Journal of Algebra 322 (2009) 2621–2630

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Embeddings of homogeneous spaces into irreducible modules

Ivan Losev

Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 24 September 2007
Available online 14 August 2009
Communicated by Peter Littelmann

Keywords:
Reductive group
Quasi-affine homogeneous space
Irreducible module
Embedding
Central automorphisms

ABSTRACT

Let G be a connected reductive algebraic group. We find a necessary and sufficient condition for a quasi-affine homogeneous space G/H to have an embedding into an irreducible G-module. For reductive H we also find a necessary and sufficient condition for a closed embedding of G/H into an irreducible module to exist. These conditions are stated in terms of the group of *central* automorphisms of G/H.

1. Introduction

The base field is the field $\mathbb{C}$ of complex numbers. Throughout the paper G denotes a connected reductive algebraic group, B a Borel subgroup of G and T a maximal torus of B.

The celebrated theorem of Chevalley states that any homogeneous space can be embedded (as a locally-closed subvariety) into the projectivization of a G-module. If H is an observable subgroup of G, that is, the homogeneous space G/H is quasi-affine, then G/H can be embedded even into a G-module itself, see, for example, [7, Theorem 1.6].

Problem 1.1. Describe all observable subgroups H such that G/H can be embedded into an *irreducible* G-module.

To state the answer to that problem we need the definition of a *central* automorphism of a G-variety. Let X be an irreducible G-variety. The subspace $\mathbb{C}(X)^{(B)}_\lambda \subset \mathbb{C}(X)$ consisting of all B-semiinvariant functions of weight $\lambda \in \mathfrak{X}(B)$ on X is stable under every G-equivariant automorphism of X. The following definition is due to Knop [2].

E-mail address: ivanlosev@math.mit.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.07.022

Definition 1.2. A G-equivariant automorphism of X is called *central* if it acts on any $\mathbb{C}(X)^{(B)}_\lambda$ by the multiplication by a constant.

We denote the group of central automorphisms of X by $\mathfrak{A}_G(X)$. We write $\mathfrak{A}_{G,H}$ instead of $\mathfrak{A}_G(G/H)$. It was shown by Knop [2, Section 5], that $\mathfrak{A}_{G,H}$ is an algebraic quasi-torus, that is, a closed subgroup of an algebraic torus.

Theorem 1.3. *Let H be an observable subgroup of G. Then the following conditions are equivalent*:

(a) *G/H can be embedded into an irreducible G-module.*
(b) *$\mathfrak{A}_{G,H}$ is a finite cyclic group or a one-dimensional torus.*

For a given subgroup $H \subset G$ the group $\mathfrak{A}_{G,H}$ can be computed using techniques from [4]. Namely, $\mathfrak{A}_{G,H}$ is the quotient of the *weight lattice* of G/H by the *root lattice* of G/H. An algorithm for computing the weight lattice is the main result of [4]. The computation of the root lattice can be reduced to that of the weight lattice by using [4, Proposition 5.2.1].

If H is a reductive subgroup of G or, equivalently, G/H is affine, then one may pose the following question:

Problem 1.4. Is there a *closed* embedding of G/H into an irreducible G-module?

Here is an answer.

Theorem 1.5. *Let H be a reductive subgroup of G. Then the following conditions are equivalent*:

(a) *There is a closed G-equivariant embedding of G/H into an irreducible G-module.*
(b) *$\mathfrak{A}_{G,H}$ is a finite cyclic group.*

We prove Theorems 1.3, resp. 1.5, in Sections 3, resp. 4. In Section 5 we present some examples of applications of our theorems.

2. Notation and conventions

$A^{(B)}_\mu$	the subspace of all B-semiinvariant functions of weight μ in a G-algebra A, where G is a connected reductive group.
$[\mathfrak{g},\mathfrak{g}]$	the derived subalgebra of a Lie algebra $\mathfrak{g}$.
G°	the connected component of unit of an algebraic group G.
$\mathrm{Gr}_i(V)$	the Grassmanian of i-dimensional subspaces in a vector space V.
$R_u(G)$	the unipotent radical of an algebraic group G.
G_x	the stabilizer of a point $x \in X$ under an action $G : X$.
$\mathrm{Int}(\mathfrak{g})$	the group of inner automorphisms of a Lie algebra $\mathfrak{g}$.
$N_G(H)$	the normalizer of a subgroup H in a group G.
$V^{\mathfrak{g}}$	$= \{v \in V \mid \mathfrak{g}v = 0\}$, where $\mathfrak{g}$ is a Lie algebra and V is a $\mathfrak{g}$-module.
$V(\mu)$	the irreducible module with highest weight μ over a reductive algebraic group or a reductive Lie algebra.
$\mathfrak{X}(G)$	the character lattice of an algebraic group G.
X^G	the fixed-point set for an action of G on X.
$\#X$	the cardinality of a set X.
$Z(G)$ (resp., $\mathfrak{z}(\mathfrak{g})$)	the center of an algebraic group G (resp., of a Lie algebra $\mathfrak{g}$).
$Z_G(\mathfrak{h})$ (resp., $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$)	the centralizer of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in an algebraic group G (resp., in its Lie algebra $\mathfrak{g}$).
λ^*	the dual weight to a dominant weight λ.

If an algebraic group is denoted by a capital Latin letter, then we denote its Lie algebra by the corresponding small fracture letter, for example, $\widehat{\mathfrak{h}}$ denotes the Lie algebra of $\widehat{H}$. All topological terms refer to the Zariski topology.

3. Proof of Theorem 1.3

First, we fix some notation and recall some definitions from the theory of algebraic transformation groups.

In this section H denotes an observable subgroup of G. The group of G-equivariant automorphisms of G/H is identified with $N_G(H)/H$. We consider $\mathfrak{A}_{G,H}$ as a subgroup in $N_G(H)/H$. Denote by H^{sat} the inverse image of $\mathfrak{A}_{G,H}$ in $N_G(H)$.

Let X be an irreducible G-variety. An element $\lambda \in \mathfrak{X}(T)$ is said to be a *weight of* X if $\mathbb{C}(X)^{(B)}_\lambda \neq 0$. Clearly, all weights of X form a subgroup of $\mathfrak{X}(T)$ called the *weight lattice* of X and denoted by $\mathfrak{X}_{G,X}$. The rank of $\mathfrak{X}_{G,X}$ is called the *rank* of X and is denoted by $\mathrm{rk}_G(X)$. We put $\mathfrak{a}_{G,X} = \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{C}$. If $X = G/G_0$, then we write $\mathfrak{X}_{G,G_0}$ instead of $\mathfrak{X}_{G,G/G_0}$. It is easy to see that the subspace $\mathfrak{a}_{G,G/G_0}$ depends only on the pair $(\mathfrak{g}, \mathfrak{g}_0)$. Thus we write $\mathfrak{a}_{\mathfrak{g},\mathfrak{g}_0}$ instead of $\mathfrak{a}_{G,G/G_0}$. If $\widehat{G}_0$ is a subgroup of G containing G_0, then there exists a dominant G-equivariant morphism $G/G_0 \to G/\widehat{G}_0$ and thence $\mathfrak{X}_{G,\widehat{G}_0} \subset \mathfrak{X}_{G,G_0}$.

The codimension of a general B-orbit in X is called the *complexity* of X and is denoted by $c_G(X)$. Again, we write $c_{\mathfrak{g},\mathfrak{g}_0}$ instead of $c_G(G/G_0)$. Let us note that $c_{\mathfrak{g},\widehat{\mathfrak{g}}_0} \leqslant c_{\mathfrak{g},\mathfrak{g}_0}$ whenever $G_0 \subset \widehat{G}_0$. For an arbitrary (not necessarily algebraic) subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we set $c_{\mathfrak{g},\mathfrak{h}} := \min_{g\in G} \dim \mathfrak{g}/(\mathrm{Ad}(g)\mathfrak{b} + \mathfrak{h})$.

Let us proceed to the proof of Theorem 1.3. The implication (a) $\Rightarrow$ (b) is easy.

Proof of (a) $\Rightarrow$ (b). By the Frobenius reciprocity, there is an $N_G(H)$-equivariant isomorphism $V(\lambda)^H \cong \mathbb{C}[G/H]^{(B)}_{\lambda^*}$. Clearly, (a) implies that the action of $N_G(H)/H$ on $V(\lambda)^H$ is effective for some λ. Now (b) follows easily from the definition of the subgroup $\mathfrak{A}_{G,H} \subset N_G(H)/H$. □

The implication (b) $\Rightarrow$ (a) will follow from the following

Proposition 3.1. *Suppose $\mathfrak{A}_{G,H}$ is a cyclic finite group or a one-dimensional torus. Then there is a dominant weight λ such that $V(\lambda)^H \neq \{0\}$ and the subset $\bigcap_{\widehat{H}\supsetneq H} V(\lambda)^{\widehat{H}}$ is not dense in $V(\lambda)^H$.*

The scheme of the proof of the proposition is, roughly, as follows. On the first step we prove that for an appropriate dominant weight λ the complexity $c_{\mathfrak{g},\mathfrak{g}_v}$ for a point $v \in V(\lambda)^H$ in general position coincides with $c_{\mathfrak{g},\mathfrak{h}}$. On the second step we check that one may choose λ such that $\mathfrak{g}_v = \mathfrak{h}$ for $v \in V(\lambda)^H$ in general position. At last, we show that $G_v = H$ for general $v \in V(\lambda)^H$.

We begin with some simple lemmas.

Lemma 3.2. $\dim V(\nu)^H \leqslant \dim V(\nu+\mu)^H$ *for any dominant weights μ, ν such that $V(\mu)^H \neq 0$.*

Proof. By the Frobenius reciprocity, $V(\nu)^H \cong \mathbb{C}[G/H]^{(B)}_{\nu^*}$, $V(\nu+\mu)^H \cong \mathbb{C}[G/H]^{(B)}_{(\nu+\mu)^*}$. The map $f_1 \mapsto ff_1 : \mathbb{C}[G/H]^{(B)}_{\nu^*} \hookrightarrow \mathbb{C}[G/H]^{(B)}_{(\nu+\mu)^*}$ is injective for any $f \in \mathbb{C}[G/H]^{(B)}_{\mu^*}$, $f \neq 0$. □

In the sequel we will need some properties of central automorphisms.

Lemma 3.3.

1. *An element $n \in N_G(H)/H$ is central iff it acts trivially on $\mathbb{C}(G/H)^B$.*
2. $\mathfrak{A}_{G,H} \subset Z(N_G(H)/H)$.

Proof. In this proof and below we will need the following standard fact which is a special case of [7, Theorem 3.3].

Lemma 3.4. *Let X be an affine G-variety with open G-orbit G/H. Then any element of $\mathbb{C}(G/H)^B$ can be represented as a fraction of two regular elements of $\mathbb{C}[X]^{(B)}$ of equal B-weights.*

So to prove assertion 1 it is enough to check that $\mathfrak{n}$ acts on $\mathbb{C}[X]^{(B)}_\lambda$ by the multiplication by a constant for any dominant weight λ provided $\mathfrak{n}$ acts trivially on $\mathbb{C}(G/H)^B$. Since X contains a dense G-orbit, we have $\mathbb{C}[X]^G = \mathbb{C}$. It follows from [7, Theorem 3.24], that $\dim \mathbb{C}[X]^{(B)}_\lambda < \infty$. Now our claim is clear.

Assertion 2 follows from [2, Corollary 5.6]. □

The following technical proposition is crucial in the proof of Proposition 3.1.

Proposition 3.5. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be proper subspaces of $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$ and $\mathfrak{X}_1, \ldots, \mathfrak{X}_l$ sublattices of $\mathfrak{X}_{G,H}$ such that $p_i := \#(\mathfrak{X}_{G,H}/\mathfrak{X}_i)$, $i = \overline{1,l}$, are pairwise different primes. Put $c := c_{\mathfrak{g},\mathfrak{h}}$. Then there exists a dominant weight λ with $V(\lambda)^H \neq \{0\}$ satisfying condition (1), when c is arbitrary, and conditions (2), (3), when $c > 0$.*

(1) $\lambda^* \notin \bigcup_{i=1}^k \mathfrak{a}_i \cup \bigcup_{i=1}^l \mathfrak{X}_i$.
(2) *The codimension of the closure of the subset $Z := (\bigcup V(\lambda)^{\widehat{\mathfrak{h}}}) \cap V(\lambda)^H$ in $V(\lambda)^H$, where the union is taken over all algebraic subalgebras $\widehat{\mathfrak{h}} \subset \mathfrak{g}$ such that $\widehat{\mathfrak{h}} \supset \mathfrak{h}$, $c_{\mathfrak{g},\widehat{\mathfrak{h}}} < c$, is strictly bigger than $2\dim G$.*
(3) *For any $f \in \mathbb{C}(G/H)^B$ there exist $f_1, f_2 \in \mathbb{C}[G/H]^{(B)}_{\lambda^*}$ such that $f = \frac{f_1}{f_2}$.*

Lemma 3.6. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k, \mathfrak{X}_1, \ldots, \mathfrak{X}_l$ be such as in Proposition 3.5. Let $\mu' \in \mathfrak{X}_{G,H}$ satisfy condition (1). Then there is $n \in \mathbb{N}$ such that for any $\lambda \in \mathfrak{X}_{G,H}$ at least one of the weights $\lambda + \mu', \lambda + 2\mu', \ldots, \lambda + n\mu'$ satisfies condition (1) of Proposition 3.5.*

Proof. Set $n := (k+1)p_1 \cdots p_l$. The proof is easy. □

Proof of Proposition 3.5. Let us choose a norm $|\cdot|$ on the space $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}(\mathbb{R}) := \mathfrak{X}_{G,H} \otimes_\mathbb{Z} \mathbb{R}$. By Timashev's theorem [8], the following assertions hold:

- There exists $A_0 \in \mathbb{R}$ such that $\dim V(\lambda)^{\widehat{\mathfrak{h}}} < A_0|\lambda|^{c-1}$ for any subalgebra $\widehat{\mathfrak{h}} \subset \mathfrak{g}$ with $c_{\mathfrak{g},\widehat{\mathfrak{h}}} < c$ and any dominant weight λ.
- For any $A \in \mathbb{R}$ there exists a dominant weight λ such that $\dim V(\lambda)^H > A|\lambda|^{c-1}$.

Denote by Y the subvariety of $\coprod_{i=\dim\mathfrak{h}}^{\dim\mathfrak{g}} \mathrm{Gr}_i(\mathfrak{g})$ consisting of all subalgebras $\widehat{\mathfrak{h}} \subset \mathfrak{g}$ containing $\mathfrak{h}$. It is clear that $Y_0 := \{\widehat{\mathfrak{h}} \in Y \mid c_{\mathfrak{g},\widehat{\mathfrak{h}}} < c\}$ is an open subvariety of Y. Put $V := V(\lambda)^H$, $\widetilde{Z} := \{(\widehat{\mathfrak{h}}, v) \in Y_0 \times V \mid v \in V(\lambda)^{\widehat{\mathfrak{h}}}\}$. The latter is a closed subvariety in $Y_0 \times V$ of dimension at most $\dim Y_0 + \max_{\widehat{\mathfrak{h}} \in Y_0} \dim V(\lambda)^{\widehat{\mathfrak{h}}}$.

Note that Z is just the image of $\widetilde{Z}$ under the projection $Y_0 \times V \to V$. Thus if $c > 0$, then the dimension of the closure of Z does not exceed $A_0|\lambda|^{c-1} + \dim Y_0$.

Note that there exists a dominant weight λ_1 satisfying condition (3). Indeed, the field $\mathbb{C}(G/H)^B$ is finitely generated, let $f_1, \ldots, f_s$ be its generators. Lemma 3.4 implies that there are $f_{i1}, f_{i2} \in \mathbb{C}[G/H]^{(B)}_{\nu_i}$, $i = \overline{1,s}$, such that $f_i = \frac{f_{i1}}{f_{i2}}$. It is enough to take $\sum_{i=1}^s \nu_i^*$ for λ_1. Note that for any dominant weight λ_2 with $\mathbb{C}[G/H]^{(B)}_{\lambda_2^*} \neq 0$ the dominant weight $\lambda_2 + \lambda_1$ also satisfies condition (3).

Note that there is a dominant weight λ_2 satisfying condition (1) and such that $V(\lambda_2)^H \neq \{0\}$. Indeed, otherwise $\bigcup_{i=1}^k \mathfrak{a}_i$ contains a subset of the form $a + \mathfrak{X}$, where $a \in \mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$ and $\mathfrak{X}$ is a lattice in $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$ of rank $\dim \mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. So in the case $c = 0$ we are done.

Now suppose $c > 0$. Let n be such as in Lemma 3.6. Choose $A > 0$ and a dominant weight ν such that $\dim V(\nu)^H > A|\nu|^{c-1}$ and $A|\nu|^{c-1} > A_0(|\nu| + |\lambda_1| + n|\lambda_2|)^{c-1} + \dim Y_0 + 2\dim G$. Further, there is $j \in \{1, \ldots, n\}$ such that $\lambda := \nu + \lambda_1 + j\lambda_2$ satisfies (1). It is easy to deduce from Lemma 3.2 that λ

satisfies condition (2). Finally λ satisfies condition (3), for it is of the form $\lambda_1+\lambda_3$ for some λ_3 with $\mathbb{C}[G/H]^{(B)}_{\lambda_3^*}\neq 0$. $\square$

The next proposition is used on the second step of the proof.

Proposition 3.7. *The set* $\{\mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}} \mid \widehat{\mathfrak{h}}=[\widehat{\mathfrak{h}},\widehat{\mathfrak{h}}]+R_u(\widehat{\mathfrak{h}})+\mathfrak{h},\ \widehat{\mathfrak{h}}$ *is algebraic*$\}$ *is finite.*

Proof. Let $\mathfrak{h}=\mathfrak{s}\oplus R_u(\mathfrak{h})$, $\widehat{\mathfrak{h}}=\widehat{\mathfrak{s}}\oplus R_u(\widehat{\mathfrak{h}})$ be Levi decompositions. We may assume that $\mathfrak{s}\subset\widehat{\mathfrak{s}}$. Denote by $\widehat{H}$, $\widehat{S}$ the connected subgroups of G corresponding to $\widehat{\mathfrak{h}}$, $\widehat{\mathfrak{s}}$. By the Weisfeller theorem, see [10], there is a parabolic subgroup $P\subset G$ and a Levi subgroup $L\subset P$ such that $\widehat{S}\subset L$, $R_u(\widehat{H})\subset R_u(P)$. Conjugating $\mathfrak{h}$ and $\widehat{\mathfrak{h}}$ by an element of G, we may assume that $T\subset L$ and that P is opposite to B. By Panyushev's theorem [6],

$$\mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}}=\mathfrak{a}_{L,L*_{\widehat{S}}(R_u(\mathfrak{p})/R_u(\widehat{\mathfrak{h}}))}.$$

There is an inclusion of $\widehat{S}$-modules $R_u(\mathfrak{p})/R_u(\widehat{\mathfrak{h}})\hookrightarrow\mathfrak{g}/\widehat{\mathfrak{s}}$. So the set of all pairs $(L,R_u(\mathfrak{p})/R_u(\widehat{\mathfrak{h}}))$ (with given $\widehat{\mathfrak{s}}$) is finite. It remains to check that $\widehat{\mathfrak{s}}$ belongs only to finitely many $\mathrm{Int}(\mathfrak{l})$-conjugacy classes. The following well-known lemma (which stems, for example, from [9, Proposition 3]) allows us to replace $\mathrm{Int}(\mathfrak{l})$-conjugacy in the previous statement with $\mathrm{Int}(\mathfrak{g})$-conjugacy.

Lemma 3.8. *Let $\mathfrak{g}_0$ be a reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_1$ a reductive subalgebra of $\mathfrak{g}_0$. The set of subalgebras of $\mathfrak{g}_0$, that are* $\mathrm{Int}(\mathfrak{g})$*-conjugate to $\mathfrak{g}_1$, decomposes into finitely many* $\mathrm{Int}(\mathfrak{g}_0)$*-conjugacy classes.*

The equality $\widehat{\mathfrak{h}}=[\widehat{\mathfrak{h}},\widehat{\mathfrak{h}}]+R_u(\widehat{\mathfrak{h}})+\mathfrak{h}$ is equivalent to $\widehat{\mathfrak{s}}=[\widehat{\mathfrak{s}},\widehat{\mathfrak{s}}]+\mathfrak{s}$. Therefore the statement on the finiteness of the set of $\mathrm{Int}(\mathfrak{g})$-conjugacy classes stems from the following lemma that finishes the proof of the proposition. $\square$

Lemma 3.9. *Let $\mathfrak{s}$ be a reductive subalgebra of $\mathfrak{g}$. The set of* $\mathrm{Int}(\mathfrak{g})$*-conjugacy classes of reductive subalgebras $\widehat{\mathfrak{s}}\subset\mathfrak{g}$ such that $\widehat{\mathfrak{s}}=[\widehat{\mathfrak{s}},\widehat{\mathfrak{s}}]+\mathfrak{s}$ is finite.*

Proof. We may replace $\mathfrak{s}$ with its Cartan subalgebra and assume that $\mathfrak{s}\subset\mathfrak{t}$. In this case the proof is in two steps.

Step 1. It is a standard fact that the set of subspaces of $\mathfrak{t}$ that are Cartan subalgebras of semisimple subalgebras of $\mathfrak{g}$ is finite. Conjugating $\widehat{\mathfrak{s}}$ by an element of $Z_G(\mathfrak{s})$, one may assume that there is a Cartan subalgebra $\mathfrak{t}_0\subset\widehat{\mathfrak{s}}$ contained in $\mathfrak{t}$. Since $\widehat{\mathfrak{s}}=[\widehat{\mathfrak{s}},\widehat{\mathfrak{s}}]+\mathfrak{s}$, we see that $\mathfrak{t}_0$ is the sum of $\mathfrak{s}$ and a Cartan subalgebra of a semisimple subalgebra of $\mathfrak{g}$. By the remark in the beginning of the paragraph, there are only finitely many possibilities for $\mathfrak{t}_0$.

Step 2. Clearly, $\mathfrak{z}(\widehat{\mathfrak{s}})=\mathfrak{t}_0\cap(\mathfrak{t}_0\cap[\widehat{\mathfrak{s}},\widehat{\mathfrak{s}}])^{\perp}$, where the orthogonal complement is taken with respect to some invariant non-degenerate symmetric form on $\mathfrak{g}$. Thus, by the previous step, there are only finitely many possibilities for $\mathfrak{z}(\widehat{\mathfrak{s}})$. Obviously, $\widehat{\mathfrak{s}}$ is a direct sum of $\mathfrak{z}(\widehat{\mathfrak{s}})$ and a semisimple subalgebra of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\widehat{\mathfrak{s}}))$. Thence, $\widehat{\mathfrak{s}}$ belongs to one of finitely many $Z_G(\mathfrak{z}(\widehat{\mathfrak{s}}))$-conjugacy classes of subalgebras. To complete the proof of the lemma it remains to apply Lemma 3.8 to $\mathfrak{g}_0=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\widehat{\mathfrak{s}}))$. $\square$

Corollary 3.10. *There are proper subspaces $\mathfrak{a}_1,\ldots,\mathfrak{a}_k\subset\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$ satisfying the following condition: if $\widehat{\mathfrak{h}}$ is an algebraic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ such that $c_{\mathfrak{g},\widehat{\mathfrak{h}}}=c_{\mathfrak{g},\mathfrak{h}}$ and $\mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}}\not\subset\mathfrak{a}_i$ for any i, then $\widehat{\mathfrak{h}}\subset\mathfrak{h}^{sat}$.*

Proof. For $\mathfrak{a}_i$ we take elements of the set $\{\mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}}\mid\widehat{\mathfrak{h}}=[\widehat{\mathfrak{h}},\widehat{\mathfrak{h}}]+R_u(\widehat{\mathfrak{h}})+\mathfrak{h},\ \mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}}\neq\mathfrak{a}_{\mathfrak{g},\mathfrak{h}},\ \widehat{\mathfrak{h}}$ is algebraic$\}$. Put $\widehat{\mathfrak{h}}_0=[\widehat{\mathfrak{h}},\widehat{\mathfrak{h}}]+R_u(\widehat{\mathfrak{h}})+\mathfrak{h}$. Clearly, $\widehat{\mathfrak{h}}_0=[\widehat{\mathfrak{h}}_0,\widehat{\mathfrak{h}}_0]+R_u(\widehat{\mathfrak{h}}_0)+\mathfrak{h}$. If $\mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}_0}$ is not contained in any $\mathfrak{a}_i$, then $\mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}_0}=\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. Moreover, since $\mathfrak{h}\subset\widehat{\mathfrak{h}}_0\subset\widehat{\mathfrak{h}}$, we get $c_{\mathfrak{g},\mathfrak{h}}=c_{\mathfrak{g},\widehat{\mathfrak{h}}}\leqslant c_{\mathfrak{g},\widehat{\mathfrak{h}}_0}\leqslant c_{\mathfrak{g},\mathfrak{h}}$. Applying the following lemma to $\mathfrak{g}_0=\widehat{\mathfrak{h}}_0,\mathfrak{h}$, we get $\widehat{\mathfrak{h}}_0=\mathfrak{h}$.

Lemma 3.11. *For any algebraic subgroup $G_0 \subset G$ we have*

$$2(\dim \mathfrak{g} - \dim \mathfrak{g}_0) \geqslant 2c_{\mathfrak{g},\mathfrak{g}_0} + 2\dim \mathfrak{a}_{\mathfrak{g},\mathfrak{g}_0} + \dim \mathfrak{g} - \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{g},\mathfrak{g}_0})$$

with the equality provided G_0 is observable.

Proof. This follows from [1, Sätze 7.1, 8.1, Korollar 8.2]. □

It follows that $\mathfrak{h}$ is an ideal of $\widehat{\mathfrak{h}}$ and that $\widehat{\mathfrak{h}}/\mathfrak{h}$ is a commutative reductive algebraic Lie algebra. Let $\widehat{H}$ denote the connected subgroup of G corresponding to $\widehat{\mathfrak{h}}$. By Proposition 4.7 from [3], $\widehat{H}/H^\circ$ acts on G/H° by central automorphisms, equivalently, $\widehat{\mathfrak{h}} \subset \mathfrak{h}^{sat}$. □

The following lemma is used on Step 3 of the proof of Proposition 3.1.

Lemma 3.12. *Let a dominant weight λ satisfy condition* (3) *of Proposition* 3.5. *Then*:

(3′) *Any subgroup $\widehat{H} \subset G$ such that $H \subset \widehat{H}$, $H^\circ = \widehat{H}^\circ$ and $V(\lambda)^H = V(\lambda)^{\widehat{H}}$ is contained in H^{sat}.*

Proof. By the Frobenius reciprocity, $\mathbb{C}[G/\widehat{H}]^{(B)}_{\lambda_*} = \mathbb{C}[G/H]^{(B)}_{\lambda_*}$. By the choice of λ_*, $\mathbb{C}(G/H)^B = \mathbb{C}(G/\widehat{H})^B$. Equivalently, $\mathbb{C}(G/B)^H = \mathbb{C}(G/B)^{\widehat{H}}$. Applying the main theorem of the Galois theory to the field $\mathbb{C}(G/B)^{H^\circ}$, we see that the images of H/H°, $\widehat{H}/H^\circ$ in $\mathrm{Aut}(\mathbb{C}(G/B)^{H^\circ})$ (or, equivalently, $\mathrm{Aut}(\mathbb{C}(G/H^\circ)^B)$) coincide. By assertion 1 of Lemma 3.3, $\widehat{H}/H^\circ = (H/H^\circ)\Gamma$, where $\Gamma \subset \mathfrak{A}_{G,H^\circ}$. Assertion 2 of Lemma 3.3 implies that H is a normal subgroup in $\widehat{H}$. In virtue of the natural inclusion $\mathbb{C}(G/H)^B \hookrightarrow \mathbb{C}(G/H^\circ)^B$, the group $\widehat{H}/H$ acts trivially on $\mathbb{C}(G/H)^B$. It remains to apply assertion 1 of Lemma 3.3 once more. □

Now we define subspaces $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ of $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$ and sublattices $\mathfrak{X}_1, \dots, \mathfrak{X}_l$ of $\mathfrak{X}_{G,H}$ satisfying the assumptions of Proposition 3.5.

Suppose that $\mathfrak{A}_{G,H}$ is a finite group. Take for $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ the subspaces found in Corollary 3.10. Let $\mathfrak{A}_{G,H} \cong \bigoplus_{i=1}^l \mathbb{Z}/p_i^{a_i}\mathbb{Z}$, where $p_1, \dots, p_l$ are distinct primes. Take for $\mathfrak{X}_i$ the lattice $\mathfrak{X}_{G,\widetilde{H}_i}$, where $\widetilde{H}_i$ denotes the unique subgroup of H^{sat} such that $\#\widetilde{H}_i/H = p_i$. Clearly, $\widetilde{H}_i/H$, $i = \overline{1,l}$, are all minimal proper subgroups of $\mathfrak{A}_{G,H}$.

Now suppose that $\mathfrak{A}_{G,H}$ is a one-dimensional torus. For $\mathfrak{a}_1, \dots, \mathfrak{a}_{k-1}$ we take subspaces found in Corollary 3.10 and for $\mathfrak{a}_k$ take the subspace $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}^{sat}}$.

Proposition 3.1 follows from Proposition 3.5, Lemma 3.12 and the following proposition.

Proposition 3.13. *Let λ be a dominant weight with $V(\lambda)^H \neq \{0\}$ satisfying conditions* (1), (2) *of Proposition* 3.5 *for $\mathfrak{a}_1, \dots, \mathfrak{a}_k, \mathfrak{X}_1, \dots, \mathfrak{X}_l$ defined above and condition* (3′) *of Lemma* 3.12 (*or only condition* (1) *if $c_{\mathfrak{g},\mathfrak{h}} = 0$*). *Then λ has the properties indicated in Proposition* 3.1.

Proof. Set $V := V(\lambda)^H$. By the choice of λ, $\mathfrak{g}_v = \mathfrak{h}$ and $G_v \cap H^{sat} = H$ for $v \in V$ in general position.

First of all, we consider the case $c_{\mathfrak{g},\mathfrak{h}} = 0$. By Lemma 3.3, $H^{sat} = N_G(H)$. Further, $N_G(H^\circ)/H^\circ$ is commutative and thence $\widehat{H} \subset N_G(H)$ for any $\widehat{H}$ with $\widehat{H}^\circ = H^\circ$. Thus $G_v \subset H^{sat}$ for a non-zero vector $v \in V$.

In the sequel we assume that $c_{\mathfrak{g},\mathfrak{h}} > 0$. Let us prove that the set

$$\bigcup_{\widetilde{H} \supset H,\, \widetilde{H}^\circ = H^\circ} V(\lambda)^{\widetilde{H}}$$

is not dense in V.

Any subgroup $\widetilde{H} \subset G$ with $\widetilde{H}^\circ = H^\circ$ lies in $N_G(H^\circ)$. Denote by Y_n the subset of $N_G(H^\circ)/H^\circ$ consisting of all elements h such that h and H/H° generate a finite subgroup in $N_G(H)$, whose order divides n. For $h \in Y_n$ we denote by $\widetilde{H}(h)$ the inverse image in $N_G(H^\circ)$ of the subgroup of $N_G(H^\circ)/H^\circ$ generated by h and H/H°.

Note that for every n the subset $Y_n \subset N_G(H^\circ)/H^\circ$ is closed. Put

$$Y_{n,i} = \bigl\{h \in Y_n \bigm| \operatorname{codim}_V V(\lambda)^{\widetilde{H}(h)} = i\bigr\}.$$

This is a locally-closed subvariety of Y_n. Lemma 3.12 implies $Y_{n,0} = \{1\}$ or $\emptyset$.

It is enough to show that for all $n, i > 0$ the subset

$$\bigcup_{h \in Y_{n,i}} V(\lambda)^{\widetilde{H}(h)} \tag{1}$$

is not dense in V.

Assume the converse: let $n, i \in \mathbb{N}$ be such that the subset (1) is dense in V. Then (compare with the proof of Proposition 3.5) $\dim Y_{n,i} \geqslant i$. It follows that $i \leqslant \dim Y_{n,i} \leqslant \dim G$. For $h_1, h_2 \in Y_{n,i}$ the inequality

$$\dim V(\lambda)^{\widetilde{H}(h_1)} \cap V(\lambda)^{\widetilde{H}(h_2)} \geqslant \dim V - 2i \geqslant \dim V - 2\dim G \tag{2}$$

holds. Let $\widetilde{H}(h_1, h_2)$ denote the algebraic subgroup of G generated by $\widetilde{H}(h_1)$ and $\widetilde{H}(h_2)$. Note that $\dim V(\lambda)^{\widetilde{H}(h_1 h_2)} = V(\lambda)^{\widetilde{H}(h_1)} \cap V(\lambda)^{\widetilde{H}(h_2)}$. In virtue of (2) and condition (2) of Proposition 3.5, $V(\lambda)^{\widetilde{H}(h_1,h_2)} \neq 0$, $c_{\mathfrak{g},\widetilde{\mathfrak{h}}(h_1,h_2)} = c_{\mathfrak{g},\mathfrak{h}}$. By the choice of λ and Corollary 3.10, $\mathfrak{a}_{\mathfrak{g},\widetilde{\mathfrak{h}}(h_1,h_2)} = \mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. Now Lemma 3.11 implies that $\dim \widetilde{\mathfrak{h}}(h_1, h_2) \leqslant \dim \mathfrak{h}$. Since $\mathfrak{h} \subset \widetilde{\mathfrak{h}}(h_1, h_2)$, we see that $\widetilde{\mathfrak{h}}(h_1, h_2) = \mathfrak{h}$ (for any $h_1, h_2 \in Y_{n,i}$). In particular, any $h_1, h_2 \in Y_{n,i}$ generate a finite subgroup in $N_G(H^\circ)/H^\circ$. Choose an irreducible component $Y' \subset Y_{n,i}$ of positive dimension. Consider the map $\rho : Y' \times Y' \to N_G(H^\circ)/H^\circ, (h_1, h_2) \mapsto h_1 h_2^{-1}$. Its image is a non-discrete constructible set, whose elements have finite order in $N_G(H^\circ)/H^\circ$. Note that 1 is a nonisolated point in $\overline{\operatorname{im}\rho}$. Thus there is a locally-closed subvariety $Z \subset \overline{\operatorname{im}\rho}$ of positive dimension, whose closure contains 1. The subsets $Z_j := \{z \in Z \mid z^j = 1\}$ are closed in Z. Thus $1 \in \overline{Z}_j$ for some j. However, 1 is an isolated point in $\{g \in N_G(H^\circ)/H^\circ \mid g^j = 1\}$. Contradiction. □

4. Proof of Theorem 1.5

Again, one implication in Theorem 1.5 is almost trivial.

Proof of (a) ⇒ (b). Let $V(\lambda)$ be an irreducible module with closed orbit G/H. By Theorem 1.3, $\mathfrak{A}_{G,H}$ is either a finite cyclic group or a one-dimensional torus. As we noted in the proof of the implication (a) ⇒ (b), $\mathfrak{A}_{G,H}$ acts on $V(\lambda)^H$ by constants. If $\mathfrak{A}_{G,H} \cong \mathbb{C}^\times$, then $0 \in \overline{\mathfrak{A}_{G,H} v}$ for any $v \in V(\lambda)^H$. Thus $0 \in \overline{N_G(H)v}$ whence $0 \in \overline{Gv}$. □

The proof of the other implication is much more complicated. Below we assume that $\mathfrak{A}_{G,H}$ is cyclic. At first, we prove (b) ⇒ (a) for reductive subgroups $H \subset G$ satisfying the following condition.

(∗) $T_0 := (N_G(H)/H)^\circ$ is a torus, equivalently, the Lie algebra $\mathfrak{g}^H$ is commutative.

The proof for H satisfying (∗) is based on the following technical proposition, which is analogous to Proposition 3.5.

Proposition 4.1. *Let H satisfy $(*)$ and $\mathfrak{a}_1,\ldots,\mathfrak{a}_k,\mathfrak{X}_1,\ldots,\mathfrak{X}_l$ be such as in Proposition* 3.5. *Then there is a dominant weight λ satisfying conditions* (1)–(3) *of Proposition* 3.5 (*only* (1) *for $c_{\mathfrak{g},\mathfrak{h}}=0$*) *and the following condition*:

(4) *The rational cone spanned by the weights of T_0 in $V(\lambda)^H$ coincides with the whole space $\mathfrak{X}(T_0)\otimes_{\mathbb{Z}}\mathbb{Q}$.*

We note that if $c_{\mathfrak{g},\mathfrak{h}}=0$, then (4) holds automatically.

Proof of (b) ⇒ (a) for *H* satisfying $(*)$. Recall a theorem by Luna, see [5] and also [7, Theorem 6.17].

Lemma 4.2. *Let V be a G-module and $v\in V$ be a point stabilized by a reductive subgroup $H\subset G$. Then Gv is closed if and only if $N_G(H)v$ is closed.*

By Lemma 3.12 and Proposition 3.13, there is a dense subset $V^0\subset V:=V(\lambda)^H$ such that $G_v=H$ for any $v\in V^0$. By condition (4) of Proposition 4.1, a general orbit for the action $T_0:V$ is closed. It follows that there is $v\in V$ such that $G_v=H$ and the orbit $N_G(H)v$ is closed. By Lemma 4.2, Gv is also closed. □

It remains to prove Proposition 4.1 only for $c_{\mathfrak{g},\mathfrak{h}}>0$.

Let us introduce some further notation. Set $L:=\mathfrak{X}(T_0)\otimes_{\mathbb{Z}}\mathbb{Q}$. Let Ψ (resp., Ψ^0) denote the set of dominant weights λ with $V(\lambda)^H\neq 0$ (resp., satisfying condition (3)). By Lemma 3.2, Ψ is a monoid. For $\lambda\in\Psi$ we denote by $S(\lambda)$ the set of weights of T_0 in $V(\lambda)^H$. Since $\mathbb{C}[G/H]^{(B)}_{\lambda^*}\mathbb{C}[G/H]^{(B)}_{\mu^*}\subset\mathbb{C}[G/H]^{(B)}_{\lambda^*+\mu^*}$, we have $S(\lambda)+S(\mu)\subset S(\lambda+\mu)$. Finally, we denote by $\widetilde{H}$ the inverse image of T_0 in $N_G(H)$ under the natural epimorphism $N_G(H)\twoheadrightarrow N_G(H)/H$.

Lemma 4.3. *There is a dominant weight ν satisfying conditions* (1), (3), (4).

Proof. *Step 1.* Let us check that $\mathfrak{a}_{\mathfrak{g},\widetilde{\mathfrak{h}}}=\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. Since $\mathfrak{A}_{G,H}$ is finite, Lemma 3.3 implies that the action $T_0:\mathbb{C}(G/H)^B$ is locally effective. It follows that $c_{\mathfrak{g},\widehat{\mathfrak{h}}}=c_{\mathfrak{g},\mathfrak{h}}-\dim T_0$. The required equality follows from the inclusion $\mathfrak{a}_{\mathfrak{g},\widetilde{\mathfrak{h}}}\subset\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$ and Lemma 3.11.

Step 2. By Step 1, elements λ_0^* with $\lambda_0\in\Psi, 0\in S(\lambda_0)$, span $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. Clearly, $\Psi^0+\Psi\subset\Psi^0$. Therefore even elements λ_0^* with $\lambda_0\in\Psi_0:=\{\lambda_0\in\Psi^0\mid 0\in S(\lambda_0)\}$ span $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. Fix $\lambda_0\in\Psi_0$. We claim that $S(\lambda_0)$ spans the vector space L. Indeed, otherwise there is a subgroup $\widetilde{H}_0\subset\widetilde{H}$ such that $\dim\widetilde{H}_0/H>0$ and $\widetilde{H}_0$ acts trivially on $V(\lambda)^H$. By (3), $\widetilde{H}_0$ acts trivially on $C(G/H)^B$, which contradicts $\#\mathfrak{A}_{G,H}<\infty$.

Step 3. Set $\nu_0:=\lambda_0+\lambda_0^*$. Clearly, $V(\lambda_0)^H\cong(V(\lambda_0^*)^H)^*$. Thus $S(\lambda_0)=-S(\lambda_0^*)$. It follows that $S(\nu_0)\supset S(\lambda_0),-S(\lambda_0)$ whence the rational cone spanned by $S(\nu_0)$ coincides with L.

Step 4. Let μ', n be such as in Lemma 3.6. For sufficiently large m the cone spanned by $mS(\nu_0)+iS(\mu')$ coincides with L for any $i=1,\ldots,n$. Thus for appropriate μ' the weight $\nu:=m\nu_0+i\mu'$ satisfies (1), (3), (4). □

Proof of Proposition 4.1. Let ν be such as in Lemma 4.3, n be such as in Lemma 3.6. We fix a norm $|\cdot|$ on $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}(\mathbb{R})$ such that $|\lambda|=|\lambda^*|$ for any $\lambda\in\mathfrak{a}_{\mathfrak{g},\mathfrak{h}}$. Let A_0, Y_0 be such as in the proof of Proposition 3.5.

We choose $\lambda\in\Psi$ and $A\in\mathbb{R}$ such that $\dim V(\lambda)^H>A|\lambda|^{c-1}$, where $c:=c_{\mathfrak{g},\mathfrak{h}}$, and

$$A|\lambda|^{c-1}>A_0\big(2|\lambda|+|\nu|n\big)^{c-1}+2\dim G+\dim Y_0.$$

By Lemma 3.6, there is $i\in\{1,2,\ldots,n\}$ such that $\widetilde{\lambda}:=\lambda+\lambda^*+i\nu$ satisfies (1) and automatically (3). As in the proof of Proposition 3.5, $\widetilde{\lambda}$ satisfies (2). Finally, note that $S(\lambda)=-S(\lambda^*)$. It follows that $S(\nu)\subset S(\widetilde{\lambda})$ whence $\widetilde{\lambda}$ satisfies (4). □

Proof of Theorem 1.5 in the general case. Now H is a subgroup of G such that $\mathfrak{A}_{G,H}$ is a finite cyclic group and the algebra $\mathfrak{g}^H$ is not commutative.

There is a finite cyclic subgroup Γ in a maximal torus of $N_G(H)/H$ such that $Z_{N_G(H)/H}(\Gamma)^\circ$ is a maximal torus of $N_G(H)/H$, $\#\Gamma$ is prime and divides neither $\#\mathfrak{A}_{G,H}$ nor $\#H/H^\circ$. Let $\overline{H}$ denote the inverse image of Γ in $N_G(H)$. Clearly, $\overline{H} \cap H^{sat} = H$. Moreover, $(N_G(\overline{H})/\overline{H})^\circ$ is a torus. Choose a dominant weight λ satisfying conditions (1)–(4) of Propositions 3.5, 4.1 (for $\overline{H}$ instead of H). Let us check that $V(\lambda)$ has the required properties.

Choose $\mathfrak{a}_1, \dots, \mathfrak{a}_k, \mathfrak{X}_1, \dots, \mathfrak{X}_l$ as in Proposition 3.13 for $\overline{H}$ instead of H. Let us check that λ satisfies conditions (1), (2) of Proposition 3.5 and condition (3′) of Lemma 3.12 for H.

Condition (1) follows from the equality $\mathfrak{A}_{G,H} = \mathfrak{A}_{G,\overline{H}}$, which, in turn, stems from [2, Theorem 6.3], and the choice of Γ. To check condition (2) it is enough to check that the subset $Z \subset V(\lambda)$ defined there is closed. This will follow if we check that $c_{\mathfrak{g},\widehat{\mathfrak{h}}} < c_{\mathfrak{g},\mathfrak{h}}$ for any algebraic subalgebra $\widehat{\mathfrak{h}} \subset \mathfrak{g}$ such that $\mathfrak{h} \subsetneq \widehat{\mathfrak{h}}$, $V(\lambda)^{\widehat{\mathfrak{h}}} \neq \{0\}$. Assume the converse: let $\mathfrak{h} \subsetneq \widehat{\mathfrak{h}}$, $V(\lambda')^{\widehat{\mathfrak{h}}} \neq \{0\}$, $c_{\mathfrak{g},\widehat{\mathfrak{h}}} = c_{\mathfrak{g},\mathfrak{h}}$. At first, suppose that $\widehat{\mathfrak{h}} = [\widehat{\mathfrak{h}}, \widehat{\mathfrak{h}}] + R_u(\widehat{\mathfrak{h}}) + \mathfrak{h}$. Then, by the choice of $\mathfrak{a}_i$, we see that $\mathfrak{a}_{\mathfrak{g},\mathfrak{h}} = \mathfrak{a}_{\mathfrak{g},\widehat{\mathfrak{h}}}$. Contradiction with Lemma 3.11. Now let $\mathfrak{s}$ denote a maximal reductive subalgebra of $\widehat{\mathfrak{h}}$ containing $\mathfrak{h}$. Then $\mathfrak{s} \supset \mathfrak{s}_0 := \mathfrak{h} + \mathfrak{z}(\mathfrak{s}) \supsetneq \mathfrak{h}$. It follows that $c_{\mathfrak{g},\mathfrak{s}_0} = c_{\mathfrak{g},\mathfrak{h}}$. Thanks to Lemma 3.3, the last equality contradicts $\#\mathfrak{A}_{G,H} < \infty$. So conditions (1), (2) for λ and H hold.

Let us check condition (3′). Let $\widehat{H}$ be a subgroup of G strictly containing H such that $H^\circ = \widehat{H}^\circ$, $V(\lambda)^H = V(\lambda)^{\widehat{H}}$. Let $\widetilde{H}$ denote the algebraic subgroup of G generated by $\overline{H}$, $\widehat{H}$. Then $V(\lambda)^{\widetilde{H}} = V(\lambda)^{\overline{H}} \cap V(\lambda)^{\widehat{H}} = V(\lambda)^{\overline{H}}$. Thanks to Lemma 3.12, $\widetilde{H} \subset \overline{H}^{sat}$. From the choice of $\mathfrak{X}_j$ it follows that $\widehat{H} \subset \widetilde{H} = \overline{H}$. By the choice of Γ, $\overline{H} = \widehat{H}$. So $V(\lambda)^H = V(\lambda)^{\overline{H}}$. Choose a nilpotent element $\xi \in \mathfrak{g}^H$. Then $\exp(t\xi)\overline{H}\exp(t\xi)^{-1} \neq \overline{H}$ but

$$\exp(t\xi)V(\lambda)^{\overline{H}} = \exp(t\xi)V(\lambda)^H = V(\lambda)^H = V(\lambda)^{\overline{H}}. \tag{3}$$

But, by Lemma 3.12 and Proposition 3.13, there is $v \in V(\lambda)^{\overline{H}}$ with $G_v = \overline{H}$. However, $G_{\exp(t\xi)v} = \exp(t\xi)G_v\exp(t\xi)^{-1}$ and so $\exp(t\xi)v \notin V(\lambda)^{\overline{H}}$. Contradiction with (3). So condition (3′) holds for λ, H. By Proposition 3.13, there is a dense open subset $V^0 \subset V(\lambda)^H$ such that $G_v = H$ for any $v \in V^0$.

It remains to prove that there is $v \in V^0$ with closed G-orbit or, equivalently (by Lemma 4.2), with closed $N_G(H)$-orbit. Let $u \in V(\lambda)^{\overline{H}}$ be such that $G_u = \overline{H}$ and $N_G(\overline{H})u$ is closed. Since $\#\Gamma$ does not divide $\#H/H^\circ$, we have $N_G(\overline{H}) \subset N_G(H)$. By Lemma 4.2, $N_G(H)u$ is closed. Since there is a closed $N_G(H)$-orbit in $V(\lambda)^H$ of dimension $\dim N_G(H)/H$, a general orbit is also closed, thanks to the Luna slice theorem. $\square$

5. Some examples

In Introduction we have remarked that the group $\mathfrak{A}_{G,H}$ can be computed for any algebraic subgroup $H \subset G$. However, in general, the computation algorithm is rather involved. In this section we give examples when the application of our theorems is easy.

Example 5.1. Let H be a spherical observable subgroup of G, the word "spherical" means $c_{\mathfrak{g},\mathfrak{h}} = 0$. In this case every automorphism of G/H is central, so $\mathfrak{A}_{G,H} = N_G(H)/H$. The classification of reductive spherical subgroups is known and in this case groups $N_G(H)/H$ are easy to compute. Note also that G/H can be embedded to any module $V(\lambda)$ provided $\lambda \notin \mathfrak{X}_{G,\widetilde{H}}$ for any subgroup $\widetilde{H} \subset G$ containing H. For example, let $G = \mathrm{SL}_{2n+1}$, $H = \mathrm{Sp}_{2n}$. In this case $N_G(H)/H$ is a one-dimensional torus. In fact, G/H can be embedded into $\bigwedge^3 \mathbb{C}^{2n+1}$ provided $n \geqslant 3$.

Example 5.2. Let H be a finite subgroup of G. It follows from results of [2] that in this case $\mathfrak{A}_{G,H} \cong Z(G)/Z(G) \cap H$. So any homogeneous space G/H, where $Z(G)$ is a cyclic group or a one-dimensional torus, can be embedded into a simple module as a closed subvariety.

Example 5.3. Let G be simple with cyclic $Z(G)$. Computations in [3,4] show that, as a rule, the lattice $\mathfrak{X}_{G,G/H^{sat}}$ coincides with the root lattice of G. For such subgroup the homogeneous space G/H admits a closed embedding into an irreducible module.

Acknowledgments

I thank Ivan Arzhantsev, who communicated this problem to me, found a gap in the proof in the previous version of the paper and made some other useful remarks. Also I would like to thank the referee for useful comments. The paper was partially written during my stay in the Fourier University, Grenoble, in June 2006. I express my gratitude to this institution and especially to Prof. Michel Brion for hospitality. The author's work was supported in part by A. Moebius foundation.

References

[1] F. Knop, Weylgruppe und Momentabbildung, Invent. Math. 99 (1990) 1–23.
[2] F. Knop, Automorphisms, root systems and compactifications, J. Amer. Math. Soc. 9 (1996) 153–174.
[3] I.V. Losev, Computation of the Cartan spaces of affine homogeneous spaces, Mat. Sb. 198 (2007) 83–108 (in Russian); English translation in: arXiv:math.AG/0606101v2, 21 pages.
[4] I.V. Losev, Computation of weight lattices of G-varieties, Sovrem. Mat. Prilozh. 60 (2008) 70–98 (in Russian); English translation in: J. Math. Sci. 161 (1) (2009), doi:10.1007/s10958-009-9537-5.
[5] D. Luna, Adhérences d'orbite et invariants, Invent. Math. 29 (1975) 231–238.
[6] D.I. Panyushev, Complexity and rank of nilpotent orbits, Manuscripta Math. 83 (1994) 223–237.
[7] V.L. Popov, E.B. Vinberg, Invariant theory, in: Itogi Nauki i Techniki, in: Sovrem. Probl. Mat. Fund. Naprav., vol. 55, VINITI, Moscow, 1989, pp. 137–309 (in Russian); English translation in: Algebraic Geometry, vol. 4, in: Encyclopaedia Math. Sci., vol. 55, Springer-Verlag, Berlin, 1994.
[8] D.A. Timashev, Complexity of a homogeneous spaces and growth of multiplicities, Transform. Groups 9 (2004) 65–72.
[9] E.B. Vinberg, On invariants of a set of matrices, J. Lie Theory 6 (1996) 249–269.
[10] B.Yu. Weisfeller, On a class of unipotent subgroups in semisimple algebraic groups, Uspekhi Mat. Nauk 22 (1966) 222–223 (in Russian).