A space-time variational approach to hydrodynamic stability theory

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We present a hydrodynamic stability theory for incompressible viscous fluid flows based on a space-time variational formulation and associated generalized singular value decomposition of the (linearized) Navier-Stokes equations. We first introduce a linear framework applicable to a wide variety of stationary or time-dependent base flows: we consider arbitrary disturbances in both the initial condition and the dynamics measured in a "data" space-time norm; the theory provides a rigorous, sharp (realizable), and efficiently computed bound for the velocity perturbation measured in a "solution" space-time norm. We next present a generalization of the linear framework in which the disturbances and perturbation are now measured in respective selected space-time *semi-norms*; the semi-norm theory permits rigorous and sharp quantification of, for example, the growth of initial disturbances or functional outputs. We then develop a (Brezzi-Rappaz-Raviart) nonlinear theory which provides, for disturbances which satisfy a certain (rather stringent) amplitude condition, rigorous finite-amplitude bounds for the velocity and output perturbations. Finally, we demonstrate the application of our linear and nonlinear hydrodynamic stability theory to unsteady moderate Reynolds-number flow in an eddy-promoter channel.

Key words: Hydrodynamic stability; nonmodal analysis; space-time variational analysis; singular-value decomposition; functional outputs; Brezzi-Rappaz-Raviart theory.

1. Introduction

The field of hydrodynamic stability theory [11] is vast with many applications from engineering to meteorology to astrophysics. Most early work of both an analytical and computational nature focused on modal analysis relevant to long-time behavior. More recently, the emphasis has turned to nonmodal analysis of finite-time stability as introduced by Butler and Farrell [9], Reddy and Henningson [30], and Trefethen *et al.* [36, 35] (see also a review by Schmid [32] and references therein). The finite-time framework can be extended to consider a variety of base flows as well as disturbances in the initial condition *and* the dynamics.

The hydrodynamic stability theory presented in this work addresses the same goals as nonmodal stability theory. However, we provide a new formulation which yields certain earlier results more easily and also admits extension in several important directions. The foundation for our computational framework is a variational formulation and associated error estimation theory for approximation

of partial differential equations. In particular, the framework is inspired by the space-time variational *a posteriori* error analysis recently introduced by Schwab and Stevenson [34] for wavelet methods and subsequently applied to finite element and reduced basis discretizations of linear parabolic equations [37], the Burgers' equation [41], and the Boussinesq equations [40]. (For previous applications of variational frameworks to hydrodynamic theory, see for example Joseph [20] and Johnson *et al.* [19].)

The linear theory developed in Section 3 provides a sharp bound for a velocity perturbation about a given base flow subject to arbitrary disturbances in both the initial condition and the dynamics over a finite time interval. The "solution" (respectively, "data") norms for the perturbation (respectively, disturbance) are the norms with respect to which the (linearized) partial differential equation is well posed and hence as strong (respectively, weak) as possible. Moreover, our global perturbation bound is *sharp* in the sense that there exists a disturbance — provided by the theory — for which the perturbation bound is achieved.

In the *absence* of disturbances to the dynamics, our method is equivalent to finite-time stability analysis based on the singular-vector structure of the linear tangent propagator as first proposed by Lorenz [24] and employed, for example, in the context of weather prediction by Buizza *et al.* [8, 7] and in the context of hydrodynamic stability by Schmid and Kytömaa [33], Barkley *et al.* [4], and Abdessemed *et al.* [1]; these singular vectors are related to the (norm-dependent) Lyapunov vectors in the infinite-time limit [23]. (For review of Lyapunov vectors, we refer to classical work by Eckmann and Ruelle [12] and recent work by Kuptsov and Parlitz [22].)

In the *presence* of disturbances to the dynamics — in which the linear tangent propagator is no longer exact — earlier analyses considered time-*independent* base flows subject to harmonic excitation [36] or temporally-white stochastic noise [14], and time-*harmonic* base flows subject to impulsive noise [16]. We generalize these results in two important ways: we permit arbitrary (well-posed) disturbances; we permit arbitrary time-*dependent* base flows (in which, for example, resonances may occur [5]). Note furthermore that we do not prescribe either the spatial or the temporal shape of the disturbances but rather deduce the most-sensitive space-time disturbance from our infimization principle.

In Section 4, we generalize the linear stability bounds to the case in which we measure the perturbation and disturbance in respective *semi*-norms. We first develop an abstract framework which provides sharp perturbation bounds for the semi-norm pair. As a first example of the semi-norm framework, we reproduce the classical initial-to-final perturbation growth analysis based on the singular value decomposition of the linear tangent propagator. As a second example, we quantify the sensitivity of a scalar output — defined by a functional of the field, which we recast as rank-one semi-norm — with respect to arbitrary disturbances in the initial condition and the dynamics: we provide conditions under which outputs can be quite insensitive to disturbances; we also discuss the connection between the semi-norm treatment of outputs and the more standard adjoint formulation of outputs — and the potential advantage of the former in the case of multiple outputs. As a third example of the semi-norm framework, we consider an application to optimal flow control. For related discussion of semi-norm-based hydrodynamic stability we refer to recent work by Foures *et al.* [15].

The quantification of the uncertainty in outputs is relevant in many settings. One example is the experimental setting, in which the analysis precisely quantifies the stability of a given measurement to perturbations; this information may then serve to choose an experimental protocol which is less sensitive to undesirable noise. Another example is the design setting, in which the analysis provides an assessment of potential degradation of performance in the presence of noise (and the development of mitigation strategies, i.e., robust design).

In Section 5, we turn our attention to the development of a *nonlinear* (finite-amplitude) perturbation theory. The additional critical ingredient of the nonlinear theory is the Brezzi-Rappaz-Raviart theory developed for nonlinear *a posteriori* error estimation of variational discretizations [6]. We exploit this theory here to provide a *rigorous* velocity perturbation bound for the full Navier-Stokes equations under a certain (unfortunately, rather stringent) condition on the magnitude of the disturbance. We also develop associated nonlinear output bounds.

Finally, in Section 6, we apply the space-time hydrodynamic stability theory to an eddy-promoter channel considered by Karniadakis *et al.* [21]. The numerical results demonstrate the capabilities — as well as the limitations — of the proposed linear and nonlinear theory to characterize hydrodynamic stability of time-dependent flows.

2. Governing equations

(*a*) *Model problem: eddy-promoter channel*

We consider flow through a planar periodic channel equipped with eddy promoters — cylindrical obstacles designed to increase mixing in the channel. This flow has been extensively studied by Karniadakis *et al.* [21]. The geometry of the channel is described in figure 1: the channel is characterized by a half-height $\tilde{h}$, cylinder separation length $\tilde{L}$, cylinder diameter $\tilde{d}$, and cylinder-center height $\tilde{b}$; in this section, $\tilde{(\cdot)}$ denotes a dimensional quantity. The flow is governed by the incompressible Navier-Stokes equations and is driven by a prescribed fixed *pressure-gradient* of magnitude $\tilde{\bar{p}}_x$; note that Karniadakis *et al.* consider a prescribed fixed *flowrate*. No-slip boundary conditions are imposed along the top (Γ_4) and bottom (Γ_3) walls, and periodic boundary conditions are imposed on Γ_1-Γ_2. (We do *not* aim to analyze the spatial growth of perturbations [18], as considered by Schatz *et al.* [31].)

We introduce the following characteristic time, length, velocity, and pressure scales for normalization: $\tilde{h}^2/\tilde{\nu}$, $\tilde{h}$, $\tilde{u}_c^{\text{Poiseuille}} \equiv (-\tilde{\bar{p}}_x \tilde{h}^2)/(2\tilde{\rho}\tilde{\nu})$, and $-\tilde{\bar{p}}_x \tilde{h}$, respectively; here, $\tilde{\nu}$ is the kinematic viscosity, and $\tilde{\rho}$ is the density. With this nondimensionalization, the geometry of the channel is described in terms of $h \equiv \tilde{h}/\tilde{h} = 1$, $L \equiv \tilde{L}/\tilde{h}$, $d \equiv \tilde{d}/\tilde{h}$, and $b \equiv \tilde{b}/\tilde{h}$. Throughout the rest of this work, we denote the nondimensionalized spatial domain of interest by Ω and the nondimensionalized time interval of interest by $I \equiv (0, T]$. Our nondimensionalization differs from that of Karniadakis *et al.* [21] in two regards: our time scale is based on diffusion (rather than convection); our velocity scale is based on the prescribed pressure gradient (rather than the prescribed flowrate).
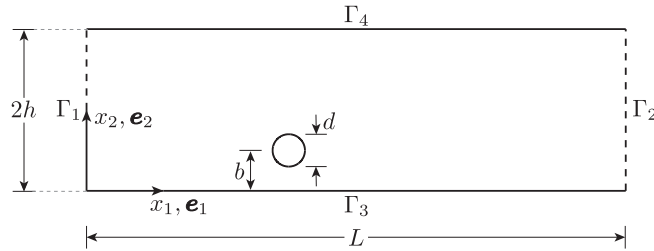
Figure 1. A single periodic section of the eddy-promoter channel with a cylinder separation $L = 6.666$, channel half-height $h = 1.0$, cylinder diameter $d = 0.4$, and cylinder-center height $b = 0.5$.

Consequently, our pressure-gradient-based Reynolds number is given by $Re \equiv \tilde{u}_c^{\text{Poiseuille}} \tilde{h}/\tilde{\nu}$.

(b) Governing equations: strong form

We first present the incompressible Navier-Stokes equations in a familiar strong form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + Re((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}) = -2\boldsymbol{\nabla}p - 2\boldsymbol{e}_1 + \nabla^2 \boldsymbol{u} \quad \text{in } \Omega \text{ for } t \in I \equiv (0, T] , \quad (2.1)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \text{ for } t \in (0, T] , \quad (2.2)$$

where $\boldsymbol{u}$ is the nondimensional velocity, p is the nondimensional (periodic part of) pressure, and $2\boldsymbol{e}_1$ (for $\boldsymbol{e}_1$ the unit vector in the first coordinate direction (x_1)) is the nondimensional prescribed mean pressure gradient. The associated boundary conditions are

$$\boldsymbol{u}(\boldsymbol{x}, t) = 0 \quad \text{on } \Gamma_3 \text{ and } \Gamma_4 ,$$

$$\boldsymbol{u}(\boldsymbol{x} + L\boldsymbol{e}_1, t) = \boldsymbol{u}(\boldsymbol{x}, t) \quad \text{on } \Gamma_1 ,$$

$$\frac{\partial \boldsymbol{u}}{\partial x_1}(\boldsymbol{x} + L\boldsymbol{e}_1, t) = \frac{\partial \boldsymbol{u}}{\partial x_1}(\boldsymbol{x}, t) \quad \text{on } \Gamma_1 , \quad (2.3)$$

$$p(\boldsymbol{x} + L\boldsymbol{e}_1, t) = p(\boldsymbol{x}, t) \quad \text{on } \Gamma_1 ;$$

the initial condition is

$$\boldsymbol{u}(\boldsymbol{x}, t = 0) = \boldsymbol{u}_0 \quad \text{in } \Omega , \quad (2.4)$$

where $\boldsymbol{u}_0$ represents the initial velocity field.

In addition, for the eddy-promoter problem of interest, we choose two specific outputs. The first output is the time-averaged flowrate through the channel,

$$J^1(\boldsymbol{u}) = \frac{1}{L} \int_I \int_\Omega \boldsymbol{u}_1 d\boldsymbol{x} dt ; \quad (2.5)$$

here, we take advantage of the divergence free condition in defining the flowrate output. The second output is the local x_1-velocity $1.5d$ downstream of the cylinder along the cylinder center line in the wake region; in particular, we measure the

regularized quantity

$$J^2(\boldsymbol{u}) = \int_I \int_\Omega \left(\frac{1}{2\pi\sigma} e^{-\frac{\|\boldsymbol{x}-\boldsymbol{x}_0\|^2}{2\sigma^2}} \right) \boldsymbol{u}_1 d\boldsymbol{x} dt \ , \tag{2.6}$$

with a standard deviation $\sigma = 0.1$ and center $\boldsymbol{x}_0 = (x_c + 2d, b)$ for x_c the x_1-coordinate of the cylinder center.

(c) Governing equations: space-time weak form

The solution to the unsteady Navier-Stokes equations is most generally described in a space-time variational setting. The multiplication of the strong form by a test function and the integration of the resulting equation over the spatial domain Ω *and* the temporal interval I yields a weak statement for the Navier-Stokes equations: Find the nondimensional velocity $\boldsymbol{u} \in \mathcal{X}$ such that

$$\mathcal{G}(\boldsymbol{u}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \mathcal{Y}, \tag{2.7}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are the space-time trial (or "solution") and test spaces, respectively, defined shortly in Section 2(d), and $\mathcal{G} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is the space-time semilinear form given by

$$\mathcal{G}(\boldsymbol{w}, \boldsymbol{v}) \equiv \dot{\mathcal{M}}(\boldsymbol{w}, \boldsymbol{v}) + \mathcal{C}(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) + \mathcal{A}(\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{w}(0), \boldsymbol{v}^{(2)})_{L^2(\Omega)} \tag{2.8}$$

$$- \langle [2\boldsymbol{e}_1, \boldsymbol{u}_0], \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}} \ . \tag{2.9}$$

Here the evolution term $\dot{\mathcal{M}}$, the convective term $\mathcal{C}$, and the diffusion term $\mathcal{A}$ are given by

$$\dot{\mathcal{M}}(\boldsymbol{w}, \boldsymbol{v}) = \int_I \int_\Omega \boldsymbol{v}_i^{(1)} \frac{\partial \boldsymbol{w}_i}{\partial t} d\boldsymbol{x} dt \tag{2.10}$$

$$\mathcal{C}(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{v}) = -\frac{Re}{2} \int_I \int_\Omega \frac{\partial \boldsymbol{v}_i^{(1)}}{\partial \boldsymbol{x}_j} (\boldsymbol{z}_i \boldsymbol{w}_j + \boldsymbol{z}_j \boldsymbol{w}_i) d\boldsymbol{x} dt \tag{2.11}$$

$$\mathcal{A}(\boldsymbol{w}, \boldsymbol{v}) = \int_I \int_\Omega \frac{\partial \boldsymbol{v}_i^{(1)}}{\partial \boldsymbol{x}_j} \frac{\partial \boldsymbol{w}_i}{\partial \boldsymbol{x}_j} d\boldsymbol{x} dt \ , \tag{2.12}$$

respectively, where $\boldsymbol{v} = [\boldsymbol{v}^1, \boldsymbol{v}^2]$ is a test function "couple" whose first component enforces the evolution equation and whose second component enforces the initial condition. Here and throughout the rest of this work, we shall assume summation over repeated indices, $\boldsymbol{a}_i \boldsymbol{b}_i \equiv \sum_{i=1}^n \boldsymbol{a}_i \boldsymbol{b}_i$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, unless stated otherwise. The above variational formulation, with the choice of $\mathcal{X}$ and $\mathcal{Y}$ to be clarified shortly, incorporates appropriate boundary conditions and the initial condition; periodicity of flux on Γ_1-Γ_2 (2.3) is a natural boundary condition (weakly imposed).

To facilitate the subsequent stability analysis in the space-time setting, we also introduce here the linear form associated with the Fréchet derivative of $\mathcal{G}$,

$$\partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) \equiv \dot{\mathcal{M}}(\boldsymbol{w}, \boldsymbol{v}) + 2\mathcal{C}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + \mathcal{A}(\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{w}(0), \boldsymbol{v}^{(2)})_{L^2(\Omega)} \ , \tag{2.13}$$

where $\boldsymbol{u} \in \mathcal{X}$ is the *base flow* — a solution of (2.7) — about which the equations are linearized. Note we take advantage here and later of the symmetry of $\mathcal{C}$ in the first two arguments.

In addition, to quantify output perturbations in the space-time setting, we introduce a (bounded) linear functional $\ell \in \mathcal{X}'$ of the form

$$\ell(\boldsymbol{w}) = \int_I \int_\Omega \boldsymbol{g}_i^I \boldsymbol{u}_i d\boldsymbol{x} dt + \int_\Omega \boldsymbol{g}_i^T \boldsymbol{u}_i(T) d\boldsymbol{x} \ ; \tag{2.14}$$

the superscripts I and T stand for the time interval I and the final time T, respectively. Note that both of the output functionals for the eddy-promoter channel problem, J^1 and J^2 given by (2.5) and (2.6), respectively, conform to the form given by (2.14) for $\boldsymbol{g}^T = 0$. In Section 5, we also consider quadratic output functionals.

(d) *Space-time spaces, inner products, and norms*

We present precise definitions of the spaces, inner products, and norms used throughout this work. We will adopt the standard notations used in the partial differential equation community (e.g., [29]). We note that the well-posedness of the space-time formulation of the Navier-Stokes equations has recently been placed on a firm theoretical foundation [17], albeit for spaces slightly different from those considered in this work.

The $L^2(\Omega)$ space over the domain $\Omega \in \mathbb{R}^d$ is equipped with an inner product $(w, v)_{L^2(\Omega)} \equiv \int_\Omega wv d\boldsymbol{x}$ and induced norm $\|w\|_{L^2(\Omega)} \equiv \sqrt{(w, w)_{L^2(\Omega)}}$ for scalar-valued functions $\{w : \|w\|_{L^2(\Omega)} < \infty\}$. The space $(L^2(\Omega))^d$ is equipped with an inner product $(\boldsymbol{w}, \boldsymbol{v})_{L^2(\Omega)} \equiv \int_\Omega \boldsymbol{w}_i \boldsymbol{v}_i d\boldsymbol{x}$ and induced norm $\|\boldsymbol{w}\|_{L^2(\Omega)} \equiv \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{L^2(\Omega)}}$ for vector-valued functions $\{\boldsymbol{w} : \|\boldsymbol{w}\|_{L^2(\Omega)} < \infty\}$; to avoid notational clutter, we denote the "vector" inner product (and norm) as $(\cdot, \cdot)_{L^2(\Omega)}$ instead of $(\cdot, \cdot)_{(L^2(\Omega))^d}$ (and $\| \cdot \|_{L^2(\Omega)}$ instead of $\| \cdot \|_{(L^2(\Omega))^d}$). The $H^1(\Omega)$ space is equipped with an inner product $(w, v)_{H^1(\Omega)} \equiv \int_\Omega \nabla w \cdot \nabla v d\boldsymbol{x}$ and induced norm $\|w\|_{H^1(\Omega)} = \sqrt{(w, w)_{H^1(\Omega)}}$ for scalar-valued functions $\{w : \|w\|_{H^1(\Omega)} < \infty\}$. The $(H^1(\Omega))^d$ space is equipped with an inner product $(\boldsymbol{w}, \boldsymbol{v})_{H^1(\Omega)} \equiv \int_\Omega \nabla \boldsymbol{w}_i \cdot \nabla \boldsymbol{v}_i d\boldsymbol{x}$ and induced norm $\|\boldsymbol{w}\|_{H^1(\Omega)} = \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{H^1(\Omega)}}$ for vector-valued functions $\{\boldsymbol{w} : \|\boldsymbol{w}\|_{H^1(\Omega)} < \infty\}$. The (non-divergent-free) velocity space W is equipped with an inner product $(\boldsymbol{w}, \boldsymbol{v})_W \equiv (\boldsymbol{w}, \boldsymbol{v})_{H^1(\Omega)}$ and induced norm $\|\boldsymbol{w}\|_W \equiv \|\boldsymbol{w}\|_{H^1(\Omega)}$ for functions $\{\boldsymbol{w} \in (H^1(\Omega))^d : \boldsymbol{w}|_{\Gamma_3} = \boldsymbol{w}|_{\Gamma_4} = 0, \ \boldsymbol{w}|_{\Gamma_1} = \boldsymbol{w}|_{\Gamma_2}\}$; here, $\boldsymbol{w}|_{\Gamma_3} = \boldsymbol{w}|_{\Gamma_4} = 0$ enforces the no-slip boundary conditions on Γ_3 and Γ_4, and $\boldsymbol{w}|_{\Gamma_1} = \boldsymbol{w}|_{\Gamma_2}$ enforces the periodic boundary conditions on Γ_1 and Γ_2.

We next introduce a divergence-constraint bilinear form

$$b(q, \boldsymbol{w}) = -\int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{w} d\boldsymbol{x}, \quad \forall q \in L^2(\Omega), \ \forall \boldsymbol{w} \in W \ .$$

The space V of divergence-free velocities is equipped with an inner product $(\boldsymbol{w}, \boldsymbol{v})_V \equiv (\boldsymbol{w}, \boldsymbol{v})_{H^1(\Omega)}$ and induced norm $\|\boldsymbol{w}\|_V \equiv \|\boldsymbol{w}\|_{H^1(\Omega)}$ for functions $\{\boldsymbol{v} \in W :$

$b(q, \boldsymbol{v}) = 0, \ \forall q \in L^2(\Omega)\}$. Note that the (square of the) V-norm of the velocity $\boldsymbol{u}$ is the volume integral of the viscous dissipation rate.

The dual space of V, denoted by V', is equipped with the induced norm $\|j\|_{V'} \equiv \sup_{\boldsymbol{w} \in V} j(\boldsymbol{w})/\|\boldsymbol{w}\|_V$ for (bounded) functionals $\{j : V \to \mathbb{R} | \|j\|_{V'} < \infty\}$. We may express the action of a linear functional in V' as either $j(\boldsymbol{v})$ or by the duality pairing $\langle j, \boldsymbol{v} \rangle_{V' \times V}$. By the Riesz representation theorem, $\|j\|_{V'} = \|Rj\|_V$ where the Riesz operator $R : V' \to V$ satisfies, for given $j \in V'$, $(Rj, v)_V = j(v), \ \forall v \in V$.

The solution to the (unsteady) Navier-Stokes equations is most succinctly described in the space-time setting. We introduce a space-time space $L^2(I; V)$ equipped with an inner product $(\boldsymbol{w}, \boldsymbol{v})_{L^2(I;V)} \equiv \int_I (\boldsymbol{w}(t), \boldsymbol{v}(t))_V dt$ and induced norm $\|\boldsymbol{w}\|_{L^2(I;V)} \equiv \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{L^2(I;V)}}$ for functions $\{\boldsymbol{w} : \|\boldsymbol{w}\|_{L^2(I;V)} < \infty\}$. We also introduce a space $H^1(I; V')$ equipped with an inner product $(\boldsymbol{w}(t), \boldsymbol{v}(t))_{H^1(I;V')} = \int_I (R\dot{\boldsymbol{w}}(t), R\dot{\boldsymbol{v}}(t))_V dt$ and induced norm $\|\boldsymbol{w}\|_{H^1(I;V')} \equiv \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{H^1(I;V')}}$ for functions $\{\boldsymbol{w} : \|\boldsymbol{w}\|_{H^1(I;V')} < \infty\}$; recall that $R : V' \to V$ is the Riesz operator.

We seek our velocity solution in the space-time trial space

$$\mathcal{X} \equiv H^1(I; V') \cap L^2(I; V)$$

equipped with an inner product

$$(\boldsymbol{w}, \boldsymbol{v})_{\mathcal{X}} \equiv (\boldsymbol{w}, \boldsymbol{v})_{H^1(I;V')} + (\boldsymbol{w}, \boldsymbol{v})_{L^2(I;V)} + (\boldsymbol{w}(T), \boldsymbol{v}(T))_{L^2(\Omega)}$$

and induced norm $\|\boldsymbol{w}\|_{\mathcal{X}} = \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{\mathcal{X}}}$. The (square of the) second term in the norm, $\|\boldsymbol{w}\|_{L^2(I;V)}^2$, measures the total viscous dissipation over the time interval I. The space $\mathcal{X}$ is continuously embedded in $C^0([0, T]; (L^2(\Omega))^d)$ [13, 2], and thus in particular $\boldsymbol{w}(0)$ and $\boldsymbol{w}(T)$ are meaningful in $(L^2(\Omega))^d$ for $\boldsymbol{w} \in \mathcal{X}$. The dual-space of $\mathcal{X}$, $\mathcal{X}'$, is equipped with norm $\|\ell\|_{\mathcal{X}'} \equiv \sup_{\boldsymbol{w} \in \mathcal{X}} \ell(\boldsymbol{w})/\|\boldsymbol{w}\|_{\mathcal{X}}$; we may express the action of a linear functional in $\mathcal{X}'$ as either $\ell(\boldsymbol{w})$ or by the duality pairing $\langle \ell, \boldsymbol{w} \rangle_{\mathcal{X}' \times \mathcal{X}}$. Note that $\|\ell\|_{\mathcal{X}'} = \|L\|_{\mathcal{X}}$ where $L \in \mathcal{X}$ is the Riesz representation satisfying $(L, \boldsymbol{w})_{\mathcal{X}} = \langle \ell, \boldsymbol{w} \rangle_{\mathcal{X}' \times \mathcal{X}}, \ \forall \boldsymbol{w} \in \mathcal{X}$.

We also introduce an associated test space

$$\mathcal{Y} \equiv L^2(I; V) \oplus (L^2(\Omega))^d$$

with inner product

$$(\boldsymbol{w}, \boldsymbol{v})_{\mathcal{Y}} \equiv (\boldsymbol{w}^{(1)}, \boldsymbol{v}^{(1)})_{L^2(I;V)} + (\boldsymbol{w}^{(2)}, \boldsymbol{v}^{(2)})_{L^2(\Omega)}$$

and induced norm $\|\boldsymbol{w}\|_{\mathcal{Y}} = \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{\mathcal{Y}}}$. We again elaborate upon the couple $[\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}] \in \mathcal{Y}$: the first component, $\boldsymbol{w}^{(1)} \in L^2(I; V)$, enforces the evolution equation; the second component, $\boldsymbol{w}^{(2)} \in (L^2(\Omega))^d$, enforces the initial condition. The dual-space of $\mathcal{Y}$, $\mathcal{Y}'$, is equipped with a norm $\|\boldsymbol{f}\|_{\mathcal{Y}'} \equiv \sup_{\boldsymbol{v} \in \mathcal{Y}} \boldsymbol{f}(\boldsymbol{v})/\|\boldsymbol{v}\|_{\mathcal{Y}}$; we may express the action of a linear functional in $\mathcal{Y}'$ as either $\boldsymbol{f}(\boldsymbol{v})$ or by the duality pairing $\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}}$. Note that $\|\boldsymbol{f}\|_{\mathcal{Y}'} = \|F\|_{\mathcal{Y}}$ where $F \in \mathcal{Y}$ is the Riesz representation satisfying

$$(F, \boldsymbol{v})_{\mathcal{Y}} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \forall \boldsymbol{v} \in \mathcal{Y}. \tag{2.15}$$

We may think of $\mathcal{Y}'$ as our "data" space.

3. Linear theory

(a) *Linearized perturbation equations*

In order to develop a linear hydrodynamic stability theory within our space-time setting we first introduce the linear perturbation equations. Given disturbances to the dynamics $\boldsymbol{f}^{(1)}$ and the initial condition $\boldsymbol{f}^{(2)}$, the linearized evolution of the velocity perturbation $\boldsymbol{u}'$ about the base flow $\boldsymbol{u}$ is governed by the usual linearized Navier-Stokes equations:

$$\boldsymbol{u}'_t + Re((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}' + (\boldsymbol{u}' \cdot \boldsymbol{\nabla})\boldsymbol{u}) = -2\boldsymbol{\nabla}p' + \nabla^2\boldsymbol{u}' + \boldsymbol{f}^{(1)} \quad \text{in } \Omega \times I ,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0 \quad \text{in } \Omega \times I ,$$

with the boundary conditions

$$\boldsymbol{u}'(\boldsymbol{x}, t) = 0 \quad \text{on } \Gamma_3 \text{ and } \Gamma_4$$

$$\boldsymbol{u}'(\boldsymbol{x} + L\boldsymbol{e}_1, t) = \boldsymbol{u}'(\boldsymbol{x}, t) \quad \text{on } \Gamma_1 ,$$

$$\frac{\partial \boldsymbol{u}'}{\partial x_1}(\boldsymbol{x} + L\boldsymbol{e}_1, t) = \frac{\partial \boldsymbol{u}'}{\partial x_1}(\boldsymbol{x}, t) \quad \text{on } \Gamma_1 ,$$

$$p'(\boldsymbol{x} + L\boldsymbol{e}_1, t) = p'(\boldsymbol{x}, t) \quad \text{on } \Gamma_1 ,$$

and the initial disturbance condition

$$\boldsymbol{u}'(\boldsymbol{x}, t = 0) = \boldsymbol{f}^{(2)} \quad \text{in } \Omega .$$

For $\boldsymbol{u} + \boldsymbol{u}'$ to approximate well the solution to the full Navier-Stokes equations, $\boldsymbol{f}^{(1)}$ and $\boldsymbol{f}^{(2)}$ must be suitably small; we address nonlinear considerations in Section 5.

We may also express the linearized Navier-Stokes equations in a space-time variational form, which is more amenable to our hydrodynamic stability analysis: Find $\boldsymbol{u}' \in \mathcal{X}$ such that

$$\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{u}', \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}} , \quad \forall \boldsymbol{v} \in \mathcal{Y} , \tag{3.1}$$

where $\boldsymbol{u}$ is the base flow field, $\boldsymbol{u}'$ is the perturbed velocity field, $\boldsymbol{f} \equiv [\boldsymbol{f}^{(1)}, \boldsymbol{f}^{(2)}] \in \mathcal{Y}'$ is the disturbance, and $\partial\mathcal{G}$ is the linearized form (2.13). Note that the right-hand side of the equation may be decomposed into

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \langle [\boldsymbol{f}^{(1)}, \boldsymbol{f}^{(2)}], \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \langle \boldsymbol{f}^{(1)}, \boldsymbol{v}^{(1)} \rangle_{L^2(I;V') \times L^2(I;V)} + (\boldsymbol{f}^{(2)}, \boldsymbol{v}^{(2)})_{L^2(\Omega)}$$

corresponding to a disturbance to the "dynamics," $\boldsymbol{f}^{(1)} \in L^2(I; V')$, and to the initial condition, $\boldsymbol{f}^{(2)} \in (L^2(\Omega))^d$. Again, periodicity of flux is weakly imposed in (3.1).

Note that (3.1) is well-posed and a unique solution exists for any $\boldsymbol{f} \in \mathcal{Y}'$ if the bilinear form $\partial\mathcal{G}(\boldsymbol{u}; \cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ satisfies the three conditions of the Banach-Nečas-Babuška theorem (also referred to as the Babuška-Lax-Milgram theorem or the Babuška-Aziz theorem) [13, 27, 3]: the boundedness condition, $\exists C(\boldsymbol{u}) < \infty$ such that $\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) \leq C(\boldsymbol{u})\|\boldsymbol{w}\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}$, $\forall\boldsymbol{w} \in \mathcal{X}$, $\forall\boldsymbol{v} \in \mathcal{Y}$; the inf-sup condition, $\exists\beta(\boldsymbol{u}) > 0$ such that $\inf_{\boldsymbol{w} \in \mathcal{X}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})/(\|\boldsymbol{w}\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}) \geq \beta(\boldsymbol{u})$; and the adjoint injectivity condition, $(\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) = 0, \ \forall\boldsymbol{w} \in \mathcal{X}) \Rightarrow (\boldsymbol{v} = 0)$.

In addition, we may express (3.1) in distributional form: Find $\boldsymbol{u}' \in \mathcal{X}$ such that

$$\langle G\boldsymbol{u}', \boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}} , \quad \forall \boldsymbol{v} \in \mathcal{Y} , \tag{3.2}$$

where $G : \mathcal{X} \to \mathcal{Y}'$ is the linearized forward operator satisfying

$$\langle G\boldsymbol{w}, \boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) , \quad \forall \boldsymbol{w} \in \mathcal{X}, \ \boldsymbol{v} \in \mathcal{Y} .$$

In operator form, (3.2) may be expressed as

$$G\boldsymbol{u}' = \boldsymbol{f} \quad \text{in } \mathcal{Y}' .$$

Note that the inverse operator, $G^{-1} : \mathcal{Y}' \to \mathcal{X}$, exists if the three conditions of the Banach-Nečas-Babuška theorem stated above are satisfied.

Remark 1. The action of (the finite element approximation of) G^{-1} can be computed very efficiently: thanks to the discontinuous-in-time test space $\mathcal{Y}$, in fact $G^{-1}f$ admits evaluation in a time-marching fashion as a sequence of decoupled spatial problems [37]. The space-time framework informs the theory but does not encumber the computations.

(b) Global stability: space-time inf-sup constant

We wish to quantify the velocity perturbation $\boldsymbol{u}'$ in the $\mathcal{X}$ norm. To this end, we introduce a critical ingredient of our hydrodynamic stability analysis: the space-time inf-sup constant

$$\beta(\boldsymbol{u}) \equiv \inf_{\boldsymbol{w}\in\mathcal{X}} \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{w}\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} . \tag{3.3}$$

We may also express the inf-sup constant in distributional form:

$$\beta(\boldsymbol{u}) = \inf_{\boldsymbol{w}\in\mathcal{X}} \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\langle G\boldsymbol{w}, \boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\|\boldsymbol{w}\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} .$$

By the definition of the dual norm,

$$\beta(\boldsymbol{u}) = \inf_{\boldsymbol{w}\in\mathcal{X}} \frac{\|G\boldsymbol{w}\|_{\mathcal{Y}'}}{\|\boldsymbol{w}\|_{\mathcal{X}}} = \inf_{\boldsymbol{f}\in\mathcal{Y}'} \frac{\|\boldsymbol{f}\|_{\mathcal{Y}'}}{\|G^{-1}\boldsymbol{f}\|_{\mathcal{X}}} = \left(\sup_{\boldsymbol{f}\in\mathcal{Y}'} \frac{\|G^{-1}\boldsymbol{f}\|_{\mathcal{X}}}{\|\boldsymbol{f}\|_{\mathcal{Y}'}}\right)^{-1} ;$$

the inverse of the inf-sup constant is thus the norm of the inverse operator $G^{-1} : \mathcal{Y}' \to \mathcal{X}$ measured in $\|\cdot\|_{\mathcal{L}(\mathcal{Y}',\mathcal{X})}$. In addition, we may introduce a linear operator $S_{\boldsymbol{u}} : \mathcal{X} \to \mathcal{Y}$ such that

$$(S_{\boldsymbol{u}}\boldsymbol{w}, \boldsymbol{v})_{\mathcal{Y}} = \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}), \quad \forall \boldsymbol{w} \in \mathcal{X}, \ \forall \boldsymbol{v} \in \mathcal{Y} , \tag{3.4}$$

and substitute the operator $S_{\boldsymbol{u}}$ into the inf-sup definition (3.3) to yield yet another expression for the inf-sup constant:

$$\beta(\boldsymbol{u}) = \inf_{\boldsymbol{w}\in\mathcal{X}} \frac{(S_{\boldsymbol{u}}\boldsymbol{w}, \boldsymbol{v})_{\mathcal{Y}}}{\|\boldsymbol{w}\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} = \inf_{\boldsymbol{w}\in\mathcal{X}} \frac{\|S_{\boldsymbol{u}}\boldsymbol{w}\|_{\mathcal{Y}}}{\|\boldsymbol{w}\|_{\mathcal{X}}} , \tag{3.5}$$

where the second equality follows from the Cauchy-Schwarz inequality. We observe that $S_{\boldsymbol{u}}$ may be interpreted as a supremizer operator.

As regards the relationship between the perturbation $\boldsymbol{u}'$ and the disturbance $\boldsymbol{f}$, we have the following proposition:

PROPOSITION 1. *For a given disturbance $\boldsymbol{f}$, the velocity perturbation $\boldsymbol{u}'$ governed by (3.1) is bounded by*

$$\|\boldsymbol{u}'\|_{\mathcal{X}} \leq \frac{1}{\beta(\boldsymbol{u})}\|\boldsymbol{f}\|_{\mathcal{Y}'} \ ,$$

where $\beta(\boldsymbol{u})$ is the space-time inf-sup constant defined by (3.3). This bound is sharp in the sense that there exists a disturbance $\boldsymbol{f}$ for which the relationship holds with equality.

Proof. For any $\boldsymbol{u}' \in \mathcal{X}$, the solution of (3.1) for given $\boldsymbol{f}$, we obtain

$$\beta(\boldsymbol{u}) = \inf_{\boldsymbol{w}\in\mathcal{X}} \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})}{\|\boldsymbol{w}\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} \leq \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{u}',\boldsymbol{v})}{\|\boldsymbol{u}'\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} = \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\langle\boldsymbol{f},\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\|\boldsymbol{u}'\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} = \frac{\|\boldsymbol{f}\|_{\mathcal{Y}'}}{\|\boldsymbol{u}'\|_{\mathcal{X}}} \ .$$

A straightforward algebraic manipulation yields the desired inequality. The sharpness of the bound follows from the Banach-Nečas-Babuška theorem which ensures the existence of $\boldsymbol{f} = \boldsymbol{f}_* \in \mathcal{Y}'$ corresponding to the infimizer $\boldsymbol{u}' = \boldsymbol{u}'_* \in \mathcal{X}$ and for which

$$\beta(\boldsymbol{u}) = \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{u}'_*,\boldsymbol{v})}{\|\boldsymbol{u}'_*\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} = \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\langle\boldsymbol{f}_*,\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\|\boldsymbol{u}'_*\|_{\mathcal{X}}\|\boldsymbol{v}\|_{\mathcal{Y}}} = \frac{\|\boldsymbol{f}_*\|_{\mathcal{Y}'}}{\|\boldsymbol{u}'_*\|_{\mathcal{X}}} \ .$$

This concludes the proof. ∎

The above proposition shows that the velocity perturbation $\boldsymbol{u}'$ measured in the $\mathcal{X}$ norm is bounded by the disturbance $\boldsymbol{f}$ measured in the $\mathcal{Y}$ norm multiplied by the reciprocal of the space-time inf-sup constant. In this sense, the inf-sup constant quantifies the global stability of the linearized flow equation (3.1). A small inf-sup constant implies that the flow is unstable or "sensitive" such that a small disturbance in the initial condition or the dynamics may lead to a large perturbation in the velocity; the sharpness of the inf-sup bound guarantees that such a large perturbation is realizable and in fact provides a construction. It follows from (2.15) that the Riesz representation of $\boldsymbol{f}_*$ in Proposition 1 is given by $F_* = S_{\boldsymbol{u}}\boldsymbol{u}'_*$. Conversely, a large inf-sup constant implies that the flow is relatively stable and insensitive to disturbances.

Proposition 1 is general in the sense that 1) it applies to perturbations linearized about *any* base flow condition, and 2) it accounts for *arbitrary* disturbances in the initial condition and the dynamics (in $\mathcal{Y}'$). The feature 1) implies that application is not limited to steady or time-periodic flows: the technique can address hydrodynamic stability of transient flows and aperiodic flows. The feature 2) implies that application is not limited to disturbances in initial conditions: the technique can address hydrodynamic stability with respect to time-dependent *forcing* (for example, harmonic forcing near resonance). Under these general conditions, the statement provides a rigorous and sharp *quantitative* bound for the resulting velocity perturbation measured in the strongest possible solution norm in terms of the disturbances measured in the weakest possible data norm. In Appendix A, for some simple but representative ODEs, we demonstrate the ability of the space-time inf-sup constant to identify the least stable mode and to quantify the perturbation growth.

(c) A space-time generalized eigenproblem

We now pose $\beta(\boldsymbol{u})$ as an eigenproblem. The solution to the infimization problem (3.5) is the square root of the minimum eigenvalue of a symmetric positive-definite eigenproblem: Find $(\boldsymbol{\xi}_i, \lambda_i) \in \mathcal{X} \times \mathbb{R}$ such that

$$(S_{\boldsymbol{u}}\boldsymbol{\xi}_i, S_{\boldsymbol{u}}\boldsymbol{w})_{\mathcal{Y}} = \lambda_i (\boldsymbol{\xi}_i, \boldsymbol{w})_{\mathcal{X}}, \quad \forall \boldsymbol{w} \in \mathcal{X} \ . \tag{3.6}$$

Without loss of generality, we order the eigenpairs such that $0 < \lambda_1 \le \lambda_2 \le \cdots$ and normalize the eigenfunctions such that $\|\boldsymbol{\xi}_i\|_{\mathcal{X}} = 1$, $i = 1, 2, \ldots$. Thus, we have $\beta(\boldsymbol{u}) = \sqrt{\lambda_1}$. Due to the symmetry of the eigenproblem, we also obtain $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)_{\mathcal{X}} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

We can also express the eigenproblem (3.6) in operator form. To this end, we introduce the following operators:

$$G^* : \mathcal{Y} \to \mathcal{X}', \quad \langle G\boldsymbol{w}, \boldsymbol{v}\rangle_{\mathcal{Y}' \times \mathcal{Y}} = \langle \boldsymbol{w}, G^*\boldsymbol{v}\rangle_{\mathcal{X} \times \mathcal{X}'}, \quad \forall \boldsymbol{w} \in \mathcal{X}, \ \boldsymbol{v} \in \mathcal{Y} \ , \tag{3.7}$$

$$Y : \mathcal{Y} \to \mathcal{Y}', \quad \langle Y\boldsymbol{w}, \boldsymbol{v}\rangle_{\mathcal{Y}' \times \mathcal{Y}} = (\boldsymbol{w}, \boldsymbol{v})_{\mathcal{Y}}, \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \mathcal{Y} \ , \tag{3.8}$$

$$X : \mathcal{X} \to \mathcal{X}', \quad \langle X\boldsymbol{w}, \boldsymbol{v}\rangle_{\mathcal{X}' \times \mathcal{X}} = (\boldsymbol{w}, \boldsymbol{v})_{\mathcal{X}}, \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \mathcal{X} \ . \tag{3.9}$$

Then, for a given $\boldsymbol{w} \in \mathcal{X}$, the supremizer defined by (3.4) may be expressed as $S_{\boldsymbol{u}}\boldsymbol{w} = Y^{-1}G\boldsymbol{w}$ (in $\mathcal{Y}$). We can then express the eigenproblem (3.6) in distributional form: Find $(\boldsymbol{\xi}_i, \lambda_i) \in \mathcal{X} \times \mathbb{R}$ such that

$$\langle G^*Y^{-1}G\boldsymbol{\xi}_i, \boldsymbol{w}\rangle_{\mathcal{X}' \times \mathcal{X}} = \lambda_i \langle X\boldsymbol{\xi}_i, \boldsymbol{w}\rangle_{\mathcal{X}' \times \mathcal{X}} \ , \quad \forall \boldsymbol{w} \in \mathcal{X} \ .$$

Equivalently, we may express the eigenproblem in an operator form: Find $(\boldsymbol{\xi}_i, \lambda_i) \in \mathcal{X} \times \mathbb{R}$ such that

$$G^*Y^{-1}G\boldsymbol{\xi}_i = \lambda_i X\boldsymbol{\xi}_i \quad \text{in } \mathcal{X}' \ . \tag{3.10}$$

We remark on the computability of the inf-sup constant.

Remark 2. The operator form of the symmetric positive-definite eigenproblem, (3.10), permits a direct transcription to an efficient computational procedure based on a Krylov method, in particular the Lanczos method with $\mathcal{X}$-orthonormalization. The Krylov space suitable for the evaluation of the *minimum* eigenvalue is generated via inverse iteration

$$\mathcal{K}(G^{-1}YG^{-*}X) \equiv \{\boldsymbol{z}^j : \boldsymbol{z}^j = G^{-1}YG^{-*}X\boldsymbol{z}^{j-1}, \ j = 1, 2, \ldots\} \ ,$$

where $\boldsymbol{z}^0$ is a random element in $\mathcal{X}$.

The generation of a new element of the Krylov space requires the following operations: the application of X; the linearized backward solve (an *adjoint* solve, as discussed in greater detail in Appendix B), G^{-*}; the application of Y; and the linearized forward solve, G^{-1}. The computation of the Krylov space invokes only a linearized forward solution and backward solution, and hence, from Remark 1, no fully coupled space-time procedures are required. Furthermore, we have observed in practice that the Lanczos method converges quite rapidly as the minimum eigenvalue is often well separated from the rest of the spectrum. The inf-sup calculation may be readily implemented as a post-processing procedure typically at cost similar to the base flow solution. (See also Buizza *et al.* [8, 7] and Barkley *et al.* [4] for a related computational method.)

(d) A space-time generalized singular value decomposition

We now describe the space-time formulation in terms of the singular value decomposition (SVD). The inf-sup constant is the minimum generalized singular value of $\partial\mathcal{G}(\boldsymbol{u};\cdot,\cdot):\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ with respect to the $\mathcal{X}$-$\mathcal{Y}$ norm pair; "generalized" refers to the fact that the trial (right) and test (left) spaces are equipped with the $\mathcal{X}$ and $\mathcal{Y}$ norm, respectively, instead of the usual ℓ_2 norm. We now make this connection explicit.

We construct above a $\mathcal{X}$-orthonormal trial space basis $\{\boldsymbol{\xi}_j\}_j$. Similarly, we can construct a $\mathcal{Y}$-orthonormal test space basis $\{\boldsymbol{\eta}_i\}_i\equiv\{\lambda_i^{-1/2}S_{\boldsymbol{u}}\boldsymbol{\xi}_i\}_i$; the $\mathcal{Y}$-orthonormality follows from $(\boldsymbol{\eta}_j,\boldsymbol{\eta}_i)_{\mathcal{Y}}=(\lambda_j^{-1/2}S_{\boldsymbol{u}}\boldsymbol{\xi}_j,\lambda_i^{-1/2}S_{\boldsymbol{u}}\boldsymbol{\xi}_i)_{\mathcal{Y}}=\lambda_j^{-1}\lambda_j(\boldsymbol{\xi}_j,\boldsymbol{\xi}_i)_{\mathcal{Y}}=\delta_{ij}$. We now express an arbitrary element $\boldsymbol{w}\in\mathcal{X}$ in the $\mathcal{X}$-orthonormal basis $\{\boldsymbol{\xi}_j\}_j$, $\boldsymbol{w}=\sum_j\alpha_j\boldsymbol{\xi}_j$, and an arbitrary element $\boldsymbol{v}\in\mathcal{Y}$ in the $\mathcal{Y}$-orthonormal basis $\{\boldsymbol{\eta}_i\}_i$, $\boldsymbol{v}=\sum_i\theta_i\boldsymbol{\eta}_i$. In these bases, our linearized form simplifies to

$$\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})=\sum_{i,j}\alpha_j\theta_i\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{\xi}_j,\boldsymbol{\eta}_i)=\sum_{i,j}\alpha_j\theta_i(S_{\boldsymbol{u}}\boldsymbol{\xi}_j,S_{\boldsymbol{u}}\lambda_i^{-1/2}\boldsymbol{\xi}_i)_{\mathcal{Y}}=\sum_i\alpha_i\theta_i\lambda_i^{1/2}\ .$$

Thus, the $\mathcal{X}$-orthonormal trial basis $\{\boldsymbol{\xi}_j\}_j$ and $\mathcal{Y}$-orthonormal test basis $\{\boldsymbol{\eta}_i\}_i$ diagonalize the bilinear form $\partial\mathcal{G}(\boldsymbol{u};\cdot,\cdot):\{\boldsymbol{\xi}_j\}_j$ is a set of trial (i.e. right) singular vectors, $\{\boldsymbol{\eta}_i\}_i$ is a set of test (i.e. left) singular vectors, and $\{\sigma_i\}_i\equiv\{\lambda_i^{1/2}\}_i$ are the singular values. In particular, $\beta(\boldsymbol{u})=\lambda_1^{1/2}=\sigma_1$.

Let us express the perturbed velocity $\boldsymbol{u}'$ in terms of the singular triple $\{\boldsymbol{\xi}_j,\boldsymbol{\eta}_j,\sigma_j\}_j$. We first expand $\boldsymbol{u}'$ in the basis of $\{\boldsymbol{\xi}_j\}_j$, $\boldsymbol{u}'=\sum_j\alpha_j\boldsymbol{\xi}_j$; the coefficient α_i then follows from testing the linearized equation against $\boldsymbol{\eta}_i$, i.e., $\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{u}',\boldsymbol{\eta}_i)=\sum_j\alpha_j\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{\xi}_j,\boldsymbol{\eta}_i)=\alpha_i\sigma_i=\langle\boldsymbol{f},\boldsymbol{\eta}_i\rangle_{\mathcal{Y}'\times\mathcal{Y}}$ (no sum on i). Hence, the perturbed velocity is given by

$$\boldsymbol{u}'=\sum_j\frac{\langle\boldsymbol{f},\boldsymbol{\eta}_j\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\sigma_j}\boldsymbol{\xi}_j\ ;$$

furthermore, from the $\mathcal{X}$-orthonormality of $\{\boldsymbol{\xi}_j\}_j$, we have $\|\boldsymbol{u}'\|_{\mathcal{X}}^2=\sum_j(\langle\boldsymbol{f},\boldsymbol{\eta}_j\rangle_{\mathcal{Y}'\times\mathcal{Y}}/\sigma_j)^2$. In addition, the perturbed output may be expressed as

$$\ell(\boldsymbol{u}')=\sum_j\frac{\langle\boldsymbol{f},\boldsymbol{\eta}_j\rangle_{\mathcal{Y}'\times\mathcal{Y}}\ell(\boldsymbol{\xi}_j)}{\sigma_j}\ . \tag{3.11}$$

Note we adopt the convection that summation over a single index without explicitly indicated limits implies summation over all modes $1,\ldots,\infty$ (or finite truncation in the case of numerical approximation).

We can also express the dual norm $\|\boldsymbol{f}\|_{\mathcal{Y}'}$ in terms of $\{\boldsymbol{\eta}_i\}_i$. By the Riesz representation theorem, there exists a unique $F\in\mathcal{Y}$ such that $(F,\boldsymbol{v})_{\mathcal{Y}}=\langle\boldsymbol{f},\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}$, $\forall\boldsymbol{v}\in\mathcal{Y}$. By $\mathcal{Y}$-orthonormality of $\{\boldsymbol{\eta}_i\}_i$, we have $F=\sum_i\langle\boldsymbol{f},\boldsymbol{\eta}_i\rangle_{\mathcal{Y}'\times\mathcal{Y}}\boldsymbol{\eta}_i$.

Thus, the dual norm of the disturbance $\boldsymbol{f}$ is

$$\|\boldsymbol{f}\|_{\mathcal{Y}'} = \|F\|_{\mathcal{Y}} = \left(\sum_i \langle \boldsymbol{f}, \boldsymbol{\eta}_i \rangle_{\mathcal{Y}' \times \mathcal{Y}}^2 \right)^{1/2} . \tag{3.12}$$

By the same argument, there exists a unique $L \in \mathcal{X}$ such that $(L, \boldsymbol{w})_{\mathcal{X}} = \ell(\boldsymbol{w})$, $\forall \boldsymbol{w} \in \mathcal{X}$. By $\mathcal{X}$-orthonormality of $\{\boldsymbol{\xi}_j\}_j$, we have $L = \sum_i \ell(\boldsymbol{\xi}_i)\boldsymbol{\xi}_i$. Thus, the dual norm of the output functional ℓ is

$$\|\ell\|_{\mathcal{X}'} = \|L\|_{\mathcal{X}} = \left(\sum_i (\ell(\boldsymbol{\xi}_i))^2 \right)^{1/2} . \tag{3.13}$$

We will appeal to this space-time generalized SVD and associated perturbed velocity, perturbed output, and dual norm representations to discuss various aspects of our space-time formulation in the subsequent sections. We note that our space-time generalized SVD is related to, but different from, the SVD of the finite-time linear tangent propagator [24, 8, 7]; the precise relationship is discussed in Section 4(b).

(e) Limitation of global stability theory: output stability

In the following section, we shall focus on the quantification of stability in norms weaker than the norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$; one particular application of such a "semi-norm" bound is uncertainty quantification of the output, which we shall characterize as a "rank-one" norm. We motivate here why this generalization is required.

Due to the linearity of our output functional, we have $\ell(\boldsymbol{u} + \boldsymbol{u}') - \ell(\boldsymbol{u}) = \ell(\boldsymbol{u}')$. Arguably the simplest way to bound the output perturbation is to first construct the global perturbation bound of Proposition 1 and to then appeal to the dual norm of the output functional,

$$|\ell(\boldsymbol{u}')| \leq \sup_{\boldsymbol{w} \in \mathcal{X}} \frac{|\ell(\boldsymbol{w})|}{\|\boldsymbol{w}\|_{\mathcal{X}}} \|\boldsymbol{u}'\|_{\mathcal{X}} = \|\ell\|_{\mathcal{X}'} \|\boldsymbol{u}'\|_{\mathcal{X}} \leq \frac{\|\ell\|_{\mathcal{X}'} \|\boldsymbol{f}\|_{\mathcal{Y}'}}{\beta(\boldsymbol{u})} \equiv \Delta_\ell^{(1)} .$$

The expression shows that the output bound scales with the inverse of the inf-sup constant $\beta(\boldsymbol{u})$: the bound considers *the least-stable mode (i.e., the inf-sup infimizer) regardless of whether this mode affects the output.* As a result, the above output bound is, in general, not sharp: there does *not* exist a perturbation $\boldsymbol{f}$ for which the bound holds with equality.

To illustrate the lack of sharpness, we express $\Delta_\ell^{(1)}$ in terms of the singular triple $\{\boldsymbol{\xi}_i, \boldsymbol{\eta}_i, \sigma_i\}_i$. We appeal to $\beta(\boldsymbol{u}) = \sigma_1$, (3.12), and (3.13), to write

$$\Delta_\ell^{(1)} = \frac{1}{\sigma_1} \left(\sum_i (\ell(\boldsymbol{\xi}_i))^2 \right)^{1/2} \left(\sum_i \langle \boldsymbol{f}, \boldsymbol{\eta}_i \rangle_{\mathcal{Y}' \times \mathcal{Y}}^2 \right)^{1/2} . \tag{3.14}$$

A comparison of the bound $\Delta_\ell^{(1)}$ with the SVD-based output representation (3.11) reveals that the lack of sharpness arises from the misalignment between the supremizer of the output functional ℓ and the infimizer of the stability constant $\beta(\boldsymbol{u})$. In the most extreme case in which $\ell(\boldsymbol{\xi}_1) = 0$, we immediately lose

σ_2/σ_1 in sharpness; more generally, the loss of sharpness will be significant if $|\ell(\boldsymbol{\xi}_1)| \ll \|\ell\|_{\mathcal{X}'}$. The following section provides a construction for sharp output perturbation bounds and, more generally, any semi-norm quantity.

4. Linear theory: semi-norm generalization

(a) Abstract formulation

Motivated by the lack of sharpness in the quantification of the output perturbation based on the global inf-sup constant, we consider a generalization of the global inf-sup that can provide a sharp bound for any semi-norm quantity, including functional outputs. Toward this end, we consider a decomposition of $\mathcal{Y}$ into $\mathcal{Y}$-orthogonal subspaces

$$\mathcal{Y}_1 = \{\boldsymbol{v} \in \mathcal{Y}: \; \boldsymbol{v} = E\boldsymbol{z}, \boldsymbol{z} \in \mathcal{D}\}$$

$$\mathcal{Y}_2 = \{\boldsymbol{v} \in \mathcal{Y}: \; (\boldsymbol{v}, \boldsymbol{z})_{\mathcal{Y}} = 0, \; \forall \boldsymbol{z} \in \mathcal{Y}_1\} \;,$$

where $E : \mathcal{D} \to \mathcal{Y}$ is a bounded linear operator for $\mathcal{D}$ some suitable Banach space. Let $|\cdot|_{\mathcal{Y}_1}$ be a semi-norm on the test space $\mathcal{Y}$ defined by $|\boldsymbol{v}|_{\mathcal{Y}_1} \equiv \|\boldsymbol{v}_1\|_{\mathcal{Y}}$, where $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with $\boldsymbol{v}_1 \in \mathcal{Y}_1$ and $\boldsymbol{v}_2 \in \mathcal{Y}_2$. In addition, let $|\cdot|_{\mathcal{X}_1}$ be a semi-norm on the trial space $\mathcal{X}$ induced by a symmetric bilinear form $(\cdot, \cdot)_{\mathcal{X}_1}$, $|\boldsymbol{w}|_{\mathcal{X}_1} = \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{\mathcal{X}_1}}$. We now introduce a semi-norm generalized space-time inf-sup constant:

$$\tilde{\beta}(\boldsymbol{u}) \equiv \inf_{\boldsymbol{w} \in \mathcal{X}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})}{|\boldsymbol{w}|_{\mathcal{X}_1} |\boldsymbol{v}|_{\mathcal{Y}_1}} \;. \tag{4.1}$$

As regards the relationship between the perturbation $\boldsymbol{u}'$ and the disturbance $\boldsymbol{f}$ measured in the respective semi-norms, we have the following proposition:

PROPOSITION 2. *Let the disturbance $\boldsymbol{f}$ be a bounded functional with respect to the $\mathcal{Y}_1$-norm, i.e., $|\boldsymbol{f}|_{\mathcal{Y}_1'} \equiv \sup_{\boldsymbol{v} \in \mathcal{Y}} \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}} / |\boldsymbol{v}|_{\mathcal{Y}_1} < \infty$. Then, the velocity perturbation $\boldsymbol{u}'$ governed by (3.1) is bounded by*

$$|\boldsymbol{u}'|_{\mathcal{X}_1} \leq \frac{1}{\tilde{\beta}(\boldsymbol{u})} |\boldsymbol{f}|_{\mathcal{Y}_1'} \;,$$

where $\tilde{\beta}(\boldsymbol{u})$ is the generalized semi-norm space-time inf-sup constant defined by (4.1). This bound is sharp in the sense that there exists a disturbance $\boldsymbol{f}$ for which the relationship holds with equality.

Proof. The replacement of the infimizer with the solution to (3.1), $\boldsymbol{u}' \in \mathcal{X}$, for $\boldsymbol{f}$ such that $\|\boldsymbol{f}\|_{\mathcal{Y}_1} < \infty$ yields

$$\tilde{\beta}(\boldsymbol{u}) \leq \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{u}', \boldsymbol{v})}{|\boldsymbol{u}'|_{\mathcal{X}_1} |\boldsymbol{v}|_{\mathcal{Y}_1}} = \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{|\boldsymbol{u}'|_{\mathcal{X}_1} |\boldsymbol{v}|_{\mathcal{Y}_1}} = \frac{|\boldsymbol{f}|_{\mathcal{Y}_1'}}{|\boldsymbol{u}'|_{\mathcal{X}_1}} \;.$$

A straightforward manipulation yields the desired inequality.

We now prove sharpness. We first introduce the supremizers associated with $\mathcal{Y}_1$ and $\mathcal{Y}_2$,

$$S_1\boldsymbol{w}\in\mathcal{Y}_1:\quad (S_1\boldsymbol{w},\boldsymbol{v})_\mathcal{Y}=\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}),\quad \forall \boldsymbol{w}\in\mathcal{X},\ \boldsymbol{v}\in\mathcal{Y}_1\ ,$$

$$S_2\boldsymbol{w}\in\mathcal{Y}_2:\quad (S_2\boldsymbol{w},\boldsymbol{v})_\mathcal{Y}=\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}),\quad \forall \boldsymbol{w}\in\mathcal{X},\ \boldsymbol{v}\in\mathcal{Y}_2\ .$$

We then express the generalized semi-norm inf-sup constant as

$$\tilde{\beta}(\boldsymbol{u})=\inf_{\boldsymbol{w}\in\mathcal{X}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}_1)+\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}_2)}{|\boldsymbol{w}|_{\mathcal{X}_1}|\boldsymbol{v}_1+\boldsymbol{v}_2|_{\mathcal{Y}_1}}$$

$$=\inf_{\boldsymbol{w}\in\mathcal{X}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{(S_1\boldsymbol{w},\boldsymbol{v}_1)_\mathcal{Y}+(S_2\boldsymbol{w},\boldsymbol{v}_2)_\mathcal{Y}}{|\boldsymbol{w}|_{\mathcal{X}_1}\|\boldsymbol{v}_1\|_\mathcal{Y}}\ .$$

We next proceed by contradiction: suppose $\boldsymbol{w}_*$ is the infimizer such that $(S_2\boldsymbol{w}_*,\boldsymbol{v}_2)_\mathcal{Y}\neq 0$ for some $\boldsymbol{v}_2\in\mathcal{Y}_2$; the corresponding supremizer is thus $S_2\boldsymbol{w}_*\in\mathcal{Y}_2$ since this yields $\|S_2\boldsymbol{w}_*\|_\mathcal{Y}^2$ for the numerator but zero for the denominator; the ratio would then be infinite, and hence $\boldsymbol{w}_*$ cannot be the infimizer. Thus, the infimizer must be in the space

$$\widetilde{\mathcal{X}}\equiv\{\boldsymbol{w}\in\mathcal{X}:\ \partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})=0,\ \forall \boldsymbol{v}\in\mathcal{Y}_2\}\ . \tag{4.2}$$

By the definition of the supremization operators S_1 and S_2, any $\boldsymbol{w}\in\widetilde{\mathcal{X}}$ is the solution to (3.1) for disturbance $\boldsymbol{f}_{\boldsymbol{w}}$ whose Riesz representation is $F_{\boldsymbol{w}}=S_1\boldsymbol{w}\in\mathcal{Y}_1$. Restricting the infimizer to $\boldsymbol{w}\in\widetilde{\mathcal{X}}$,

$$\tilde{\beta}(\boldsymbol{u})=\inf_{\boldsymbol{w}\in\widetilde{\mathcal{X}}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{(S_1\boldsymbol{w},\boldsymbol{v}_1)_\mathcal{Y}}{|\boldsymbol{w}|_{\mathcal{X}_1}\|\boldsymbol{v}_1\|_\mathcal{Y}}=\inf_{\boldsymbol{w}\in\widetilde{\mathcal{X}}}\frac{(S_1\boldsymbol{w},S_1\boldsymbol{w})_\mathcal{Y}}{|\boldsymbol{w}|_{\mathcal{X}_1}\|S_1\boldsymbol{w}\|_\mathcal{Y}}=\inf_{\boldsymbol{w}\in\widetilde{\mathcal{X}}}\frac{\|S_1\boldsymbol{w}\|_\mathcal{Y}}{|\boldsymbol{w}|_{\mathcal{X}_1}}=\frac{\|\boldsymbol{f}_{\boldsymbol{w}_{**}}\|_{\mathcal{Y}'}}{\|\boldsymbol{w}_{**}\|_\mathcal{X}} \tag{4.3}$$

for $\boldsymbol{w}_{**}$ the infimizer and $\boldsymbol{f}_{\boldsymbol{w}_{**}}$ satisfying $\langle\boldsymbol{f}_{\boldsymbol{w}_{**}},\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}=(S_1\boldsymbol{w}_{**},\boldsymbol{v})_\mathcal{Y}$, $\forall \boldsymbol{v}\in\mathcal{Y}$. Hence the bound is sharp. ∎

While the above proof is mathematically straightforward, the infimization problem posed on the constrained space $\widetilde{\mathcal{X}}$ defined by (4.3) is computationally cumbersome for some choices of $\mathcal{D}$ and E. In the following proposition (and its corollary), we present a form of the generalized semi-norm inf-sup constant that is more amenable to computation.

PROPOSITION 3. *The generalized inf-sup constant associated with the semi-norms $|\cdot|_{\mathcal{Y}_1}$ and $|\cdot|_{\mathcal{X}_1}$ is equivalent to*

$$\tilde{\beta}(\boldsymbol{u})=\inf_{F\in\mathcal{Y}_1}\frac{\|F\|_\mathcal{Y}}{|G^{-1}YF|_{\mathcal{X}_1}}\ .$$

Proof. For a given $\boldsymbol{w}\in\mathcal{X}$, let $F_{\boldsymbol{w}}\in\mathcal{Y}$ satisfy $(F_{\boldsymbol{w}},\boldsymbol{v})_\mathcal{Y}=\langle G\boldsymbol{w},\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}$, $\forall \boldsymbol{v}\in\mathcal{Y}$; in other words, $G\boldsymbol{w}=YF_{\boldsymbol{w}}$ in $\mathcal{Y}'$. Then,

$$\tilde{\beta}(\boldsymbol{u})\equiv\inf_{\boldsymbol{w}\in\mathcal{X}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})}{|\boldsymbol{w}|_{\mathcal{X}_1}|\boldsymbol{v}|_{\mathcal{Y}_1}}=\inf_{\boldsymbol{w}\in\mathcal{X}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{\langle G\boldsymbol{w},\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{|\boldsymbol{w}|_{\mathcal{X}_1}|\boldsymbol{v}|_{\mathcal{Y}_1}}$$

$$=\inf_{F\in\mathcal{Y}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{(F,\boldsymbol{v})_\mathcal{Y}}{|G^{-1}YF|_{\mathcal{X}_1}|\boldsymbol{v}|_{\mathcal{Y}_1}}=\inf_{F\in\mathcal{Y}}\sup_{\boldsymbol{v}\in\mathcal{Y}}\frac{(F,\boldsymbol{v})_\mathcal{Y}}{|G^{-1}YF|_{\mathcal{X}_1}\|\boldsymbol{v}_1\|_\mathcal{Y}}\ .$$

We again proceed by contradiction: assume the infimizer $F_* = F_1 + F_2 \notin \mathcal{Y}_1$ (that is, $F_2 \neq 0$); then the supremizer is $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ for $\boldsymbol{v}_1 = 0$ and $\boldsymbol{v}_2 = F_2$ since the numerator would be $\|F_2\|_{\mathcal{Y}}^2$ while the denominator would vanish; the ratio would thus be infinite, and hence the infimizer must be in $\mathcal{Y}_1$. In addition, if $F \in \mathcal{Y}_1$, then $(F, \boldsymbol{v})_{\mathcal{Y}} = (F, \boldsymbol{v}_1)_{\mathcal{Y}}$, $\forall \boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \in \mathcal{Y}$, by orthogonality. Thus,

$$\tilde{\beta}(\boldsymbol{u}) = \inf_{F \in \mathcal{Y}_1} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{(F, \boldsymbol{v}_1)_{\mathcal{Y}}}{|G^{-1}YF|_{\mathcal{X}_1} \|\boldsymbol{v}_1\|_{\mathcal{Y}}} = \inf_{F \in \mathcal{Y}_1} \frac{\|F\|_{\mathcal{Y}}}{|G^{-1}YF|_{\mathcal{X}_1}} \ ,$$

where the last equality follows from the Cauchy-Schwarz inequality. ∎

We have the following corollary to Proposition 3:

COROLLARY 1. *The generalized inf-sup constant associated with the semi-norms $|\cdot|_{\mathcal{Y}_1}$ and $|\cdot|_{\mathcal{X}_1}$ may be expressed as*

$$\tilde{\beta}(\boldsymbol{u}) = \inf_{\boldsymbol{d} \in \mathcal{D}} \frac{\|E\boldsymbol{d}\|_{\mathcal{Y}}}{|G^{-1}YE\boldsymbol{d}|_{\mathcal{X}_1}} \ , \tag{4.4}$$

where $E : \mathcal{D} \to \mathcal{Y}$ is the operator associated with $\mathcal{Y}_1$ as introduced in Proposition 2.

Proof. The equivalence follows from the fact that every member $F \in \mathcal{Y}_1$ may be expressed as $F = E\boldsymbol{d}$ for some $\boldsymbol{d} \in \mathcal{D}$ by the construction of $\mathcal{Y}_1$. ∎

The inf-sup problem (4.4) is associated with a symmetric positive-definite eigenproblem: Find $(\boldsymbol{\chi}_i, \tilde{\lambda}_i) \in \mathcal{D} \times \mathbb{R}$ such that

$$(E\boldsymbol{\chi}_i, E\boldsymbol{d})_{\mathcal{Y}} = \tilde{\lambda}_i (G^{-1}YE\boldsymbol{\chi}, G^{-1}YE\boldsymbol{d})_{\mathcal{X}_1}, \quad \forall \boldsymbol{d} \in \mathcal{D} \ . \tag{4.5}$$

The inf-sup constant is the square root of the minimum eigenvalue, i.e., $\tilde{\beta}(\boldsymbol{u}) = \tilde{\lambda}_i^{1/2}$. In addition, if we introduce operators

$$E^* : \mathcal{Y}' \to \mathcal{D}', \quad \langle E\boldsymbol{d}, \boldsymbol{v} \rangle_{\mathcal{Y} \times \mathcal{Y}'} = \langle \boldsymbol{d}, E^*\boldsymbol{v} \rangle_{\mathcal{D} \times \mathcal{D}'} \ , \quad \forall \boldsymbol{d} \in \mathcal{D}, \ \boldsymbol{v} \in \mathcal{Y}' \ ,$$

$$X_1 : \mathcal{X} \to \mathcal{X}', \quad \langle X_1 \boldsymbol{w}, \boldsymbol{v} \rangle_{\mathcal{X}' \times \mathcal{X}} = (\boldsymbol{w}, \boldsymbol{v})_{\mathcal{X}_1} \ , \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \mathcal{X} \ ,$$

then we can write the eigenproblem in operator form: Find $(\boldsymbol{\chi}_i, \tilde{\lambda}_i) \in \mathcal{D} \times \mathbb{R}$ such that

$$E^*YE\boldsymbol{\chi}_i = \tilde{\lambda}_i E^*YG^{-*}X_1 G^{-1}YE\boldsymbol{\chi}_i \quad \text{in} \quad \mathcal{D}' \ . \tag{4.6}$$

Let us make a few remarks.

Remark 3. The "full-norm" $\mathcal{X}$-$\mathcal{Y}$ inf-sup considered in Section 3 corresponds in (4.6) to the spaces and operators $\mathcal{D} = \mathcal{Y}$, $E = Id$, and $X_1 = X$. In this case, a straightforward manipulation reduces (4.6) to (3.10), and, in particular, the eigenfunction $\boldsymbol{\chi}_i$ of (4.6) is the test (i.e. left) singular vector of $\partial \mathcal{G}$ in $(\mathcal{X}, \mathcal{Y})$, $\boldsymbol{\eta}_i$.

Remark 4. The operator form (4.6) admits direct transcription to a computational procedure. Similarly to the full-norm $\mathcal{X}$-$\mathcal{Y}$ inf-sup eigenproblem (3.10), the semi-norm eigenproblem (4.6) may be solved efficiently by a Krylov space method.

(b) Initial disturbance-to-final perturbation stability

As the first application of the semi-norm generalized inf-sup framework, we consider the classical hydrodynamic stability problem: bound the perturbation in the final condition as a function of the initial disturbance. To this end, we choose our disturbance space $\mathcal{D} = (L^2(\Omega))^d$ and the extension operator $E : \mathcal{D} \to \mathcal{Y}$ with $E\boldsymbol{z} = [0, \boldsymbol{z}]$, where we recall that, for $\boldsymbol{v} \in \mathcal{Y}$, $\boldsymbol{v} = [\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}] \in L^2(I; V) \oplus (L^2(\Omega))^d \equiv \mathcal{Y}$. The corresponding space $\mathcal{Y}_1 \subset \mathcal{Y}$ is

$$\mathcal{Y}_1 = \{\boldsymbol{v} \in \mathcal{Y} : \boldsymbol{v} = [0, \boldsymbol{v}^{(2)}], \ \boldsymbol{v}^{(2)} \in (L^2(\Omega))^d\} \ ,$$

$$\mathcal{Y}_2 = \{\boldsymbol{v} \in \mathcal{Y} : \boldsymbol{v} = [\boldsymbol{v}^{(1)}, 0], \ \boldsymbol{v}^{(1)} \in L^2(I; V)\} \ .$$

The associated semi-norm is

$$|\boldsymbol{v}|_{\mathcal{Y}_1} = \|[0, \boldsymbol{v}^{(2)}]\|_{\mathcal{Y}} = \|\boldsymbol{v}^{(2)}\|_{L^2(\Omega)} \ .$$

We choose for the trial-space semi-norm

$$|\boldsymbol{w}|_{\mathcal{X}_1} = \|\boldsymbol{w}(T)\|_{L^2(\Omega)} \ ,$$

which measures the perturbation at the final time.

We now note that the associated supremizer $S_1\boldsymbol{w} \in \mathcal{Y}_1 = (L^2(\Omega))^d$ is

$$(S_1\boldsymbol{w}, \boldsymbol{v}^{(2)})_{L^2(\Omega)} = \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, [0, \boldsymbol{v}^{(2)}]) = (\boldsymbol{w}(0), \boldsymbol{v}^{(2)})_{L^2(\Omega)}, \quad \forall \boldsymbol{v}^{(2)} \in (L^2(\Omega))^d \ ,$$

which implies $S_1\boldsymbol{w} = \boldsymbol{w}(0)$. By (4.1) and (4.3), the associated inf-sup constant for the initial-final stability is

$$\tilde{\beta}_{\mathrm{IF}}(\boldsymbol{u}) = \inf_{\boldsymbol{w}\in\mathcal{X}} \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{w}(T)\|_{L^2(\Omega)}\|\boldsymbol{v}^{(2)}\|_{L^2(\Omega)}} = \inf_{\boldsymbol{w}\in\widetilde{\mathcal{X}}_{\mathrm{IF}}} \frac{\|\boldsymbol{w}(0)\|_{L^2(\Omega)}}{\|\boldsymbol{w}(T)\|_{L^2(\Omega)}} \ , \qquad (4.7)$$

where the application of (4.2) to this case yields

$$\widetilde{\mathcal{X}}_{\mathrm{IF}} = \{\boldsymbol{w} \in \mathcal{X} : \ \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, [\boldsymbol{v}^{(1)}, 0]) = 0, \ \boldsymbol{v}^{(1)} \in L^2(I; V)\} \ . \qquad (4.8)$$

It follows that $\boldsymbol{w} \in \widetilde{\mathcal{X}}_{\mathrm{IF}}$ solves the homogeneous equation subject to any given initial condition $\boldsymbol{w}(0) \equiv \boldsymbol{f}^{(2)}$. By Proposition 2, the semi-norm inf-sup provides a sharp bound

$$\|\boldsymbol{u}'(T)\|_{L^2(\Omega)} \leq \frac{1}{\tilde{\beta}_{\mathrm{IF}}(\boldsymbol{u})}\|\boldsymbol{f}^{(2)}\|_{L^2(\Omega)} = \frac{1}{\tilde{\beta}_{\mathrm{IF}}(\boldsymbol{u})}\|\boldsymbol{u}'(0)\|_{L^2(\Omega)} \ ,$$

with the equality realized for $\boldsymbol{f}_*^{(2)} = \boldsymbol{w}_*(0)$ for $\boldsymbol{w}_*$ the infimizer.

Remark 5. The eigenmodes and eigenvalues of the initial-final inf-sup stability problem are related to the singular values and singular vectors, respectively, of the linear tangent propagator over $(0, T]$.

(c) Output stability: rank-one inf-sup constant

We now revisit the problem that motivated the general semi-norm inf-sup stability framework: output uncertainty quantification. Specifically, we wish to bound the perturbation in a functional output $\ell(\boldsymbol{u}')$ due to disturbances in both the initial condition and the dynamics. Since we consider the effect of all

disturbances, we have $E = Id$, $\mathcal{Y}_1 = \mathcal{Y}$, and $\mathcal{Y}_2 = \emptyset$. The associated norm for the test space and semi-norm for the trial space are

$$|\boldsymbol{v}|_{\mathcal{Y}_1} = \|\boldsymbol{v}\|_{\mathcal{Y}} \quad \text{and} \quad |\boldsymbol{w}|_{\mathcal{X}_1} = |\ell(\boldsymbol{w})| \ .$$

The resulting *rank-one inf-sup constant* for the output is

$$\beta_\ell(\boldsymbol{u}) \equiv \inf_{\boldsymbol{w} \in \mathcal{X}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})}{|\ell(\boldsymbol{w})| \|\boldsymbol{v}\|_{\mathcal{Y}}} \ . \tag{4.9}$$

We suggest the name rank-one inf-sup for the stability constant because the output $|\ell(\cdot)|$ plays the role of a rank-one semi-norm.

The rank-one inf-sup constant is the square root of the minimum eigenvalue of the eigenproblem: Find $(\hat{\boldsymbol{\xi}}_i, \hat{\lambda}_i) \in \mathcal{X} \times \mathbb{R}$ such that

$$(S_{\boldsymbol{u}} \hat{\boldsymbol{\xi}}_i, S_{\boldsymbol{u}} \boldsymbol{w})_{\mathcal{Y}} = \hat{\lambda}_i \ell(\hat{\boldsymbol{\xi}}_i) \ell(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \mathcal{X} \ . \tag{4.10}$$

Without loss of generality, we scale $\hat{\boldsymbol{\xi}}_i$ such that $\ell(\hat{\boldsymbol{\xi}}_i) = 1$. As shown in Appendix B, the eigenfunction associated with the minimum eigenvalue is closely related to the adjoint of the output.

By Proposition 2, the rank-one inf-sup provides a sharp bound

$$|\ell(\boldsymbol{u}')| \leq \frac{1}{\beta_\ell(\boldsymbol{u})} \|\boldsymbol{f}\|_{\mathcal{Y}'} \equiv \Delta_\ell^{(2)} \ . \tag{4.11}$$

To compare the sharpness of the output perturbation bound based on the rank-one inf-sup, $\Delta_\ell^{(2)}$ defined by (4.11), and the output perturbation bound based on the global inf-sup, $\Delta_\ell^{(1)}$ defined by (3.14), we express $\Delta_\ell^{(2)}$ in terms of the singular triple $\{\boldsymbol{\xi}_i, \boldsymbol{\eta}_i, \sigma_i\}_i$. Toward this end, we express $\boldsymbol{w} \in \mathcal{X}$ in terms of $\mathcal{X}$-orthonormal trial basis $\{\boldsymbol{\xi}_i\}_i$, $\boldsymbol{w} = \sum_i \alpha_i \boldsymbol{\xi}_i$. We then write

$$\beta_\ell(\boldsymbol{u})^{-2} = \sup_{\boldsymbol{w} \in \mathcal{X}} \frac{\ell(\boldsymbol{w})^2}{(S_{\boldsymbol{u}} \boldsymbol{w}, S_{\boldsymbol{u}} \boldsymbol{w})_{\mathcal{Y}}} = \sup_{\alpha} \frac{\left(\sum_j \alpha_j \ell(\boldsymbol{\xi}_j) \right)^2}{\sum_{i,j} \alpha_i \alpha_j (S_{\boldsymbol{u}} \boldsymbol{\xi}_j, S_{\boldsymbol{u}} \boldsymbol{\xi}_i)_{\mathcal{Y}}}$$

$$= \sup_{\alpha} \frac{\left(\sum_j \alpha_j \ell(\boldsymbol{\xi}_j) \right)^2}{\sum_i \sigma_j^2 \alpha_j^2} \ .$$

We can construct an upper bound for $\beta_\ell(\boldsymbol{u})^{-2}$: we invoke the Cauchy-Schwarz inequality to the numerator to obtain $(\sum_j \alpha_j \ell(\boldsymbol{\xi}_j))^2 = (\sum_j \sigma_j \alpha_j \sigma_j^{-1} \ell(\boldsymbol{\xi}_j))^2 \leq (\sum_j \sigma_j^2 \alpha_j^2)(\sum_j \sigma_j^{-2} \ell(\boldsymbol{\xi}_j)^2)$; we then cancel the first term with the denominator to arrive at the upper bound $\beta_\ell(\boldsymbol{u})^{-1} \leq \sum_j \ell(\boldsymbol{\xi}_j)^2 / \sigma_j^2$. On the other hand, we can construct a lower bound: we choose a particular candidate $\alpha_i = \ell(\boldsymbol{\xi}_i)/\sigma_i^2$ to deduce $\beta_\ell(\boldsymbol{u})^{-2} \geq (\sum_j \ell(\boldsymbol{\xi}_j)^2 / \sigma_j^2)^2 / (\sum_j \ell(\boldsymbol{\xi}_j)^2 / \sigma_j^2) = \sum_j \ell(\boldsymbol{\xi}_j)^2 / \sigma_j^2$. As upper and lower bound coincide, we conclude

$$\beta_\ell(\boldsymbol{u})^{-1} = \left(\sum_j \frac{\ell(\boldsymbol{\xi}_j)^2}{\sigma_j^2} \right)^{1/2} \ .$$

Thus, our upper bound (4.11) can be expressed as

$$\Delta_\ell^{(2)} \equiv \left(\sum_i \frac{1}{\sigma_i^2} (\ell(\boldsymbol{\xi}_i))^2 \right)^{1/2} \left(\sum_i \langle \boldsymbol{f}, \boldsymbol{\eta}_i \rangle_{\mathcal{Y}' \times \mathcal{Y}}^2 \right)^{1/2}. \tag{4.12}$$

Unlike $\Delta_\ell^{(1)}$, the perturbation bound $\Delta_\ell^{(2)}$ does not amplify the energy in the disturbance $\|\boldsymbol{f}\|_{\mathcal{Y}'}$ by the stability constant of the least-stable mode; rather, each mode of the output is attenuated by the respective singular value. This bound can be significantly sharper than $\Delta_\ell^{(1)}$ if the output functional is insensitive to the least-stable modes, i.e. $|\ell(\boldsymbol{\xi}_i)| \ll \|\ell\|_{\mathcal{X}'}$ for those modes with small σ_i.

Remark 6. The generalized semi-norm formulation permits construction of a sharp error bound for multiple functional outputs based on a *single* inf-sup constant. For instance, given linear output functionals $\ell^{(j)} \in \mathcal{X}$, $j = 1, \dots, M$, we may form a semi-norm $\|\boldsymbol{w}\|_{\mathcal{X}_1}^2 = \sum_{j=1}^M (\ell^{(j)}(\boldsymbol{w}))^2$, find the semi-norm inf-sup constant $\tilde{\beta}_{\{\ell^{(j)}\}_j}(\boldsymbol{u})$, and construct a bound $\|\boldsymbol{f}\|_{\mathcal{Y}'}/\tilde{\beta}_{\{\ell^{(j)}\}_j}(\boldsymbol{u})$. In this sense the semi-norm is more attractive than the more standard adjoint approach discussed in Appendix B.

Remark 7. We can further improve the stability of the output by "filtering" the least stable modes. Our strategy derives directly from the space-time output bound modal decomposition. By the $\mathcal{X}$-orthonormality of the space-time eigenmodes $\{\boldsymbol{\xi}\}_i$, we can define an M-mode-filtered output functional $\hat{\ell}_M \in \mathcal{X}'$ as

$$\hat{\ell}_M = \ell - \sum_{i=1}^M \ell(\boldsymbol{\xi}_i) X \boldsymbol{\xi}_i, \quad \text{in } \mathcal{X}',$$

where $X : \mathcal{X} \to \mathcal{X}'$ is the operator defined in (3.9). For such a filtered output, we realize

$$\Delta_{\hat{\ell}_M}^{(2)} = \left(\sum_{M+1} \frac{1}{\sigma_i^2} (\ell(\boldsymbol{\xi}_i))^2 \right)^{1/2} \left(\sum_i \langle \boldsymbol{f}, \boldsymbol{\eta}_i \rangle_{\mathcal{Y}' \times \mathcal{Y}}^2 \right)^{1/2}.$$

The modified output is significantly less sensitive to the perturbation $\boldsymbol{f}$ if the singular values $\sigma_1 \leq \cdots \leq \sigma_M$ are well-separated from $\sigma_{M+1} \leq \dots$. It would remain to identify the *relevance* of the filtered output in any particular application.

(d) Optimal control

We consider an application of the semi-norm generalized inf-sup stability framework to optimal control. Here, $\mathcal{D}$ is the space of control functions, and the operator $E : \mathcal{D} \to \mathcal{Y}$ defines the realizable shape of the control inputs. The linear perturbation equation (3.1) for a given control $\boldsymbol{c} \in \mathcal{D}$ is as follows: Find $\boldsymbol{u}' \in \mathcal{X}$ such that

$$\partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{u}', \boldsymbol{v}) = (E\boldsymbol{c}, \boldsymbol{v})_{\mathcal{Y}}, \quad \forall \boldsymbol{v} \in \mathcal{Y};$$

our goal is to control a single output quantity $\ell(\boldsymbol{u}')$. Accordingly, we set $\|\boldsymbol{w}\|_{\mathcal{X}_1} = |\ell(\boldsymbol{w})|$.

To deduce the space-time optimal control, we reinterpret the least-stable mode and the associated disturbance as the most-sensitive mode and the associated control input. The generalized space-time inf-sup constant (4.1) in this context yields

$$\tilde{\beta}_{\mathrm{OC}}(\boldsymbol{u}) \equiv \inf_{\boldsymbol{w}\in\mathcal{X}} \sup_{\boldsymbol{v}\in\mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})}{|\ell(\boldsymbol{w})||\boldsymbol{v}|_{\mathcal{Y}_1}} = \inf_{\boldsymbol{w}\in\mathcal{X}} \sup_{\boldsymbol{d}\in\mathcal{D}} \frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{w},E\boldsymbol{d})}{|\ell(\boldsymbol{w})|\|E\boldsymbol{d}\|_{\mathcal{Y}}} \leq \sup_{\boldsymbol{d}\in\mathcal{D}} \frac{\partial\mathcal{G}(\boldsymbol{u};\boldsymbol{u}',E\boldsymbol{d})}{|\ell(\boldsymbol{u}')|\|E\boldsymbol{d}\|_{\mathcal{Y}}}$$

$$= \sup_{\boldsymbol{d}\in\mathcal{D}} \frac{(E\boldsymbol{c},E\boldsymbol{d})_{\mathcal{Y}}}{|\ell(\boldsymbol{u}')|\|E\boldsymbol{d}\|_{\mathcal{Y}}} = \frac{\|E\boldsymbol{c}\|_{\mathcal{Y}}}{|\ell(\boldsymbol{u}')|} \ .$$

It follows that $|\ell(\boldsymbol{u}')| \leq \tilde{\beta}_{\mathrm{OC}}(\boldsymbol{u})^{-1}\|E\boldsymbol{c}\|_{\mathcal{Y}}, \forall \boldsymbol{c}\in\mathcal{D}$. By Proposition 2, there exists a control (direction) $\boldsymbol{c}_*$ for which the relationship holds with equality: $|\ell(\boldsymbol{u}'_*)| = \tilde{\beta}_{\mathrm{OC}}(\boldsymbol{u})^{-1}\|E\boldsymbol{c}_*\|_{\mathcal{Y}}$. This optimal control $\boldsymbol{c}_*$ is precisely the supremizer of the inf-sup infimizer; the associated change in the flow takes the shape of the inf-sup infimizer. Note that, by appealing to Corollary 1, the optimal control (and the associated sensitivity) may be computed directly from the eigenproblem (4.5) (or (4.6)).

We note that $\tilde{\beta}_{\mathrm{OC}}(\boldsymbol{u})^{-1} = |\ell(\boldsymbol{u}'_*)|/\|\boldsymbol{y}_*\|_{\mathcal{Y}}, \ \boldsymbol{y}_* = E\boldsymbol{c}_*$, quantifies the largest possible change in $|\ell(\boldsymbol{u}')|$ for given control effort (measured as $\|\boldsymbol{y}_*\|_{\mathcal{Y}}$). Within linear theory, if we wish to achieve a prescribed change in the output of magnitude δ, the most effective control is $\alpha\boldsymbol{y}_* \in \mathcal{D}$, where the scaling factor $\alpha \in \mathbb{R}$ satisfies $|\alpha| = \beta_{\mathrm{OC}}(\boldsymbol{u})\delta$. The sign of α must be deduced independently: one sign will give the largest per-unit-control increase and the other sign the largest per-unit-control decrease. Of course, within the real (nonlinear) flow control context, our prediction for the amplitude will only be accurate if the prescribed change in the output is small enough such that nonlinear terms are unimportant.

5. Nonlinear theory

(a) Nonlinear perturbation equations

We now wish to develop a theory which characterizes nonlinear propagation of perturbations. We consider the fully-nonlinear perturbation dynamics governed by

$$\mathcal{G}(\tilde{\boldsymbol{u}},\boldsymbol{v}) = \langle\boldsymbol{f},\boldsymbol{v}\rangle_{\mathcal{Y}'\times\mathcal{Y}}, \quad \forall v\in\mathcal{Y} \ , \tag{5.1}$$

where, as before, $\boldsymbol{f} = [\boldsymbol{f}^{(1)},\boldsymbol{f}^{(2)}] \in \mathcal{Y}'$ for $\boldsymbol{f}^{(1)} \in L^2(I;V')$ and $\boldsymbol{f}^{(2)} \in (L^2(\Omega))^d$ the disturbances to the dynamics and the initial condition, respectively.

(b) Nonlinear global bound: Brezzi-Rappaz-Raviart theory

We wish to quantify the upper bound of the velocity perturbation $\|\tilde{\boldsymbol{u}} - \boldsymbol{u}\|_{\mathcal{X}}$ in terms of the energy in the disturbance $\|\boldsymbol{f}\|_{\mathcal{Y}'}$. To this end, we appeal to the Brezzi-Rappaz-Raviart *a posteriori* error estimation theory [6]. Note that, in the space-time context, unlike in the space-only context, the theory applies without complications arising from branch isolation of nonlinear solution trajectories.

PROPOSITION 4 (Brezzi-Rappaz-Raviart perturbation bound). *Let γ be the continuity constant that satisfies*

$$\mathcal{C}(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{v}) \le \gamma^2 \|\boldsymbol{w}\|_{\mathcal{X}} \|\boldsymbol{z}\|_{\mathcal{X}} \|\boldsymbol{v}\|_{\mathcal{Y}}, \quad \forall \boldsymbol{w}, \boldsymbol{z} \in \mathcal{X}, \ \boldsymbol{v} \in \mathcal{Y} . \tag{5.2}$$

Suppose the disturbance $\|\boldsymbol{f}\|_{\mathcal{Y}'}$ is sufficiently small in the sense that

$$\|\boldsymbol{f}\|_{\mathcal{Y}'} < \frac{\beta^2(\boldsymbol{u})}{4\gamma^2} , \tag{5.3}$$

where $\beta(\boldsymbol{u})$ is the inf-sup constant defined in (3.3). Then, the perturbation in the solution is bounded by

$$\|\tilde{\boldsymbol{u}} - \boldsymbol{u}\|_{\mathcal{X}} \le \frac{2\|\boldsymbol{f}\|_{\mathcal{Y}'}}{\beta(\boldsymbol{u})} . \tag{5.4}$$

Proof. The proposition is a specialization of the more general result of Brezzi *et al.* [6] to a quadratic nonlinearity; the proof follows accordingly as shown for example in Veroy and Patera [38]. We provide the proof in Appendix C. ∎

The Brezzi-Rappaz-Raviart statement provides a rigorous nonlinear bound of the velocity perturbation for a given small but finite-amplitude disturbance $\boldsymbol{f}$ in the initial condition or the dynamics. The amplitude of the disturbance for which the theory is valid is precisely governed by (5.3); note that the maximum amplitude of the perturbation is a function of the space-time inf-sup constant, which measures the stability of the flow, and the continuity constant, which measures the extent of nonlinearity. As the bound statement (5.4) is of the same form as that of the linear theory (identical save a factor of two), we can interpret (5.3) as providing a condition under which the linear theory is valid. In essence, the proposition permits a non-asymptotic (finite-amplitude) interpretation of linear stability.

For the Navier-Stokes equations with the quadratic convection operator and in particular our $\mathcal{C}$ as expressed in (2.11), the continuity constant of (5.2) is given by

$$\gamma^2 = Re \cdot \rho^2 ,$$

where ρ^2 is the $L^4(I; L^4(\Omega))$-$\mathcal{X}$ Sobolev embedding constant

$$\rho = \sup_{\boldsymbol{w} \times \mathcal{X}} \frac{\|\boldsymbol{w}\|_{L^4(I; L^4(\Omega))}}{\|\boldsymbol{w}\|_{\mathcal{X}}} , \tag{5.5}$$

and $\|\boldsymbol{w}\|_{L^4(I; L^4(\Omega))}^4 \equiv \int_I \int_\Omega (\boldsymbol{w}_i \boldsymbol{w}_i)^2 d\boldsymbol{x} dt$. For our choice of the $\mathcal{X}$ norm, it can be shown [25] (and numerically confirmed in the sense of asymptotic mesh independence [40]) that, in two dimensions, the continuous embedding indeed holds and hence ρ is bounded; we restrict attention in this paper to two-dimensional flows. However, in three dimensions, the continuous embedding no longer holds; future work will consider alternative norms. An efficient fixed-point algorithm for evaluating the constant has been devised by Deparis [10] and in particular employed in the space-time setting in [40].

Remark 8. The BRR perturbation bound permits a rigorous quantification of nonlinear hydrodynamic stability by identifying the amplitude condition under which

which the linear theory is valid. The approach is different from a direct nonlinear analysis of the most sensitive finite-amplitude initial disturbance conducted, for example, by Pringle *et al.* [28] and Monokrousos *et al.* [26]. The latter, unlike our BRR perturbation bound, on the one hand incorporates fully nonlinear information, but on the other hand does not provide a rigorous global bound statement in the presence of multiple local optima.

(*c*) *Nonlinear output bounds for quadratic outputs*

For our nonlinear theory, we consider a quadratic output functional $\ell : \mathcal{X} \to \mathbb{R}$ of the form

$$\ell(\boldsymbol{w}) = \ell^1(\boldsymbol{w}) + \ell^2(\boldsymbol{w}, \boldsymbol{w}) \ ,$$

where $\ell^1 \in \mathcal{X}'$ and $\ell^2 : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a symmetric, bounded bilinear form with a continuity constant $\gamma_{\ell^2}^2$, i.e.,

$$\ell^2(\boldsymbol{w}, \boldsymbol{w}) \le \gamma_{\ell^2}^2 \|\boldsymbol{w}\|_{\mathcal{X}}^2 \ , \quad \forall \boldsymbol{w} \in \mathcal{X} \ . \tag{5.6}$$

The output functional linearized about $\boldsymbol{z}$ is

$$\partial\ell(\boldsymbol{z}; \boldsymbol{w}) = \ell^1(\boldsymbol{w}) + 2\ell^2(\boldsymbol{z}, \boldsymbol{w}) \ .$$

Accordingly, we redefine the rank-one inf-sup constant as

$$\beta_\ell(\boldsymbol{u}) \equiv \inf_{\boldsymbol{w} \in \mathcal{X}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})}{|\partial\ell(\boldsymbol{u}; \boldsymbol{w})| \|\boldsymbol{v}\|_{\mathcal{Y}}} \ . \tag{5.7}$$

This generalization of the output to the quadratic form enables us to consider quantities such as the time-averaged dissipation, $\int_I \int_\Omega \nabla\boldsymbol{u}_i \cdot \nabla\boldsymbol{u}_i d\boldsymbol{x}dt$.

The following proposition establishes a bound on output perturbation due to finite-amplitude disturbances in the initial condition and the dynamics.

PROPOSITION 5. *Suppose the disturbance is sufficiently small in the sense that* (5.3) *is satisfied. Then, the perturbation in the output quantity is bounded by*

$$|\ell(\tilde{\boldsymbol{u}}) - \ell(\boldsymbol{u})| \le \frac{\|\boldsymbol{f}\|_{\mathcal{Y}'}}{\beta_\ell(\boldsymbol{u})} + 4 \left(\frac{\gamma^2}{\beta_\ell(\boldsymbol{u})} + \gamma_{\ell^2}^2 \right) \frac{\|\boldsymbol{f}\|_{\mathcal{Y}'}^2}{\beta(\boldsymbol{u})^2} \ . \tag{5.8}$$

where $\beta_\ell(\boldsymbol{u})$ is the rank-one inf-sup constant, (5.7), γ is the continuity constant for $\mathcal{C}$, (5.2), γ_{ℓ^2} is the continuity constant for ℓ^2, (5.6), and $\beta(\boldsymbol{u})$ is the inf-sup constant.

Proof. See Appendix D. ∎

6. Demonstration: eddy-promoter channel

We apply the hydrodynamic stability analysis developed in the previous sections to the eddy-promoter channel flow described in Section 2(a). The unsteady Navier-Stokes equations are discretized by a $\mathbb{P}^2$-$\mathbb{P}^1$ Taylor-Hood continuous Galerkin discretization in space and a $\mathbb{P}^2$ discontinuous Galerkin discretization in time; details are provided in [40]. The space-time L^4-$\mathcal{X}$ embedding constant for the eddy-promoter channel geometry is $\rho \approx 0.420$. We note that many of the earlier

(a) Case I ($Re = 1$)

(b) Case II ($Re = 150$)

(c) Case III-A ($Re = 300$, steady-periodic)

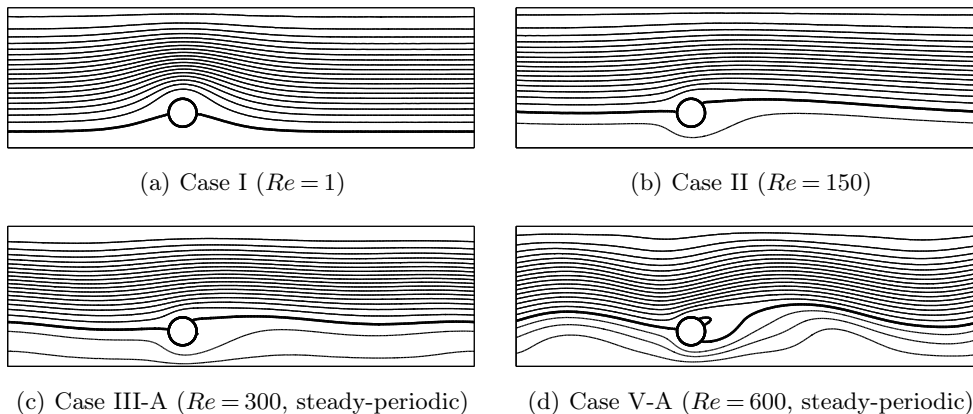(d) Case V-A ($Re = 600$, steady-periodic)

Figure 2. Snapshots of several base states: velocity streamlines.

finite-time stability analyses focus on three-dimensional flow instability [9, 30, 36, 35, 32]. Our purpose here is only to demonstrate the generality of our formulation and in particular to emphasize disturbances to dynamics especially for unsteady base flows; hence we restrict attention to less computationally intensive two-dimensional flow. Future work will address the three-dimensional case.

We study variation in the space-time stability constant for five different values of the Reynolds number, with each case assigned a Roman numeral: $Re = 1$, which results in essentially Stokes flow (Case I); $Re = 150$, which results in a moderate Reynolds number steady flow as $t \to \infty$ (Case II); and $Re = 300$, 450, and 600, which exhibit steady-periodic behavior in time as $t \to \infty$ (Case III, IV, and V, respectively). For Cases I and II, the base flow $\boldsymbol{u}$ is taken to be the respective steady state solution. For Cases III, IV, and V, we consider two different base flows $\boldsymbol{u}$, distinguished by letters A and B: the steady-periodic state (for some arbitrary phase) that naturally arises from a long-time integration of the Navier-Stokes equations (Sub-case A); and the steady but unstable equilibrium state (Sub-case B). The combination of the Reynolds number and sub-case designations yields the particular case number; for example, the $Re = 450$ case with the steady-periodic base state is denoted as Case IV-A. Note that Sub-case A is the physically meaningful analysis; Sub-case B is considered for purposes of comparison and illustration. The time integration is carried out to $T = 1/4$ diffusive time units, which corresponds to many convective time units (in all cases except for Case I, which is not of primary interest). Representative velocity fields for selected cases are shown in Figure 2. (Note that Cases III-B, IV-B, and V-B do correspond to a solution of the initial value problem (2.7), though in practice these unstable states are obtained by direct calculation of the steady equations.)

Figure 3 shows the variation in the global inf-sup constant, defined by (3.3), with the Reynolds number. The space-time inf-sup constant is unity for the Stokes flow (c.f. proof in [40]) and, in general, decays with the Reynolds number. The result demonstrates that a small disturbance $\boldsymbol{f}$ can be more rapidly amplified in higher Reynolds number flows, as expected. The inf-sup constant associated with the unstable equilibrium base flow decays rapidly with the Reynolds number,
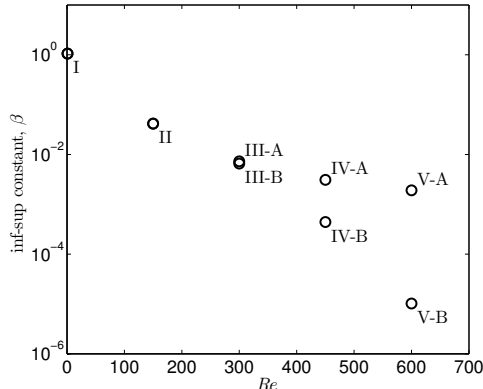
Figure 3. The (global) inf-sup stability constant for different base flows considered.

decreasing to $\beta(\boldsymbol{u}) \approx 1.0 \times 10^{-5}$ for Case V-B. On the other hand, the inf-sup constant for the (more physically relevant) steady-periodic base flow — which in fact corresponds to nonlinear saturation of the linear unstable mode — is much better controlled: $\beta(\boldsymbol{u}) \approx 1.7 \times 10^{-3}$ for Case V-A. The result confirms that the linearized theory based on the unstable equilibrium condition grossly overestimates the sensitivity of (saturated) periodic flows, as might be expected.

In order to understand the difference in the inf-sup constant behavior for the two base flows, we show in Figure 4 the inf-sup infimizer (i.e., the least-stable mode) for Case V-A and Case V-B. Both infimizers are effectively traveling waves. The snapshot of the inf-sup infimizer for Case V-B at $t = T$ is similar to the least-stable (Tollmien-Schlichting-like) normal mode associated with the unstable equilibrium condition reported by Karniadakis *et al.* [21]; the temporal history demonstrates that this mode grows *exponentially* in time. The spatial structure of the inf-sup infimizer for Case V-A still bears some resemblance to a Tollmien-Schlichting wave; however, and more importantly, the time history demonstrates that the mode grows *linearly* in time — indicative of resonance.

Figure 5 shows the variation in the rank-one inf-sup constant associated with the two outputs. Recall that the first output is the flowrate represented as the integral of the x-velocity over the entire domain, and the second output is the regularized aft-cylinder local x-velocity. We report *relative* sensitivity by scaling each rank-one inf-sup constant by the respective output value. Not surprisingly, the flowrate output is less sensitive to disturbances than the local velocity output. Both output inf-sup constants are significantly less sensitive to the Reynolds number than the global inf-sup constant. The result suggests that, while the growth of the least-stable mode increases rapidly with Reynolds number (as implied by the rapid decay of $\beta(\boldsymbol{u})$), the impact of this growth on time-averaged outputs is relatively small (as implied by the slow decay of $\beta_\ell(\boldsymbol{u})$). The result confirms our assertion that time-averaged outputs are not strongly influenced by the least-stable, traveling-wave mode; for instance, for the flowrate output of Case V-A, $|\ell(\boldsymbol{\xi}_1)| \approx 2.3 \times 10^{-4} \ll \|\ell\|_{\mathcal{X}'} \approx 9.5 \times 10^{-2}$.

The Brezzi-Rappaz-Raviart threshold value, $\beta^2/(4\gamma^2)$, varies from $\mathcal{O}(1)$ for low Reynolds number to $\mathcal{O}(10^{-8})$ for $Re = 600$. While the BRR theory provides
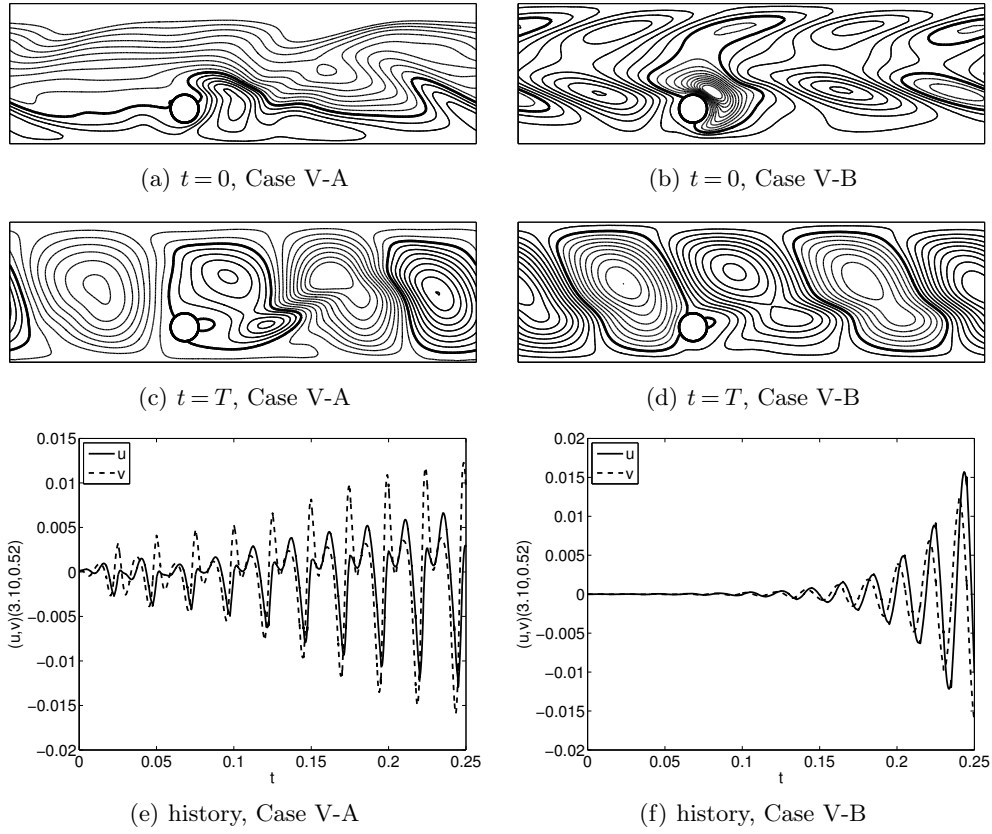
(a) $t = 0$, Case V-A

(b) $t = 0$, Case V-B

(c) $t = T$, Case V-A

(d) $t = T$, Case V-B

(e) history, Case V-A

(f) history, Case V-B

Figure 4. The (global) inf-sup infimizers $\boldsymbol{\xi}_1$ for the steady-periodic base flow (Case V-A) and the unstable equilibrium base flow (Case V-B) at $Re = 600$.
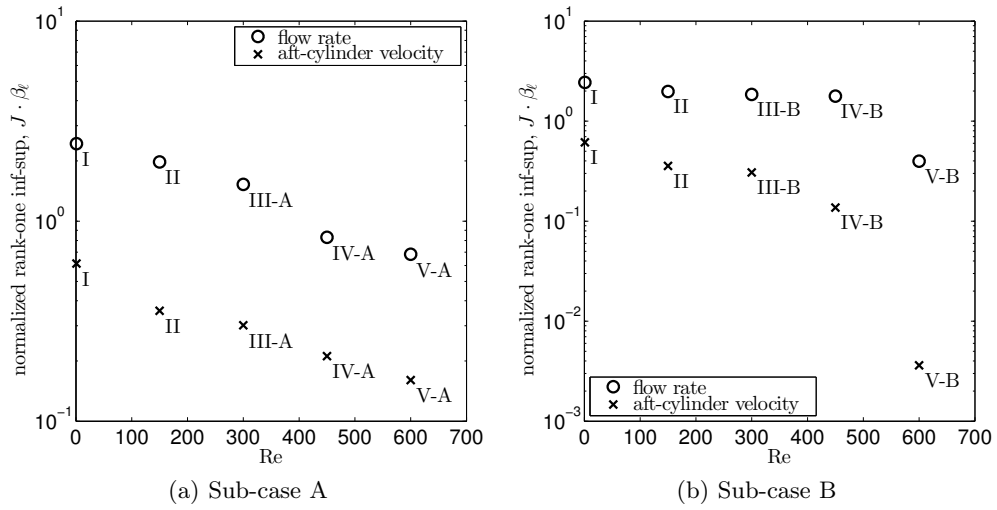

(a) Sub-case A

(b) Sub-case B

Figure 5. Output rank-one inf-sup constants for the eddy-promoter channel flow.

rigorous bounds for the velocity perturbation governed by the full nonlinear dynamics, for high Reynolds number flows, application is clearly limited to *very* small disturbances.

Remark 9. As both the global and output bounds developed are (at least asymptotically) sharp, they cannot be improved for the particular norms considered in this work. Specifically, the decrease of the BRR threshold with the Reynolds number suggests that there may be a limit in bounding nonlinear growth of perturbations in the deterministic sense. Thus, statistical quantification of the perturbation — for example by incorporating ergodic theory (e.g. [12, 22]) or by appealing to recent work on sensitivity calculation for chaotic systems [39] — may be crucial to constructing output perturbation bounds for higher Reynolds number and eventually turbulent flows.

Acknowledgments

A. Asymptotic behaviors of the space-time inf-sup constant for simple representative ODEs

In this appendix, we consider the asymptotic behavior of the space-time inf-sup constant for some simple ODEs using analytical tools. Through the analysis, we will demonstrate that the behavior of the inf-sup constant indeed agrees with the known worst-case perturbation growth behavior for the ODEs considered.

(a) Preliminary

The ODEs we consider are of the form

$$\dot{w} + \alpha w = 0, \quad t \in I \equiv (0, T] ,$$

for some fixed parameter $\alpha \in \mathbb{C}$. The appropriate function space for analyzing the ODE is $V = \ell_2$ (the vector ℓ_2 norm). To simplify the analysis, we restrict the trial space to functions that vanish at $t = 0$; consequently, $\mathcal{X} = H^1_{(0)}(I; \ell_2)$ and

$\mathcal{Y} = L^2(I; \ell_2)$. We have

$$\partial \mathcal{G}(w, v) = \int_0^T v^*(\dot{w} + \alpha w) dt,$$

$$\|w\|_{\mathcal{X}}^2 = \int_0^T \left(|\dot{w}|^2 + |w|^2\right) dt + |w(T)|^2,$$

$$\|v\|_{\mathcal{Y}}^2 = \int_0^T |v|^2 dt,$$

where v^* is the complex conjugate of v. We then identify the supremizer as $v = \dot{w} + \alpha w$ to express the inf-sup constant as

$$\beta^2 = \inf_{w \in \mathcal{X}} \frac{\int_0^T (\dot{w}^* + \alpha^* w^*)(\dot{w} + \alpha w) dt}{\int_0^T \left(|\dot{w}|^2 + |w|^2\right) dt + |w(T)|^2} .$$

We have the associated eigenproblem: Find $(\xi, \lambda) \in C^1(I) \times \mathbb{R}$ such that

$$\int_0^T \dot{w}^* \dot{\xi} + \alpha^* w^* \dot{\xi} + \alpha \dot{w}^* \xi + |\alpha|^2 w^* \xi dt = \lambda \left(\int_0^T \left(\dot{w}^* \dot{\xi} + w^* \xi \right) dt + w^*(T)\xi^*(T) \right),$$

$$\forall w \in C^1(I) .$$

We then invoke integration by parts to arrive at the strong form of the eigenproblem

$$-(1 - \lambda)\ddot{\xi} + (\alpha^* - \alpha)\dot{\xi} + (|\alpha|^2 - \lambda)\xi = 0 , \tag{A.1}$$

$$\xi(0) = 0, \quad (1 - \lambda)\dot{\xi}(T) + (\alpha - \lambda)\xi(T) = 0 . \tag{A.2}$$

We now consider two special cases.

(b) Exponentially unstable system: $\alpha = \rho \in \mathbb{R}$, $\rho < 0$, $T \to \infty$

The first case we consider is unstable dynamics with $\alpha = \rho \in \mathbb{R}$, $\rho < 0$, with $T \to \infty$. Let us represent eigenvectors as $\xi(t) = e^{\sigma t}$ for some σ and eigenvalues as $\lambda = \epsilon$, where we expect $\epsilon \to 0$ as $\rho T \to -\infty$. Then, (A.1) becomes

$$-(1 - \epsilon)\sigma^2 + (\rho^2 - \epsilon) = 0 .$$

Solving the quadratic equation and using asymptotic expansions ($\epsilon \to 0$), we deduce $\sigma_\pm$ is

$$\sigma_\pm = \pm \sigma = \pm \left(\frac{\rho^2 - \epsilon}{1 - \epsilon} \right)^{1/2} \approx \pm|\rho| \left(1 + \frac{\epsilon}{2} \left(1 - \frac{1}{\rho^2} \right) \right) .$$

Given $\sigma_\pm \in \mathbb{R}$, we represent eigenvectors as $\xi(t) = c_1 \sinh(\sigma t) + c_2 \cosh(\sigma t)$. The initial condition requires $c_2 = 0$, hence, without loss of generality, $\xi(t) = \sinh(\sigma t)$.

The final condition (A.2) requires

$$(1 - \epsilon)\sigma \cosh(\sigma T) + (\rho - \epsilon)\sinh(\sigma T) = 0 \ .$$

We rearrange the expression and substitute the approximation for σ to obtain

$$\tanh\left(|\rho|T\left(1 + \frac{\epsilon}{2}\left(1 - \frac{1}{\rho^2}\right)\right)\right) \approx (1 - \epsilon)\left(1 - \frac{\epsilon}{|\rho|}\right)\left(1 + \frac{\epsilon}{2}\left(1 - \frac{1}{\rho^2}\right)\right) \ .$$

We then obtain the asymptotic expansion (as $T \to \infty$, $\epsilon \to 0$)

$$1 - 2e^{-2|\rho|T}e^{-|\rho|T\epsilon(1-\rho^{-2})} \approx 1 - \frac{\epsilon}{2}\left(1 + \frac{1}{|\rho|}\right)^2 \ .$$

If we *assume* $|\rho|T\epsilon \to 0$, then

$$\epsilon = 4(1 + |\rho|^{-1})^{-2}e^{-2|\rho|T} \ ;$$

we subsequently confirm that $|\rho|T\epsilon = 4|\rho|T(1 + |\rho|^{-1})^{-2}e^{-2|\rho|T} \to 0$ as $T \to \infty$. Consequently, the space-time inf-sup constant behaves asymptotically as

$$\beta = \lambda^{1/2} = \epsilon^{1/2} = 2(1 + |\rho|^{-1})^{-1}e^{-|\rho|T} = \mathcal{O}(e^{-|\rho|T}), \quad \rho < 0, \ T \to \infty \ .$$

Thus, for a system in which the perturbation grows exponentially with time, the space-time inf-sup constant decays exponentially with final time. We also note that the infimizer, $\sinh(\sigma t)$, grows exponentially in time.

$$(c) \textit{ Purely oscillatory system: } \alpha = i\zeta \in i\mathbb{R}, \ \zeta > 0, \ T \to \infty$$

The second case we consider is neutrally stable oscillatory dynamics: $\alpha = i\zeta \in i\mathbb{R}$, $\zeta > 0$, and $T \to \infty$. As before, substituting $\xi(t) = e^{\sigma t}$ and taking $\lambda = \epsilon$, the eigenproblem (A.1) becomes

$$-(1 - \epsilon)\sigma^2 - 2i\zeta\sigma + (\zeta^2 - \epsilon) = 0 \ .$$

Evaluating the quadratic equation and making asymptotic approximations,

$$\sigma_\pm = \frac{-i\zeta \pm \sqrt{-\zeta^2 + (1 - \epsilon)(\zeta^2 - \epsilon)}}{1 - \epsilon} \approx (1 + \epsilon)(-i(\zeta \pm \sqrt{\epsilon(\zeta^2 + 1)})) \ .$$

Enforcing the condition $\xi(0) = 0$, our eigenfunction is of the form

$$\xi(t) = e^{-i\zeta(1+\epsilon)t}\sin(\sqrt{\epsilon(\zeta^2 + 1)}(1 + \epsilon)t) \ .$$

Substitution of the expression to the final condition (A.2) yields, after tedious but straightforward manipulation,

$$i\zeta\epsilon^2 \sin(\theta) + (1 - \epsilon^2)\sqrt{\epsilon(\zeta^2 + 1)}\cos(\theta) - \epsilon\sin(\theta) = 0 \ ,$$

where $\theta = \sqrt{\epsilon(\zeta^2 + 1)}(1 + \epsilon)T$. As $\epsilon \to 0$, the second term is dominant. Thus, we require $\cos(\theta) = 0$ as $\epsilon \to 0$. In other words,

$$\sqrt{\epsilon(\zeta^2 + 1)}(1 + \epsilon)T = \pi\left(\tfrac{1}{2} + n\right), \quad n \in \mathbb{Z} \ .$$

We invoke asymptotic approximations and chose the minimizer $n = 0$ to obtain

$$\beta = \lambda^{1/2} = \epsilon^{1/2} = \frac{\pi}{2\sqrt{\zeta^2 + 1}\,T} = \mathcal{O}(T^{-1}), \quad \text{as } \zeta \to \infty \ .$$

Thus, the inverse of the space-time inf-sup constant grows linearly with time. This is consistent with the fact that excitation of a purely oscillatory system leads to resonance, in which the amplitude of the oscillation grows linearly with time. Indeed, the frequency of oscillation associated with the eigenfunction is ζ.

B. Relationship between the rank-one inf-sup and adjoint

We present another sharp output bound based on an adjoint. Recall that, for a given output characterized by (2.14) with $\boldsymbol{g} = (\boldsymbol{g}^I, \boldsymbol{g}^T)$, the adjoint Navier-Stokes equations written in the strong form is

$$-\boldsymbol{\psi}_t + Re(-(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\psi} + \boldsymbol{\nabla}\boldsymbol{u} \cdot \boldsymbol{\psi}) = -2\boldsymbol{\nabla}q + \nabla^2\boldsymbol{\psi} + \boldsymbol{g}^I, \quad \text{in } \Omega \times I$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0, \quad \text{in } \Omega \times I,$$

where $(\boldsymbol{\nabla}\boldsymbol{u} \cdot \boldsymbol{\psi})_i \equiv \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\boldsymbol{\psi}_j$, and q is the continuity adjoint. The associated boundary conditions are

$$\boldsymbol{\psi}(\boldsymbol{x}, t) = 0 \quad \text{on } \Gamma_3 \text{ and } \Gamma_4$$

$$\boldsymbol{\psi}(\boldsymbol{x} + L\boldsymbol{e}_1, t) = \boldsymbol{\psi}(\boldsymbol{x}, t), \quad \text{on } \Gamma_1$$

$$\frac{\partial \boldsymbol{\psi}}{\partial x_1}(\boldsymbol{x} + L\boldsymbol{e}_1, t) = \frac{\partial \boldsymbol{\psi}}{\partial x_1}(\boldsymbol{x}, t), \quad \text{on } \Gamma_1$$

$$q(\boldsymbol{x} + L\boldsymbol{e}_1, t) = q(\boldsymbol{x}, t), \quad \text{on } \Gamma_1 \ ,$$

and the terminal condition is

$$\boldsymbol{\psi}(\boldsymbol{x}, t = T) = \boldsymbol{g}^T \quad \text{in } \Omega \ .$$

The adjoint equation is solved backward in time starting from the terminal condition $\boldsymbol{g}^T$ and forced by $\boldsymbol{g}^I$.

We may express the adjoint Navier-Stokes equations in a space-time weak form: Find $\boldsymbol{\psi} \in \mathcal{Y}$ such that

$$\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{\psi}) = \ell(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \mathcal{X} \ . \tag{B.1}$$

Note the reversal of the trial-test roles played by the spaces $\mathcal{X}$ and $\mathcal{Y}$. We assume well-posedness.

The operator form of this equation is

$$G^*\boldsymbol{\psi} = \ell \quad \text{in } \mathcal{X}' \ ,$$

where $G^* : \mathcal{Y} \to \mathcal{X}'$ is the adjoint operator defined already in (3.7).

Using the adjoint, we arrive at the following output perturbation bound:

PROPOSITION 6. *For any arbitrary disturbance $\boldsymbol{f}$ and solution $\boldsymbol{u}'$ to (3.1), the perturbation in the output $\ell(\boldsymbol{u}')$ is bounded by*

$$|\ell(\boldsymbol{u}')| \leq \|\boldsymbol{\psi}\|_{\mathcal{Y}} \|\boldsymbol{f}\|_{\mathcal{Y}'} \equiv \Delta_\ell^{(3)} \ .$$

The bound is sharp in the sense that, for any ℓ, there exists a disturbance $\boldsymbol{f}$ for which the relationship holds with equality.

Proof. The bound is a direct consequence of the definition of the adjoint,

$$|\ell(\boldsymbol{u}')| = |\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{u}', \boldsymbol{\psi})| = |\langle \boldsymbol{f}, \boldsymbol{\psi} \rangle_{\mathcal{Y}' \times \mathcal{Y}}| \leq \|\boldsymbol{\psi}\|_{\mathcal{Y}} \|\boldsymbol{f}\|_{\mathcal{Y}'} \ .$$

The sharpness follows from the definition of the dual norm and the Riesz representation theorem. ∎

While derived from a different principle, we have the following proposition regarding the rank-one inf-sup and adjoint-based bound:

PROPOSITION 7. *The rank-one inf-sup output bound $\Delta_\ell^{(2)}$ and the adjoint-based output bound $\Delta_\ell^{(3)}$ are identical. In particular, $\beta_\ell(\boldsymbol{u}) = \|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-1}$.*

Proof. By definition of the adjoint and the supremizing operator,

$$\beta_\ell(\boldsymbol{u}) = \inf_{\boldsymbol{w} \in \mathcal{X}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})}{|\partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{\psi})| \|\boldsymbol{v}\|_{\mathcal{Y}}} = \inf_{\boldsymbol{w} \in \mathcal{X}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{(S_{\boldsymbol{u}}\boldsymbol{w}, \boldsymbol{v})_{\mathcal{Y}}}{|(S_{\boldsymbol{u}}\boldsymbol{w}, \boldsymbol{\psi})_{\mathcal{Y}}| \|\boldsymbol{v}\|_{\mathcal{Y}}}$$

$$= \inf_{\boldsymbol{w} \in \mathcal{X}} \frac{\|S_{\boldsymbol{u}}\boldsymbol{w}\|_{\mathcal{Y}}}{|(S_{\boldsymbol{u}}\boldsymbol{w}, \boldsymbol{\psi})_{\mathcal{Y}}|} = \frac{1}{\|\boldsymbol{\psi}\|_{\mathcal{Y}}} \ ,$$

where the last equality follows from noting that $S_{\boldsymbol{u}}\boldsymbol{w} = \boldsymbol{\psi}$ is the infimizer. ∎

Remark 10. Note that the infimizer of the rank-one inf-sup constant, $\hat{\boldsymbol{\xi}}_1$ defined by the eigenproblem (4.10), is related to the adjoint by $S_{\boldsymbol{u}}\hat{\boldsymbol{\xi}}_1 = \|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-2}\boldsymbol{\psi}$, and hence furthermore the adjoint is the supremizer associated with the rank-one inf-sup infimizer. (The scaling factor of $\|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-2}$ is required to satisfy the normalization condition: $(S_{\boldsymbol{u}}\hat{\boldsymbol{\xi}}_1, S_{\boldsymbol{u}}\hat{\boldsymbol{\xi}}_1)_{\mathcal{Y}} = \|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-2} = \hat{\lambda}_1^2$.) This also implies that the least stable mode may be explicitly expressed as $\hat{\boldsymbol{\xi}}_1 = \|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-2} S_{\boldsymbol{u}}^{-1}\boldsymbol{\psi}$. In other words, we do not need an iterative procedure to locate the minimum eigenvalue (and the associated eigenfunction) of (4.10).

Remark 11. We may consider an alternative proof of the β_ℓ-$\|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-1}$ equivalence by appeal to the generalized SVD. We have already shown that $\beta_\ell^{-2} = \sum_i \sigma_i^{-2}\ell(\boldsymbol{\xi}_i)^2$. We now express the adjoint $\boldsymbol{\psi} \in \mathcal{Y}$ in terms of the $\mathcal{Y}$-orthonormal test basis $\{\boldsymbol{\eta}_i\}_i$, $\boldsymbol{\psi} = \sum_i \theta_i \boldsymbol{\eta}_i$. From the definition of the adjoint and the SVD triple, $\ell(\boldsymbol{\xi}_j) = \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{\xi}_j, \boldsymbol{\psi}) = \sum_i \theta_i \partial\mathcal{G}(\boldsymbol{u}; \boldsymbol{\xi}_j, \boldsymbol{\eta}_i) = \theta_j \sigma_j$. Thus, $\boldsymbol{\psi} = \sum_i \sigma_i^{-1} \boldsymbol{\eta}_i \ell(\boldsymbol{\xi}_j)$ with norm $\|\boldsymbol{\psi}\|_{\mathcal{Y}}^2 = \sum_i \sigma_i^{-2}(\ell(\boldsymbol{\xi}_j))^2$. Thus, $\beta_\ell^{-1} = \|\boldsymbol{\psi}\|_{\mathcal{Y}}$.

C. Proof of Brezzi-Rappaz-Raviart perturbation bounds

We prove Proposition 4. Let $H : \mathcal{X} \to \mathcal{X}$ be a map satisfying

$$\partial \mathcal{G}(\boldsymbol{u}; H(\boldsymbol{w}), \boldsymbol{v}) = \partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) - \mathcal{G}(\boldsymbol{w}, \boldsymbol{v}) + \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \forall \boldsymbol{v} \in \mathcal{Y} .$$

The above equation is well-posed since $\partial \mathcal{G}(\boldsymbol{u}; \cdot, \cdot)$ is inf-sup stable; $\tilde{\boldsymbol{u}}$ is a fixed point of H.

We note that

$$\partial \mathcal{G}(\boldsymbol{u}; H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2), \boldsymbol{v}) = \partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}^1 - \boldsymbol{w}^2, \boldsymbol{v}) - (\mathcal{G}(\boldsymbol{w}^1, \boldsymbol{v}) - \mathcal{G}(\boldsymbol{w}^2, \boldsymbol{v}))$$

$$= \partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}^1 - \boldsymbol{w}^2, \boldsymbol{v}) - \partial \mathcal{G} \left(\frac{1}{2}(\boldsymbol{w}^1 + \boldsymbol{w}^2); \boldsymbol{w}^1 - \boldsymbol{w}^2, \boldsymbol{v} \right)$$

$$= \mathcal{C}(2\boldsymbol{u} - (\boldsymbol{w}^1 + \boldsymbol{w}^2), \boldsymbol{w}^1 - \boldsymbol{w}^2, \boldsymbol{v}) .$$

By the definition of the inf-sup constant and the continuity constant, it follows that

$$\beta(\boldsymbol{u}) \| H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2) \|_{\mathcal{X}} \leq \gamma^2 \| 2\boldsymbol{u} - (\boldsymbol{w}^1 + \boldsymbol{w}^2) \|_{\mathcal{X}} \| \boldsymbol{w}^1 - \boldsymbol{w}^2 \|_{\mathcal{X}} .$$

Thus, H is a contraction mapping for $\gamma^2 \beta(\boldsymbol{u})^{-1} \| 2\boldsymbol{u} - (\boldsymbol{w}^1 + \boldsymbol{w}^2) \|_{\mathcal{X}} < 1$ or, conservatively, for $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ in the ball $B(\boldsymbol{u}; \alpha_1)$ with $\alpha_1 \in [0, \beta(\boldsymbol{u})/(2\gamma^2))$.

In addition, we have

$$\partial \mathcal{G}(\boldsymbol{u}; H(\boldsymbol{w}) - \boldsymbol{u}, \boldsymbol{v}) = \partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w} - \boldsymbol{u}, \boldsymbol{v}) - (G(\boldsymbol{w}, \boldsymbol{v}) - G(\boldsymbol{u}, \boldsymbol{v})) + \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}}$$

$$= \partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w} - \boldsymbol{u}, \boldsymbol{v}) - \partial \mathcal{G} \left(\frac{1}{2}(\boldsymbol{w} + \boldsymbol{u}); \boldsymbol{w} - \boldsymbol{u}, \boldsymbol{v} \right) + \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}}$$

$$= \mathcal{C}(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{u}, \boldsymbol{v}) + \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathcal{Y}' \times \mathcal{Y}} .$$

By the definition of the dual norm, the inf-sup constant, and the continuity constant, we have

$$\beta(\boldsymbol{u}) \| H(\boldsymbol{w}) - \boldsymbol{u} \|_{\mathcal{X}} \leq \| \boldsymbol{f} \|_{\mathcal{Y}'} + \gamma^2 \| \boldsymbol{w} - \boldsymbol{u} \|_{\mathcal{X}}^2 .$$

Simple algebraic manipulation shows that, provided that $4\gamma^2 \beta(\boldsymbol{u})^{-2} \| \boldsymbol{f} \|_{\mathcal{Y}'} < 1$ — this condition is precisely the BRR condition (5.3) — we may choose $\alpha_2 \in \frac{1}{2} \beta(\boldsymbol{u}) \gamma^{-2} [1 - \sqrt{1 - 4\gamma^2 \beta(\boldsymbol{u})^{-2} \| \boldsymbol{f} \|_{\mathcal{Y}'}}, 1 + \sqrt{1 - 4\gamma^2 \beta(\boldsymbol{u})^{-2} \| \boldsymbol{f} \|_{\mathcal{Y}'}}]$ such that H maps $B(\boldsymbol{u}; \alpha_2)$ to itself. By the contraction mapping theorem, it follows that $\tilde{\boldsymbol{u}}$ is unique in the ball $B(\boldsymbol{u}; \beta(\boldsymbol{u})/(2\gamma^2))$ and furthermore

$$\| \tilde{\boldsymbol{u}} - \boldsymbol{u} \|_{\mathcal{X}} \leq \frac{\beta(\boldsymbol{u})}{2\gamma^2} \left(1 - \sqrt{1 - \frac{4\gamma^2 \| \boldsymbol{f} \|_{\mathcal{Y}'}}{\beta(\boldsymbol{u})^2}} \right) \leq \frac{2 \| \boldsymbol{f} \|_{\mathcal{Y}'}}{\beta(\boldsymbol{u})} ,$$

which is the BRR bound (5.4).

D. Proof of quadratic output bounds

We prove Proposition 5. To this end, we appeal to the equivalence of $\beta_\ell(\boldsymbol{u})$ and $\|\boldsymbol{\psi}\|_{\mathcal{Y}}^{-1}$ and prove (5.8) expressed in terms of the adjoint:

$$|\ell(\tilde{\boldsymbol{u}}) - \ell(\boldsymbol{u})| \leq \|\boldsymbol{\psi}\|_{\mathcal{Y}} \|\boldsymbol{f}\|_{\mathcal{Y}'} + 4 \frac{\gamma^2 \|\boldsymbol{\psi}\|_{\mathcal{Y}} + \gamma_{\ell^2}^2}{\beta(\boldsymbol{u})^2} \|\boldsymbol{f}\|_{\mathcal{Y}'}^2 \; ;$$

here, the adjoint is redefined as the solution to the dual problem: Find $\boldsymbol{\psi} \in \mathcal{Y}$ such that

$$\partial \mathcal{G}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{\psi}) = \partial \ell(\boldsymbol{u}; \boldsymbol{w}), \quad \forall \boldsymbol{w} \in \mathcal{X} \; . \tag{D.1}$$

For notational convenience, let $\delta \boldsymbol{u} \equiv \tilde{\boldsymbol{u}} - \boldsymbol{u}$. Because the semilinear form $\mathcal{G}$ is quadratic,

$$\mathcal{G}(\tilde{\boldsymbol{u}}, \boldsymbol{v}) - \mathcal{G}(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2} \left(\partial \mathcal{G}(\tilde{\boldsymbol{u}}; \delta \boldsymbol{u}, \boldsymbol{v}) + \partial \mathcal{G}(\boldsymbol{u}; \delta \boldsymbol{u}, \boldsymbol{v}) \right), \quad \forall \boldsymbol{v} \in \mathcal{Y} \; ,$$

Similarly, for the quadratic output,

$$\ell(\tilde{\boldsymbol{u}}) - \ell(\boldsymbol{u}) = \partial \ell(\boldsymbol{u}; \delta \boldsymbol{u}) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u}) \; .$$

The combination of the above two equations, the definition of the adjoint (D.1), and $\mathcal{G}(\boldsymbol{u}, \boldsymbol{v}) = 0$, $\forall \boldsymbol{v} \in \mathcal{Y}$, yields an output error representation formula,

$$\ell(\tilde{\boldsymbol{u}}) - \ell(\tilde{\boldsymbol{u}}) = \partial \ell(\boldsymbol{u}; \delta \boldsymbol{u}) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u}) = \partial \mathcal{G}(\boldsymbol{u}; \delta \boldsymbol{u}, \boldsymbol{\psi}) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u})$$

$$= \left[\mathcal{G}(\tilde{\boldsymbol{u}}, \boldsymbol{\psi}) - \frac{1}{2} (\partial \mathcal{G}(\tilde{\boldsymbol{u}}; \delta \boldsymbol{u}, \boldsymbol{\psi}) + \partial \mathcal{G}(\boldsymbol{u}; \delta \boldsymbol{u}, \boldsymbol{\psi})) \right]$$

$$+ \partial \mathcal{G}(\boldsymbol{u}; \delta \boldsymbol{u}, \boldsymbol{\psi}) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u})$$

$$= \mathcal{G}(\tilde{\boldsymbol{u}}, \boldsymbol{\psi}) + \frac{1}{2} \left(\partial \mathcal{G}(\boldsymbol{u}; \delta \boldsymbol{u}, \boldsymbol{\psi}) - \partial \mathcal{G}(\tilde{\boldsymbol{u}}; \delta \boldsymbol{u}, \boldsymbol{\psi}) \right) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u})$$

$$= \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle_{\mathcal{Y}' \times \mathcal{Y}} + \mathcal{C}(\delta \boldsymbol{u}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{\psi}) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u})$$

$$= \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle_{\mathcal{Y}' \times \mathcal{Y}} - \mathcal{C}(\delta \boldsymbol{u}, \delta \boldsymbol{u}, \boldsymbol{\psi}) + \ell^2(\delta \boldsymbol{u}, \delta \boldsymbol{u}) \; ,$$

where the second-to-last equality follows from the linearized form (2.13). The definitions of the continuity constants γ and γ_{ℓ^2} and the application of the (global) BRR perturbation bound on $\|\delta \boldsymbol{u}\|_{\mathcal{X}}$ yield the desired result.

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