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Acute and nonobtuse triangulations of polyhedral surfaces

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ABSTRACT

In this paper, we prove the existence of acute triangulations for general polyhedral surfaces. We also show how to obtain nonobtuse subtriangulations of triangulated polyhedral surfaces.

1. Introduction

A *triangulation* of a two-dimensional surface is a subdivision of the surface into nonoverlapping triangles performed in such a way that the intersection of any two distinct triangles is either empty or consists of a vertex or an edge. We are interested in constructing triangulations of a surface with *acute* or *nonobtuse* triangles. The construction arises in connection with discretization of partial differential equations, finite element mesh generation, and other related areas of computer graphics and simulations.

The construction of nonobtuse triangulations of polygons has been investigated in numerous articles, such as those of Baker, Grosse and Rafferty [1], and those of Bern, Mitchell and Ruppert [2]. Later, Maehara [6] and Yuan [7] obtained results on acute triangulations of polygons using a linear number of triangles. Zamfirescu's survey [8] gives a good introduction to the topic.

As early as 1960, Burago and Zalgaller [3] proved the existence of acute triangulations of general two-dimensional polyhedral surfaces. The present paper reproves the same result using completely different methods and by a very elementary construction. In contrast to the original proof, it provides a significantly simple proof of the result. Our result is complementary to the results obtained by Maehara and Yuan, and is motivated by many of the techniques used by them. As a corollary of our construction, we also obtain a nonobtuse *subtriangulation* of polygons with holes and surfaces of polyhedra that have been subdivided into polygonal regions. For the special case of triangulated

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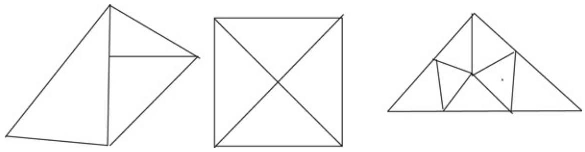

Fig. 1. An invalid triangulation, a nonobtuse triangulation of a square, and an acute triangulation of a triangle.

polygons with no Steiner points[1] in the interior, a construction was given by Bern and Eppstein [4], who proposed this more general result as a conjecture.

In this paper, we have not addressed the issue of the complexity of the number of acute triangles needed to triangulate a polyhedral surface. It is known that a polygon can be triangulated using a linear number of triangles, and it would be interesting to see whether such a bound can be achieved in the case of polyhedral surfaces as well.

Another interesting generalization of this problem is to consider its analog in higher dimensions. Very little is known about acute tilings of three-dimensional Euclidean space. Very recently it was discovered that it is possible to tile three-dimensional space using tetrahedra having acute dihedral angles; see [5]. However, it is still not even known whether a cube can be subdivided into acute tetrahedra.

In Section 2, we summarize the main results proved in this paper. In Section 3, we present the preliminary constructions of Steiner points on the edges and the circles. These points are useful in the construction of the final triangulation. In Section 4, we present the final construction of the nonobtuse triangulation using a grid imposition.

2. Main results

By a (general) *polyhedral surface*, we mean a two-dimensional simplicial complex obtained as a finite union of triangles given with their edge lengths, and a map identifying pairs of edges. There is an intrinsic metric on the surface. Clearly, when two edges are identified, they have the same edge length.

An *acute (nonobtuse) triangulation* of a polygon or polyhedral surface is a subdivision of the surface into nonoverlapping acute (nonobtuse) triangles such that any two distinct triangles are either disjoint, or share a common vertex or a common edge. In Fig. 1, we see a few examples illustrating the notion of a triangulation.

The main result of this paper is a new proof of a special case of the Burago–Zalgaller theorem.

Theorem 2.1. *Every two-dimensional polyhedral surface can be triangulated into nonobtuse triangles.*

Maehara [6] and Yuan [7] show how to obtain an acute triangulation of a polygon given a nonobtuse triangulation. In their construction, the number of triangles increases by a constant factor. Combined with Theorem 2.1, these results give a new and elementary proof of the Burago–Zalgaller theorem.

Theorem 2.2. *Every two-dimensional polyhedral surface can be triangulated into acute triangles.*

It is clear that every polygon can be triangulated if there are no restrictions on the angles of the triangles. Also, every obtuse triangle can be subdivided into two nonobtuse triangles by drawing the altitude from the vertex forming the obtuse angle to the opposite side. The difficulty arises when we try to fit these triangles together. To obtain a nonobtuse triangulation of a triangle, we may have to add vertices in the interior of its edges. These vertices need to match those for any other triangle sharing the same edge.

[1] By Steiner points, we just mean additional points that are not given as part of the input and serve as vertices in the triangulation. No other special properties are assumed for these points.

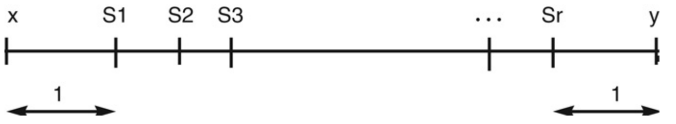

Fig. 2. Picking points in the interior of an edge.

There are several elementary constructions of a nonobtuse triangulation of every (not necessarily convex) polygon. An interesting related question is the following. Given a polygon that has been subdivided into polygonal regions, can we obtain a nonobtuse triangulation of all the regions in such a way that the regions fit together to give a nonobtuse triangulation of the original polygon? In this paper, we prove that we can.

We employ the "divide-and-conquer" technique to obtain a nonobtuse triangulation of a general polyhedral surface, by subtriangulating each triangle separately with matching Steiner points on the common edges. As we mentioned earlier, this construction resolves the Bern–Eppstein conjecture, but is also of independent interest.

We define a *subdivided polyhedral surface* to be a two-dimensional polyhedral surface that has been subdivided into polygonal regions.

Theorem 2.3. *Every subdivided polyhedral surface can be subtriangulated into nonobtuse triangles respecting the boundaries of the polygonal regions.*

Theorem 2.3 follows from the proof of Theorem 2.2. Unfortunately, our methods do not allow us to obtain *acute* subtriangulations of subdivided polyhedral surfaces.

3. Preliminary constructions

Let $\mathcal{P}$ be a polyhedral surface. Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be the set of its vertices, $\mathcal{E} = \{e_1, \ldots, e_m\}$ the set of its edges, and $\mathcal{F} = \{F_1, \ldots, F_k\}$ the set of its faces.

We need to obtain a triangulation of $\mathcal{P}$ satisfying the required conditions. In order to do so, in addition to the vertices $\mathcal{V}$ of $\mathcal{P}$, we might have to add to the interiors of the faces and edges of $\mathcal{P}$ additional vertices that, together with $\mathcal{V}$, comprise the vertices of the triangles in our triangulation. We call these additional points the *Steiner* points.

We first give a procedure for choosing vertices in the interiors of all the edges in $\mathcal{E}$. In our final triangulation, these vertices are the Steiner points on $\mathcal{E}$. We then show how to triangulate any polygonal face $F \in \mathcal{F}$ using nonobtuse triangles. The procedure ensures that the Steiner points on the boundary of F are precisely those vertices that are chosen initially in the interiors of the edges in $\mathcal{E}$ that bound F.

For any two faces F_1 and F_2 with a common edge, the Steiner points along that edge are the same for both of those faces. Hence, once we obtain such a triangulation, all the triangulated faces fit together to give a triangulation of the entire polyhedron.

We show how to add Steiner points to the face boundaries. Since $\mathcal{P}$ is finite, without loss of generality we may assume that the polygon is scaled such that the distance between any two vertices is at least 10. (A smaller number would have sufficed.) Let θ be such that, for each face $F \in \mathcal{P}$, every angle determined by two adjacent edges in F is at least θ. Pick t such that $0 < t < \sin(\theta/2)$.

Let xy be any edge of $\mathcal{P}$. We pick a finite number of points, say $s_1, \ldots, s_r$ in that order, in the interior of xy such that $\ell(xs_1) = \ell(s_r y) = 1$ and

$$t < \ell(s_i s_{i+1}) < t\sqrt{2} \quad \text{for } 1 \le i \le r - 1.$$

We pick such points in the interior of each edge in P. Fig. 2 illustrates this picking of points. This choice of points will be useful later on in the proof of Proposition 4.2.

Let F be any given face in $\mathcal{P}$. We show how to obtain a nonobtuse triangulation of F where the vertices on the boundary are precisely the original vertices of $\mathcal{P}$ in F along with the points added in the interior of each boundary edge. All the rest of the Steiner points are in the interior of F.

We construct circles padding the boundary of F in the following manner. We may assume that F is a convex polygon. Let V_F be the set of its original vertices, and S_F the set of its Steiner points added in the interiors or its boundary edges. For each $v \in V_F$, we construct a circle of radius 1 with v as center, and for each $s \in S_F$, we construct a circle of radius t with s as center.

By choice of S_F and t, these circles cover the entire boundary of F. Moreover, any two adjacent circles intersect at an angle greater than $\frac{\pi}{2}$. Let K be a point of intersection of two adjacent circles. The angle between the two circles is defined as the smallest nonnegative angle through which one of the circles has to be rotated about K so that it becomes externally tangent to the other circle at the point K.

We now show how to add *nodes* to the circle boundaries that will be Steiner points in our triangulation of the face F. Consider the inner boundary B of the region covered by the circles. By this we mean: look at the complement (in F) of the disks that are bounded by the circles that we constructed. Then B is the boundary of the resulting region. We identify all the points of intersection of adjacent circles on B as nodes. If the arc of a circle between any two consecutive nodes subtends an angle greater than $\frac{\pi}{2}$ at the center of the circle, add a node to the mid-point of the arc which divides the arc into smaller arcs, each subtending a nonobtuse angle at the center. We set up a coordinate system and identify the X and Y directions. For each circle, if the lines through the center in the X and Y directions intersect the inner boundary of the region covered by circles, then we add these points of intersections as nodes as well. We do so to ensure that later on, in Proposition 4.3, certain boundary cases do not arise, thus simplifying the analysis.

4. Nonobtuse triangulation

In this section, we show how to impose a grid, and then use it to obtain the desired nonobtuse triangulation.

Given a set H of lines in the horizontal direction and a set V of lines in the vertical direction, we say they form a *rectangular grid*. The lines are called the *lattice lines*, and the points of intersection of the lines are called the *lattice points*. The rectangles defined by two consecutive horizontal lines and two consecutive vertical lines are called the *lattice cells*. The *aspect ratio* of the grid is defined to be the supremum, taken over all lattice cells, of the ratio of the length of the cell to its breadth.

Lemma 4.1. *Given a set of n points $\{(x_i, y_i) \mid 1 \le i \le n\}$ in $\mathbb{R}^2$, and given $\epsilon > 0$, we can impose a rectangular grid of aspect ratio at most $1 + \epsilon$ such that the n points are lattice points of the grid.*

Proof. We'll prove the lemma by giving a construction of such a grid. First draw lines in the x and y directions through each of the n points to form some rectangular grid G. Our final grid will be a refinement of G. Let s be the smallest distance between any two consecutive parallel lines of G. Let m be natural number such that $1/m < \epsilon$. Choose $\delta > 0$ such that $\delta < s/(m + 1)$.

Claim: We can get a refinement G' of G, by adding additional lines in the X and Y directions, such that if d is the distance between some two consecutive parallel lines, then $\delta < d < \delta(1 + \frac{1}{m})$.

Proof of Claim: Consider any two consecutive parallel lines with distance $s' \ge s$ between them. Then $\frac{s'}{\delta} - \frac{s'}{\delta(1+\frac{1}{m})} = \frac{s'}{\delta}(1 - \frac{m}{m+1}) > \frac{s}{\delta}(\frac{1}{m+1}) > 1$. Hence there exists an integer k that lies strictly between $\frac{s'}{\delta}$ and $\frac{s'}{\delta(1+\frac{1}{m})}$. Therefore, if $k - 1$ equally spaced lines are added between the two originally given lines, then the distance between any two consecutive ones is between δ and $\delta(1 + \frac{1}{m})$, as was needed. We repeat this procedure for every pair of consecutive parallel lines in the grid G. This proves the claim.

By the choice of d, this new grid G' clearly has aspect ratio less than $1 + \epsilon$, and by construction it passes through the given n points. □

Applying Lemma 4.1 to the set of current nodes of F, we can impose a rectangular grid G of aspect ratio at most 1.2 such that each node is a lattice point. We add each point of intersection of the grid lines with the inner boundary B as a node.

Let Q be the polygon whose boundary edges are the straight line segments joining any two adjacent nodes. Let R be the region of the face excluding the interior of Q. We next show how to get a nonobtuse triangulation of R and of Q.

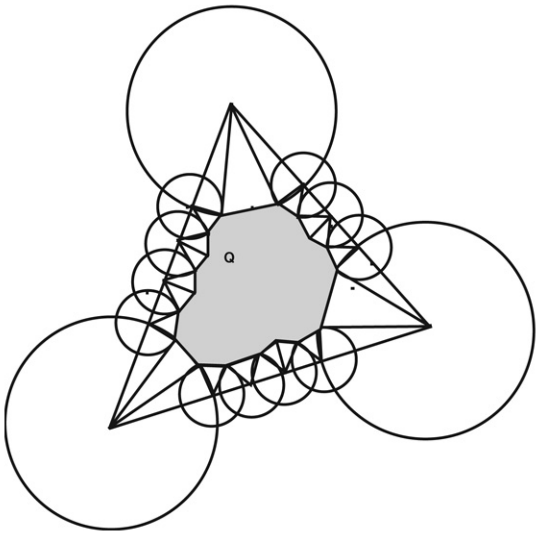

Fig. 3. The shaded region in the interior is Q. The figure illustrates the covering of the face boundary with circles.

Proposition 4.2. *The region R can be triangulated with nonobtuse triangles without adding any Steiner points in addition to those constructed originally on the edges of the face boundary and those constructed on the boundary B of the circles.*

Proof. By our choice of θ and the points s_i that are picked in the interiors of the edges of $\mathcal{P}$ (in Section 3), we ensure that adjacent circles intersect at an angle greater than $\frac{\pi}{2}$. Hence the angle formed between segments joining centers of adjacent circles to their common point of intersection is less than $\frac{\pi}{2}$. Also, for any two adjacent nodes, the arc between them subtends a nonobtuse angle at the center of the corresponding circle. Hence, when we add segments between the centers of the circles and the nodes on the circumferences of the respective circles, we get a triangulation of R by nonobtuse isosceles triangles without adding any additional Steiner points; see Fig. 3. $\square$

We say that Q' is a *refinement* of Q if in addition to the nodes of Q, we pick additional nodes on B, and consider the polygon formed by connecting any two adjacent nodes. Let R' be the region of the face F excluding the interior of Q'. Then just like R, also R' can be triangulated by nonobtuse triangles. Therefore, it remains to triangulate Q or a refinement of Q.

Proposition 4.3. *By adding points to the boundary B the region Q can be refined to a region Q', such that Q' can be triangulated with nonobtuse triangles. The vertices of this triangulation are the lattice points of the imposed grid G and points on the boundary B that were added to obtain the refinement.*

Proof. We introduce all the lattice points lying in the interior of Q as Steiner points. Every grid cell lying wholly in the interior of Q is a rectangle. The rectangles can be divided into two right angled triangles by a diagonal. At the boundary of Q, the grid cells intersect the boundary of Q to form right triangles, trapezoids, and pentagons.

Fig. 4 shows all the configurations in which the arc of a circle can intersect a grid cell.

The grid can be chosen fine enough that no two nonintersecting circles intersect the same grid cell. By construction, any two intersecting circles that make up the boundary B intersect at an angle greater than $\frac{\pi}{2}$. Also, there is a grid line passing through all the points of intersections of the circles that lie on B. Together, these conditions imply that no grid cell is intersected by the arc of more than one circle.

The points of intersection of the diameters in the X and Y directions with the boundaries of the circles were added as Steiner points; hence, clearly Cases 1, 3 and 6 of Fig. 4 cannot arise. For Cases 2, 4, 5, 7 and 8, the natural nonobtuse triangulations of the intersection of the grid cell with the boundary

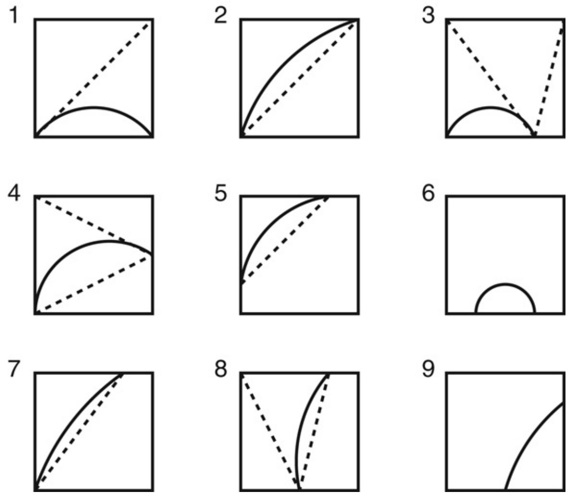

Fig. 4. Boundary cases.

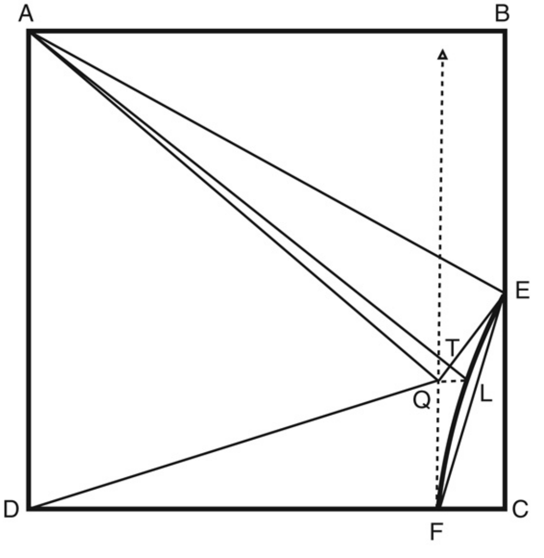

Fig. 5. Triangulating the pentagon.

of Q are shown in Fig. 4. The only case that needs to be resolved is 9, in which the intersection of the grid cell with the boundary results in a pentagon; see Fig. 5. This case is handled in Lemma 4.4. □

Lemma 4.4. *Suppose a grid cell intersects the boundary of Q to form a pentagon as in Fig. 4, Case 9. Then it is possible either to triangulate the pentagon with nonobtuse triangles or to refine the pentagonal region by adding one more Steiner point to the arc of B contained in the pentagon and then triangulate the resulting region with nonobtuse triangles.*

Proof. Let $ABCD$ be the grid cell, and EF be the part of the boundary of Q that is intersected by $ABCD$. As in Fig. 5, we have arc EF intersecting the grid cell $ABCD$. We need to obtain a nonobtuse triangulation of the pentagonal region $ABEFD$ or a region bounded after taking a refinement of the arc EF by adding points in its interior. However, we cannot add points in the interiors of the segments BE, AB, DA, FD.

The triangles $\triangle ABE$ and $\triangle ADF$ are right and hence nonobtuse. If $\triangle AEF$ is nonobtuse, then we are done. Consider the case when $\triangle AEF$ is obtuse. Without loss of generality, let $\angle AEF$ be greater than $\frac{\pi}{2}$. At F, we construct a ray perpendicular to DC. Since the points of intersection of the diameters in the X and Y directions with the boundaries of the circles were added as Steiner points, we note that the constructed ray at F lies in the exterior of the arc EF.

Let L be a (moving) point on the arc EF, and let Q be the intersection of the line through L parallel to DC and the line through F parallel to DA. Let T be the intersection of the lines AL and QE. We assert that when L moves along the arc FE from F to E, then, at some stage, the angle $\angle ATE$ becomes the right angle. If $L = F$, then $Q = T = F$ and $\angle ATE = \angle AFE < \frac{\pi}{2}$. When $L \to E$, then $AL \to AE$, and QL becomes parallel to DC. In this case, the angle α between AL and QE is clearly obtuse. Hence, at some intermediate point, α must equal $\frac{\pi}{2}$. Fix L, Q, T at this stage.

We introduce the vertex L on the arc to get a refinement of the arc. We now show how to obtain a nonobtuse triangulation of the hexagonal region $ABELFD$ without introducing any Steiner points on its boundary. It is clear that $\triangle ABE$, $\triangle ATE$, $\triangle ETL$, $\triangle ATQ$, $\triangle TQL$, $\triangle QLF$, and $\triangle DQF$ are all right triangles, and hence nonobtuse. Hence it remains to show that $\triangle AQD$ is nonobtuse. By assumption, $\angle AEF > \frac{\pi}{2}$. So, $\angle BAE > \angle FEC$. Hence, $\frac{|BE|}{|AB|} > \frac{|FC|}{|EC|}$. Hence,

$$|FC| < \frac{|BE| \cdot |EC|}{|AB|} \leq \frac{(\frac{1}{2}(|BE| + |EC|))^2}{|AB|} = \frac{|BC|^2}{4|AB|}.$$

Now, by Lemma 4.1, we may assume that the aspect ratio of the rectangular grid is 1.2. Then, $\frac{1}{1.2} \leq \frac{|BC|}{|AB|} \leq 1.2$, and we can conclude that $|FC| < \frac{1.2|BC|}{4} = 0.3|BC| = 0.3|AD|$. Since $\frac{1}{1.2} < \frac{|DF|+|FC|}{|AD|} < \frac{|FD|}{|AD|} + 0.3$, we have $|FD| > 0.533|AD|$. Thus the entire ray FQ lies outside the circle with diameter AD, and hence so does Q. Therefore $\angle AQD < \pi/2$. But $\angle ADQ$ and $\angle DAQ$ are less than $\pi/2$. Therefore $\triangle AQD$ is nonobtuse. The proof of Lemma 4.4 is now complete. □

Together with triangulations of the other boundary cases as shown in Fig. 4, Lemma 4.4 completes the proof of the existence of a nonobtuse triangulation of the faces of $\mathcal{P}$ with common Steiner points on the edges separating two adjacent faces. Hence, we get a nonobtuse triangulation of $\mathcal{P}$, and thus have proved Theorem 2.1.

As an immediate corollary of our proof of Theorem 2.1, we obtain the following result, which doesn't seem to have been proved previously.

Theorem 4.5. *Every polyhedral surface that has been divided into polygonal regions can be subtriangulated into nonobtuse triangles respecting the boundaries of the polygonal regions.*

Proof. We can view the polygonal regions as the "faces" of the polyhedron, and then apply Theorem 2.1. Thus we obtain a nonobtuse triangulation of the individual faces with the same set of Steiner points on the common edge between any two adjacent faces. Hence the faces can be put together to give a nonobtuse triangulation of the entire surface. □

Once we obtain a nonobtuse triangulation of a polyhedral surface, we can also employ the techniques of Maehara [6] to extend it to give an acute triangulation of the surface and obtain Theorem 2.2. We skip the details here; they are found in Maehara [6]. Once we have a nonobtuse triangulation of the polyhedral surface, Maehara's method of proof very naturally extends to the case of a polyhedral surface. However, in the construction of an acute triangulation, the triangles might lie across edges separating two faces, and hence, might get folded and no longer be planar. Therefore, we only get an acute triangulation using geodesic acute triangles. It is not yet known whether one can get an acute subtriangulation of a polygon that has been subdivided into polygonal regions.

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