# Electromagnetic channel capacity for practical purposes

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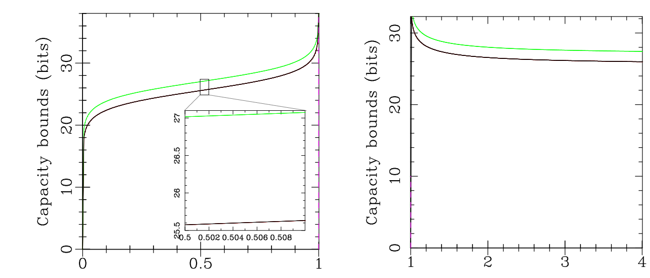
We give analytic upper bounds to the channel capacity $C$ for transmission of classical information in electromagnetic channels (bosonic channels with thermal noise). In the practically relevant regimes of high noise and low transmissivity, by comparison with know lower bounds on $C$, our inequalities determine the value of the capacity up to corrections which are irrelevant for all practical purposes. Examples of such channels are radio communication, infrared or visible-wavelength free space channels. We also provide bounds to active channels that include amplification.

Shannon [1] famously proved that the maximum number of bits transmitted through a narrowband Gaussian-noise channel is $C = \log_2(1 + S/N)$ for each use of the channel, where $S/N$ is the signal to noise ratio. At bottom, the noise has a quantum origin, and the calculation of the capacity requires a quantum description of the channel. Accordingly, one of the oldest questions in quantum information theory is the calculation of channel capacities [2, 3].




(a) (b)

FIG. 1: Plots of the bounds for the passive channel with thermal noise (a) and for the amplifier (b) in the practically relevant regimes of high thermal noise (large $N$): the upper and lower bounds (red and black curves of Figs. 2b and 2c) are basically coincident here. (a) Plots of the bounds (1), (4) and of the bound of Koenig and Smith [4] as a function of the transmissivity $\eta$ for a microwave radio communication, $\mathcal{E}_\eta^N$, with $N = 2000$ at room temperature, $\bar{N} = 10^{11}$ for $\sim 1$ mW transmission power (assuming that one channel use lasts for one oscillation period of the radiation). In this regime the lower bound (1) is indistinguishable from our upper bound (4) (black curve) (see also the magnification in the insert), whereas the gap with Koenig and Smith's bound (green curve) is evident. Typical transmissivities are very low for these channels (e.g., $\eta = 0.04$ for a 14 dB attenuation). (b) Capacity of the amplifier $\mathcal{A}_\kappa^N$ as a function of the gain $\kappa$: here the lower bound (1) is indistinguishable from the upper bound (6) (black line), whereas the upper bound (28) (green line) is not useful. Here again $\bar{N} = 10^{11}$ and $N = 2000$.

Some of the most practically relevant communication channels are active and passive bosonic channels with thermal noise, e.g. radio or infrared-light communication. In this paper we provide upper and lower bounds for their capacity. In the case of the passive bosonic channel, our bounds supersede the recent one given by Koenig and Smith [4]: in particular in contrast to their bound, for the practically relevant regimes of large thermal noise or low transmissivities (where each channel use can convey only small fractions of a bit) our bounds are sufficiently tight to constitute an expression for the capacity which is good for practical purposes. These findings are consistent with the Holevo-Werner conjecture [5], that Gaussian mixtures of coherent states achieve capacity and that these channels are additive. In other words, in situations of practical interest, quantum effects (such as entanglement among subsequent channel uses) do not give any advantage and a coding alphabet composed of coherent states (e.g., the is output from a maser or laser) achieves capacity. It is important to stress, however, that there are other regimes in which our inequalities are not tight: (slight) quantum advantages in low-noise regimes might be still possible.

We consider two passive channels: the thermal bosonic channel $\mathcal{E}_\eta^N$ that can be modeled by a beam splitter of transmissivity $\eta$ that mixes the signal with a thermal state with mean photon number $N$ (the capacity of this channel for $N = 0$ is already known [9]), and the classical additive noise channel $\mathcal{N}_n$ in which the signal is randomly displaced in the complex phase-space according to a Gaussian probability distribution of variance $n$. We also consider phase-insensitive amplifiers $\mathcal{A}_\kappa^N$ with gain $\kappa \geqslant 1$, whose additional input mode (required to ensure the correct commutation relations of the fields) is in a thermal state of mean photon number $N$. Using a signal of $\bar{N}$ average photons, with an alphabet of Gaussian-distributed coherent states, one find the following lower bounds for their capacities $C$:

$$C(\mathcal{E}_\eta^N) \geqslant g(\eta\bar{N} + (1-\eta)N) - g((1-\eta)N) , \quad (1)$$

$$C(\mathcal{N}_n) \geqslant g(\bar{N} + n) - g(n) , \quad (2)$$

$$C(\mathcal{A}_\kappa^N) \geqslant g(\kappa\bar{N} + (\kappa-1)(N+1)) - g((\kappa-1)(N+1)), \quad (3)$$

where $g(x) := (x+1)\log_2(x+1) - x\log_2 x$. (A proof of the Holevo-Werner conjecture would turn these into equalities, but it has been elusive even after a decade of concerted efforts [5–8].) The main result of our paper is a collection of upper bounds which asymptotically match the above lower bounds in the practically relevant regimes of high noise, or low transmissivity, or high amplification (see Fig. 1). In particular we show that the following inequalities apply

$$\begin{aligned} C(\mathcal{E}_\eta^N) &\leqslant g(\eta\bar{N}+(1-\eta)N) - g((1-\eta)N-\eta) , \quad (4)\\ &\quad \text{for } \eta \leqslant \tfrac{N}{N-1} , \\ C(\mathcal{N}_n) &\leqslant g(\bar{N}+n) - g(n-1) \text{ for } n \geqslant 1, \quad (5)\\ C(\mathcal{A}_\kappa^N) &\leqslant g(\kappa\bar{N}+(\kappa-1)(N+1)) - g(N(\kappa-1)-1) , \\ &\quad \text{for } \kappa \geqslant \tfrac{N+1}{N} . \quad (6) \end{aligned}$$

In addition to these simple bounds (which are nonetheless good enough for many applications) we also derive other, even tighter, bounds in what follows. All our bounds apply to narrowband channels, but they can be extended to broadband channels using the variational techniques we detailed in [10]. The remainder of the paper is devoted to the proof of these and of the further bounds.

*On Bounds and Conjectures:—* To characterize the channels one can use their action on the state's characteristic function $\chi(\mu) := \mathrm{Tr}[\rho\ e^{\mu a^\dagger - \mu^* a}]$ ($a$ being the annihilation operator of the mode): $\chi(\mu)$ is transformed by the channels $\mathcal{E}_\eta^N$, $\mathcal{N}_n$, and $\mathcal{A}_\kappa^N$ as (see, e.g., Ref. [6])

$$\begin{aligned} \chi(\mu) &\xrightarrow{\mathcal{E}_\eta^N} \chi(\eta\mu)\ e^{-(1-\eta)(N+1/2)|\mu|^2} , \quad (7)\\ \chi(\mu) &\xrightarrow{\mathcal{N}_n} \chi(\mu)\ e^{-n|\mu|^2} , \\ \chi(\mu) &\xrightarrow{\mathcal{A}_\kappa^N} \chi(\kappa\mu)\ e^{-(\kappa-1)(N+1/2)|\mu|^2} . \end{aligned}$$

The main difficulty in the calculation of the classical capacity of a quantum channel $\Phi$ is superadditivity [3]: there exist channels [11] in which the alphabet that achieves capacity must be composed by entangled quantum states that span multiple channel uses. Accordingly, one must regularize as follows [3]

$$C(\Phi) = \lim_{m\to\infty} C_\chi(\Phi^{\otimes m})/m , \quad (8)$$

where $\Phi^{\otimes m}$ indicates $m$ uses of the channel $\Phi$, and [12, 13]

$$C_\chi(\Psi) = \max_{\{p_i,\rho_i\}} S\Big(\sum_i p_i\Psi[\rho_i]\Big) - \sum_i p_i S(\Psi[\rho_i]) . \quad (9)$$

Here $S(\rho) := -\mathrm{Tr}\rho\log_2\rho$ is the von Neumann entropy, $\Psi[\rho_i]$ is the output state from the channel $\Psi$ (that may represent multiple uses of $\Phi$), and the maximization is performed over the set of ensembles $\{p_i, \rho_i\}$ formed by density matrices $\rho_i$ and probabilities $p_i$ that may satisfy some resource constraint (such as on the average photon number $\bar{N}$ discussed above). Lower bounds to $C(\Phi)$ can be obtained by calculating the right hand side of (9) for a specific encoding alphabet, i.e. fixing the value of $m$ (say $m = 1$) and using a specific choice of for $p_i$ and $\rho_i$, as was done to obtain the inequalities (1)-(3). In contrast, an upper bound for $C(\Phi)$ is provided by

$$C(\Phi) \leqslant S_{max}(\Phi) - \lim_{m\to\infty} S_{min}(\Phi^{\otimes m})/m , \quad (10)$$

where $S_{max}(\Phi) = \max_\rho S(\Phi(\rho))$ is the maximum output entropy for a single channel use [using the same restrictions in the maximization as in the definition of $C(\Phi)$], and $S_{min}(\Psi) = \min_\rho S(\Psi(\rho))$ is the (unrestricted) minimum output entropy of the channel $\Psi$. The regularization over $m$ in (10) is required by the superadditivity of the minimum output entropy [11], and constitutes the main difficulty in deriving bounds through (10). However, if $\Phi = \Phi_{EB}$ is entanglement-breaking [14, 15], the regularization is unnecessary [16] and (10) can be replaced by

$$C(\Phi_{EB}) \leqslant S_{max}(\Phi_{EB}) - S_{min}(\Phi_{EB}) , \quad (11)$$

[notice that *both* $S_{min}(\Phi_{EB}^{\otimes m})$ and $C_\chi(\Phi_{EB}^{\otimes m})$ are additive quantities].

For $\Phi = \mathcal{E}_\eta^N$, $\mathcal{N}_n$, or $\mathcal{A}_\kappa^N$ the first term on the right of inequality (10), $S_{max}(\Phi)$, is easily computed by exploiting the fact that the thermal state maximizes the entropy for fixed average photon number $\bar{N}$:

$$\begin{aligned} S_{max}(\mathcal{E}_\eta^N) &= g(\eta\bar{N}+(1-\eta)N) , \quad (12)\\ S_{max}(\mathcal{N}_n) &= g(\bar{N}+n) , \\ S_{max}(\mathcal{A}_\kappa^N) &= g(\kappa\bar{N}+(\kappa-1)(N+1)) . \end{aligned}$$

In contrast, evaluating the second term on the right of inequality (10) is extremely demanding: the Holevo-Werner conjecture can be rephrased into a conjecture on the values of $S_{min}(\Phi^{\otimes m})$ [6, 7, 17, 18], which states that the *min* is achieved by a vacuum state $|0\rangle^{\otimes m}$. If this were true, one could use (10) to provide upper bounds that exactly match the lower bounds (1)-(3). A proof of this is lacking, but in Ref. [6] several bounds were obtained for the special case of $m = 1$: they constrain $S_{min}(\mathcal{E}_\eta^N)$ and $S_{min}(\mathcal{N}_n)$ close to their conjectured values of $g((1-\eta)N)$ and $g(n)$, respectively. Using (11), such bounds can be immediately translated into constraints on the capacity $C$ whenever the maps are entanglement-breaking, i.e. when $\eta \leqslant N/(N+1)$ for $\mathcal{E}_\eta^N$, when $n \geqslant 1$ for $\mathcal{N}_n$, and when $N \geqslant 1/(\kappa-1)$ for $\mathcal{A}_\kappa^N$ [15]. For instance, exploiting this fact, inequality (4) can be derived by replacing the term $S_{min}(\mathcal{E}_\eta^N)$ of (11) with the single-mode lower bound A of Ref. [6]. More generally, the same approach exploited in [6] can be adapted to the multi-channel use scenario to construct tight inequalities directly for the quantities $S_{min}([\mathcal{E}_\eta^N]^{\otimes m})/m$ and $S_{min}([\mathcal{N}_n]^{\otimes m})/m$. When substituted into (10) together with the identities (12) these then translate into

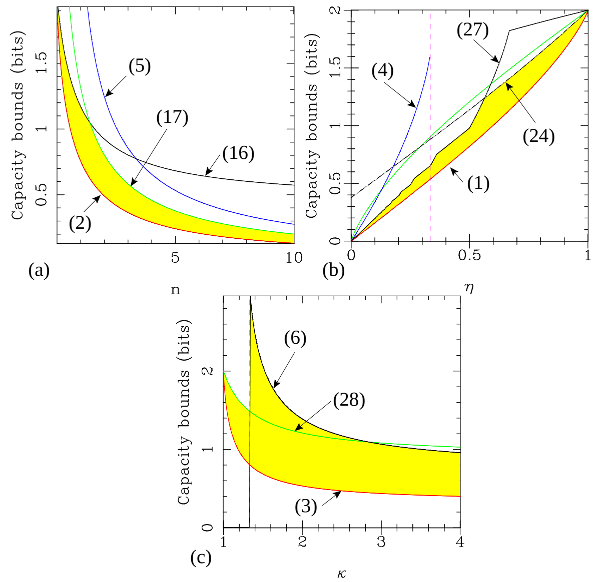



FIG. 2: Plots of the bounds in regimes that emphasize the gap between the lower and the upper bounds (these regimes are typically not interesting in practical applications). (a) Capacity of the Gaussian channel $\mathcal{N}_n$ for $\bar{N}=1$. Red curve: lower bound (3); blue, black, green curves: upper bounds (5), (16), and (17), respectively. The yellow area emphasizes the gap between the best upper and lower bounds. (b) Capacity of the passive electromagnetic channel $\mathcal{E}_\eta^N$. Red curve: lower bound (1); blue curve: upper bound (4) (valid only for $\eta \leqslant N/(N+1)$ shown as a vertical dashed line where the channel becomes entanglement breaking); green curve: Koenig and Smith's bound from [4]; black line: upper bound (27); black dashed line: upper bound (24). Here $N=0.5$ thermal photons and $\bar{N}=1$ average photons in the signal (which gives bits-per-photon for each channel use). (c) Plots of the bounds for the amplifying channel $\mathcal{A}_\kappa^N$ with gain $\kappa$. Red curve: lower bound (2); black curve: upper bound (6) (valid only for $\kappa \geqslant (N+1)/N$); green curve: upper bound (28). The discontinuity for $\kappa=(N+1)/N$ (vertical dashed line) separates the entanglement breaking regime on the right from the pure-loss regime on the left. Here $N=3$ and $\bar{N}=1$.

a collection of upper bounds for $C$ that hold beyond the entanglement-breaking regime detailed above.

*Bounds for the Additive Classical noise channel* $\mathcal{N}_n$*:—* As detailed below, the bounds $a$, $b$, and $d$ of Ref. [6] for $m=1$ can be generalized to arbitrary $m$ as follows

$$S_{min}(\mathcal{N}_n^{\otimes m})/m \;\geqslant\; g(n-1)\,, \quad [\forall n \geqslant 1] \tag{13}$$
$$S_{min}(\mathcal{N}_n^{\otimes m})/m \;\geqslant\; \log_2(2n+1)\,, \tag{14}$$
$$S_{min}(\mathcal{N}_n^{\otimes m})/m \;\geqslant\; 1+\log_2(n)\,, \tag{15}$$

whence, using (10), Eq. (13) gives (5), while Eqs. (14) and (15) respectively give the further bounds

$$C(\mathcal{N}_n) \;\leqslant\; g(\bar{N}+n)-\log_2(2n+1)\,, \tag{16}$$
$$C(\mathcal{N}_n) \;\leqslant\; g(\bar{N}+n)-1-\log_2(n)\,, \tag{17}$$

[the generalization of the bound $c$ of [6] is not reported here since it converges to Eq. (14) for $m \to \infty$]. These bounds are compared to the lower bound (2) in Fig. 2(a): note how the gap between the upper and lower bounds closes asymptotically for high noise, $n \to \infty$.

The proof of Eq. (13), and hence of the bound (5), was given in Ref. [17] by expanding a generic input state $\rho$ in terms of its multi-mode Husimi distribution function and applying the concavity of von Neumann entropy. An alternative proof follows from inequality $a$ of Ref. [6] and from (11), using the fact that the channel $\mathcal{N}_n$ is entanglement breaking for $n \geqslant 1$ [15].

The proof of Eq. (14) exploits the fact that the von Neumann entropy is never smaller than the Rényi entropy of order 2 [19, 20] i.e. $S(\rho) \geqslant S_2(\rho) := -\log_2 \mathrm{Tr}[\rho^2]$. Thus, for all input density matrices $\rho$ of $m$ channel uses we have

$$S(\mathcal{N}_n^{\otimes m}(\rho)) \geqslant S_2(\mathcal{N}_n^{\otimes m}(\rho)) \geqslant m\,\log_2(2n+1)\,, \tag{18}$$

where the last inequality follows from the fact that the minimum Rényi entropy of integer order at the output of the channel $\mathcal{N}_n$ is additive and saturated by the vacuum input state [21]. The bound (14), and hence (16), follow by minimizing with respect to $\rho$.

The proof of Eq. (15) closely follows the proof of bound $d$ in Ref. [6] for $m=1$. Indeed, given a generic pure input state $|\psi\rangle$, the eigenvalues $\gamma_k$ of the relevant output state $\rho' = \mathcal{N}_n^{\otimes m}(|\psi\rangle\langle\psi|)$ can be expressed as

$$\gamma_k = \int d^{2m}\vec{\mu}\; P_n^{(m)}(\vec{\mu})|\langle\gamma_k|D(\vec{\mu})|\psi\rangle|^2\,, \tag{19}$$

where $|\gamma_k\rangle$ is the corresponding eigenvector of $\rho'$, $D(\vec{\mu})$ is the $m$-mode displacement operator, $P_n^{(m)}(\vec{\mu}) := \exp[-|\vec{\mu}|^2/n]/(\pi n)^m$, and the integral is performed over the $m$-dimensional complex vectors $\vec{\mu} \in \mathbb{C}^m$. By convexity, for all $z \geqslant 1$ one can write

$$\begin{aligned}\mathrm{Tr}[(\rho')^z] \;&\leqslant\; \sum_k \int \frac{d^{2m}\vec{\mu}}{\pi^m}\,[\pi^m P_n^{(m)}(\vec{\mu})]^z|\langle\gamma_k|D(\vec{\mu})|\psi\rangle|^2 \\ &=\; 1/(zn^{z-1})^m\,,\end{aligned} \tag{20}$$

which gives inequality (15), and hence (17), by remembering that $S(\rho') = \lim_{z\to 1^+}\log_2 \mathrm{Tr}[(\rho')^z]/(1-z)$ [19, 20].

*Bounds for the Lossy Thermal channel* $\mathcal{E}_\eta^N$*:—* The bounds (13)-(15) can be immediately turned into inequalities for $S_{min}([\mathcal{E}_\eta^N]^{\otimes m})$ by exploiting the compositions rules [6] that link $\mathcal{E}_\eta^N$ and $\mathcal{N}_n$ that also apply to the multi-use scenario $m>1$. In particular, $[\mathcal{E}_\eta^N]^{\otimes m} = \mathcal{N}_{(1-\eta)N}^{\otimes m} \circ [\mathcal{E}_\eta^0]^{\otimes m}$. Hence, following the same reasoning of Ref. [6], we find

$$S_{min}([\mathcal{E}_\eta^N]^{\otimes m}) \geqslant S_{min}(\mathcal{N}_{(1-\eta)N}^{\otimes m})\,. \tag{21}$$

In particular, from (14) and (15) we obtain the multi-use versions of the bounds B and C of Ref. [6], i.e.,

$$S_{min}([\mathcal{E}_\eta^N]^{\otimes m})/m \;\geqslant\; \log_2(2(1-\eta)N+1)\,, \tag{22}$$
$$S_{min}([\mathcal{E}_\eta^N]^{\otimes m})m \;\geqslant\; 1+\log_2((1-\eta)N)\,, \tag{23}$$

which give rise to the following bounds for the capacity

$$C(\mathcal{E}_\eta^N) \leqslant g((1-\eta)\bar{N}+N) - \log_2(2(1-\eta)N+1) \,, \quad (24)$$

$$C(\mathcal{E}_\eta^N) \leqslant g((1-\eta)\bar{N}+N) - 1 - \log_2((1-\eta)N) \,, \quad (25)$$

[the bound obtained from (13) is not reported here as it is always subsided by the inequality (4)]. Further bounds can be obtained by generalizing the inequalities E and F of Ref. [6]: for all integers $k$, we have for inequality E:

$$\frac{S_{min}([\mathcal{E}_\eta^N]^{\otimes m})}{m} \geqslant \tfrac{k-1}{k}\, g(\tfrac{k}{k-1}(1-\eta)N) \text{ for } \eta \leqslant 1/k,$$
$$\frac{S_{min}([\mathcal{E}_\eta^N]^{\otimes m})}{m} \geqslant \tfrac{k-1}{k}\, g\left(\tfrac{k}{k-1}\left[(1-\eta)N - \eta + \frac{1}{k}\right]\right) \text{ for } \eta \geqslant 1/k, \quad (26)$$

and for inequality F:

$$\frac{S_{min}([\mathcal{E}_\eta^N]^{\otimes m})}{m} \geqslant \tfrac{k-1}{k}\, g((1-\eta)N) + \tfrac{1}{k}\tfrac{S_{min}([\mathcal{N}_{(1-\eta)N}]^{\otimes m})}{m} \text{ for } \eta \leqslant 1/k,$$
$$\frac{S_{min}([\mathcal{E}_\eta^N]^{\otimes m})}{m} \geqslant \tfrac{k-1}{k}\, g((1-\eta)N - \eta + \frac{1}{k}) + \tfrac{1}{k}\frac{S_{min}([\mathcal{N}_{n'}]^{\otimes m})}{m} \text{ for } \eta \geqslant 1/k, \quad (27)$$

with $n' = (1-\eta)N - \eta + 1/k$ [see the supplementary material for the proof]. The associated bounds for $C(\mathcal{E}_\eta^N)$ are obtained by substituting the above expressions into Eq. (10) together with the identities (12). In Fig. 2(b) we report the one associated to (27), together with the bounds (4) and (24), for a direct comparison with Koenig and Smith's inequality [4], and with the lower bound (1). In the low-noise regime of small $N$ a capacity bound was presented also in [8].

*Bounds for the Amplifying channel* $\mathcal{A}_\kappa^N$*:—* We now prove that the capacity of the channel $\mathcal{A}_\kappa^N$ satisfies inequality (6) and

$$C(\mathcal{A}_\kappa^N) \leqslant g(\kappa\bar{N}/[N(\kappa-1)+\kappa]), \text{ for } N < \tfrac{1}{\kappa-1} \,, \quad (28)$$

see Figs. 1(b) and 2(c). The proof of inequality (6) can be obtained by adapting the derivation of (13) provided in Ref. [17]. Specifically, the Husimi distribution of a generic input state $\rho$ for $m$ channel uses is

$$Q(\vec{\alpha}) = \langle\vec{\alpha}|\rho|\vec{\alpha}\rangle/\pi^m \,, \quad \rho = \int d^{2m}\vec{\alpha}\, Q(\vec{\alpha})\, \sigma(\vec{\alpha}) \,, \quad (29)$$

where $|\vec{\alpha}\rangle$ is a $m$-mode coherent state and $\sigma(\vec{\alpha}) := \int \frac{d^{2m}\vec{\mu}}{\pi^m}\; D(\vec{\mu})\; e^{\vec{\mu}^\dagger\cdot\vec{\mu} - \vec{\mu}^T\cdot\vec{\mu}^* - \vec{\mu}^\dagger\cdot\vec{\mu}/2}$. The corresponding output state can then be expressed as $[\mathcal{A}_\kappa^N]^{\otimes m}[\rho] = \int d^{2m}\vec{\alpha}\; Q(\vec{\alpha})\; [\mathcal{A}_\kappa^N]^{\otimes m}[\sigma(\vec{\alpha})]$, while the concavity of the output entropy implies

$$S([\mathcal{A}_\kappa^N]^{\otimes m}[\rho]) \geqslant \int d^{2m}\vec{\alpha}\, Q(\vec{\alpha}) S([\mathcal{A}_\kappa^N]^{\otimes m}[\sigma(\vec{\alpha})]) \,, \quad (30)$$

which is meaningful only when $[\mathcal{A}_\kappa^N]^{\otimes m}[\sigma(\vec{\alpha})]$ is a quantum state, i.e. if $N(\kappa-1) \geqslant 1$, when $[\mathcal{A}_\kappa^N]^{\otimes m}[\sigma(\vec{\alpha})]$ is an $m$-mode thermal state with average photon number $N(\kappa-1)-1$ per mode, so that $S([\mathcal{A}_\kappa^N]^{\otimes m}[\sigma(\vec{\alpha})]) = m\, g(N(\kappa-1)-1)$. Substituting it into (30) we find

$$\tfrac{1}{m} S_{min}([\mathcal{A}_\kappa^N]^{\otimes m}) \geqslant g(N(\kappa-1)-1) \,, \text{ for } N \geqslant \tfrac{1}{\kappa-1} \quad (31)$$

which, through (10), implies (6).

Finally, the proof of the bound (28) uses the concatenation $\mathcal{A}_\kappa^N = \mathcal{A}_G^0 \circ \mathcal{E}_\eta^0$ with $G = N(\kappa-1)+\kappa$ and $\eta = \kappa/G$, the fact that the capacity is always degraded under channel multiplication, and the fact that $C(\mathcal{E}_\eta^0) = g(\eta\bar{N})$ [9].

*Conclusions:—* We have given upper and lower bounds for the classical capacity of important active and passive bosonic channels, and we have shown that these bounds asymptotically coincide (yielding the actual capacity) in the regimes of practical interest, i.e. for low transmissivity, high thermal noise, or high amplification.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## Supplemental Material

Here we provide explicit derivations of the inequalities (26) and (27).

*Proof of Eq. (26):—* This inequality can be easily obtained by generalizing to the multimode scenario the beam-splitter decomposition of the channel $\mathcal{E}_\eta^N$ detailed in the Appendix D1 of Ref. [6] [the same decomposition was also exploited in Ref. [7]]. Consider first the case $\eta = 1/k$ with $k$ an integer [generalization to arbitrary $\eta$ will given later]. The basic idea is to express the transformation induced by the map $\mathcal{E}_{1/k}^N$ in terms of a sequence of $k-1$ beam-splitter interactions that couple the incoming signal mode state $\rho$ with $k$ independent bosonic thermal baths states $\rho_{th}$ characterized by the same photon number $N$. As discussed in Ref. [6] this can be done in such a way that local observers located at each of the $k$ outputs of the array will receive [up to an irrelevant local unitary transformation] the *same* output signal $\mathcal{E}_{1/3}^N[\rho]$. For instance for $k = 3$ this can be obtained by setting the transmissivity of the first beam splitter equal to $\eta_1 = 2/3$ and the second one to $\eta_2 = 1/2$. The same construction clearly can be applied to channel $[\mathcal{E}_{1/k}^N]^{\otimes m}$ of the $m$-channel use scenario by repeating the decomposition for each channel independently. An example of the resulting scheme for $k = 3$ and $m = 2$ is shown in Fig. 3: here $A_1$ and $A_2$ represent the two channel inputs that in principle can be loaded with a non separable state $\rho$; $B_1$, $C_1$ are instead the two thermal bath modes needed to represent the first channel use, while $B_2$ and $C_2$ are those associated with the second channel use [all of them being initialized in thermal states having average photon-number $N$]. In this extended configuration one can easily verify that the two-mode states at the ports $A'_1A'_2$, $B'_1B'_2$, and $C'_1C'_2$ of the figure are all unitarily equivalent to the density matrix $[\mathcal{E}_{1/3}^N]^{\otimes 2}(\rho)$ [in other words, up to local unitary transformations, each one of those output couples yields a unitary dilation [3] of the same channel $[\mathcal{E}_{1/3}^N]^{\otimes 2}$]. This in particular implies that the associated output entropies must be identical, i.e. $S(A'_1A'_2) = S(B'_1B'_2) = S(C'_1C'_2) = S([\mathcal{E}_{1/3}^N]^{\otimes 2}(\rho))$. Exploiting the sub-additivity of the von Neumann entropy [3] we can hence write

$$S(A'_1A'_2B'_1B'_2C'_1C'_2) \leqslant 3S([\mathcal{E}_{1/3}^N]^{\otimes 2}(\rho)) \ , \tag{32}$$

where $S(A'_1A'_2B'_1B'_2C'_1C'_2)$ is the entropy of the joint state at the output of the device. Observing that the transformation [i.e., the beam-splitter couplings] that takes the input modes of the system $A_1A_2B_1B_2C_1C_2$ to their associated output $A'_1A'_2B'_1B'_2C'_1C'_2$ configuration is unitary, we can then identify $S(A'_1A'_2B'_1B'_2C'_1C'_2)$ with the input entropy $S(A_1A_2B_1B_2C_1C_2)$. The latter can easily be computed by noticing that the incoming state is just a tensor product of $\rho$ with $m(k-1) = 4$ bosonic thermal states with mean photon-number $N$, i.e., $S(A_1A_2B_1B_2C_1C_2) = S(\rho) + 4g(N)$. Substituting this into Eq. (32) we finally get

$$S([\mathcal{E}_{1/3}^N]^{\otimes 2}(\rho)) \geqslant S(\rho) + \frac{4}{3}g(N) \geqslant \frac{4}{3}g(N) \ . \tag{33}$$

The same argument can be easily repeated for arbitrary $m$ and $k$ integers: in this case, we use $m(k-1)$ local bath modes organized in $m$ parallel rows, each containing $k-1$ beam-splitter transformations whose transmissivities are determined as in Ref. [6]. Similarly to the case explicitly discussed above, an inequality for $S([\mathcal{E}_{1/k}^N]^{\otimes m}(\rho))$ can be obtained via sub-additivity by grouping the $mk$ output modes into $k$ subsets of $m$ elements each. The resulting expression is

$$S([\mathcal{E}_{1/k}^N]^{\otimes m}(\rho)) \geqslant m \ \frac{(k-1)}{k} \ g(N) \ . \tag{34}$$

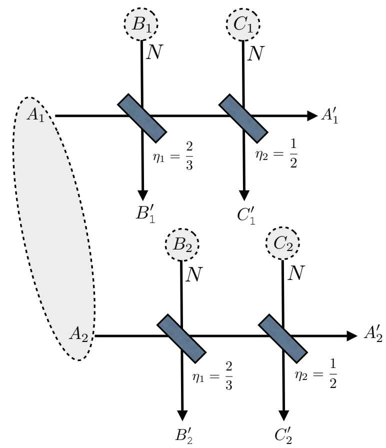


FIG. 3: Beam splitter-decomposition scheme for the channel $[\mathcal{E}_\eta^N]^{\otimes m}$ with $\eta = 1/3$ and $m = 2$. Thermal states of mean photon-number $N$ are injected at the input ports $B_1$, $C_1$, $B_2$, and $C_2$.

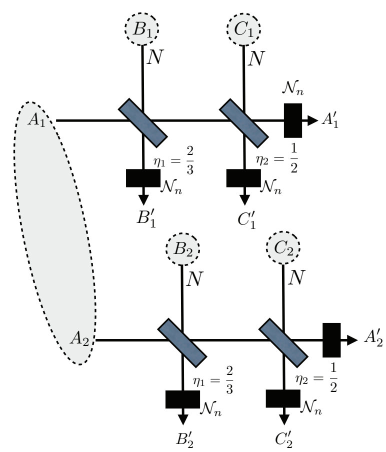


FIG. 4: Beam-splitter array used to derive Eq. (27), depicted for $m = 2$.

Generalization of this inequality to $\eta \leqslant 1/k$ can finally be obtained along the same lines used in Ref. [6] by exploiting the following composition rules

$$[\mathcal{E}_{\eta_2}^{N_2}]^{\otimes m} \circ [\mathcal{E}_{\eta_1}^{N_1}]^{\otimes m} = [\mathcal{E}_{\eta_1\eta_2}^{N'}]^{\otimes m} \,, \tag{35}$$

which is a trivial multi-mode generalization of the identity (19) from [6] [here $N' = [\eta_2(1-\eta_1)N_1 + (1-\eta_2)N_2]/(1-\eta_1\eta_2)$]. The reader can check that the resulting expression coincides with the first part of the inequality (26). Similarly, Eq. (34) can be used to induce a bound for $\eta \geqslant 1/k$ by following the same line of reasoning presented in [6] while exploiting the composition rule

$$[\mathcal{E}_{\eta}^{N}]^{\otimes m} = [\mathcal{E}_{\eta'}^{N'}]^{\otimes m} \circ [\mathcal{A}_{\eta/\eta'}^{0}]^{\otimes m} \,, \qquad \text{for } \eta \geqslant \eta', \tag{36}$$

which is the $m$-mode counterpart of the identity (B3) of [6]. The resulting inequality yields the second part of (26).

*Proof of Eq. (27):—* The $m = 1$ version of this inequality was derived in [6], by exploiting a beam-splitter array obtained by applying a channel $\mathcal{N}_n$ at each of the output ports of the scheme used to derive the $m = 1$ equivalent of Eq. (26) [see Fig. 12 of [6]]. As for Eq. (26), the main difficulty in applying the same argument to arbitrary $m$ is generalizing such an array to the multi-mode case scenario and properly grouping the corresponding output ports. This can be done as sketched in Fig. 4: i.e., adding $\mathcal{N}_n$ to each of the ports in Fig. 3 and by keeping the same grouping scheme as before. With this guidance the reader can now closely follow the same derivation given in Ref. [6] [the steps are rather cumbersome, but basically coincide with those we have discussed in the previous section].