Magnetohydrodynamic Stability in a Levitated Dipole

D. T. Garnier,* J. Kesner, and M. E. Mauel*

Plasma Science and Fusion Center,
Massachusetts Institute of Technology,
Cambridge, MA 02139

February 23, 1999

Abstract

Plasma confined by a magnetic dipole is stabilized, at low beta, by magnetic compressibility. The ideal magnetohydrodynamic (MHD) requirements for stability against interchange and high-n ballooning modes are derived at arbitrary beta for a fusion grade laboratory plasma confined by a levitated dipole. A high beta MHD equilibrium is found numerically with a pressure profile near marginal stability for interchange modes, a peak local beta of $\beta \sim 10$, and volume averaged beta of $\overline{\beta} \sim 0.5$. This equilibrium is demonstrated to be ballooning stable on all field lines.

The dipole magnetic field is the simplest and most common magnetic field configuration in the universe. It is the magnetic far-field of a single, circular current loop, and it represents the dominate structure of the middle magnetospheres of magnetized planets and neutron stars. The use of a dipole magnetic field generated by a levitated ring to confine a hot plasma for fusion power generation was first considered by Akira Hasegawa [1, 2]. In order to eliminate losses along the field lines Hasegawa suggested the use of a levitated ring. He postulated that if a hot plasma having pressure profiles similar to those observed in nature could be confined by a laboratory dipole magnetic field, this plasma might also be immune to anomalous (outward) transport of plasma energy and particles.

The dipole confinement concept is based on the idea of generating pressure profiles near marginal stability for low-frequency magnetic and electrostatic fluctuations. From ideal MHD marginal stability results when the pressure profile satisfies the adiabaticity condition [3, 4], $\delta(pV^\gamma) = 0$, where p is plasma pressure, V is the flux tube volume ($V \equiv \oint d\ell/B$) and $\gamma = 5/3$. This condition leads to dipole pressure profiles that scale with radius as $r^{-20/3}$, similar to energetic particle pressure profiles observed in the Earth's magnetosphere. This condition limits the peak pressure *i.e.* $p_{peak} \leq p_{edge}(V_{edge}/V_{peak})^\gamma$ and a

relatively low pressure at the plasma edge requires a large flux expansion, i.e. $V_{edge}/V_{peak} \gg 1$.

At low beta the magnetic field in the plasma will closely approximate the vacuum field. At finite beta the equilibrium field can be determined from a solution of the Grad-Shafranov equation. At sufficiently high beta the stability of MHD ballooning modes needs to be examined.

The high beta MHD stability limit has been examined by several authors [5, 6, 7] in the magnetospheric context. For the magnetospheric problem it is necessary to consider rotation, anisotropy ($p_\perp \neq p_\parallel$) as well as the boundary condition where the field lines enter the conducting regions near the planetary poles. Chan *et al.*[7] utilize a low beta equilibrium expansion in the ballooning calculation and their results are suspect at high beta[8].

As a laboratory approach to controlled fusion a circular magnet that is located within a plasma will generate a dipole configuration. To avoid losses on supports the ring needs to be superconducting and be magnetically levitated within the vacuum chamber. Since a large flux expansion is necessary to obtain a fusion grade plasma (for fixed edge plasma parameters) the configuration tends to require a small coil that is levitated within a relatively large vacuum chamber. An initial test of this concept is embodied in the Levitated Dipole Experiment[9] (LDX)

---

*Also at: Department of Applied Physics, Columbia University, New York, NY 10027
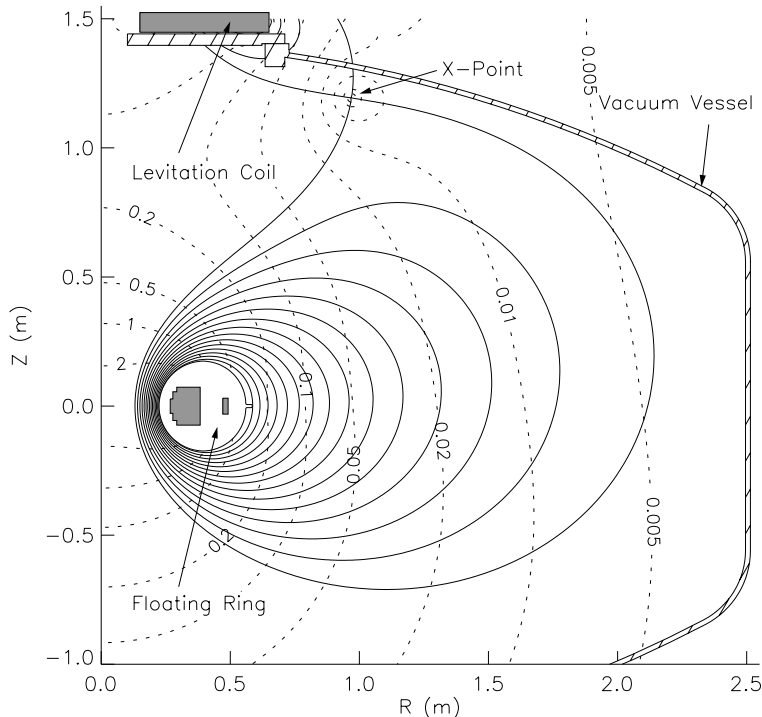
Figure 1: Vacuum field in LDX.

which is being built jointly by Columbia University and MIT. An important goal of this experiment is to study the beta limits of the dipole configuration.

In this article we consider the beta limits imposed by ideal MHD in a dipole configuration as formed by a floating ring. We will first find the numerical solution to the finite-β equilibrium. We will then use the equilibrium solution to evaluate interchange and ballooning stability. We will show that when an equilibrium is obtained for a plasma profile that is stable to interchange modes it will also be stable to high-n ballooning modes. The reason for the unusually high ballooning limit is that, in order to minimize the stabilizing plasma compressibility, the most unstable modes will be up-down anti-symmetric. As a result the ballooning eigenmodes are forced to have a node at the point of maximum beta (near the outer midplane of the configuration) and the region of strong bending is forced to occur at relatively low beta.

We consider first the MHD equilibrium of a fusion grade laboratory plasma confined by a levitated superconducting magnet. Since the currents in the floating ring and external coils are toroidal and we assume that no poloidal currents are driven in the plasma, the magnetic field will be entirely poloidal. We can then write a somewhat simplified Ampere's

law as[10]

$$\mu_0 \vec{J} = -\frac{1}{R}\Delta^*\psi\vec{e}_\phi \qquad (1)$$

where ψ the flux function, $\psi = (1/2\pi)\int \vec{B}\cdot d\vec{A}$, $\vec{B} = \nabla\psi \times \vec{e}_\phi/R$, the elliptic operator Δ^*, $\Delta^*\psi \equiv R^2\nabla\cdot(\nabla\psi/R^2)$, and R, Z, ϕ are cylindrical coordinates.

Since plasma field lines pass through the open bore of the magnet the field lines are closed. Unlike magnetospheric plasmas particles are confined on closed field lines and we can assume an isotropic plasma pressure. We will also neglect gravitational and rotational forces that are significant in space plasmas (e.g. in the Jovian magnetosphere rotational pressure is larger than the kinetic pressure). Force balance then yields a reduced Grad-Shafranov equation

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} \qquad (2)$$

where the plasma pressure, p, is a flux function, *i.e.* $p = p(\psi)$. Unless otherwise specified we will define beta as the local beta throughout this article, i.e.

$$\beta \equiv 2\mu_0 \frac{p(\psi)}{B^2(R,Z)} \qquad (3)$$

The vacuum field, which closely resembles the low beta equilibrium field is shown in Fig. 1. Notice that
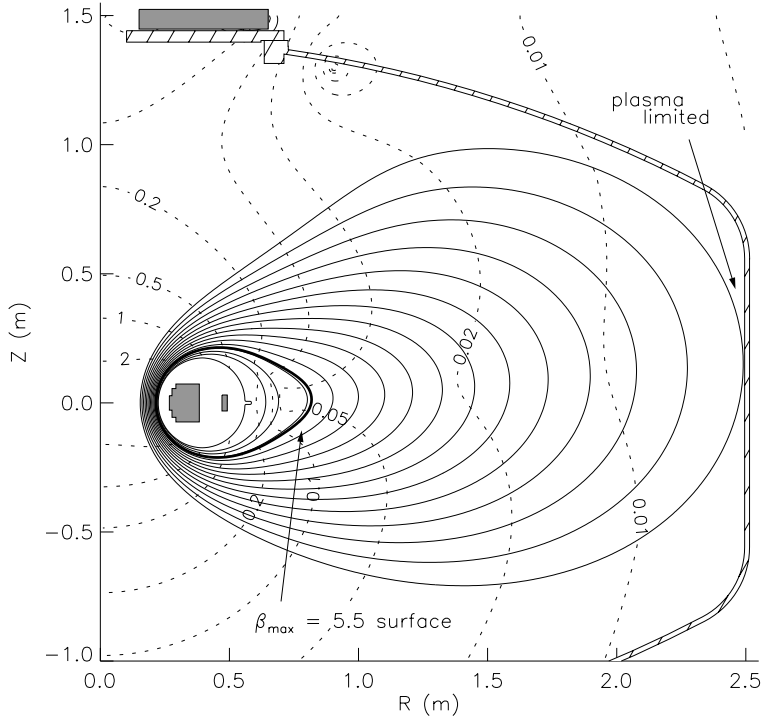
2

Figure 2: High β equilibrium ($\beta_{max} = 10$) solution in the LDX geometry.

an x-point and a separatrix can form due to the presence of a coil that is attractive to and supports the weight of the floating coil.

We have solved the equilibrium equation numerically using a modified version of the TokaMac code [11]. Similar to the EFIT [12] equilibrium fitting code, the code finds a free boundary solution to Eq. (2) by iteratively updating the grid boundary conditions and then solving the appropriate fixed boundary problem for ψ at each iteration using a multi grid relaxation method. On each iteration, the Dirichelet boundary condition is computed from Green's functions for each coil and the plasma current is determined from the right hand side of Eq. (2).

For a fixed edge pressure the highest beta value is obtained for a pressure profile that is marginally stable to interchange modes. Therefore, we will search for high beta equilibria that are obtained as follows: We specify a zero pressure at the surface of the floating ring and a fixed value, p_{edge} at the outer flux surface (defined to be either the vacuum chamber wall or a magnetic separatrix, whichever is located closer in). We also specify the spacial location of the pressure peak. Between the inner zero pressure surface and the pressure peak we specify that the pressure rise in a cosinusoidal fashion, i.e. $p(\psi) = 0.5(1 - cos[2\pi(\psi - \psi_0)/(\psi_{peak} - \psi_0)])$. This

choice provides a zero gradient at the inner and the peak pressure locations. Between the location of the pressure peak and the outer flux surface we specify that the pressure rises according to the finite beta interchange constraint $p \propto 1/V^\gamma$. This profile is applied in a iterative fashion. The pressure profile will therefore be marginally stable to interchange modes between the pressure peak and the outer flux tube.

We demonstrate high-β equilibrium by choosing a pressure profile that is defined by an edge pressure of 7 Pa and the peak pressure location of R=0.75 m. The floating coil extends between R=0.2235 m and R=0.585 m and the pressure is set to zero on the field line that passes through the R=0.2235 m location. The resulting equilibrium, shown in Fig. 2 has a peak local beta of $\beta_{max} \simeq 10$. The high-β equilibrium will be used to evaluate the field line curvature as is required for the solution of the ballooning equation [Eq. (11)] which is discussed below. Notice that the equilibrium has moved out radially and the plasma is now limited on an outer limiter and not by the magnetic separatrix.

The stability of a dipole to ideal MHD interchange and ballooning modes can be evaluated from the MHD energy principle. We will consider the confinement of a high-β plasma. For a levitated dipole the field is poloidal and the curvature is in the $\nabla\psi$

3

direction, i.e. $\vec{\kappa} = \kappa_\psi \nabla \psi$ with $\vec{\kappa} = \vec{b} \cdot \nabla \vec{b}$. The energy principle gives:[10]

$$\delta W_F = \frac{1}{2\mu_0} \int_p d^3 r \left[|\vec{Q}_\perp|^2 + B^2|\nabla \cdot \vec{\xi}_\perp + 2\vec{\xi}_\perp \cdot \vec{\kappa}|^2 + \gamma \mu_0 p \langle \nabla \cdot \vec{\xi}_\perp \rangle^2 - (2\mu_0 \vec{\xi}_\perp \cdot \nabla p)(\vec{\kappa} \cdot \vec{\xi}_\perp^*) \right]. \tag{4}$$

with the flux average defined as $\langle c \rangle = (\oint c \; d\ell/B)/(\oint d\ell/B)$. In Eq. (4) $\vec{Q} = \nabla \times (\vec{\xi} \times \vec{B})$ and $\vec{\xi}_\perp$ is the amplitude of the perpendicular displacement. The first term in Eq. (4) represents the energy associated with field-line-bending. The second term magnetic compression and the third term is plasma compression. The plasma compression term only appears for a closed field line system. The last term is the curvature (instability) drive. There is no kink drive term because only diamagnetic current is present in the equilibrium, i.e. $j_\parallel = 0$. We can minimize the sum of the stabilizing plasma + magnetic compression terms to obtain [13]

$$B^2|\nabla \cdot \vec{\xi}_\perp + 2\vec{\xi}_\perp \cdot \vec{\kappa}|^2 + \gamma \mu_0 p \langle \nabla \cdot \vec{\xi}_\perp \rangle^2 \rightarrow \frac{4\gamma \mu_0 p \langle \vec{\xi}_\perp \cdot \vec{\kappa} \rangle^2}{1 + \gamma \langle \beta \rangle / 2}. \tag{5}$$

Consider highly localized modes. We approximate $\vec{\xi}_\perp = \vec{\eta}_\perp e^{iS}$, where $\nabla S \equiv \vec{k}_\perp$ and $\vec{B} \cdot \nabla S = 0$. Following Freidberg[10], we can obtain to lowest order (in $1/k_\perp a$), $\vec{\eta}_{\perp 0} = (X/B)\vec{b} \times \vec{k}_\perp$. The second order contribution to δW then becomes

$$\delta W_2 = \frac{1}{2\mu_0} \int d\psi d\phi W \tag{6}$$

with

$$W(\psi, \phi, \zeta) = \int J d\zeta \left[\frac{k_\perp^2}{J^2 B^2} \left(\frac{\partial X}{\partial \zeta} \right)^2 - 2\mu_0 \left(\frac{\partial S}{\partial \phi} \right)^2 \frac{dp}{d\psi} \kappa_\psi X^2 + 4\gamma \mu_0 p \left(\frac{\partial S}{\partial \phi} \right)^2 \frac{\langle \kappa_\psi X \rangle^2}{1 + \gamma \langle \beta \rangle / 2} \right]. \tag{7}$$

In the region between the pressure peak and the wall of the vacuum chamber the curvature drive [second term in Eq. (7)] is always destabilizing since $p_\psi < 0$ and $\kappa_\psi < 0$. The compressibility term (the third term in Eq. (7)) is always stabilizing provided $\langle X \kappa_\psi \rangle \neq 0$ although the stabilization becomes inefficient at sufficiently high $\langle \beta \rangle$. Since $k_\perp$ only occurs in the stabilizing field-line-bending term we can further minimize Eq. (7) by taking $\vec{k}_\perp = k_\phi \nabla \phi + k_\psi \nabla \psi \approx (1/R)\partial S/\partial \phi = n/R$. We can now minimize Eq. (7) to obtain the balloon equation (at marginal stability):

$$B\frac{d}{d\ell}\frac{1}{BR^2}\frac{dX}{d\ell} + 2\mu_0 \kappa_\psi p_\psi X - 4\gamma \mu_0 p \kappa_\psi \frac{\langle X \kappa_\psi \rangle}{1 + \gamma \langle \beta \rangle / 2} = 0 \tag{8}$$

with $p_\psi \equiv dp/d\psi$. Equation (8) together with the Grad-Shafranov equation [Eq. (2)] determine the marginal stability of both high-n interchange and ballooning modes. The continuity of the eigenfunction and its derivative provide the boundary condition for integrating around a closed field line.

Consider first the stability of interchange modes, i.e. modes with $X = constant$. Equation (8) then indicates stability for

$$\frac{2\gamma p \langle \kappa_\psi \rangle}{1 + \gamma \langle \beta \rangle / 2} \geq p_\psi \tag{9}$$

We can simplify Eq. (8) for ballooning modes by considering the flux tube average to obtain a constraint on the solution. The flux tube average of Eq. (8) is

$$\langle X \kappa_\psi \rangle \left[\frac{2\gamma p \langle \kappa_\psi \rangle}{1 + \gamma \langle \beta \rangle / 2} - p_\psi \right] = 0. \tag{10}$$

Equation (10) indicates that when interchange modes are stable the solution to Eq. (8) requires that $\langle X \kappa_\psi \rangle = 0$. Thus we can simplify Eq. (8) for ballooning modes:

$$B\frac{d}{d\ell}\frac{1}{BR^2}\frac{dX}{d\ell} + 2\alpha \mu_0 \kappa_\psi p_\psi X = 0. \tag{11}$$

We have added a multiplicative constant, α, to the second term in Eq. (11) and therefore this equation reduces to the balloon equation when $\alpha = 1$. The condition $\langle X \kappa_\psi \rangle = 0$ is satisfied by both even and odd modes but the odd modes are found to be the most unstable.

Once the equilibrium is solved we can determine κ_ψ and Eq. (11) can be easily solved with a shooting technique. In the solution we set X=0 at the minimum magnetic field point and integrate around the (closed) field line of length L. We vary α until X(L)=X(0). Further, since a real (experimental) configuration is not up-down symmetric we need to vary the X=0 starting point location until we find $X'(0) = X'(L)$. The X=0 point is found to be close
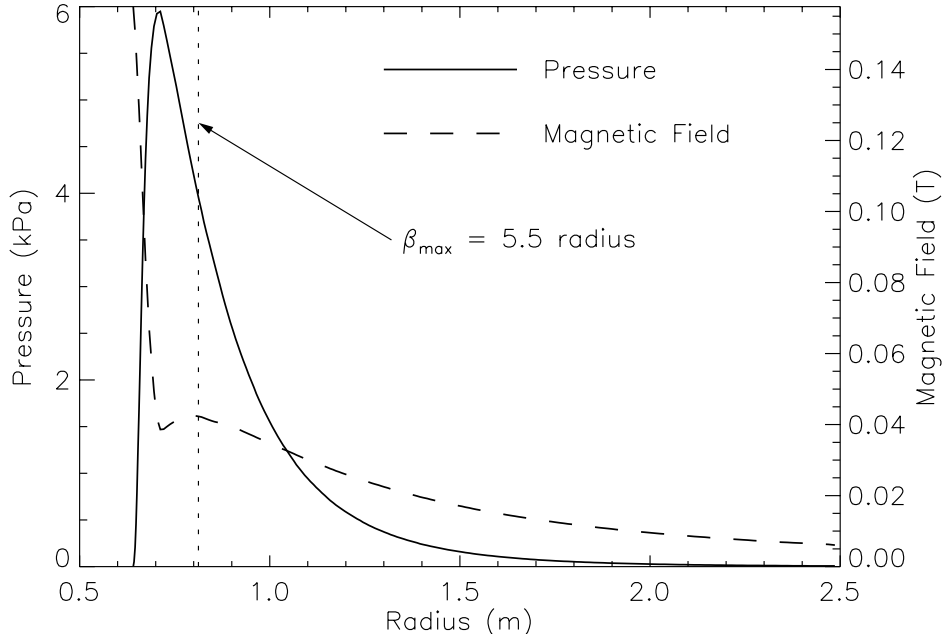
Figure 3: Pressure profile and midplane magnetic field for an equilibrium with ($\beta_{max} = 10$) solution in the LDX geometry.

the the magnetic field minimum at the outer midplane. The value of α is a measure of how stable (or unstable) the field line is to ballooning with $\alpha > 1$ indicating stability. This method of stability analysis is similar to the Newcomb theorem[14].

Consider the stability of a high β equilibrium. To obtain the maximum peak pressure (for a given edge pressure) we will choose the pressure profile $p(\psi)$ to satisfy marginal interchange stability. Figure 3 displays the pressure profile for the high β equilibrium having a maximum local β, $\beta_{max} > 10$, flux average β, $\langle\beta\rangle_{max} = 3.9$ and volume average β, $\overline{\beta} \sim 0.50$. In the stability calculation we considered 62 flux surfaces. The inner 5 surfaces are stable since $p_\psi > 0$ while the outer 45 surfaces have $p_\psi < 0$. The magnetic field on the outer midplane is displayed in Fig. 3. All surfaces are found to be stable to ballooning modes.

The marginally stable eigenmode can be determined as the point where α has been adjusted to satisfy the boundary conditions. The resulting eigenmodes are not the true MHD eigenmode since Eq. (11) is only equivalent to the balloon equation when $\alpha = 1$. For $\alpha > 1$ the pressure gradient in the instability drive term has been effectively increased without re calculating the equilibrium. In a typical calculation we find that, for the odd mode, $\alpha = 70$, while $\alpha = 119$ for the even mode, confirming our in-

tuition the the even mode is more stable. The odd and even eigenmodes as well as the along-the-field-line β profile are shown in Fig. 4. Considering the odd mode we observe that the mode amplitude is small in the region of high beta (near $s \sim 0$) and the field line must bend in a region where beta is reduced to $< 10\%$ of its peak value. The even mode is always seen to be more stable because it is forced to bend (and change signs) in the high beta region and it therefore requires a larger bending energy. In conclusion we have shown that the equilibrium and high-n ballooning equations become relatively simple in a dipole configuration. The steepest pressure profile that is consistent with interchange stability provides the maximum peak-local-beta for a given radial extent of the confined plasma. An evaluation of equilibrium and stability indicates that equilibria can be found at very high β and we have shown a case that was solved for $\beta \sim 10$. We have also found that high-n ballooning modes remain stable in the high β regime.

It has been pointed out that in a similar configuration a separatrix can be formed with x-points on the magnetic axis such that $\oint dl/B \rightarrow \infty$. In this configuration the edge pressure gradient can become very large and ballooning modes will limit β in the vicinity of the separatrix [15, 16]. We have not examined this possibility.
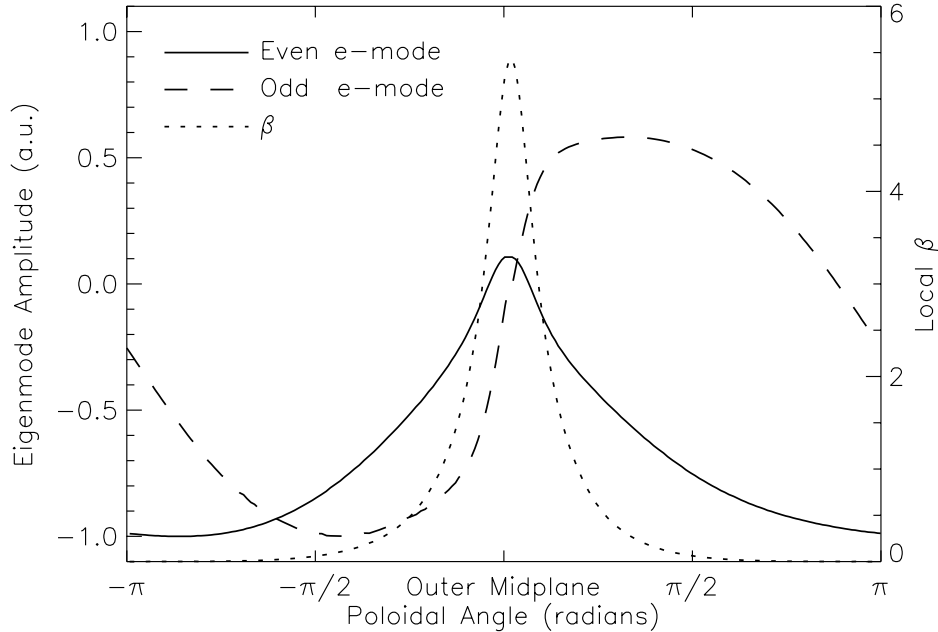
5

Figure 4: The approximate eigenmodes for the $\beta = 5$ field line (in the high-β equilibrium). The beta profile along-the-field-line is also shown.

Acknowledgements

References

[1] A. Hasegawa, Comm. Pl. Phys. & Cont. Fus. **1**, 147 (1987).

[2] A. Hasegawa, L. Chen and M. Mauel, Nucl. Fusion **30**, 2405 (1990).

[3] M.N. Rosenbluth and C.L. Longmire, Ann. Phys. **1**, 120 (1957).

[4] I.B. Bernstein, E. Frieman, M. Kruskal, R. Kulsrud, Proc. R. Soc. Lond. A **244** 17 (1958).

[5] E. Hameiri, P. Laurence, M. Mond, J. Geophys. Res. **96** 1513 (1991).

[6] C.Z. Cheng and Q. Qian, J. Geophys. Res. **99** 11193 (1994).

[7] A. Chan, M. Xia and L. Chen, J. Geophys. Res. **99** 17351 (1994).

[8] S. Krasheninnikov, P. Catto and R.D. Hazeltine, Phys. Rev. Lett. **82**, 2689 (1999).

[9] J. Kesner, L. Bromberg, D. Garnier, M. Mauel, 17th IAEA Fusion Energy Conf, Paper IAEA-F1-CN-69-ICP/09 Yokohama, Japan (1998).

[10] J.P. Freidberg, *Ideal Magnetohydrodynamics*, (Plenum Press, New York, 1987) pp. 110, 259, 372.

[11] Interpretive equilibrium code written for tokamak geometry was obtained from M. Mauel.

[12] L.L. Lao, H. StJohn, R.D. Stambaugh, A.G. Kellman, W. Pfeiffer, Nucl. Fusion **25** 1611 (1985).

[13] S. Krasheninnikov, private communication, 1999.

[14] W.A. Newcomb, Ann. Phys. **10** 232 (1960).

[15] B. Lehnert, Nature **181** 331 (1958), J. Nucl. Energy (Part J) 40 (1959).

[16] T. Hellsten, Physica Scripta, **9** 313 (1976).