a bi-clique of size $\delta n/\log n$. Here δ can be taken to be a polynomial in ε. We have the following immediate corollary of Theorem 1.1 and this result.

Corollary 1.2 *For every $\varepsilon > 0$, there exists $\delta > 0$ such that every string graph with n vertices and at least εn^2 edges contains a bi-clique with parts of size $\delta n/\log n$.*

In this corollary, δ again can be taken to be a polynomial in ε. In other words, every collection C of n curves in the plane with at least εn^2 intersecting pairs has two subcollections A and B each of size at least $\delta n/\log n$ such that every curve in A intersects every curve in B. By the construction in [11], the dependence on n in Corollary 1.2 is tight.

Pach and G. Tóth [37] conjectured that for every collection C of n curves in the plane, any pair of which intersect in at most k points, the intersection (string) graph of C or its complement contains a bi-clique of size $c_k n$, where $c_k > 0$ depends only on k. This conjecture was proved by Fox, Pach, and Cs. Tóth [13]. The main ingredient of the proof was a variant of Corollary 1.2 for intersection graphs of curves with a bounded number of intersection points per pair. A similar result for intersection graphs of convex sets was established in [14].

The importance of arrangements of curves and Theorem 1.1 in particular is exhibited in its applications to graph drawing problems. A *topological graph* is a graph drawn in the plane with vertices as points and edges as curves connecting corresponding endpoints. The well known Crossing Lemma discovered by Ajtai, Chvatal, Newborn, and Szemerédi [3] and independently by Leighton [32] says that every topological graph with n vertices and $m \geq 4n$ edges has $\Omega\left(m^3/n^2\right)$ pairs of crossing edges. By induction, this is equivalent to the statement that every topological graph with n vertices and $m \geq 3n$ edges has an edge that intersects $\Omega\left(m^2/n^2\right)$ other edges. An *ℓ-grid* in a topological graph is a pair of disjoint edge subsets E_1, E_2 such that every edge in E_1 crosses every edge in E_2. Is the following strengthening of the Crossing Lemma true: every topological graph with n vertices and $m \geq 3n$ edges contains an ℓ-grid with $\ell = \Omega\left(m^2/n^2\right)$? With Cs. Tóth in [15], we show that the answer is yes if we assume that every pair of curves in the topological graph intersect in at most a fixed constant number of points, but no in general. Indeed, we construct a drawing of the complete bipartite graph $K_{n,n}$, which does not contain an ℓ-grid with $\ell \geq cn^2/\log n$, where c is an absolute constant. This counterexample cannot be substantially improved: using Corollary 1.2 together with a result of Kolman and Matoušek [24] relating the bisection width and the pairwise crossing number of a graph, we prove in [15] that every topological graph with n vertices and $m \geq 3n$ edges contains an ℓ-grid with $\ell = \Omega\left(\frac{m^2/n^2}{\log^c m/n}\right)$. It was proved in [34] that for each positive integer ℓ there is a constant c_ℓ

[1] Throughout this paper, for the sake of simplicity, we systematically omit floor and ceiling signs whenever they are not crucial. All logarithms are base 2.

such that every topological graph with n vertices and at least $c_\ell n$ edges contains an ℓ-grid. Their proof gives that we may take $c_\ell = 16 \cdot 24^{4^\ell}\ell$, which is double-exponential in ℓ, while the above mentioned result shows that we may take $c_\ell = \sqrt{\ell}\log^c \ell$ for some absolute constant c, which is best possible up to the polylogarithmic factor. With Ackerman and Suk in [2], we show using Corollary 1.2 that every topological graph with n vertices and no ℓ-grid with *distinct vertices* has at most $c_\ell n \log^* n$ edges, where $c_\ell = \ell^{O(\log\log \ell)}$ and $\log^*$ is the iterated logarithm function.

It is a general question in geometric graph theory to investigate how much one can relax planarity while still ensuring that the graph is sparse? To formalize this question we need the following definition. A topological graph is *k-quasi-planar* if no k edges pairwise intersect. A well known conjecture states that every k-quasi-planar topological graph on n vertices has at most $c_k n$ edges for some constant c_k depending only on k. This conjecture is only known for $k \leq 4$. The case $k = 2$ follows easily from Euler's polyhedral formula, the case $k = 3$ was proved by Pach, Radoičić, and G. Tóth [35], and the case $k = 4$ was proved by Ackerman [1]. The best known upper bound on the number of edges of a k-quasi-planar topological graph on n vertices is by Ackerman, who gave an upper bound of the form $c_k n(\log n)^{4k-16}$ for $k \geq 4$. In another paper [12], we again use Corollary 1.2 together with the result of Kolman and Matoušek [24] to obtain a new upper bound on the number of edges in a k-quasi-planar topological graph. We show that every k-quasi-planar topological graph on n vertices has at most $n(\log n)^{c\log k}$ edges, where c is an absolute constant. In particular for each $\varepsilon > 0$ there is $\delta > 0$ and n_0 such that every topological graph on $n \geq n_0$ vertices and at least $n^{1+\varepsilon}$ edges has $n^{\delta/\log\log n}$ pairwise crossing edges. This is a significant improvement on the previous bound of $\delta \log n/\log\log n$.

To make our paper self-contained, in the next section we present the (few lines long) proof of the fact discovered by Golumbic, Rotem, and Urrutia [18] and Lovász [33] that every incomparability graph is a string graph. In Section 3, we establish a simple lemma showing that every dense graph has a cubic number of triangles K_3 or a quartic number of induced claws $K_{1,3}$. In the proof of Theorem 1.1, we will distinguish between the case that the string graph of the collection of curves has few triangles and the case in which it has many triangles. These two proofs will be presented in Sections 5 and 7, respectively.

In Section 4, we introduce some important notions: we define "grounded", "double-grounded", "strongly double-grounded", and "split" families of curves. For instance, roughly speaking, a family C is called grounded if there is a special curve, a so-called "ground" curve γ, with the property that all elements of C have an endpoint on γ, but otherwise they are disjoint from it. (The other definitions will be similar.) We show that the intersection graphs of split families of curves are incomparability graphs (Lemma 4.2). This elementary fact plays an important role in our arguments. The proof of Theorem 1.1 in the case where the string graph has many triangles is reduced in Section 7 to proving Theorem 1.1 in the special case where the collection of curves is grounded. This special case is settled in Section 6.

In most parts of this paper, we assume for simplicity that all families of curves we consider are in "general position", i.e., no point belongs to *three* curves of the family. In Section 8, we outline how to get rid of this assumption. Finally, Section 9 includes a brief discussion of a variant of Theorem 1.1 when each pair of curves in the collection intersect in at most a fixed constant number of points. For the clarity of the presentation, we do not make any serious attempt to optimize the absolute constants appearing in our statements and proofs.

2 Incomparability graphs as string graphs

We use the notation $[n] = \{1, \ldots, n\}$. A *linear extension* of a poset $(P, \prec)$ on n elements is a one-to-one map $\pi : P \to [n]$ such that if $y \prec z$, then $\pi(y) < \pi(z)$. The *intersection* of a set Π of one-to-one maps from a set P on n elements to $[n]$ is the poset $(P, \prec)$ such that $y \prec z$ if and only if $\pi(y) < \pi(z)$ for every $\pi \in \Pi$ and for all $y, z \in P$. It is straightforward to show that every poset is the intersection of its linear extensions. The *dimension* of a poset is the minimum number of linear extensions whose intersection is that poset. An old result of Hiraguchi [21] states that every poset on $n \geq 4$ elements has dimension at most $n/2$. See the book [42] by Trotter for more on the dimension theory for posets.

If $f_1, \ldots, f_n : [0, 1] \to \mathbb{R}$ are continuous functions, we can define a partial order $\prec$ on these functions by $f_i \prec f_j$ if $f_i(x) < f_j(x)$ for all $x \in [0, 1]$. The following proposition implies that every partially ordered set can be represented in this way. In particular, it implies that the a graph is an incomparability graph if and only if it is the intersection graph of a collection of curves given by continuous functions defined on $[0, 1]$.

Proposition 2.1 *([18], [33]) For each partial order $\prec$ on $[n]$, there is a family of continuous functions $f_1, \ldots, f_n : [0, 1] \to \mathbb{R}$ such that $i \prec j$ if and only if $f_i(x) < f_j(x)$ for each $x \in [0, 1]$.*

Proof. Let $([n], \prec)$ be a poset with dimension d, and let $\Pi = \{\pi_1, \ldots, \pi_d\}$ denote a set of d linear extensions whose intersection is the poset. Assign to each π_k a distinct point x_k of the interval $[0, 1]$, so that

$$0 = x_1 < x_2 < \ldots < x_d = 1.$$

For each i $(1 \leq i \leq n)$, define a continuous, piecewise linear function $f_i(x)$, as follows. For any k $(1 \leq k \leq d)$, set $f_i(x_k) = \pi_k(i)$, and let $f_i(x)$ change linearly over the interval $[x_k, x_{k+1}]$ for $k < m$.

Obviously, whenever $i \prec j$ for some $i \neq j$, we have that $\pi_k(i) < \pi_k(j)$ for every k, and hence $f_i(x) < f_j(x)$ for all $x \in [0, 1]$. On the other hand, if i and j are incomparable with respect to the ordering $\prec$, we find that there are indices k and k' $(1 \leq k \neq k' \leq m)$ such that $f_i(x_k) < f_j(x_k)$ and $f_i(x_{k'}) > f_j(x_{k'})$, therefore, by continuity, the curves of f_i and f_j must cross at least once in the interval $(x_k, x_{k'})$. This completes the proof. $\Box$

For an illustration of the proof of Proposition 2.1, see Figure 1.

The proof of Proposition 2.1, together with Hiraguchi's theorem mentioned in the first paragraph of this section, implies that every incomparability graph with n vertices is the intersection graph of a collection of curves given by continuous functions defined on the interval $[0, 1]$ such that every pair intersect in at most $n/2 - 1$ points. Every bi-clique is the incomparability graph of a 2-dimensional poset and is the intersection graph of a collection of segments (intersecting in at most one point per pair).

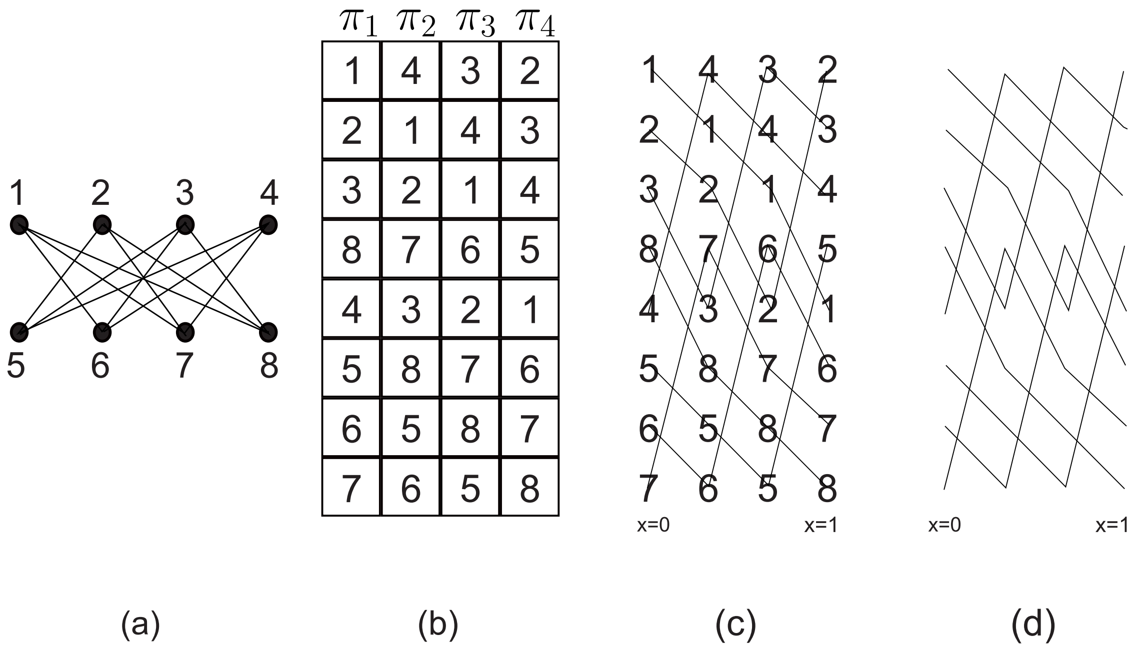

Figure 1: (a) depicts the Hasse diagram of a poset on 8 elements. In (b), four linear extensions are exhibited, each from greatest element down to smallest element, whose intersection is the poset in (a). In (c) and (d), we represent each element of the poset by a piecewise linear function defined on $[0, 1]$.

On the other hand, according to a result of Kratochvíl and Matoušek [29], there are string graphs with n vertices that require an exponential number of intersection points in any of their realizations. Consequently, the "geometric complexity" of a string graph may be much larger than the complexity of the canonical substructures whose existence is guaranteed by Theorem 1.1 and Corollary 1.2.

3 Triangles and Claws

The *clique multiplicity* $k_s(G)$ is the number of cliques of size s in graph G. The *Ramsey multiplicity* $k_s(n) = \min\{k_s(G) + k_s(\bar{G}) : |G| = n\}$ is the minimum number of cliques or independent sets of size s over all graphs G on n vertices. The exact value of $k_3(n)$ was determined by Goodman [19]:

$$k_3(n) = \begin{cases} \frac{n(n-2)(n-4)}{24} & \text{if } n \text{ is even} \\ \frac{n(n-1)(n-5)}{24} & \text{if } n \equiv 1 \pmod 4 \\ \frac{(n+1)(n-3)(n-4)}{24} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Note that $k_3(n)$ is asymptotic to $n^3/24$ and we will use the estimate $k_3(n) \geq n^3/32$ for $n \geq 24$. See the paper [6] by Conlon for more details on $k_s(n)$ with $s > 3$.

For any two subsets of vertices, $A, B \subseteq V(G)$, let $e(A, B)$ denote the number of edges of G with one endpoint in A and the other in B. A *claw* is a graph with *four* vertices and *three* edges, having a vertex, the *root*, which is adjacent to the remaining *three* vertices.

Lemma 3.1 *Let G be a graph, and let A and B be (not necessarily disjoint) subsets of $V(G)$ with $e(A,B) \geq \varepsilon n^2$ and $\varepsilon \geq 48/n$. Then B contains at least $2^{-9}\varepsilon^3 n^3$ triangles or there are at least $2^{-9}\varepsilon^3 n^4$ induced claws whose root is in A and whose* three *other vertices are in B.*

Proof. Delete all vertices from A that are adjacent to at most $\varepsilon n/2$ vertices in B, and let A' be the resulting subset of vertices. Since $e(A \setminus A', B) \leq (\varepsilon n/2)|A \setminus A'| \leq \varepsilon n^2/2$, then there are at least $\varepsilon n^2 - \varepsilon n^2/2 = \varepsilon n^2/2$ edges with one vertex in A' and the other vertex in B.

For $a \in A'$, let $N(a)$ denote the subset of vertices in B adjacent to a, so $|N(a)| \geq \varepsilon n/2 \geq 24$. By Goodman's theorem, $N(a)$ contains at least $|N(a)|^3/64$ triangles or at least $|N(a)|^3/64$ independent sets of size three. In the latter case, $N(a)$ and hence B contains at least $|N(a)|^3/64 \geq (\varepsilon n/2)^3/64 = 2^{-9}\varepsilon^3 n^3$ triangles, and we are done. So we may suppose that there are at least $|N(a)|^3/64$ independent sets of size 3 in $N(a)$ for each $a \in A'$. Hence, the number of induced claws whose root is in A and whose other three vertices are in B is at least

$$\sum_{a \in A'} |N(a)|^3/64 \geq |A'| \left(\sum_{a \in A'} |N(a)|/|A'| \right)^3 /64 = \frac{e(A,B')^3}{64|A'|^2} \geq \frac{(\varepsilon n^2/2)^3}{64|A'|^2} \geq 2^{-9}\varepsilon^3 n^4,$$

where the first inequality is by Jensen's inequality for the convex function $f(x) = x^3$ and the last inequality follows from $|A'| \leq n$. $\square$

The complete bipartite graph $K_{a,b}$ on $a+b$ vertices has parts of size a and b with all edges between them. In particular, $K_{1,3}$ is the claw. The above lemma can be easily extended to show that for any fixed a, b, s, every dense graph on n vertices contains $\Omega(n^s)$ copies of the complete graph K_s or $\Omega(n^{a+b})$ induced copies of the complete bipartite graph $K_{a,b}$.

4 Special string graphs that are incomparability graphs

The purpose of this section is to develop notation and terminology necessary for the proof of Theorem 1.1 and to prove a simple lemma showing that the intersection graphs of rather special collections of curves are incomparability graphs.

Throughout this paper, unless it is stated otherwise, we always assume that the curves we consider are in *general position*, i.e., that no point belongs to three of them. After completing the proof of Theorem 1.1 for collections of curves in general position, in Section 8 we discuss how our arguments can be modified to deal with the degenerate cases.

Recall that a curve γ is a subset of the plane homeomorphic to the unit interval $[0,1]$. That is, a curve γ is the image of a homeomorphism f from $[0,1]$ to a subset of the plane $\mathbb{R}^2$. We associate the function f with the curve γ. In particular, each curve comes with an orientation from the *starting point* $f(0)$ to the *final point* $f(1)$. The points $f(0)$ and $f(1)$ are *endpoints of* γ. The other points of γ are *interior points of* γ. For distinct points $f(a), f(b)$ of γ, $f(a)$ *comes before* $f(b)$ *along* γ if $a < b$, and otherwise $f(a)$ *comes after* $f(b)$ *along* γ. A *subcurve* of a curve γ is the image of the function f restricted to a subinterval $[a,b]$ of $[0,1]$. In particular, a subcurve is a curve if $a < b$, it consists of a single point if $a = b$, and it is the empty set if $a > b$.

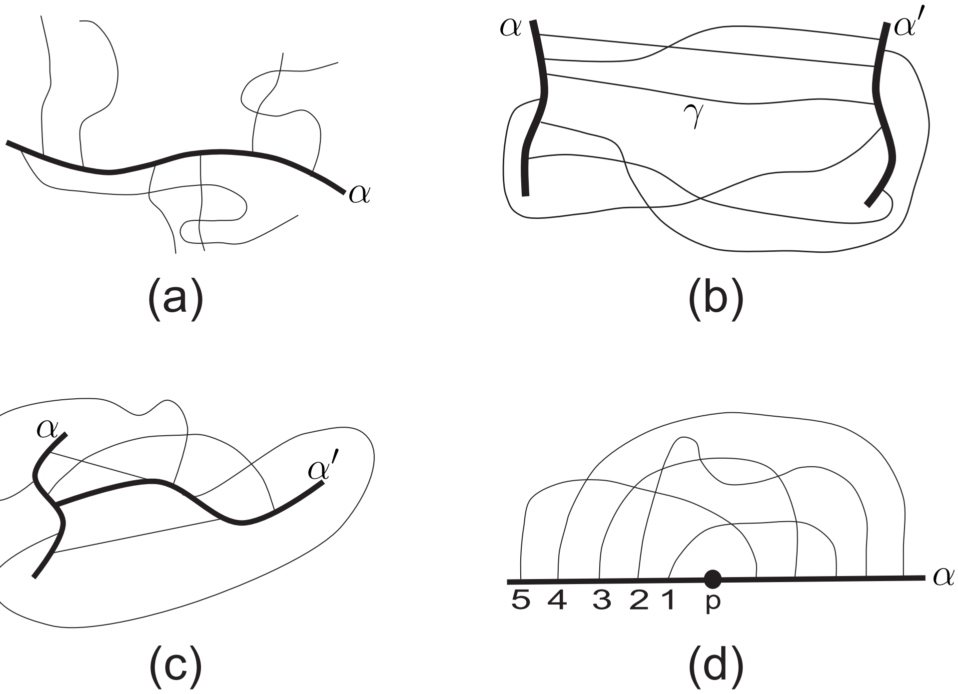

Figure 2: In (a), a grounded collection of curves. In (b), a double-grounded collection of curves with separator γ. In (c), a strongly double-grounded collection of curves. In (d), a split collection of curves with middle point p.

A collection C of curves in the plane is said to be *grounded* if there is a curve α such that every member γ of C has precisely one endpoint on α and the rest of γ is disjoint from α (see Figure 2(a)). The curve α is called a *ground* for C. Since we consider curves in general position, in a grounded collection C, no point of the ground can belong to *two* elements of C.

A collection C of curves is *double-grounded* if there are disjoint curves α, α' such that every member of C has one endpoint on α, the other endpoint on α', and the rest of the curve is disjoint from $\alpha \cup \alpha'$. A collection C of curves is *double-grounded with a separator* if there is a curve γ such that $C \cup \{\gamma\}$ is double-grounded and every curve in C is disjoint from γ (see Figure 2(b)). We then call γ a *separator* for C. A collection C of curves is *strongly double-grounded* if there is an ordered pair (α, α') of curves with no interior point in common such that one of the endpoints of α' lies on α, and every member of C has one endpoint on α, the other on α', and the rest of it is disjoint from $\alpha \cup \alpha'$ (see Figure 2(c)). Finally, we call a collection C of curves *split* if there is a curve α and a point p in the interior of α such that every member γ in C has one endpoint before p along α, the other endpoint after p along α, the interior of γ is disjoint from α, and the ends of curves on C all lie on the same side of α (see Figure 2(d)). In this case we say that α splits C and call p a middle point for C.

By tracing along the exterior of $\alpha \cup \alpha' \cup \gamma$ of a double grounded collection with grounds α, α' and separator γ, we see that every double-grounded collection of curves with a separator is split. Similarly, by tracing along the two grounds of a strongly double-grounded collection of curves, we see that every strongly double-grounded collection of curves is split. One can easily check the other cases of the following simple proposition.

Proposition 4.1 *A collection of curves is double-grounded with a separator if and only if it is strongly double-grounded if and only if it is split.*

We are implicitly assuming that the collection of curves are in general position. The above proposition fails to hold if this assumption is not made, as there could be a pair of curves in the collection which share an endpoint on different sides of a ground, which would make it impossible for the collection of curves to be split. This technical issue is discussed in detail in Section 8.

For any pair of intersecting (oriented) curves (α, β), let $p(\alpha, \beta)$ denote the first point along α that belongs to β. (The existence of such a point follows from the fact that β is a closed set and α is homeomorphic to the unit interval.) Furthermore, let $\alpha(\beta)$ denote the subcurve of α with the same starting point as α and with final point $p(\alpha, \beta)$.

We finish this section with the following lemma followed by a useful remark.

Lemma 4.2 *The intersection graph of every split collection S of curves is an incomparability graph.*

Proof. Let α be a ground for S with middle point p. We can label the curves in S according to the order of their endpoints along α, as $\{\gamma_1, \ldots, \gamma_n\}$, starting at the middle point p and increasing label in the direction toward the starting point, so the curve γ_i has label i. See Figure 2(d). Define a binary relation $\prec$ on S, as follows. Let $\gamma_i \prec \gamma_j$ if and only if γ_i is disjoint from γ_j and $i < j$. For each curve γ_j, there is a closed Jordan curve β_j which consists of γ_j together with the subcurve of α whose endpoints are the endpoints of γ_j. By the Jordan curve theorem, if $i < j < k$, curve i is disjoint from curve j and curve j is disjoint from curve k, then the interior of curve i lies in the interior of the Jordan region bounded by β_j and the interior of curve k lies in the exterior of the Jordan region bounded by β_j, so curve i is disjoint from curve k. Thus, $\prec$ is a partial order and the intersection graph of S is an incomparability graph. $\square$

By Proposition 4.1 and Lemma 4.2, the intersection graphs of double-grounded collections of curves with a separator and strongly double-grounded collections of curves are incomparability graphs.

The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph union $G_1 \cup G_2$ together with all edges between V_1 and V_2. It is easy to check the following useful remark.

Remark: The join of two incomparability graphs is an incomparability graph. Similarly, the disjoint union of two incomparability graphs is an incomparability graph. In particular, an incomparability graph with added isolated vertices is also an incomparability graph, so a graph with just one edge is an incomparability graph.

In the proof of Theorem 1.1, we are allowed to take the empty subcurve for some of the curves in C, and as long as the the intersection graph of the nonempty subcurves is a dense incomparability graph, we are done, as the empty subcurves are isolated vertices in the intersection graph. So if C is a collection of curves in the plane whose intersection graph is sparse but has at least one edge, then we will take C' to be the collection of subcurves of the elements of C consisting of a pair of intersecting

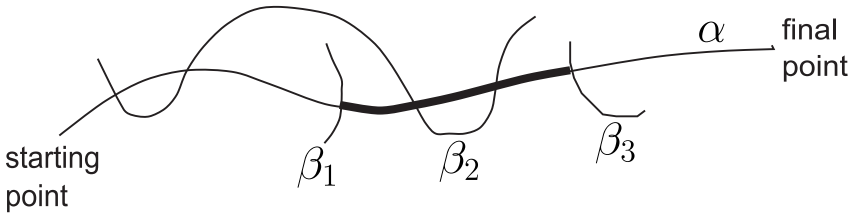

Figure 3: A very nice quadruple $(\alpha, \beta_1, \beta_2, \beta_3)$. If $\beta = \beta_3$ and $\beta' = \beta_1$, then the subcurve α' is the bold subcurve of α.

curves in C and the *empty* subcurve for each of the remaining elements. The intersection graph of C', consisting of a single edge and $|C| - 2$ isolated vertices, is clearly an incomparability graph.

5 String graphs with few triangles

The aim of this section is to prove Theorem 1.1 for every (nondegenerate) collection C of curves whose string graph has few triangles (see Theorem 5.4). This is the first half of the proof of Theorem 1.1. We start with three useful lemmas.

Lemma 5.1 *Let A, B be collections of curves such that there are q ordered quadruples $(\alpha, \beta_1, \beta_2, \beta_3) \in A \times B \times B \times B$ such that $\beta_1, \beta_2, \beta_3$ are pairwise disjoint and α intersects β_1, β_2, and β_3. Then there are disjoint curves $\beta, \beta' \in B$, a subcollection $B' \subset B$ such that every curve in B' is disjoint from β and β', and a double-grounded collection A' of subcurves of curves of A with grounds β, β' such that there are more than $\frac{q}{6|B|^2}$ intersecting pairs of curves in $A' \times B'$.*

Proof. Pick an ordered pair (β, β') of distinct curves of B at random. Note that there are $|B|(|B|-1)$ such ordered pairs. Let $N(\beta)$ denote the set of curves in A that intersect β. For each curve $\alpha \in N(\beta)$, $\alpha(\beta)$ is the subcurve of α that has the same starting point as α and ends at $p(\alpha, \beta)$, the first point of β along α. Let $N = \{\alpha(\beta) : \alpha \in N(\beta)\}$.

For each curve $\alpha(\beta) \in N$ that intersects β', let α' denote the subcurve of $\alpha(\beta)$ with the same final point $p(\alpha, \beta)$ as $\alpha(\beta)$ and whose starting point is the last intersection point of β' with $\alpha(\beta)$ along $\alpha(\beta)$. Let $A' = \{\alpha' : \alpha(\beta) \in N\}$.

Call a quadruple $Q = (\alpha, \beta_1, \beta_2, \beta_3) \in A \times B \times B \times B$ *nice* if $\beta_1, \beta_2, \beta_3$ are pairwise disjoint and α intersects β_1, β_2, and β_3. By symmetry, if $Q = (\alpha, \beta_1, \beta_2, \beta_3)$ is nice, then for any of the *six* permutations π of $1, 2, 3$, the quadruple $Q_\pi = (\alpha, \beta_{\pi(1)}, \beta_{\pi(2)}, \beta_{\pi(3)})$ is also nice. Call a nice quadruple $(\alpha, \beta_1, \beta_2, \beta_3)$ *very nice* if it also satisfies the following two properties:

1. the point $p(\alpha, \beta_3)$ comes after the points $p(\alpha, \beta_1)$ and $p(\alpha, \beta_2)$ along α, and
2. along α the last point of β_2 before $p(\alpha, \beta_3)$ comes after the last point of β_1 before $p(\alpha, \beta_3)$.

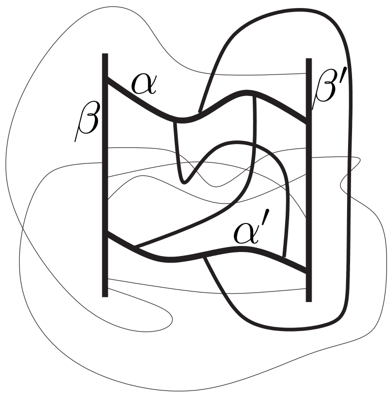

Figure 4: Curves $\alpha, \alpha', \beta, \beta'$ are in bold. Disjoint curves $\{\alpha, \alpha'\}$ are grounded with grounds β, β'. A collection A' of double-grounded curves (thin) with grounds β, β' such that every curve in A' is disjoint from α and from α'. A collection B' of double-grounded curves (medium thickness) with grounds α, α' such that every curve in A' is disjoint from β and from β'.

See Fig. 3 for an example of a very nice quadruple. It is easy to see that for each nice quadruple Q, exactly one of the *six* quadruples Q_π is very nice. For each very nice quadruple $Q = (\alpha, \beta_1, \beta_2, \beta_3)$, let E_Q denote the event that $\beta = \beta_3$ and $\beta' = \beta_1$. The probability of E_Q is clearly $\frac{1}{|B|(|B|-1)}$.

By linearity of expectation, the expected number of events E_Q that hold is

$$\frac{q/6}{|B|(|B|-1)} > \frac{q}{6|B|^2}.$$

Therefore, there exists a pair (β, β') of curves in B such that for more than $\frac{q}{6|B|^2}$ very nice quadruples Q, the event E_Q holds. Pick such a pair (β, β'). The collection A' is double-grounded with grounds β and β'. Let B' denote the set of curves in B that intersect neither β nor β'. Note that for each very nice Q for which E_Q holds, the subcurve α' of the curve α is in A' and intersects the curve $\beta_2 \in B'$. Hence, there are more than $\frac{q}{6|B|^2}$ intersecting pairs in $A' \times B'$. $\square$

The last lemma is crucial for the proof of the following statement.

Lemma 5.2 *Let A be a double-grounded collection of at most n curves with grounds β, β', and let B be a collection of at most n curves disjoint from β, β' such that the number of intersecting pairs in $A \times B$ is at least εn^2 with $\varepsilon \geq 48/n$, and the number of pairwise intersecting triples in A is less than $2^{-9}\varepsilon^3 n^3$. Then we can find disjoint curves $\alpha, \alpha' \in A$ and a double-grounded collection B' of subcurves of curves in B with grounds α, α' such that the subcollection A' consisting of all curves in A that are disjoint from α and α' has the property that the the number of intersecting pairs of curves in $A' \times B'$ is at least $2^{-9}\varepsilon^3 n^4/|A|^2$. See Fig. 4.*

Proof. Since there are less than $2^{-9}\varepsilon^3 n^3$ triangles in the intersection graph of A, then by Lemma 3.1 we have that there are at least $2^{-9}\varepsilon^3 n^4$ induced claws in the intersection graph of $A \cup B$ with root in B and the other three vertices in A. Applying Lemma 5.1 with A and B switched, there are disjoint curves $\alpha, \alpha' \in A$, a subcollection $A' \subset A$ such that every curve in A' is disjoint from α, α', and a double-grounded collection B' of subcurves of curves of B with grounds α, α' such that there are more than $2^{-9}\varepsilon^3 n^4/|A|^2$ intersecting pairs in $A' \times B'$. $\Box$

We need one more lemma before we prove Theorem 1.1 in the case that the intersection graph of the collection of curves has few triangles.

Lemma 5.3 *Let $A \cup \{\alpha, \alpha'\}$ be a double-grounded collection of curves with grounds β, β', and let B be a double-grounded collection of curves with grounds α, α' such that every curve in A is disjoint from α and α', and every curve in B is disjoint from β and β'. Then $A \cup B$ is the disjoint union of two collections of split curves such that every curve in the first split collection is disjoint from every curve in the second split collection.*

Proof. By the Jordan curve theorem, $\alpha \cup \alpha' \cup \beta \cup \beta'$ partitions the plane into two regions, an inside region I and an outside region O. Each curve in $A \cup B$ either lies entirely in the closure $\bar{I}$ of I or the closure $\bar{O}$ of O. A curve γ in $A \cup B$ that lies entirely in $\bar{I}$ is an *inside curve*, otherwise γ is an *outside curve.* Since our curves are in general position, every outside curve is disjoint from every inside curve. By tracing along the outside, we see that the collection of outside curves is split. Similarly tracing along the inside, we see that the collection of inside curves is split. $\Box$

We next establish Theorem 1.1 for collections of curves whose intersection graphs have few triangles.

Theorem 5.4 *Let C be a collection of curves such that the intersection graph of C has at least εn^2 edges and fewer than $2^{-36}\varepsilon^9 n^3$ triangles. Then for each $\gamma \in C$, there is a subcurve γ' of γ such that the intersection graph of the collection $\{\gamma' : \gamma \in C\}$ is an incomparability graph with at least $2^{-45}\varepsilon^9 n^2$ edges.*

Proof. By the remark at the end of Section 4, we may assume that $2^{-45}\varepsilon^9 n^2 > 1$, so that $\varepsilon > 2^5 n^{-2/9}$.

By Lemma 3.1, since there are fewer than $2^{-9}\varepsilon n^3$ triangles in the intersection graph of C, letting $A = B = C$, there are at least $q = 6 \cdot 2^{-9}\varepsilon^3 n^4$ ordered quadruples $(\alpha, \beta_1, \beta_2, \beta_3) \in A \times B \times B \times B$ such that $\beta_1, \beta_2, \beta_3$ are pairwise disjoint and α intersects β_1, β_2, and β_3. By Lemma 5.1, there are disjoint curves $\beta, \beta' \in B$, a subcollection $B' \subset B$, and a double-grounded collection A' of subcurves of curves of A with grounds β, β' such that every curve in B' is disjoint from β, β' and the number of intersecting pairs in $A' \times B'$ is larger than $\frac{q}{6|B|^2} \geq 2^{-9}\varepsilon^3 n^2$. Let $\varepsilon_1 = 2^{-9}\varepsilon^3$, so

$$\varepsilon_1 > 2^{-9}\left(2^5 n^{-2/9}\right)^3 > 2^6 n^{-2/3} > 48/n.$$

Since there are fewer than $2^{-36}\varepsilon^9 n^3 = 2^{-9}\varepsilon_1^3 n^3$ triangles in the intersection graph of C (and hence in the intersection graph of A') and $\varepsilon_1 > 48/n$, by Lemma 5.2, there are disjoint curves $\alpha, \alpha' \in A'$, and a double-grounded collection B'' of subcurves of curves in B' with grounds α, α' such that the subcollection A'' of all curves in A' that are disjoint from α and α' has the property that the number of intersecting pairs of curves in $A'' \times B''$ is at least $\varepsilon_2 n^2$, for $\varepsilon_2 = 2^{-9}\varepsilon_1^3 n^4/|A'|^2 \geq 2^{-36}\varepsilon^9 n^2$. By Lemma 5.3, the collection $A'' \cup B''$, which has at least $2^{-36}\varepsilon^9 n^2$ edges in its intersection graph, is the disjoint union of two split collections such that every curve in the first split collection is disjoint from every curve in the second split collection. Since the intersection graph of every split collection of curves is an incomparability graph (Proposition 4.2), and the disjoint union of two incomparability graphs is an incomparability graph as remarked at the end of Section 4, then the intersection graph of $A'' \cup B''$ is an incomparability graph. This completes the proof. $\square$

6 Grounded collections of curves

The aim of this section is to prove Theorem 1.1 for (nondegenerate) grounded collections of curves (see Theorem 6.6). This is an important step toward the proof of Theorem 1.1 in the general case, which will be presented in the next section. We first collect several useful lemmas for families of grounded curves.

In the previous section we proved Theorem 1.1 in the case when the string graph has few triangles. To prove Theorem 1.1 in the case when the string graph has many triangles, we have to classify the different ways how *three* curves can pairwise intersect. We start with a simple observation.

Proposition 6.1 *Let $T = \{\alpha_1, \alpha_2, \alpha_3\}$ be a set of* three *pairwise intersecting curves. There is a permutation $\gamma_1, \gamma_2, \gamma_3$ of $\alpha_1, \alpha_2, \alpha_3$ such that $p(\gamma_3, \gamma_1)$ comes before $p(\gamma_3, \gamma_2)$ along γ_3 and $p(\gamma_1, \gamma_2)$ comes before $p(\gamma_1, \gamma_3)$ along γ_1. In particular, if γ is a subcurve of γ_3 with the same starting point as γ_3 and whose final point is at least $p(\gamma_3, \gamma_1)$ and before $p(\gamma_3, \gamma_2)$ along γ_3, then γ intersects γ_1 and is disjoint from γ_2, and γ_2 intersects $\gamma_1(\gamma)$. We call such a curve γ_3* nice *for the triple T.*

Proof. We may suppose by symmetry that $p(\alpha_3, \alpha_1)$ comes before $p(\alpha_3, \alpha_2)$ along α_3.
Case 1: $p(\alpha_1, \alpha_2)$ comes before $p(\alpha_1, \alpha_3)$ along α_1. Then we may let $\gamma_i = \alpha_i$ for $i = 1, 2, 3$ and the lemma follows. Indeed, in this case $\gamma_1(\gamma_3)$ intersects γ_2, and if γ is a subcurve of γ_3, then $\gamma_1(\gamma_3)$ is a subcurve of $\gamma_1(\gamma)$ and so $\gamma_1(\gamma)$ intersects γ_2.

Thus, we may assume that we are in the following case.
Case 2: $p(\alpha_1, \alpha_3)$ comes before $p(\alpha_1, \alpha_2)$ along α_1. This case has two subcases.
Case 2a: Point $p(\alpha_2, \alpha_1)$ comes before $p(\alpha_2, \alpha_3)$ along α_2. Letting $\gamma_3 = \alpha_2$, $\gamma_2 = \alpha_3$, and $\gamma_1 = \alpha_1$, as in Case 1, the proposition follows.
Case 2b: Point $p(\alpha_2, \alpha_3)$ comes before $p(\alpha_2, \alpha_1)$ along α_2. Letting $\gamma_3 = \alpha_2$, $\gamma_2 = \alpha_1$, and $\gamma_1 = \alpha_3$, as in Case 1, the proposition follows. $\square$

Proposition 6.1 is needed for the proof of the next lemma.

Lemma 6.2 *If C is a collection of at most n curves with at least εn^3 triples of pairwise intersecting members, then there is a subcurve γ' of a curve $\gamma \in C$ with the same starting point as γ and disjoint subcollections A and B of C such that each curve in A intersects γ', no curve in B intersects γ', and the following holds. Letting A' denote the set of curves $\alpha(\gamma')$ with $\alpha \in A$, γ' is a ground for each curve in A' and there are more than $\frac{\varepsilon^2}{2} n^2$ pairs in $A' \times B$ that intersect.*

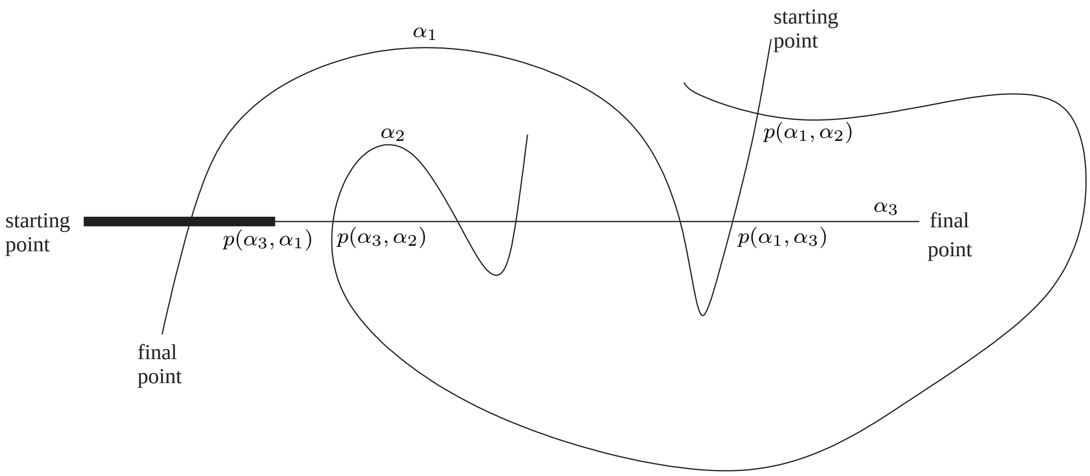

Figure 5: Three curves $\alpha_1, \alpha_2, \alpha_3$ that pairwise intersect such that $p(\alpha_3, \alpha_1)$ comes before $p(\alpha_3, \alpha_2)$ along α_3 and $p(\alpha_1, \alpha_2)$ comes before $p(\alpha_1, \alpha_3)$ along α_1. Letting $\gamma_i = \alpha_i$ for $i = 1, 2, 3$, and γ be the bold subcurve of γ_3 whose final point lies between $p(\alpha_3, \alpha_1)$ and $p(\alpha_3, \alpha_2)$, we have that γ intersects γ_1 and is disjoint from γ_2, and the subcurve $\gamma_1(\gamma)$ of γ_1 intersects γ_2.

Proof. By Lemma 6.1, in each triple of curves that pairwise intersect, there is a curve that is nice for that triple. By averaging, there is a curve $\gamma \in C$ that is nice for at least $\varepsilon n^3/|C| \geq \varepsilon n^2$ triples. A triple T of curves is *helpful* if $\gamma \in T$, the curves in T pairwise intersect, and γ is nice for T. So the number of helpful triples is at least εn^2. Let α_i denote the i^{th} curve that intersects γ along γ, breaking ties arbitrarily. The number of quadruples $\{\alpha_i, \alpha_k, \alpha_j, \gamma\}$ with $i \leq k < j$ and $T = \{\alpha_i, \alpha_j, \gamma\}$ a helpful triple is the sum of $j - i$ over all helpful triples $T = \{\alpha_i, \alpha_j, \gamma\}$. Let $f(d)$ denote the number of helpful triples $T = \{\alpha_i, \alpha_j, \gamma\}$ with $d = j - i$. The number of helpful triples is $\sum_d f(d)$, which is at least εn^2. We also have $f(d) \leq n$ for each d. Hence, the number of quadruples $\{\alpha_i, \alpha_k, \alpha_j, \gamma\}$ with $i \leq k < j$ and $T = \{\alpha_i, \alpha_j, \gamma\}$ a helpful triple is $\sum_d df(d) \geq n\left(1 + 2 + \cdots + \frac{\varepsilon n^2}{n}\right) > \frac{\varepsilon^2 n^3}{2}$. By averaging, there is a value of k such that there are more than $\frac{\varepsilon^2 n^3/2}{n} = \frac{\varepsilon^2}{2} n^2$ helpful triples $T = \{\alpha_i, \alpha_j, \gamma\}$ such that $p(\gamma, \alpha_i)$ is at most $p(\gamma, \alpha_k)$ along γ and $p(\gamma, \alpha_k)$ comes before $p(\gamma, \alpha_j)$ along γ. Let γ' denote the subcurve of γ with the same starting point as γ and final point $p(\gamma, \alpha_k)$. Let A be those curves α_h that intersect γ with $h \leq k$. Let B be those curves α_h that intersect γ with $h > k$. By construction, every curve in A intersects γ' and no curve in B intersects γ'. Also, γ' is a ground for A', and more than $\frac{\varepsilon^2}{2} n^2$ pairs of curves in $A' \times B$ intersect. □

We apply the last lemma to a grounded collection of curves, in the second case of the next lemma.

Lemma 6.3 *Let C be a grounded collection of at most n curves with ground γ, for which there are εn^2 pairs of curves in C that intersect with $\varepsilon \geq 1/n$. There is a subcurve α' of a curve $\alpha \in C$ with the same starting point on γ as α, a subcollection A of C, a strongly double-grounded collection A' of subcurves of A with grounds (γ, α'), and a subcollection $B \subset C$ of curves disjoint from α' such that there are at least $\frac{\varepsilon^4}{72} n^2$ pairs of curves in $A' \times B$ that intersect.*

Proof. We may assume that the starting point of each curve in C is its endpoint on γ. Let X denote the set of ordered triples $(\gamma_1, \gamma_2, \gamma_3)$ of distinct curves in C with γ_3 intersecting γ_1 and γ_2 and $p(\gamma_3, \gamma_1)$ coming before $p(\gamma_3, \gamma_2)$ along γ_3. For each curve $\gamma_3 \in C$, let $C(\gamma_3)$ denote the collection of curves in C that intersect γ_3. So $\sum_{\gamma_3 \in C} |C(\gamma_3)| = 2\varepsilon n^2$. By convexity of the function $f(x) = x^2$, we have

$$\begin{aligned} |X| &= \sum_{\gamma_3 \in C} \binom{|C(\gamma_3)|}{2} = \frac{1}{2} \sum_{\gamma_3 \in C} |C(\gamma_3)|^2 - |C(\gamma_3)| = -\varepsilon n^2 + \frac{1}{2} \sum_{\gamma_3 \in C} |C(\gamma_3)|^2 \\ &\geq -\varepsilon n^2 + \frac{|C|}{2} \left(\sum_{\gamma_3 \in C} |C(\gamma_3)|/|C| \right)^2 = -\varepsilon n^2 + \frac{1}{2|C|}(2\varepsilon n^2)^2 \geq \varepsilon^2 n^3. \end{aligned}$$

Let X_1 denote the collection of ordered triples $(\gamma_1, \gamma_2, \gamma_3)$ in X for which γ_2 is disjoint from γ_1, and let $X_2 = X \setminus X_1$. The proof splits into two cases depending on which of the sets X_1 and X_2 is larger.

Case 1: $|X_1| \geq |X_2|$. In this case we have $|X_1| \geq |X|/2 \geq \frac{\varepsilon^2}{2} n^3$. By averaging, there is a curve α that is the second coordinate for at least $|X_1|/|C| \geq \frac{\varepsilon^2}{2} n^2$ triples in X_1. Let A denote the collection of those curves in C that intersect α, and let A' be the collection of curves $\kappa(\alpha)$ with $\kappa \in A$. So A' is strongly double grounded with grounds (γ, α). Let B be the subcollection of curves in C that are disjoint from α. Each triple $(\beta, \alpha, \kappa) \in X_1$ with second coordinate α satisfies $\kappa \in A$, $\beta \in B$, and $\kappa(\alpha)$ intersects β. Hence, there are at least $\frac{\varepsilon^2}{2} n^2$ pairs in $A' \times B$ that intersect.

Case 2: $|X_2| > |X_1|$. In this case we have $|X_2| > |X|/2 \geq \frac{\varepsilon^2}{2} n^3$. So there are at least $\frac{\varepsilon^2}{6} n^3$ unordered triples of pairwise intersecting curves in C. By Lemma 6.2, there is a subcurve α' of a curve $\alpha \in C$ with the same starting point as α (which is on γ) and there are disjoint subcollections A and B of C for which each curve in A intersects α' and no curve in B intersects α', and letting A' denote the set of curves $\kappa(\alpha')$ with $\kappa \in A$, the collection A' is strongly double-grounded with grounds (γ, α') and there are at least $(\frac{\varepsilon^2}{6})^2 n^2/2 = \frac{\varepsilon^4}{72} n^2$ pairs in $A' \times B$ that intersect. $\square$

Lemma 6.4 *Let A be a strongly double-grounded collection of at most n curves with grounds (γ, α). Let B be a grounded collection of at most n curves also with ground γ such that every curve in B is disjoint from α. If the number of pairs in A that intersect is at most $\frac{\varepsilon^2}{8} n^2$ and the number of pairs of curves in $A \times B$ that intersect is εn^2 with $\varepsilon \geq 2/n$, then there is a curve $\kappa \in A$ satisfying the following property. Letting A' denote the set of curves in A that are disjoint from κ, and letting B' denote the set of subcurves $\beta(\kappa)$ for which $\beta \in B$ intersects κ, B' is strongly double-grounded with grounds (γ, κ) and there are at least $\frac{\varepsilon^2}{8} n^2$ pairs of curves in $A' \times B'$ that intersect.*

Proof. Let X denote the set of ordered triples $(\beta, \alpha_1, \alpha_2) \in B \times A \times A$ with α_1 and α_2 disjoint, β intersecting α_1 and α_2, and $p(\beta, \alpha_1)$ coming before $p(\beta, \alpha_2)$ along the curve β. For each curve $\beta \in B$, let $A(\beta)$ denote the collection of curves in A that intersect β. So $\sum_{\beta \in B} |A(\beta)| = \varepsilon n^2$. Since there are fewer than $\frac{\varepsilon^2}{8} n^2$ pairs of curves in A that intersect, then for a given β, there are at least $\binom{|A(\beta)|}{2} - \frac{\varepsilon^2}{8} n^2$ ordered triples $(\beta, \alpha_1, \alpha_2) \in B \times A \times A$ with α_1 and α_2 disjoint, β intersecting α_1 and α_2, and $p(\beta, \alpha_1)$

coming before $p(\beta, \alpha_2)$ along the curve β. By convexity of the function $f(x) = x^2$, we have

$$\sum_{\beta \in B} \binom{|A(\beta)|}{2} = -\frac{\varepsilon}{2}n^2 + \frac{1}{2}\sum_{\beta \in B} |A(\beta)|^2 \geq -\frac{\varepsilon}{2}n^2 + \frac{|B|}{2}\left(\sum_{\beta \in B} |A(\beta)|/|B|\right)^2 = -\frac{\varepsilon}{2}n^2 + \frac{1}{2|B|}\left(\varepsilon n^2\right)^2 \geq \frac{\varepsilon^2}{4}n^3.$$

Hence,

$$|X| \geq \frac{\varepsilon^2}{4}n^3 - |B|\frac{\varepsilon^2}{8}n^2 \geq \frac{\varepsilon^2}{8}n^3.$$

So there is a curve κ that is the second member in at least $\frac{\varepsilon^2}{8}n^3/|A| \geq \frac{\varepsilon^2}{8}n^2$ ordered triples in X. It follows that $A' \times B'$ has at least $\frac{\varepsilon^2}{8}n^2$ intersecting pairs. □

The next lemma is very similar to Lemma 5.3.

Lemma 6.5 *Suppose $A \cup \{\kappa\}$ is a strongly double-grounded collection of curves with grounds (γ, α) and κ is disjoint from every curve in A. Suppose B is a strongly double-grounded collection of curves with grounds (γ, κ). Then $A \cup B$ is the disjoint union of two collections of split curves such that every curve in the first split collection is disjoint from every curve in the second split collection.*

Proof. By the Jordan curve theorem, $\gamma \cup \alpha \cup \kappa$ partitions the plane into two regions, an inside region I and an outside region O. Each curve in $A \cup B$ either lies entirely in the closure $\bar{I}$ of I or the closure $\bar{O}$ of O. A curve β in $A \cup B$ that lies entirely in $\bar{I}$ is an *inside curve*, otherwise β is an *outside curve.* Since our curves are in general position, every outside curve is disjoint from every inside curve. By tracing along the outside, we see that the collection of outside curves is split. Similarly tracing along the inside, we see that the collection of inside curves is split. □

Putting together the last three lemmas, we obtain the main result of this section, according to which Theorem 1.1 is true for grounded collections.

Theorem 6.6 *Suppose C is a grounded collection of at most n curves whose intersection graph has at least εn^2 edges. Then for each $\gamma \in C$, there is a subcurve γ' of γ such that the intersection graph of the collection $\{\gamma' : \gamma \in C\}$ is an incomparability graph with at least $2^{-16}\varepsilon^8 n^2$ edges.*

Proof. By the remark at the end of Section 4, we may assume that $2^{-16}\varepsilon^8 n^2 > 1$ and hence $\varepsilon > 4n^{-1/4}$.

Let ψ be a ground for C. By Lemma 6.3, we can find a curve $\alpha \in C$ that has a subcurve α' with the same starting point on C as α, a subcollection A of C, a strongly double-grounded collection A' of subcurves of A with grounds (ψ, α'), and a subcollection $B \subset C$ consisting of curves disjoint from α' such that there are at least $\varepsilon_1 n^2$ intersecting pairs in $A' \times B$, where $\varepsilon_1 = \frac{\varepsilon^4}{72}$. Note that $\varepsilon_1 > \frac{(4n^{-1/4})^4}{72} = \frac{32}{9n} > \frac{2}{n}$.

The intersection graph of A' is split by Proposition 4.1. Therefore, we may assume that the intersection graph of A' has fewer than $2^{-16}\varepsilon^8 n^2 < \frac{\varepsilon_1^2}{8}$ edges, otherwise we are done. By Lemma 6.4, there is a curve $\kappa \in A'$ for which the following holds. Let A'' denote the family of all curves in A' that are disjoint from κ, and let B'' denote the family of subcurves $\beta(\kappa)$ with $\beta \in B'$ intersecting κ. The family B'' is strongly double-grounded with grounds (ψ, κ), and there are at least $\varepsilon_2 n^2$ pairs of curves in $A'' \times B''$ that intersect, where $\varepsilon_2 = \frac{\varepsilon_1^2}{8} = \frac{1}{8}\left(\frac{\varepsilon^4}{72}\right)^2 > 2^{-16}\varepsilon^8$. Lemma 6.5 implies that $A'' \cup B''$ is the disjoint union of two split collections such that every curve in the first split collection is disjoint from every curve in the second split collection. Since the intersection graph of every split collection of curves is an incomparability graph (Proposition 4.2), and the disjoint union of two incomparability graphs is an incomparability graph as remarked at the end of Section 4, then the intersection graph of $A'' \cup B''$ is an incomparability graph. This completes the proof. $\square$

7 Proof of Theorem 1.1 for nondegenerate collections

In this section, after proving an auxiliary lemma, we prove Theorem 1.1 for nondegenerate collections of curves whose intersection graph has many triangles (Theorem 7.2). Recall that in Section 5 we proved Theorem 1.1 for nondegenerate collections of curves whose intersection graph have few triangles. At the end of this section, we put these two results together to establish a quantitative version of Theorem 1.1 for nondegenerate collections of curves (Theorem 7.3). In the next section, we discuss how the proof can be modified to handle degenerate collections of curves.

Lemma 7.1 *Let n be a positive integer and $\varepsilon \geq \frac{4}{n}$. Let A be a grounded collection of at most n curves with ground γ. Let B be a collection of at most n curves that are disjoint from γ such that there are εn^2 pairs of intersecting curves in $A \times B$. Then the number of intersecting pairs of curves in A is at least $\frac{\varepsilon^2}{4}n^2$, or we can find a subcollection $A' \subset A$ and a collection B' of subcurves of curves of B such that $A' \cup B'$ is grounded and there are at least $\frac{\varepsilon^2}{8}n^2$ intersecting pairs of curves in $A' \times B'$.*

Proof. For each curve $\beta \in B$, let $d(\beta)$ be the number of curves in A that intersect β, so that we have $\sum_{\beta \in B} d(\beta) = \varepsilon n^2$. Let β_i denote the i^{th} curve in A that intersects β along β. The number of pairs of curves (β_i, β_j) in A that intersect β with $i < j$ is $\binom{d(\beta)}{2}$. Call a triple $(\beta_i, \beta_j, \beta) \in A \times A \times B$ with $i < j$ *great* if β_i is disjoint from β_j.

We may assume that the number of intersecting pairs of curves in A is less than $\frac{\varepsilon^2}{4}n^2$, so the number of great triples is at least

$$\begin{aligned}\sum_{\beta \in B}\binom{d(\beta)}{2} - \frac{\varepsilon^2}{4}n^2 &= -|B|\frac{\varepsilon^2}{4}n^2 + \frac{1}{2}\sum_{\beta \in B} d(\beta)^2 - d(\beta) \geq -\frac{\varepsilon^2}{4}n^3 - \frac{\varepsilon}{2}n^2 + \frac{1}{2}|B|(\sum_{\beta \in B} d(\beta)/|B|)^2 \\ &= -\frac{\varepsilon^2}{4}n^3 - \frac{\varepsilon}{2}n^2 + \frac{1}{2|B|}(\varepsilon n^2)^2 \geq \frac{\varepsilon^2}{8}n^3.\end{aligned}$$

Here we used the convexity of the function $f(x) = x^2$.

Hence, there is a curve $\alpha \in A$ that is represented at least $\frac{\varepsilon^2}{8}n^3/|A| \geq \frac{\varepsilon^2}{8}n^2$ times as the second member of a great triple. Let B' denote the collection of subcurves $\beta(\alpha)$ with $\beta \in B$ intersecting α. Let A' denote the collection of curves in A that are disjoint from α. By construction, there are at least $\frac{\varepsilon^2}{8}n^2$ intersecting pairs of curves in $A' \times B'$. Tracing around $\gamma \cup \alpha$, we see that $A' \cup B'$ is a grounded collection of curves with at least $\frac{\varepsilon^2}{8}n^2$ intersecting pairs. $\square$

Theorem 7.2 *Let C be a collection of curves in the plane such that the intersection graph of C has at least εn^3 triangles. Then for each $\gamma \in C$, there is a subcurve γ' of γ such that the intersection graph of the collection $\{\gamma' : \gamma \in C\}$ is an incomparability graph with at least $2^{-56}\varepsilon^{32}n^2$ edges.*

Proof. By the remark at the end of Section 4, we may assume that $2^{-56}\varepsilon^{32}n^2 > 1$ and hence $\varepsilon > 2^{7/4}n^{-1/16}$.

By Lemma 6.2, there is a subcurve κ' of a curve $\kappa \in C$ with the same starting point as κ and disjoint subcollections A and B of C such that each curve in A intersects κ', no curve in B intersects κ', and the following holds. Letting A' denote the set of curves $\alpha(\kappa')$ with $\alpha \in A$, the collection A' is grounded with ground κ' and there are at least $\varepsilon_1 n^2$ pairs in $A' \times B$ that intersect, where $\varepsilon_1 = \frac{\varepsilon^2}{2}$. Note that $\varepsilon_1 = \frac{\varepsilon^2}{2} > 4/n$.

By Lemma 7.1, there are $\frac{\varepsilon_1^2}{4}n^2$ intersecting pairs of curves in A', or there is a subcollection A'' of A' and a collection B'' of subcurves of curves of B' such that $A'' \cup B''$ is grounded and has at least $\frac{\varepsilon_1^2}{8}n^2$ intersecting pairs of curves. In either case, we have a grounded collection of subcurves with at least $\frac{\varepsilon_1^2}{8}n^2$ intersecting pairs of curves. By Theorem 6.6, for each curve γ in this grounded collection, there is a subcurve γ' of γ such that the intersection graph of the collection $\{\gamma' : \gamma \in C\}$ is an incomparability graph with at least

$$2^{-16}(\varepsilon_1^2/8)^8 n^2 = 2^{-40}\varepsilon_1^{16}n^2 = 2^{-40}(\varepsilon^2/2)^{16}n^2 = 2^{-56}\varepsilon^{32}n^2$$

edges. □

We can now establish Theorem 1.1 for nondegenerate collections of curves.

Theorem 7.3 *Let C be a collection of curves such that the intersection graph of C has εn^2 edges. Then for each $\gamma \in C$, there is a subcurve γ' of γ such that the intersection graph of the collection $\{\gamma' : \gamma \in C\}$ is an incomparability graph with at least $2^{-1212}\varepsilon^{288}n^2$ edges.*

Proof. The proof splits into two cases depending on the number of triangles in the intersection graph of C. If the intersection graph of C has fewer than $2^{-36}\varepsilon^9 n^3$ triangles, then we are done by Theorem 5.4. If the intersection graph of C has at least $2^{-36}\varepsilon^9 n^3$ triangles, then we are done by Theorem 7.2, noting that $2^{-56}\left(2^{-36}\varepsilon^9\right)^{32} - 2^{-1212}\varepsilon^{288}$. □

8 Proof of Theorem 1.1 for degenerate collections

So far we have made the assumption that the curves we consider are in general position, i.e., no point belongs to *three* or more curves. As we have indicated before, this assumption is not essential for the proof of Theorem 1.1, and it was made only for the clarity of the presentation. In this section, we discuss how the proof can be modified to handle collections of curves that are not necessarily in general position.

For any curve γ and any interior point p of γ, every sufficiently small neighborhood of p is partitioned into two regions by γ. If α is a curve that contains p as an endpoint and is otherwise disjoint

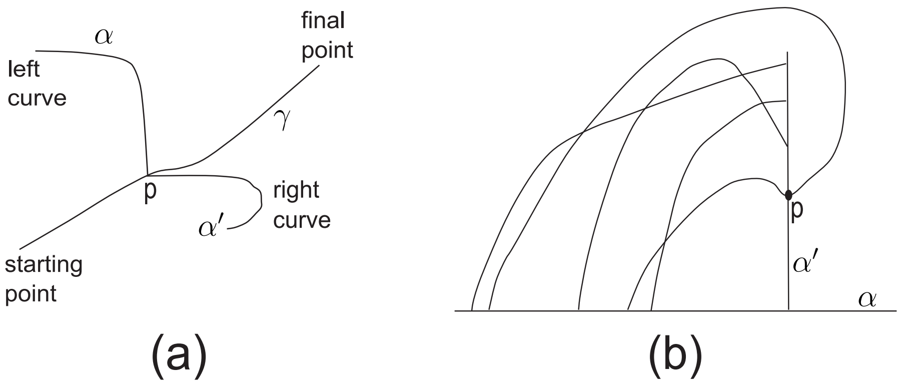

Figure 6: (a) Left curve α and right curve α' of ground γ at two-sided point p. (b) A degenerate collection of five curves with two-sided point p that is strongly double-grounded with grounds (α,α') whose intersection graph is a cycle with five vertices, which is not an incomparability graph.

from γ, α is either a *left curve* of γ at p or a *right curve* of γ at p, depending on which side of γ the curve α intersects. See Fig. 6(a). For a grounded collection C of curves with ground γ, an interior point p of γ is called *two-sided* with respect to C if there are $\alpha, \alpha' \in C$ such that α is a left curve of p and α' is a right curve of p.

We now discuss how to prove Theorem 1.1 in the case where the collection of curves is not necessarily in general position. Let C be a collection of n curves with εn^2 intersecting pairs. If there is a point p in the plane which belongs to at least $\delta_1 n$ members of C, where $\delta_1 = \varepsilon^c$ and c is a sufficiently large constant, then we may simply take the set consisting of the point p as the subcurve for each curve containing p and the empty set as the subcurve for each curve not containing p. In this special case, the intersection graph of these subcurves consists of a clique with a quadratic number of edges and the remaining vertices are isolated. It is easy to see that this intersection graph is an incomparability graph. Therefore, we may assume that no point p belongs to at least $\delta_1 n$ members of C. With this assumption, only one minor issue arises when trying to use the same argument as in the case where the curves were in general position. The problem is that Proposition 4.1, which says that a collection of curves is double-grounded with a separator if and only if it is strongly double-grounded if and only if it is split, does not hold in this case.

While the intersection graph of every split collection of curves is an incomparability graph, the intersection graph of a degenerate, strongly-double grounded collection of curves is not necessarily an incomparability graph. Indeed, Figure 6(b) gives an example of such a collection of curves whose intersection graph is the cycle on *five* vertices, which is not an incomparability graph. Proposition 4.1 is not true if we allow for there to be a point p of one of the two grounds which is two-sided with respect to the collection of curves which is either strongly-double grounded or double-grounded with a separator. If we try to repeat the argument as we did for non-degenerate collections, as we trace along the union of the two grounds of a strongly double grounded collection or a double-grounded collection with a separator to make a curve α which verifies the collection is split, the curve α will have to touch the two-sided point p twice, so that α is not a simple curve. The same problem arises in Lemmas 5.3

and 6.5, when claiming that there are no intersecting pairs of curves between the two split collections (the inside collection and the outside collection) of curves.

To handle these issues, we can use essentially the same proof of Theorem 1.1 as we did for collections of curves in general position, except that whenever we obtain a grounded collection of curves, we have to use the following lemma to find a grounded subcollection of curves with no two-sided points and we can still retain a constant fraction of the intersecting pairs.

Lemma 8.1 *Let A be a grounded collection of at most n curves with ground α, and let B be a (possibly empty) collection of curves disjoint from α such that $A \times (A \cup B)$ has m intersecting unordered pairs of distinct curves and no point of α is contained in more than m/n elements of A. Then there exists a subcollection $A' \subset A$ such that no interior point of α is two-sided with respect to A' and at least $m/8$ unordered pairs of distinct curves in $A' \times (A' \cup B)$ intersect.*

Proof. By convexity and by the fact that no point of α is contained in more than m/n elements of A, we obtain that the number of pairs of distinct curves in A that share an endpoint on α is at most $\binom{m/n}{2}n^2/m < m/2$. Thus, there are at least $m/2$ unordered pairs of distinct curves in $A \times (A \cup B)$ that intersect and do not share an endpoint on α. For each interior point p of α that is two-sided with respect to A, either we keep all left curves of p in A or all right curves of p in A, each with equal probability. All other curves we keep. The curves in A we keep make up A'. For each pair of distinct curves in A that do not share an endpoint on α, the probability that they are both kept in A' is at least $1/4$. Also, the probability that a pair of curves in $A \times B$ is in $A' \times B$ is at least $1/2$. Hence, by linearity of expectation, there is a choice of A' for which the number of unordered pairs of distinct curves in $A' \times (A' \cup B)$ that intersect is at least $\frac{1}{4}m/2 = m/8$. $\square$

9 Concluding remarks

In this paper we established Theorem 1.1, demonstrating a surprisingly close relationship between string graphs and incomparability graphs. In this final section, we discuss a variant of Theorem 1.1, the proof of which can be obtained by a straightforward modification of the original argument and is therefore left to the reader.

A collection of curves is *k-intersecting* if each pair of curves in the collection intersect in at most k points and every crossing is proper. The *x-monotone crossing dimension* xcr-dim(P) of a poset P is the minimum k such that there is a realization of P as a k-intersecting collection of curves of functions defined on the interval $[0, 1]$ (as in Section 2). The *crossing dimension* cr-dim(P) of a poset P is the minimum k such that there is a realization of P as a k-intersecting collection of curves such that each curve in P lies in the closed vertical strip $[0, 1] \times \mathbb{R}$, has one endpoint on the line $x = 0$, the other endpoint on the line $x = 1$, and the rest of the curve lies in the open vertical strip $(0, 1) \times \mathbb{R}$. By definition, we have cr-dim$(P) \leq$ xcr-dim(P) for every poset P.

It follows from the proof of Proposition 2.1 that xcr-dim$(P) \leq \dim(P) - 1$. It is easy to show that cr-dim$(P) =$ xcr-dim$(P) = \dim(P) - 1$ if $\dim(P) = 1$ or 2. However, the standard example (see [18]) demonstrates for $n \geq 3$ that there is a poset on $2n$ elements with dimension n and x-monotone crossing dimension only 2. The proof of Theorem 1.1 can easily be modified to prove the following theorem.

Theorem 9.1 *For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If C is a k-intersecting collection of curves whose string graph has at least $\varepsilon|C|^2$ edges, then one can select a subcurve γ' of each $\gamma \in C$ such that the string graph of the collection $\{\gamma' : \gamma \in C\}$ has at least $\delta|C|^2$ edges and is an incomparability graph with crossing dimension at most k.*

It follows that every dense string graph of a k-intersecting collection of curves has a dense spanning subgraph (with a different ε), which is an incomparability graph with crossing dimension at most k.

Since cr-dim$(P) = 1$ if and only if $\dim(P) = 2$, we have the following corollary. A 1-intersecting collection of curves is commonly known as an *an arrangement of pseudosegments.*

Corollary 9.2 *For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If C is a collection of pseudosegments whose string graph has at least $\varepsilon|C|^2$ edges, then one can select a subcurve γ' of each $\gamma \in C$ such that the string graph of the collection $\{\gamma' : \gamma \in C\}$ has at least $\delta|C|^2$ edges and is the incomparability graph of a 2-dimensional poset.*

By the proof of Proposition 2.1, the incomparability graph of a 2-dimensional poset is the intersection graph of a collection of segments that have one endpoint on the line $x = 0$ and the other endpoint on the line $x = 1$. Hence, for every collection of pseudosegments whose intersection graph is dense, we can pick a subcurve of each pseudosegment such that the intersection graph of the resulting collection of subcurves is a dense intersection graph of segments.

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